# SPIN MANIFOLDS AND POSITIVE SCALAR CURVATURE METRICS

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Abstract. In this expository paper, we explain some results about spin manifolds that admit positive scalar curvature metric. Namely, we discuss the work of Stolz which shows that a simply connected spin manifold of dimension $n \geq 5$ has a positive scalar curvature metric if the Clifford index vanishes and the converse due to Hitchin.

Contents



## 1. Introduction

The goal of the paper is to explain the work of Stolz characterizing when a spin manifold of dimension $n \geq 5$ admits a positive scalar curvature metric. We begin by introducing the notions of spin groups and spin structures on a manifold in section 2. We then shift our focus to describing some results that characterize when a spin or non-spin simply connected manifold has positive scalar curvature metric. The most striking feature of these results, which we introduce in section 3, is that the cobordism class of a manifold determines whether it has positive scalar curvature metric. Then in sections 4 and 5, we use generators of the cobordism ring in the non-spin case to show that all non-spin manifolds of dimension $n \geq 5$ admit positive scalar curvature metrics. In section 6, we explain how we can get an analogous result for spin manifolds. In the final two sections, we focus on Stolz' result. Here the interesting feature is that we do not have explicit spin manifolds with cobordism classes that generate the spin cobordism group. To work around this, we now need tools in algebraic topology such as the Adams spectral sequence.

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## 2. Preliminaries on Spin groups and structures

Let $V$ be a vector space over a field $k$ ($k = \mathbb{R}$ or $\mathbb{C}$) and let $q$ be a quadratic form on $V$. The Clifford algebra $Cl(V, q)$ is the tensor algebra on $V$ modulo the relation:

$$v^2 = q(v)$$

for all $v \in V$. The Clifford algebra admits a $\mathbb{Z}/2$ grading owing to the fact that the anti-automorphism

$$V \mapsto V$$
$$v \mapsto -v$$

uniquely extends to an automorphism

$$\alpha : Cl(V, q) \to Cl(V, q)$$

let $Cl^0(V, q)$ be the subset of $Cl(V, q)$ of elements fixed under $\alpha$ and $Cl^1(V, q)$ be the elements $\phi \in Cl(V, q)$ for which $\alpha(\phi) = -\phi$. As $\alpha^2 = Id$, we have:

$$Cl(V, q) = Cl^0(V, q) \oplus Cl^1(V, q) \tag{2.1}$$

Since $\alpha$ is a ring homomorphism,

$$Cl^i(V, q) \cdot Cl^j(V, q) \subset Cl^{i+j}(V, q) \tag{2.2}$$

which makes $Cl(V, q)$ a $\mathbb{Z}/2$ graded $k$-algebra.

We now consider $Cl^*(V, q)$,the group which consists of those elements $x \in Cl(V, q)$ with an inverse $x^{-1} \in Cl(V, q)$. We have a homomorphism:

$$Ad_x : Cl^*(V, q) \to Gl(Cl(V, q))$$
$$y \to \alpha(x) y x^{-1}$$

a straightforward computation can verify that if $q(v) \neq 0$ then

$$Ad_v(w) = w - 2\frac{q(v, w)}{q(v)} v$$

So $Ad_x(V) \cong V$ if $x$ satisfies $q(x) \neq 0$. In fact, $Ad_v$ preserves the quadratic form. Then if we define $P(v, q)$ to be the subgroup of $Cl^*(v, q)$ with $q(v) \neq 0$, we have a representation

$$Ad : P(V, q) \to O(V, q)$$

Note that for each $v \in P(V, q)$, $Ad_v$ is reflection about $v^\perp$. It is a fact that the orthogonal group is generated by projections so we actually have that $Ad : P(V, q) \to O(V, q)$ is a surjection.

