SCHUBERT CALCULUS

JOHN NAUGHTON

Abstract. In this paper we motivate and develop the basic machinery of Schubert calculus as a method for counting intersections of subspaces, an important problem historically in enumerative geometry. After introducing basic objects of study such as Schubert cells and Schubert varieties in the Grassmannian - and showing how intersections of these varieties can express the motivating enumerative questions - we discuss how properties of these objects can be interpereted algebraically using cohomology. We then discuss symmetric function theory in order to establish an isomorphism from the cohomology ring of the Grassmannian to a quotient of the ring of symmetric functions. Finally, we discuss the Littlewood-Richardson rule for multiplying arbitrary Schur polynomials, which via the aformentioned isomorphism allows us to compute in the cohomology ring - thus solving our original enumerative geometry problems.

Contents

1. Introduction

Schubert calculus was first developed in 1879 in Hermann Schubert's book "Kalkül die abzahlenden Geometrie" [9] (roughly, Calculus of Enumerative Geometry) as a means for investigating the number of subspaces of a particular dimension intersecting certain given subspaces in a given dimension. We will focus on the following questions, in particular the second:

(1) How many planes in $\mathbb{P}^3$ contain a given line and point?
(2) How many lines intersect four given lines in $\mathbb{P}^3$?
(3) How many $k-1$-dimensional subspaces of $\mathbb{P}^{n-1}$ intersect $k(n-k)$ given subspaces of dimension $n-k-1$ nontrivially?

Date: August 14, 2021.

The first question can be answered quite easily. If the point is not contained in the line, the plane is unique. If, however, the point lies on the line then there are infinitely many such planes.

The second question is more difficult, and Schubert answered it by finding a particular orientation of the four lines such that the solution was two [8]. In particular, suppose l_1 and l_2 intersect at P, l_3 and l_4 intersect in a point Q, none of the other lines intersect, and the planes defined by each pair of intersecting lines are not parallel. In this orientation the planes defined by l_1, l_2 and l_3, l_4 intersect in a line α which intersects all four lines. The line β through P and Q also intersects all four lines, and no other line does so. Schubert then appealed to the "Prinzip des Erhaltung der Anzahl" - the principle of conservation of number. Essentially, this principle states that if there is a configuration of the fixed subspaces for which the solution set is finite, then the answer is that same constant for every configuration which gives a finite solution. However, we can clearly see that there are degenerate cases in which the answer is zero or infinite. For example if the four lines are mutually parallel and not all contained in a plane then no line can intersect all four. On the other hand, if the lines coincide, there are infinitely many solutions. To solve this problem, Schubert argued that the properties would hold for lines in sufficiently "general" position.

Hilbert's 15^{th} problem asked to put Schubert's method on a more rigorous grounding, which led to what we know today as Schubert calculus. In this modern interpretation, we consider a collection of varieties, each consisting of all subspaces that intersect a given subspace in a specified manner. For example, in (2) we can let X_i denote the space of lines intersecting l_i. The intersection of these varieties then gives the solution set to our enumerative question. These varieties, known as Schubert varieties, can be understood algebraically in the cohomology ring of a space called the Grassmannian. It turns out we can view this ring as a quotient of the ring of symmetric functions. Therefore, if we understand the ring of symmetric functions, we will understand the cohomology ring, and thus be able to solve our enumerative questions.

This paper was inspired by Maria Gillespie's paper [5], and is intended to provide clarification.

2. Translating the Problem: The Grassmannian

As we mentioned in the introduction, it can be difficult to solve the type of intersection problems which we are interested in geometrically. Therefore, it will be beneficial to translate our geometric problems to an algebraic setting. Since we are interested in understanding how certain k-dimensional subspaces of n-dimensional space interact, the *Grassmannian* is a good place to begin our investigation.

Definition 2.1. The Grassmannian $Gr(n,k)$ is the set of k-dimensional subspaces of an n-dimensional vector space.

Remark 2.2. In this paper we will always work over $\mathbb{C}$.

Each point in the Grassmannian can be represented as the span of some full rank $k \times n$ matrix. In particular, since the span of a matrix is unchanged under elementary row operations, every point can be seen as the row span of a unique full-rank $k \times n$ matrix in reduced row echelon form.

Example 2.3. The following matrix represents a point in $Gr(12,5)$.

$$V = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & 0 & * & * & 0 & * & * \\ 0 & 1 & * & 0 & 0 & * & 0 & * & * & 0 & * & * \end{bmatrix}$$

Where the $*$ entries are independent complex numbers.

Since our reason for introducing the Grassmannian is to understand how many k-dimensional subspaces intersect some collection of fixed subspaces in given dimensions, we should consider the locus of k-planes that meet a given subspace in a given dimension. To construct this object, called a Schubert cell, we need a way to talk about how subspaces V intersect a given basis.

Definition 2.4. A <u>partition</u> $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a weakly decreasing sequence of nonnegative integers. We denote the size of the partition by $|\lambda| = \Sigma_i \lambda_i$.

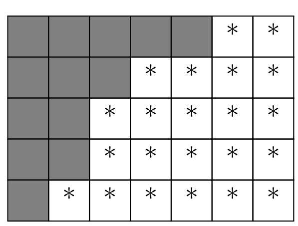
We can associate a partition to each point $V \in Gr(n,k)$ by setting λ_i to be the distance from the pivot (that is, the first nonzero entry from the left) in row i of the reduced echelon matrix representing V to the edge of a $k \times k$ staircase cut from the upper left corner of the matrix.

In the example above we find $\lambda = (5,3,2,2,1)$.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & 0 & * & * & 0 & * & * \\ 0 & 1 & * & 0 & 0 & * & 0 & * & * & 0 & * & * \end{bmatrix}$$

Definition 2.5. The <u>Young Diagram</u> of a partition λ is the left-aligned $k \times (n-k)$ grid of boxes in which row i has λ_i boxes. Equivalently, it is the complement of the right-aligned grid of starred entries in the $k \times (n-k)$ grid. Because any partition must fit within this $k \times (n-k)$ grid, we will call this the <u>ambient rectangle</u> and denote it B.

In the above example we have a 5×12 matrix with partition $\lambda = (5,3,2,2,1)$. Therefore, the associated Young Diagram will be contained in a 5×7 grid, where row 1 contains 5 shaded boxes, row two contains 3 shaded boxes, rows 3 and 4 each contain 2 shaded boxes and row 5 contains one shaded box. Equivalently, we see that this is the complement in the ambient rectangle of the right aligned grid of starred entries from the example matrix above. So, the Young Diagram associated to our above example is:



The Schubert cell Ω_λ° is the set of points in $Gr(n,k)$ whose reduced matrix has partition λ. Since each $*$ can be any complex value, we have $\Omega_\lambda^\circ \cong \mathbb{C}^{k(n-k)-|\lambda|}$. Explicitily, we have the following definition:

Definition 2.6. The standard Scubert cell corresponding to a partition λ fitting inside the ambient rectangle is:

$$\Omega_\lambda^\circ = \{V \in Gr(n,k) | \dim(V \cap \langle e_1, \ldots, e_r\rangle) = i \text{ for } n-k+i-\lambda_i \leq r \leq n-k+i-\lambda_{i+1} \text{ for all } i\}.$$

Notice that the position $n-k+i-\lambda_i$ is the location of the pivot in row i, counted from the right.[1] So, this condition says that the dimension of the intersection of V with the span of the standard basis vectors $\langle e_1, \ldots, e_r\rangle$ will be constantly equal to i from the time that e_r reaches the i^{th} pivot column, until it reaches the $(i+1)^{st}$ pivot column.

The Grassmannian can be viewed as a projective variety via the Plücker embedding. To see this, consider an ordering of the k-element subsets of $\{1, 2, \ldots, n\}$. Given a point $V \in Gr(n,k)$ represented by a reduced echelon matrix, let x_S be the determinant of the $k \times k$ minor defined by taking the columns in the subset S. Because V is full rank, not all of these determinants will be zero, and row reduction only changes the determinants up to a constant, so this process determines a point in homogeneous coordinates in $\mathbb{P}^{\binom{n}{k}-1}$. The image in $\mathbb{P}^{\binom{n}{k}-1}$ is a projective variety [4] cut out by the set of polynomials

$$\sum_{l=1}^{k+1} (-1)^l x_{(i_1, \ldots i_{k-1} j_l)} x_{j_1, \ldots \hat{j_l}, \ldots, j_{k+1}}.$$

Example 2.7. In $Gr(4,2)$ the matrix

$$\begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & -3 & 0 & 3 \end{bmatrix}$$

will embed via the Plücker map to the point in homogeneous coordinates given by the determinants of all 2×2 minors. We pick an arbitrary ordering of these minors, say $(x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34})$. The Plücker embedding then gives the point $(0 : -1 : -2 : 3 : 6 : 3)$. In this case the set of defining polynomials is a single polynomial in the determinants of the minors, namely

$$x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = (0)(3) - (-1)(6) + (-2)(3) = 0.$$

We can now define the standard Schubert varieties to be the closed subvarieties of the Grassmannian. Explicitly, we have the following definintion.

Definition 2.8. The standard Schubert variety corresponding to a partition λ is given by

$$\Omega_\lambda = \{V \in Gr(n,k) | \dim(V \cap \langle e_1, \ldots, e_{n-k+i-\lambda_i}\rangle) \geq i \text{ for all } i\}.$$

Two special cases occur: First, when the partition is empty (i.e. $\lambda = (0, 0, \ldots, 0)$) and second, when the partition occupies the entire ambient rectangle (i.e. $\lambda = (n-k, n-k, \ldots, n-k)$). In the former case, the dimension condition that $\dim(V \cap \langle e_1, \ldots, e_{n-k+i}\rangle) \geq i$ is trivially fulfilled since V is k-dimensional and $k+(n-k+i) = n+i$, and in $\mathbb{C}^n$ this guarantees the intersection has dimension at least i, so that

[1] We use the convention that $e_1 = (0, \ldots, 1)$ and e_{n-i+1} is the i^{th} standard unit vector. This is intended to make the subscripts less burdensome later on.

$\Omega_\emptyset = Gr(n,k)$. In the latter case we see Ω_B is a single point in the Grassmannian, namely, the span of the first k basis vectors.

Working with the standard basis vectors, as we have done above, allows us to easily see what we mean when we say that all of the points in a Schubert cell intersect a given subspace (in this case the span of $\langle e_1, \dots, e_r\rangle$) in given dimensions. We may inspect the locations of pivot columns, and if two matrices have the same pivot positions, they are in the same cell. However, we do not want to restrict our attention to a particular basis. This motivates the following definition:

Definition 2.9. A *complete flag* F in $\mathbb{C}^n$ is a chain of subspaces

$$\{0\} = F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^n$$

in which the dimension of F_i is i.

Using flags, we can present a more general definition of Schubert cells and Schubert varieties which depends on our choice of flag rather than our choice of basis.

Definition 2.10. For a complete flag F and a partition λ, the associated Schubert cell is defined as

$$\begin{aligned}\Omega^\circ_\lambda(F) = \{V \in Gr(n,k) |\, &\dim(V \cap F_r) = i \\ &\text{for } n-k+i-\lambda_i \le r \le n-k+i-\lambda_{i+1} \text{ for all } i\}\end{aligned}$$

and similarly the Schubert variety with respect to F and λ is defined

$$\Omega_\lambda(F) = \{V \in Gr(n,k) |\, \dim(V \cap F_{n-k+i-\lambda_i}) \ge i \text{ for all } i\}.$$

Notice that if λ and μ are partitions with $\lambda_i \le \mu_i$ for all i, then $\Omega_\lambda \supset \Omega_\mu$. In fact, Ω_λ is the disjoint union of all Schubert cells Ω°_μ where $\mu_i \ge \lambda_i$ for all i.

Example 2.11 (Translating Question 2)**.** We can translate the question of how many lines intersect four given lines in $\mathbb{P}^3$ into the languge of Schubert calculus as follows. We first consider a flag F in the higher-dimensional affine space $\mathbb{C}^4$. The variety $\Omega_1(F) \subset Gr(4,2)$ consists of the two dimensional subspaces V such that $\dim(V \cap F_{2+i-\lambda_i}) \ge i$ for all i. Working out the subscripts, this is the set of V with $\dim(V \cap F_2) \ge 1$ and $\dim(V \cap F_4) \ge 2\}$. Notice the second condition is trivially fulfilled in $Gr(4,2)$, so $\Omega_1(F)$ is the set of planes in $\mathbb{C}^4$ intersecting the given plane F_2 in at least a line. We can now quotient $\mathbb{C}^4$ by scalar multiplication to obtain the set of lines in $\mathbb{P}^3$ intersecting a given line in at least a point.

Considering four flags F^1, F^2, F^3, F^4 in $\mathbb{C}^4$, Question 2 becomes equivalent to computing the intersection

$$\Omega_1(F^1) \cap \Omega_1(F^2) \cap \Omega_1(F^3) \cap \Omega_1(F^4).$$

Example 2.12 (Translating Question 3)**.** Our next motivating question asks how many $k-1$-dimensional subspaces of $\mathbb{P}^{n-1}$ intersect $k(n-k)$ fixed subspaces of dimension $n-k-1$ nontrivially. Once again we consider a complete flag F in one-dimension higher affine space $\mathbb{C}^n$. We want to find those subspaces V for which $\dim(V \cap F_{n-k}) \ge 1$. For this to be the case, we consider the partition $\lambda = (1, 0, \dots, 0)$. With this partition, we have

$$\begin{aligned}\Omega_\lambda(F) &= \{V \in Gr(n,k) |\, \dim(V \cap F_{n-k+i-\lambda_i}) \ge i \text{ for all } i\} \\ &= \{V \in Gr(n,k) |\, \dim(V \cap F_{n-k}) \ge 1 \text{ and } \dim(V \cap F_{n-k+i}) \ge i \text{ for all } i\}.\end{aligned}$$

Since V is k-dimensional and we are in $\mathbb{C}^n$, we find $\dim(V \cap F_{n-k+i}) \geq i$ trivially, as $k + (n-k+i) = n+i$. Therefore, $\Omega_1(F)$ is the set of k-planes intersecting a given $(n-k)$-plane in at least a line. Taking $k(n-k)$ generic flags, the set of k-planes intersecting $k(n-k)$ fixed $n-k$-planes in at least a line is

$$\Omega_1(F^1) \cap \cdots \cap \Omega_1(F^{k(n-k)}).$$

Again, we quotient by scalar multiplication to obtain the set of $(k-1)$-planes intersecting $k(n-k)$ fixed $(n-k-1)$-planes nontrivially in $\mathbb{P}^{n-1}$.

We will spend the remainder of the paper working to understand how to compute the size of these type of intersections. First, we have a brief discussion of some properties of flags.[2]

2.1. **A Digression: Flags.** The standard flag F is the complete flag for which F_i is the span of the first i standard basis vectors. Similarly, the opposite flag E is the complete flag for which E_i is the span of the last i standard basis vectors.

Definition 2.13. We call a pair of subspaces of $\mathbb{C}^n$ transverse if their intersection has the expected dimension. Explicitly,

$$\dim(V \cap W) = \max(0, \dim(V) + \dim(W) - n).$$

We call a pair of flags F and E transverse if F_i and E_j are transverse for all i and j.

There is actually a weaker condition under which two flags are transverse.

Proposition 2.14. *Two complete flags F and E in $\mathbb{C}^n$ are transverse if and only if $F_{n-i} \cap E_i = \{0\}$ for all i.*

Proof. In the forward direction suppose that F and E are transverse. Then, for all i and j, we have

$$\dim(F_j \cap E_i) = \max(0, j + i - n).$$

Setting $j = n - i$ we find that the dimension of this intersection is 0. Since $F_0 = E_0 = \{0\}$ the intersection $F_{n-i} \cap E_i$ is $\{0\}$.