**Definition 2.3.** We define the group

$$Pin(V, q) := \{x \in Cl(V, q) : q(x) = \pm 1\} \tag{2.4}$$

When we restrict the representation defined above to $Pin(V, q)$, we get a homomorphism

$$Pin(V, q) \to O(V, q) \tag{2.5}$$

It's kernel is $\{\pm 1\}$ and so we have a double cover:

$$1 \to \mathbb{Z}/2 \to Pin(V, q) \to O(V, q) \tag{2.6}$$

We set $Spin(V,q)$ to be the preimage of $SO(V,q)$ under the adjoint representation. By definition we have the double cover:

$$1 \to \mathbb{Z}/2 \to Spin(V,q) \to SO(V,q) \tag{2.7}$$

Let $p : E \to X$ be an $n$-dimensional bundle with a Riemannian structure. Now consider the orthonormal frame bundle $O(n) \to P_O(E) \to X$, namely the principal $O(n)$ bundle over $X$ whose fiber at a point $x \in X$ is the set of all orthonormal bases of $E_x$.

If $E$ is oriented then we can define the principal $SO(n)$ bundle $P_{SO}(E) \subset P_O(E)$ of oriented orthonormal frames. Let $Spin(n)$ denote $Spin(R^n, q)$ where $q(v) = -|v|^2$. Spin structures are obtained by lifting the bundle $P_{SO}(E)$ along the double covering $p : Spin(n) \to SO(n)$:

**Definition 2.8.** A spin structure on an $n$-dimensional oriented bundle $E \to X$ is a principal $Spin(n)$ bundle $P_{Spin}(E)$ together with a double cover $p : P_{Spin}(E) \to P_{SO}(E)$ that is compatible with the double cover $p : Spin(n) \to SO(n)$, i.e. such that:

$$p(xg) = p(x)p(g)$$

for $x \in P_{Spin}(E)$ and $g \in Spin(n)$.

A spin manifold is an orientable Riemannian manifold with a spin structure on its tangent bundle.

We will need one more fact about the Clifford algebra. Let $Cl_n = Cl(\mathbb{R}^n, q)$ where $q(v) = -|v|^2$. We define $M_n$ to be the Grothendieck group of $\mathbb{Z}/2$-graded modules over $Cl_n$ (it is free abelian on the irreducible modules). The usual inclusion $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ induces a homomorphism $Cl_n \to Cl_{n+1}$ which further induces a homomorphism $M_{n+1} \to M_n$. The cokernel of this map can be identified with the real $K$-theory of a point $KO_n = KO_n(*)$. This identification is due to Atiyah-Bott-Shapiro [3].

## 3. Cobordism and positive scalar curvature

Two manifolds $M, N$ of dimension $n$ are cobordant if there is an $n+1$ manifold $W$ such that

$$\partial W = M \sqcup N$$

If $M$ and $N$ are oriented or spin manifolds, then we additionally require that $W$ have an orientation or spin structure such that

$$\partial W = M \sqcup -N$$

respectively ($-N$ is $N$ with the opposite orientation/spin structure).
Finally, if $M$ and $N$ are oriented or spin manifolds with maps $f : M \to X$, $g : N \to X$ to a space $X$, then the pairs $[M, f]$ and $[N, g]$ are cobordant if there is some $W$ as in the previous case which additionally admits a map $F : W \to X$ that restricts to $f$ on $M \subset W$ and restricts to $g$ on $N \subset W$.

The set of all cobordism classes of manifolds is denoted by $\Omega_n^G$, where $G = SO(n), Spin(n)$ for oriented/spin cobordism respectively. We can add cobordism classes via disjoint union and multiply them via Cartesian product, making $\Omega_*^G$ a graded ring.

The Gromov-Lawson theorem ([8]) asserts that, under certain assumptions, admitting a positive scalar curvature metric is a cobordism invariant. This theorem is the basis for many of the results in the rest of the paper.