In the other direction we induct on n. Indeed in the case $n = 1$ we have that $F_i \cap E_j = \{0\}$ for i, j not both 1, while if i and j are both 1, we see $F_1 \cap E_1 = \mathbb{C} \cap \mathbb{C} = \mathbb{C}$. This satisfies the dimension conditions in Definition 2.13.

For induction we assume:

(1) if $F_{n-i-1} \cap E_i = \{0\}$ for all i, then F and E are transverse
(2) $F_{n-i} \cap E_i = \{0\}$ for all i.

In particular $F_{n-1} \cap E_1 = \{0\}$ and $\mathbb{C}^n = F_n$, we see $F_n = F_{n-1} \oplus E_1$. Now we can quotient both flags by E_1, reducing F_n to F_{n-1} and reducing the dimension of each E_i by 1 to get a new pair of flags

$$E' = \{0 = E_1/E_1 \subset E_2/E_1 \subset \cdots \subset E_n/E_1\}$$
$$F' = \{0 = F_0 \subset F_1 \subset \cdots \subset F_{n-1}\}.$$

[2]For example, did you know that the flag of Ohio is the only state flag in the United States to be shaped as a pennant?

These new flags are transverse, since $F'_{n-i-1} \cap E'_i = F_{n-i-1} \cap E_{i+1}/E_1$. Since $F_{n-i-1} \cap E_{i+1} = \{0\}$ by assumption, we have $F'_{n-i-1} \cap E'_i = \{0\}$. By the inductive hypothesis F' and E' are transverse, so

$$\dim(F_i \cap E_j/E_1) = \max(0, i+j-1-(n-1)) = \max(0, i+j-n).$$

Now we will show that F and E are transverse. For any $i \neq n$ we know $E_1 \not\subset F_i$, so

$$\begin{aligned}\dim(F_i \cap E_j) &= \dim(F_i \cap E_j/E_1) \\ &= \max(0, i+j-n).\end{aligned}$$

On the other hand, if $i = n$ and $j \neq 0$ then the intersection should have dimension j. Indeed, $E_1 \subset F_n$, so here we have

$$\begin{aligned}\dim(F_n \cap E_j) &= 1 + \dim(F_{n-1} \cap E_j/E_1) \\ &= 1 + \max(0, n-1+j-n) \\ &= 1 + j - 1 = j.\end{aligned}$$

The case where $i = n$ and $j = 0$ is trivial, as the intersection is simply $\{0\}$. Therefore, for all i and j we have

$$\dim(F_i \cap E_j) = \max(0, i+j-n).$$

And so F and E are transverse. This completes the proof. $\square$

One reason that we like to work with flags, in addition to not being restricted to picking a basis, is to formalize Schubert's notion of "general" position. The complete flag variety $Fl(\mathbb{C}^n)$ is the collection of all complete flags in $\mathbb{C}^n$. It turns out that the matrices defining a complete flag are equivalent up to the action of the group B of upper triangular matrices. Further, $Fl(\mathbb{C}^n) \cong Gl_n(\mathbb{C})/B$ has the structure of an algebraic variety[1]. From this perspective we say that a property holds for a "general" collection of flags if it holds for all tuples of flags in some Zariski open dense subset of $Fl(\mathbb{C}) \times \cdots \times Fl(\mathbb{C})$.

As a final remark, we are justified in working with the standard and opposite flags when considering an arbitrary pair of transverse flags, because the general linear group GL_n acts transitively on complete flags. If F denotes the standard complete flag and V is any other complete flag, pick a basis for V so that $V_k = \langle v_1, \ldots, v_k \rangle$. Then the matrix $g \in GL_k$ whose columns are the vectors v_i will take F_k to V_k. Therefore, to understand the intersection of Schubert varieties with respect to transverse flags F' and E', we can do the computation using the standard and opposite flags and simple multiply the result by an appropriate $g \in GL_n$.

3. Adventures in Cohomology

The main result of this section is that the cohomology ring of the Grassmannian $H^*(Gr(n,k))$ has a $\mathbb{Z}$ basis given by the classes $\sigma_\lambda = [\Omega_\lambda]$ in the group $H^{2|\lambda|}(Gr(n,k))$, and that the product, called the *cup product*, of two classes of Schubert varieties in this ring corresponds to the class of the intersection of those varieties. This will allow us to restate our motivating questions in terms of products of classes in cohomology.

The Schubert cells, being aptly named, give a cell complex structure on the Grassmannian. We can define the 0-skeleton of $Gr(n,k)$ to be the zero dimensional Schubert variety $X^0 = \Omega^\circ_B$, where B is the ambient rectangle. Since we are working over $\mathbb{C}$ there are no odd-dimensional cells. We next have $X^2 = X^0 \cup \Omega^\circ_{\lambda^1}$ where $\lambda^1 = (n-k, \cdots, n-k, n-k-1)$ is obtained from B by removing the bottom right corner from the ambient rectangle. The closure of $\Omega^\circ_{\lambda^1}$ is $\Omega^\circ_B = X^0$, so the closure maps the boundary Ω°_B of X^2 to X^0.[3] Similarly, we define X^4 by attaching the two four cells obtained by removing the two corner squares from λ^1. Continuing in this manner, so that the $2m^{th}$ cell is formed by attaching the Schubert cells whose partition has size $|\lambda| = k(n-k) - m$, we obtain a CW decomposition of the Grassmannian into cells

$$X^0 \subset X^2 \subset \cdots \subset X^{2k(n-k)}.$$

Example 3.1. The Grassmannian $Gr(4,2)$ has a cellular decomposition

$$Gr(4,2) = \Omega^\circ_{(2,2)}(F) \cup \Omega^\circ_{(2,1)}(F) \cup \Omega^\circ_{2}(F) \cup \Omega^\circ_{(1,1)}(F) \cup \Omega^\circ_{1}(F) \cup \Omega^\circ_{\emptyset}(F),$$

where the zero-skeleton is the point $X_0 = \Omega^\circ_{(2,2)}$. We then attach the two-skeleton $X^2 = \Omega^\circ_{(2,1)}$ to X^0 via the closure map. We see that a point in X^2 has the form

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & * & 0 \end{bmatrix}.$$

The unique point in the boundary of this cell is

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

which is precisely X_0. Similarly, X^4 is formed by attaching $\Omega^\circ_{(1,1)}$ and Ω°_2 with their boundaries mapping to X^2. X^6 is formed by attaching Ω_1 and X^8 by attaching $\Omega_\emptyset$.

Now that we have defined a CW structure on the Grassmannian, we can discuss the homology of $Gr(n,k)$. Fixing a flag [4] F, we recognize that the CW structure had no cells of odd degree, so the resulting cellular chain groups will be nonzero only in even degrees. The homology groups will then be equivalent to the chain groups in even degrees, and zero otherwise. Since the Schubert varieties are in some chain group, they each define a class $[\Omega_\lambda]$ in homology. By Poincaré Duality, this gives a correspondence to classes in cohomology. We define $\sigma_\lambda := [\Omega_\lambda] \in H^{2|\lambda|}(Gr(n,k))$.

It is a fact (see [2], for instance) that because we have a CW decomposition $Gr(n,k) = X^{2k(n-k)} \supset \cdots \supset X^2 \supset X^0$ for which $X^{2m} \backslash X^{2(m-1)}$ is the disjoint union of open Schubert cells Ω°_λ (for which $|\lambda| = k(n-k) - m$), the classes σ_λ form an integral basis for $H^*(Gr(n,k))$.

It is also a fact of algebraic topology (see [6] for details) that under nice enough conditions, including when considering Schubert varieties of generic flags, the cup product of a pair of classes in cohomology corresponds to the class of the intersection of the varieties defining them. Thus, we have the following theorem:

[3] Really what we need to show is that there is a map q from the space of orthonormal k-tuples to $Gr(n,k)$ which sends topological $2m$-disks to the Schubert varieties Ω_λ, while mapping the interior of the disk to the Schubert cell Ω°_λ. For the explicit construction of these maps, see [11] section 3.2

[4] As we saw at the end of Section 2, GL_n acts transitively on complete flags. Therefore, Ω_λ will determine a unique class in homology regardless of the flag used to define it.

Theorem 3.2. *The classes σ_λ form a $\mathbb{Z}$-basis for the cohomology ring $H^*(Gr(n,k))$. Further, $H^*(Gr(n,k))$ is a graded ring and the cup product of two classes in cohomology is equivalent to the class of the intersection of the Schubert varieties defining those classes. Explicitly, for partitions λ and μ*

$$\sigma_\lambda \sigma_\mu = [\Omega_\lambda \cap \Omega_\mu] \in H^{2(|\lambda|+|\mu|)}(Gr(n,k)).$$

This brings us one step closer to solving our intersection problems. Consider partitions $\lambda^1, \dots, \lambda^m$ with $\Sigma_i |\lambda^i| = k(n-k)$. Since the cohomology is graded, $\sigma_{\lambda^1} \cdot \sigma_{\lambda^2} \cdots \sigma_{\lambda^m} \in H^{2k(n-k)}(Gr(n,k))$. However, the only generator of this top homology group is σ_B, which is the class of a single point $\Omega_B(F)$. So the intersection of the Schubert varieties corresponding to the λ^i for m generic flags is a finite union of points. In particular, the number of points is the coefficient c in the product

$$\sigma_{\lambda^1} \cdot \sigma_{\lambda^2} \cdots \sigma_{\lambda^m} = c^B_{\lambda^1,\dots,\lambda^m} \sigma_B.$$

Our next goal will be to find a way to compute these coefficients. For now, we will give an update on our motivating questions.

Example 3.3 (Questions Revisited)**.** As we saw in Example 2.10, Question 2 is equivalent to computing the number of elements of the intersection

$$\Omega_1(F^1) \cap \Omega_1(F^2) \cap \Omega_1(F^3) \cap \Omega_1(F^4).$$

This is equivalent to computing the coefficient c in the product

$$\sigma_1 \cdot \sigma_1 \cdot \sigma_1 \cdot \sigma_1 = c^{(2,2)}_{(1,1,1,1)} \sigma_{(2,2)}$$

in $H^*(Gr(4,2))$.

Similarly, given $k(n-k)$ generic flags, Question 3 becomes equivalent to finding the coefficient c in the expansion

$$\sigma_1 \cdot \cdots \cdot \sigma_1 = c^B_{(1,\dots,1)} \sigma_B$$

in $H^*(Gr(n,k))$.

4. Symmetric Functions

In the previous section we saw how our intersection questions may be restated in terms of products in the cohomology ring. In this section we will develop some important results about symmetric functions, which we will use in the next section to show that the cohomology ring of the Grassmannian can be modelled as a quotient of the ring of symmetric functions.

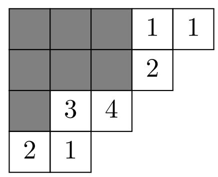
Definition 4.1. The ring of (complex) Symmetric functions Λ is the subring of the ring of formal power series $\mathbb{C}[[x_1, \dots]]$ whose elements have bounded degree and are invariant under permutation of the $x_i's$.

There is a special class of symmetric functions, called Schur functions, which are particularly useful in Schubert calculus and can be defined using Young Tableaux.

Definition 4.2. A skew shape is the difference ν/λ formed by removing the Young diagram of a partition λ from that of a strictly larger partition ν. A skew shape is a horizontal strip if no column contains more than one box.

Definition 4.3. A Semistandard Young tableau of skew shape ν/λ is a filling of the boxes of the Young diagram of shape ν/λ with positive integers such that within each row the integers weakly increase from left to right and within each column the integers strictly increase from top to bottom. The content of an SSYT is $\mu = (\mu_1, \ldots, \mu_m)$ if there are μ_i boxes labelled i. The reading word of the tableau is the word formed by concatenating rows from bottom to top.

Example 4.4. Consider the following SSYT. It has shape $(5, 4, 3, 2)/(3, 3, 1)$. Its content is $\mu = (3, 2, 1, 1)$, since there are three cells labelled 1, two cells labelled 2, and one cell each labelled 3 and 4. The reading word is 2134211.



Definition 4.5. Given a skew shape λ/μ, the associated skew Schur function is given by

$$s_{\lambda/\mu} = \sum_T x^T,$$

where the sum ranges over all SSYT's of skew shape λ/μ and $x^T = x_1^{m_1} x_2^{m_2} \ldots$ where m_i is the number of occurrences of the integer i in T.

When μ is empty, we call s_λ the Schur function of shape λ.

The first thing to check is that these s_λ really are symmetric and that they generate the ring of symmetric functions.

Proposition 4.6. *For any skew shape λ/μ the skew Schur function $s_{\lambda/\mu}$ is symmetric.*

Proof. Because S_n can be generated by transpositions, it suffices to check that $s_{\lambda/\mu}$ is invariant under the transposition $i \to i + 1$.

Suppose that λ/μ has size n and let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a sequence of nonnegative integers such that there exists an SSYT of shape λ/μ and content α. Define $\alpha' = (\alpha_1, \ldots, \alpha_{i+1}, \alpha_i, \ldots)$ by permuting α_i and α_{i+1} in α. We wish to construct a bijection $\varphi : \mathcal{T}^{\alpha}_{\lambda/\mu} \to \mathcal{T}^{\alpha'}_{\lambda/\mu}$ from the set of semistandard Young Tableaux of shape λ/μ and content α to that of content α'. This bijection guarantees that the sum $\sum_T x^T$ is invariant under transposition, and so $s_{\lambda/\mu}$ is as well.

Consider a semistandard Young tableau $T \in \mathcal{T}^{\alpha}_{\lambda/\mu}$. Since the numbering of the cells of T must strictly increase down columns we can not change cells numbered i and $i+1$ occurring in the same column. Therefore, we disregard any column which contains both i and $i+1$ cells, as well as any column which contains neither. The resulting columns will each contain exactly one of i or $i+1$. These columns form a diagram whose rows contain some number of cells labelled i and some number of cells labelled $i+1$. In each row swap the i's for $i+1$'s and the $i+1$'s for i's. Then, reorder the row to respect the weak increasing condition. This process clearly does not change the shape of T, and repeating this process for each row results in a tableau $\varphi(T) \in \mathcal{T}^{\alpha'}_{\lambda/\mu}$.

φ is injective since if $\varphi(T) = \varphi(S)$ we can simply reverse the process described above to see that $T = S$. Similarly, φ is surjective because for any tableau T of content α' and skew shape λ/μ applying φ twice gives T, and so $\varphi(T) \in \mathcal{T}^{\alpha}_{\lambda/\mu}$ maps by φ to T. In short, φ is a bijection because it is an involution. $\square$

Taking μ to be empty shows that the Schur functions are symmetric. However, it is not immediately clear that they should form a basis for Λ. There is an equivalent way of defining the Schur functions which makes this fact easier to see.

Definition 4.7. Given a polynomial in n variables $f \in \Lambda^n$, define the antisymmetrisation of f:

$$a(f) = \sum_{\sigma \in S_n} sign(\sigma)\sigma(f)$$

Such a function is antisymmetric, since a permutation of the variables will change the sign of $a(f)$ by the sign of the permutation.

For example, if $f = x^3y^2z$ then $a(f) = x^3y^2z - y^3x^2z - z^3y^2x - x^3z^2y + z^3x^2y + y^3z^2x$. Notice that if we have $\lambda = (\lambda_1, \ldots, \lambda_n)$ and there is some pair $\lambda_i = \lambda_j$, then $a(x^\lambda) = 0$ because each term containing $x_i^{\lambda_i} x_j^{\lambda_j}$ will cancel with the corresponding term containing $-x_j^{\lambda_i} x_i^{\lambda_j}$. It is also worth noting that if we take $\delta = (n-1, n-1, \ldots, 1, 0)$ then $a(x^\delta)$ is the Vandermonde polynomial

$$a(x^\delta) = \prod_{1 \leq i \leq j \leq n} (x_i - x_j).$$

If $f = x^\mu$ is a monomial, $a(x^\mu)$ is called a monomial antisymmetric function. Up to the sign of the function, we can order $\mu = (\mu_1, \mu_2 \ldots, \mu_n)$ to be weakly decreasing. As we saw, if $a(x^\mu)$ is to be nonzero, it can have no repeated exponents. Thus, the nonzero monomial antisymmetric functions are precisely those for which there exists a partition λ such that $\mu = \lambda + \delta$. Every antisymmetric function can be written as a linear combination of these $a(x^{\lambda+\delta})$. Further, if f is antisymmetric, setting $x_i = x_j$ makes f zero, and so f is divisible by the Vandermonde polynomial $a(x^\delta)$. We can now give an alternate definition of the Schur function[3]

$$s_\lambda(x) := \frac{a(x^{\lambda+\delta})}{a(x^\delta)}$$

Since $a(x^{\lambda+\delta})$ is antisymmetric, this quotient is a polynomial. In particular, it is symmetric as both the numerator and denominator are antisymmetric. Because $\{a(x^{\lambda+\delta})\}_\lambda$ forms a basis for the antisymmetric functions, and every antisymmetric function is divisible by $a(x^\delta)$, the set $\{s_\lambda\}$ forms a basis for the symmetric polynomials.

There is some nuance required in extending this basis from the ring of symmetric polynomials in n variables to Λ, but this can be justified by thinking of Λ as a direct limit of $\Lambda[x_1, \ldots, x_n]$.

Now that we have shown that the Schur functions generate the ring of symmetric functions, we would like to know how multiplication behaves in this basis. Later on we will discuss the Littlewood Richardson rule for multiplying arbitrary Schur functions, but first we will state a special case known as Pieri's rule.

Theorem 4.8. *Given a partition λ and a one-row shape (r) the product of the associated Schur functions is*

$$s_{(r)} \cdot s_\lambda = \sum_\nu s_\nu,$$

where the sum ranges over all partitions ν with ν/λ a horizontal strip of size r.

The proof of this theorem requires a good amount of technical machinery, as such we refer the reader to Section 7.15 of [7]. However, we will later give a proof of a stronger result from which this follows as a corollary.

5. The Isomorphism

We are now in a good position to prove that the cohomology ring of the Grassmannian can be described as a quotient of a ring of symmetric functions. This is a very important result, and the main theorem of our paper, as it allows us to compute products in the cohomology ring by working with symmetric polynomials, which are easier to understand.

Theorem 5.1. *There is an isomorphism*

$$H^*(Gr(n,k)) \cong \Lambda/(s_\lambda | \lambda \not\subset B),$$

where the quotient is by the ideal generated by those Schur functions corresponding to partitions which do not fit in the ambient rectangle B.

As we saw in the previous sections, the Schubert classes σ_λ form a basis for $H^*(Gr(n,k))$ while the Schur functions corresponding to partitions fitting in the ambient rectangle form a basis for the right hand side. Therefore, the map $\sigma_\lambda \to s_\lambda$ is an isomorphism of the underlying vector spaces. To show that this is indeed an isomorphism of rings we need only show that this map carries the cup product of cohomology classes to the product of Schur functions.

The Schur functions satisfy the Pieri rule. Since $s_{(r)}$ and $\sigma_{(r)}$ each form a basis for the respective rings, to establish the isomorphism we only need to show that the Schubert classes also satisfy the Pieri rule:

$$\sigma_{(r)}\sigma_\lambda = \sum_\nu \sigma_\nu,$$

where the sum is taken over all partitions ν fitting within the ambient rectangle such that ν/λ is a horizontal strip of size r. The restriction that ν fit within the ambient rectangle is not required in general for Schur functions, but as we are considering the quotient of Λ by those Schur functions corresponding to inadmissible partitions this is justified. To prove the Pieri rule for Schubert classes, we will need a few preliminary results.

Lemma 5.2. *Let F and E be transverse flags in $\mathbb{C}^n$, and let λ and μ be partitions with $|\lambda| + |\mu| = k(n-k)$. If $\mu_{k+1-i} + \lambda_i > n-k$ for some i, then the intersection $\Omega_\lambda(F) \cap \Omega_\mu(E)$ is empty.*

Proof. Suppose for some i that $\mu_{k+1-i} + \lambda_i > n-k$. Suppose for contradiction that there is some subspace $V \in \Omega_\lambda(F) \cap \Omega_\mu(E)$. Then $\dim(V) = k$, and we see

$$\dim(V \cap \langle e_1, \ldots, e_{n-k+i-\lambda_i} \rangle) \geq i,$$

while at the same time

$$\dim(V \cap \langle e_n, e_{n-1}, \ldots, e_{n+1-(n-k+(k+1-i)-\mu_{k+1-i})} \rangle) \geq k+1-i.$$

The second statement simplifies to

$$\dim(V \cap \langle e_{i+\mu_{k+1-i}}, \ldots, e_n \rangle) \geq k+1-i.$$

However, the subspaces spanned by $\langle e_n \ldots e_{n-k+i-\lambda_i} \rangle$ and $\langle e_{i+\mu_{k+1-i}}, \ldots, e_n \rangle$ are disjoint, as by our assumption $\mu_{k+1-i} + \lambda_i > n-k$ which implies $\mu_{k+1-i} + i >$

$n-k-\lambda_i+i$. Therefore, the dimension of V is at least $(k+1-i)+i=k+1$, a contradiction. □

Definition 5.3. Two partitions, λ and μ are said to be complementary if $\lambda_i + \mu_{k+1-i} = n-k$ for all i.

Lemma 5.4 (Duality Theorem)**.** *Let F and E be transverse flags in $\mathbb{C}^n$, and let λ and μ be partitions with $|\lambda|+|\mu| = k(n-k)$. The intersection $\Omega_\lambda(F)\cap\Omega_\mu(E)$ in $Gr(n,k)$ has one element if λ and μ are complementary, and zero elements otherwise.*

Proof. If μ and λ are not complementary then there exists some i such that $\lambda_i + \mu_{k+1-i} > n-k$. By the lemma the intersection $\Omega_\lambda(F)\cap\Omega_\mu(E)$ is empty.

On the other hand, if λ and μ are complementary then the dimension equations seen in the previous lemma still hold

$$\dim(V\cap\langle e_1,\ldots,e_{n-k+i-\lambda_i}\rangle)\geq i$$
$$\dim(V\cap\langle e_{i+\mu_{k+1-i}},\ldots,e_n\rangle)\geq k+1-i$$

However, we now have that $\lambda_i+\mu_{k+1-i}=n-k$ and in particular $\mu_{k+1-i}+i = n-k+i-\lambda_i$. Therefore, $\dim(V\cap\langle e_{n-k+i-\lambda_i}\rangle)=1$ for all $i=1,2,\ldots,k$. Since V is k-dimensional

$$V=\langle e_{n-k+1-\lambda_1}, e_{n-k+2-\lambda_2},\ldots,e_{n-\lambda_k}\rangle$$

Therefore the intersection contains a unique element. □

Remark 5.5. The following example, perhaps, will clarify why the Duality Theorem should hold. Consider again our example cell in $Gr(12,5)$, corresponding to the partition $\lambda=(5,3,2,2,1)$. The complementary partition is $\mu=(6,5,5,4,2)$. The ambient rectangle is then filled by placing λ in the top left and rotating μ and placing it in the bottom right. The corresponding Schubert cells are given by

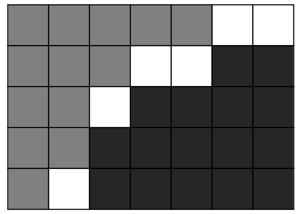
$$\begin{bmatrix} 0&0&0&0&0&0&0&0&0&1&*&* \\ 0&0&0&0&0&0&1&*&*&0&*&* \\ 0&0&0&0&1&*&0&*&*&0&*&* \\ 0&0&0&1&0&*&0&*&*&0&*&* \\ 0&1&*&0&0&*&0&*&*&0&*&* \end{bmatrix} \& \begin{bmatrix} *&0&*&0&0&*&0&*&*&1&0&0 \\ *&0&*&0&0&*&1&0&0&0&0&0 \\ *&0&*&0&1&0&0&0&0&0&0&0 \\ *&0&*&1&0&0&0&0&0&0&0&0 \\ *&1&0&0&0&0&0&0&0&0&0&0 \end{bmatrix}$$

These cells meet in exactly one point, namely the subspace spanned by the basis vectors corresponding to the 1's in the matrices. The $*$'s in each matrix define a neighborhood of this point in the Grassmannian. To be contained in both cells, then, is equivalent to setting all of the $*$'s to zero, and so the intersection is exactly one point.

We can now prove the Pieri rule for multiplying Schubert classes.

For any μ with $|\mu| = k(n-k)-|\lambda|-k$ we must show that both $\sigma_{(r)}\sigma_\lambda$ and $\sum_\nu \sigma_\nu$ have the same intersection with σ_μ. In particular, the Duality Theorem tells us that we can multiply each side by σ_{μ^c} to extract the coefficient of σ_μ on the right hand side. Therefore, if we place λ in the upper left corner of the ambient rectangle and rotate μ to set it in the bottom right, Pieri's rule is equivalent to the assertion that $\sigma_\lambda\sigma_\mu\sigma_{(r)}$ is equal to 1 when the diagrams for μ and λ do not overlap and no two of the r boxes between them are in the same column, and 0 otherwise.

Example 5.6. Consider the case in $Gr(12,5)$ when $\lambda = (5,3,2,2,1)$, and $\mu = (5,5,4,2,0)$. In the Young diagram we place λ in the top left and μ in the bottom right



We see that the diagrams do not overlap, and none of the boxes between them are contained in the same column. Therefore, Pieri's rule says that $\sigma_\lambda \sigma_\mu \sigma_{(6)} = 1$.

We get the following reformulation by recalling that the product of Schubert classes corresponds to the class of the intersection of the associated Schubert varieties.

Theorem 5.7 (Pieri Rule for Schubert Classes)**.** *Let λ and μ be partitions with size $|\lambda| + |\mu| = k(n-k) - r$. Let F and E be the standard and opposite flags, and H a generic complete flag. Then the intersection*

$$\Omega_\lambda(F) \cap \Omega_\mu(E) \cap \Omega_{(r)}(H)$$

has one element if μ^c/λ has length r and no two boxes in the same column, and is empty otherwise.

We will follow the proof in [2], filling in the details omitted therein.

In this case, the matrices corresponding to μ and λ will no longer have pivots in the same position as was the case in the Duality Theorem, and so there may be $*$ entries which occur in the matrix representations of both Ω_λ and Ω_μ. The subspaces which occur in the intersection $\Omega_\lambda \cap \Omega_\mu$ are spanned by the rows of a matrix that has nonzero entries only in the intermediary sites between pivots.

Notice that the condition that the diagrams for λ and μ do not intersect and that no two of the r boxes between them are contained in the same column is equivalent to the conditions

$$n - k - \lambda_k \geq \mu_1 \geq n - k - \lambda_{k-1} \geq \cdots \geq n - k - \lambda_1 \geq \mu_k \geq 0 \tag{5.8}$$

Set $A_i = F_{n-k+i-\lambda_i}$, $B_i = E_{n-k+i-\mu_i}$ and $C_i = A_i \cap B_{k+1-i}$. The C_i's are the subspaces spanned by the matrices with $*$'s in the i^{th} row between the partitions. Let C be the span of all C_i's.

Example 5.9. Continuing with our running example in $Gr(12,5)$, we have

$$\begin{array}{lll}
A_1 = F_3 & B_5 = E_{12} & C_1 = \langle e_1, e_2, e_3 \rangle \\
A_2 = F_6 & B_4 = E_9 & C_2 = \langle e_4, e_5, e_6 \rangle \\
A_3 = F_8 & B_3 = E_6 & C_3 = \langle e_7, e_8 \rangle \\
A_4 = F_9 & B_2 = E_4 & C_4 = \langle e_9 \rangle \\
A_5 = F_{11} & B_1 = E_3 & C_5 = \langle e_{10}, e_{11} \rangle
\end{array}$$

The subspaces V in the intersection $\Omega_\lambda \cap \Omega_\mu$ will be spanned by the rows of a matrix of the form

$$\begin{bmatrix} * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 \end{bmatrix}$$

We see that in this example the inequalities in (5.8) are satisfied, which guarantees that no two $*$'s are in the same column. So, (nonzero) vectors taken from these rows will always be linearly independent.

Now we will apply this idea in more generality to prove the Pieri rule. A few lemmas never hurt.

Lemma 5.10. *(a) $C = \bigcap_{i=1}^k (A_i + B_{k-i})$; (b) $\Sigma_{i=1}^k \dim(C_i) = k + r$; and (c) the sum $C = C_1 + \cdots + C_k$ is a direct sum of nonempty subspaces if and only if the inequalities in (5.8) hold.*

Proof. For (a), notice that the basis vector e_m is in C exactly when it is in some C_j. This occurs when $j + \mu_{k+1-j} \leq m \leq n - k + j - \lambda_j$ for some j. On the other hand, e_m is in $C = \bigcap_{i=1}^k (A_i + B_{k-i})$ when either $m \leq n-k+i-\lambda_i$ or $m > i+\mu_{k-i}$ for all i. We use the convention here that $\lambda_0 = \mu_0 = n - k$.

These two conditions are equivalent. Suppose $j+\mu_{k+1-j} \leq m \leq n-k+j-\lambda_j$ for some j. Then for all $i < j$ we have $i + \mu_{k+1-j} \leq j + \mu_{k+1-j} \leq m$, and for all $i > j$ we have $m \leq n-k+j-\lambda_j \leq n-k+i-\lambda_i$. In the latter case, take the smallest j so that $m \leq n+j-\lambda_j$. Since this is the smallest such j, we see $m > (j-1)+\mu_{k-(j-1)}$, which is what the first condition required. Therefore $C = \bigcap_{i=1}^k (A_i + B_{k-i})$.

For (b), notice that $C_i = F_{n-k+i-\lambda_i} \cap E_{n-k+i-\mu_i}$ is spanned by the vectors $e_{k-i+\mu_i+1}, \ldots, e_{n-k+i-\lambda_i}$. Therefore, C_i has dimension $(n - k + i - \lambda_i) - (k - i + \mu_i + 1) + 1 = n - 2k + 2i - \lambda_i - \mu_i$. Summing from $i = 1, 2, \ldots, k$ we find

$$\begin{aligned} \Sigma \dim(C_i) &= \Sigma_{i=1}^k n - 2k + 2i - \lambda_i - \mu_i \\ &= kn - 2k^2 + 2\frac{k(k+1)}{2} - (|\lambda| + |\mu|) \\ &= kn - 2k^2 + k(k+1) - (k(n-k) - r) \\ &= k + r \end{aligned}$$

Finally, to see (c) notice that if the inequalities in (5.8) fail then there is some column containing at least two stars, and the corresponding C_i's will intersect in a line so the sum is not direct. On the other hand, if the inequalities hold then the columns each contain at most one star, and so the intersection $C_i \cap C_j$ is trivial for all i and j. $\square$

Lemma 5.11. *If $V \in Gr(n, k)$ is in the intersection $\Omega_\lambda \cap \Omega_\mu$, then $V \subset C$. If, in addition, $C_1, \ldots, C_k$ are linearly independent, then $\dim(V \cap C_i) = 1$ for all i and $V = V \cap C_1 \oplus \cdots \oplus V \cap C_k$.*

Proof. By the previous lemma, we can show that $V \subset A_i+B_{k-i}$. If $A_i \cap B_{k-i} \neq \{0\}$ then $A_i + B_{k-i} = \mathbb{C}^n$ and so V is trivially contained in $A_i + B_{k-i}$. Therefore, we can suppose that $A_i \cap B_{k-i} = \{0\}$. Since $V \subset \Omega_\lambda \cap \Omega_\mu$, we know $\dim(V \cap A_i) \geq i$ and $\dim(V \cap B_{k-i}) \geq k - i$. Since V is k-dimensional, $V = V \cap A_i \oplus V \cap B_{k-i}$.

Now assume further that the C_i's are linearly independent. We know $\dim(V \cap C_i) \geq 1$, since $V \cap A_i$ and $V \cap B_{k+1-i}$ have dimension at least i and $k+1-i$ respectively, so that A_i and B_{k+1-i} must intersect nontivially in V. Since the C_i's are independent, V contains the direct sum $\bigoplus_i (V \cap C_i)$ which has at least dimension k. Since V has dimension k, we have $V = \bigoplus_i (V \cap C_i)$ and each summand has dimension one. □

We are finally able to prove the Pieri rule for Schubert classes.

Proof (Theorem 5.7). If the inequalities in (5.8) fail, then by (c) of Lemma 5.10 the space C is not a direct sum of the C_i's, and by (b) of the same lemma its dimension is at most $r+k-1$. In this case, a general space H of dimension $n-k+1-r$ will intersect C trivially, as $(r+k-1)+(n-k+1-r) = n$ and we are in $\mathbb{C}^n$. Therefore, no $V \subset \Omega_\lambda \cap \Omega_\mu$ is in $\Omega_{(r)}$, and so $\Omega_\lambda(F) \cap \Omega_\mu(E) \cap \Omega_{(r)}(H) = \emptyset$.

On the other hand, if the inequalities in (5.8) hold, then $C = \bigoplus C_i$ and a generic subspace L of dimension $n-k+1-r$ will intersect C in a line. Since C decomposes as a sum of C_i, the line can be written as the span of the vector $v = u_1 \oplus \cdots \oplus u_k$, with $u_i \in C_i/\{0\}$. Since we are considering subspaces $V \subset C$ intersecting L in at least a line, we must have $v \in V$. Writing $V = \bigoplus (V \cap C_i)$ we see that $u_i \in V$ for all i, and so $V = \langle u_1, \ldots, u_k \rangle$. This point is unique, which proves that the intersection $\Omega_\lambda(F) \cap \Omega_\mu(E) \cap \Omega_{(r)}(H)$ contains a single point. □

Thus, we have proven the Pieri rule and, as a consequence, established the isomorphism claimed in Theorem 5.1.

6. The Littlewood Richardson Rule

The isomorphism established in the previous section tells us that if we understand how to multiply Schur functions, we will understand how to intersect Schubert varieties. There is a combinatorial rule, known as the Littlewood Richardson rule, which generalizes the Pieri rule for multiplying Schur functions. To state the rule we will need to develop some terminology.

Definition 6.1. A word $w_1 w_2, \ldots, w_n$ is called a reverse lattice word if, when read backwards from the nth to the $(n-m)^{\text{th}}$ term, the sequence always contains at least as many i's as $i+1$'s.

Definition 6.2. A Littlewood Richardson Tableau is a semistandard Young tableau whose reading word is reverse lattice.

Definition 6.3. A sequence of skew tableau form a chain if if their shapes do not overlap and $T_1 \cup \cdots \cup T_i$ is a partition shape for all i.

We can now state the Littlewood-Richardson Rule.

Theorem 6.4. *Given partitions λ and μ, the product of the corresponding Schur functions can be written in the basis of Schur functions via the formula*

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu$$

Where $c^\nu_{\lambda,\mu}$ is the number of Littlewood-Richardson Tableaux of skew shape ν/λ and content μ.

Dr. Fulton, in chapter 5 of [2] provides an enlightening combinatorial proof of this using products of tableaux in the tableau ring and developing a homomorphism to the ring of symmetric functions. We present here a more concise proof using a sign-reversing involution due to Stembridge[10]. In the following, $T_{>j}$ denotes the subtableau of T consisting of the columns $j, j+1, \dots$, and $\omega(T)$ is the content of T, so that $\omega_k(T)$ is the number of k's in T.

Theorem 6.5. *Let λ, μ and ν be partitions such that μ/ν is defined, and let $\delta = (n-1, n-2, \dots, 1, 0)$ as before. Then*

$$a(x^{\lambda+\delta})s_{\mu/\nu} = \sum_T a(x^{\lambda+\omega(T)+\delta})$$

where the sum ranges over all tableaux of shape μ/ν for which $\lambda + \omega(T_{\geq j})$ is a partition for all j.

Proof. As we noticed in the proof of Theorem 4.6, if we consider the entries labelled i and $i+1$ for which exactly one appears in a column, we can interchange these entries within rows in a manner that maintains the weak increase of the row. We denote this by $\sigma_k(T)$, which is a semistandard Young tableau. Since S_n is generated by transpositions, we can define $\sigma(T)$ for any permutation. The maps σ_k were introduced by Bender and Knuth, who showed that $\omega(\sigma_k(T)) = s_k\omega(T)$ where s_k is the permutation $(k \leftrightarrow k+1) \in S_n$.

Since Schur functions are symmetric, for each $\omega \in S_n$ the quantities $\omega(\lambda+\delta) + \omega(T)$ and $\omega(\lambda+\delta+\omega(T))$, while not being equal, take on all of the same values as T varies over SSYT's of shape μ/ν. Therefore,

$$a(x^{\lambda+\delta})s_{\mu/\nu} = \sum_{\omega \in S_n}\sum_T sign(\omega)x^{\omega(\lambda+\delta+\omega(T))} = \sum_T a(x^{\lambda+\delta+\omega(T)})$$

Now we simply need to show that if T is a tableau such that $\lambda + \omega(T_{\geq j})$ fails to be a partition for some j, i.e.

$$\lambda_k + \omega_k(T_{\geq j}) < \lambda_{k+1} + \omega_{k+1}(T_{\geq j})$$

for some pair of k and j, then the effect of this tableau in the sum will be negated.

There are finitely many pairs (k, j), among those which violate the partition condition above. Choose the smallest k that maximizes j. Since we have maximized j, we must have that $\lambda + \omega(T_{>j})$ is a partition. Further, $w_k(T_{\geq j}) - w_{k+1}(T_{\geq j})$ can change by at most 1 as we increment or decrement j. So there must be a single $k+1$ and no k in column j, and

$$\lambda_k + \omega_k(T_{\geq j}) + 1 = \lambda_{k+1} + \omega_{k+1}(T_{\geq j})$$

Let T^* be the tableau abtained by applying the involution σ_k to $T_{<j}$ and leaving $T_{\geq j}$ unchanged. Since there are no k's in column j, but there is a $k+1$ the construction will still result in an SSYT, as it will respect the weak ordering across rows. From this construction we see $T_{\geq j} = T^*_{\geq j}$, so the map $T \to T^*$ is an involution on the set of tableaux for which $\lambda + \omega(T_{\geq j})$ fails to be a partition.

We can compare the contributions of T and T^* in the sum above. Notice that $s_k\omega(T_{<j}) = \omega(T^*_{<j})$ and s_k fixes $\lambda + \omega(T_{\geq j}) + \delta$ while mapping $\lambda + \omega(T) + \delta$ to $\lambda + \omega(T^*) + \delta$. Thus we have

$$a(x^{\lambda+\omega(T)+\delta}) = -a(x^{\lambda+\omega(T^*)+\delta})$$

Thus the contributions of T and T^* cancel. $\square$

Recalling that $s_\lambda = \frac{a(x^{\lambda+\delta})}{a(x^\delta)}$, we find as a corollary

$$s_\lambda s_{\mu/\nu} = \Sigma s_{\lambda+\omega(T)},$$

where the sum ranges over all tableaux of shape μ/ν such that $\lambda + \omega(T_{\geq j})$ is a partition for all j. Taking ν to be the empty partition proves the Littlewood-Richardson rule.

We now have a method for multiplying Schubert classes.

Theorem 6.6. *In $H^*(Gr(n,k))$ we have*

$$\sigma_{\lambda^{(1)}} \cdots \sigma_{\lambda^{(m)}} = \sum_\nu c^\nu_{\lambda^{(1)},\ldots,\lambda^{(m)}} \sigma_\nu$$

Where the sum is restricted to partitions ν fitting within the ambient rectangle.

Proof. By Theorem 6.4 and induction on the number of Schur polynomials in the product we have

$$s_{\lambda^{(1)}} \cdots s_{\lambda^{(m)}} = \sum_\nu c^\nu_{\lambda^{(1)},\ldots,\lambda^{(m)}} s_\nu$$

where $c^\nu_{\lambda^{(1)},\ldots,\lambda^{(m)}}$ is the number of chains of Littlewood Richardson tableaux of contents $\lambda^{(i)}$ with total shape ν. By Theorem 5.1 the result follows. □

In the case of a zero-dimensional intersection of Schubert varieties we have $\Sigma|\lambda^{(i)}| = k(n-k)$, and the Littlewood Richardson coefficients are zero whenever $\Sigma|\lambda^{(i)}| \neq |\nu|$, so the only possible partition is the entire ambient rectangle B. We know that there is only one generator, σ_B, of the top cohomology group, and this is the class of a single point $\Omega_B(F)$ for some flag F. Thus, we achieve the following corollary:

Corollary 6.7. *Let $\lambda^{(1)}, \ldots, \lambda^{(m)}$ be partitions fitting in the ambient rectangle B such that $\Sigma|\lambda^{(i)}| = k(n-k)$. Let $F^1, \dot{;} F^m$ be generic flags. Then we have*

$$c^B_{\lambda^{(1)},\ldots,\lambda^{(m)}} := |\Omega_{\lambda^{(1)}} \cap \cdots \cap \Omega_{\lambda^{(m)}}|$$

is equal to the number of chains of Littlewood-Richardson tableaux of contents $\lambda^{(i)}$ with total shape B.

7. Computations

We will now give a combinatorial rule for determining the number of tableaux of a certain shape.

Definition 7.1. A standard Young tableau of shape λ where $|\lambda| = n$ is a semistandard Young tableau numbered by the integers $1, 2, \ldots, n$.

Definition 7.2. For a square s in a Young diagram, define the hook length to be the sum of the number of squares strictly below s, plus the number of squares strictly to the right of s, plus one for s itself.

There is an explicit formula in terms of the hook length for the number of standard Young tableaux of a particular shape.

Theorem 7.3 (Hook Length Formula)**.** [7] *The number of standard Young tableaux of shape λ is*

$$\frac{|\lambda|!}{\prod_{s\in\lambda} hook(s)}$$

There is a nice heuristic[5] argument for this formula. Let $|\lambda| = n$ and consider the $n!$ ways to order the n integers in the tableau. A numbering will be a tableau exactly when it is strictly increasing across rows and down columns, i.e. when each square s contains the largest integer in its hook. The probability of this is $\frac{1}{hook(s)}$. If we assume these probabilities are independent for all s, then the probability that a given numbering is a tableau is given by the hook length formula.

We are finally in a position to solve our motivating problems explicitly!

Example 7.4 (Question 2 Solved)**.** As we saw in Example 3.3, Question 2 is equivalent to computing the product σ_1^4 in $H^*(Gr(4,k))$. The constant $c_{(1,1,1,1)}^{(2,2)}$ is the number of ways to fill a 2×2 rectangle with a chain of Littlewood Richardson tableaux each consisting of a single box. There are two ways to do this

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & 1 \\ \hline\end{array} \text{ and } \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & 1 \\ \hline\end{array}.$$

Because each tableau in the chain must contain a single 1, we have colored the boxes to differentiate between different tableau in the chains. The condition that $T_1 \cup \cdots \cup T_i$ be a partition shape for all i necessitates that T_1, the red tableau, is always in the top left corner. The blue tableau, T_2 can then be placed in either the top right or bottom left corner, with the green tableau T_3 taking the other corner. Finally, T_4, the yellow tableau, must be placed in the bottom right corner. Therefore, the two fillings of a 2×2 ambient rectangle by chains of Littlewood Richardson tableaux shown above are unique.

Therefore, given four lines in general position in 3-dimensional space, there are two lines which intersect all four nontrivially. This is exactly the answer we found via Schubert's method!

Example 7.5 (Question 3 Solved)**.** Similarly, Question 3 becomes equivalent to computing the coefficient in the expansion

$$\sigma_1 \cdot \cdots \cdot \sigma_1 = c_{(1,\ldots,1)}^{B} \sigma_B$$

This is the number of ways to fill the ambient rectangle with content $(1, 1, \ldots, 1)$, i.e. the number of standard Young tableaux of shape $(k(n-k))^k$. This is given by the hook-length formula applied to the entire $k \times (n-k)$ ambient rectangle. In the table below, the integers in each box denote the hook length of that box.

$$\begin{array}{|c|c|c|c|c|}\cline{1-2}\cline{4-5} n-1 & n-2 & \multicolumn{1}{c|}{} & k+1 & k \\ \cline{1-2}\cline{4-5} n-2 & n-3 & \multicolumn{1}{c|}{\ldots} & k & k-1 \\ \cline{1-2}\cline{4-5} \multicolumn{1}{c}{} & \multicolumn{1}{c}{\vdots} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\vdots} & \multicolumn{1}{c}{} \\ \cline{1-2}\cline{4-5} n-k+1 & n-k & \multicolumn{1}{c|}{\ldots} & 3 & 2 \\ \cline{1-2}\cline{4-5} n-k & n-k-1 & \multicolumn{1}{c|}{} & 2 & 1 \\ \cline{1-2}\cline{4-5}\end{array}$$

[5]By heuristic, I mean incorrect. The probabilities are not at all independent!

The product of the hook lengths along the top and right sides of the box, starting from the first box in row i to the $n-k+1-i^{\text{th}}$ box in row i, and then down the $n-k+1-i^{\text{th}}$ column, is given by

$$\frac{(n-i)!}{(i-1)!}$$

So the number of standard Young tableaux with the shape of the ambient rectangle is

$$c^B_{(1,\ldots,1)} = \frac{(k(n-k))!(k-1)!(k-2)!\cdots 2!1!}{(n-1)!(n-2)!\cdots(n-k)!}$$

This gives the number of $k-1$-dimensional subspaces of $\mathbb{P}^{n-1}$ which intersect each of $k(n-k)$ fixed subspaces of dimension $n-k-1$ nontrivially. Certainly this is not a result that would be easily determined by the elementary reasoning used originally in Question 2.

Acknowledgments

It is a pleasure to thank my mentor, Pallav Goyal, for many engaging mathematical discussions as well as for his help and guidance throughout the program. I would also like to thank all of the wonderful speakers who volunteered their time and expertise to give insight into many interesting areas of math which I had never before seen. Thank you also to the Ryerson Refugees for many fun memories of math on the quad, trips to the Med, and game nights. Finally, I would like to express my sincere gratitude to Peter May for all of the work that makes this program possible, and for giving me the opportunity to participate.

References

[1] M. Brion, *Lectures on the Geometry of Flag Varieties*, Birkhäuser Basel, 2005
[2] W. Fulton, *Young Tableaux, with Applications to Representation Theory and Geometry*, Cambridge University Press, 1997
[3] W. Fulton and J. Harris, *Representation Theore. A First Course*, Graduate Texts in Mathematics, Springer-Verlag, New York 1991
[4] A. Gathmann, *Chapter 6 : in Algebraic Geometry Coursenotes*, Retrieved August 2021 from (https://www.mathematik.uni-kl.de/ gathmann/de/alggeom.php)
[5] M. Gillespie, *Variations on a Theme of Schubert Calculus*, (arXiv:1804.08164v3), Jan 2021
[6] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2001
[7] R. Stanley, *Enumerative Combinatorics Vol 2*, Cambridge University Press, 1999
[8] F. Ronga *Schubert Calculus According to Schubert*, (arXiv:math/0608784v1) Oct 2018
[9] H. Schubert, *Kalkül der abzählenden Geometrie*, Teubner Verlag, Leipzig, 1798
[10] J. Stembridge, *A Concise Proof of the Littlewood Richardson Rule*, The Electronic Journal of Combinatorics, Vol 9, 2002
[11] T. Tajakka, *Cohomology of the Grassmannian*, Master's Thesis, Aalto University, 2015