**Theorem 3.1** (Gromov-Lawson)**.** *Let $M$ be an oriented manifold of dimension $n \geq 5$ and*

$$\tilde{M} \to M \xrightarrow{u} B\pi$$

*be the fiber sequence for its universal cover $\tilde{M}$. If $M$ is not spin and $\tilde{M}$ is spin or if $M$ is spin, then $M$ admits a positive scalar curvature metric if and only if $[M, u]$ is cobordant in $\Omega_n^G(B\pi)$ to a pair $[N, v]$ such that $N$ admits a positive scalar curvature metric. Here, $G = Spin(n)$ if $M$ is spin and $G = SO(n)$ if $M$ is not spin.*

In the special case of simply connected manifolds we have the following Corollary:

**Corollary 3.2.** *Let $M$ be a simply connected manifold of dimension $n \geq 5$. If either*

- *$M$ is spin and cobordant in $\Omega_n^{Spin}$ to a manifold admitting a positive scalar curvature metric, OR*
- *$M$ is not spin but is cobordant in $\Omega_n^{SO}$ to a manifold admitting a positive scalar curvature metric,*

*then $M$ admits a positive scalar curvature metric.*

## 4. Non spin manifolds

Let us examine the case of non-spin manifolds more closely. The ring $\Omega_*^{SO}$ is generated by the total spaces of fiber bundles whose fibers are projective spaces. Projective spaces themselves admit metrics of positive scalar curvature, and the following fact shows that the total spaces also do:

**Fact 4.1.** *If a manifold $M$ admits a positive scalar curvature metric $g$ then so does any manifold $E$ that satisfies either of the following conditions:*

- *$E = M \times N$ for some manifold $N$ with $dim(N) > 0$, or*
- *$E$ is the total space of some fiber bundle $M \to E \to N$ with $dim(N) > 0$ whose transition functions are isometries for $(M, g)$.*

To illustrate how this fact might be proven, let us consider the product bundle $M \to M \times N \to N$. If $h$ is a metric on $N$, then the metric $g \times h$ on $M \times N$ has scalar curvature:

$$Sc_{g\times h}((x, y)) = Sc_g(x) + Sc_h(y)$$

which can be negative if $Sc_h(y)$ is. The idea is to 'shrink' $M$ by replacing $g$ with $tg$ and allowing $t$ to approach 0. We have:

$$Sc_{tg\times h}((x, y)) = \frac{1}{t}Sc_g(x) + Sc_h(y)$$

As $M \times N$ is compact, we can choose some positive $t$ to ensure that the right hand side is always positive. This idea can be generalized to any fiber bundle $M \to E \to N$.

By combining corollary 3.2 and the facts above, we get:

**Proposition 4.2.** *Every simply connected, closed, non-spin manifold of dimension $n \geq 5$ admits a positive scalar curvature metric.*

## 5. Spin manifolds and the $\widehat{A}$ genus

The $\widehat{A}$ genus is an obstruction to spin manifolds admitting positive scalar curvature metrics.

**Definition 5.1.** If $L \to X$ is a line bundle define:

$$\widehat{A}(L) = \frac{c_1(L)/2}{\sinh(c_1(L)/2)} \in H^1(X; \mathbb{Q})$$

We extend this definition to sums of line bundles by:

$$\widehat{A}(E \oplus F) = \widehat{A}(E) \cdot \widehat{A}(F)$$

and further define $\widehat{A}$ on all bundles by naturality and the splitting principle.

For an oriented manifold $M$ of dimension $n = 4k$ we define

$$\widehat{A}(M) = \langle \widehat{A}(TM), [M] \rangle \tag{5.2}$$

where $TM$ is the tangent bundle, $[M]$ is the fundamental class in $H_n(M, \mathbb{Q})$ and $\langle -, - \rangle$ is the evaluation pairing

$$H^n(M; \mathbb{Q}) \otimes H_n(M; \mathbb{Q}) \to \mathbb{Q}$$

**Theorem 5.3** (Lichnerowicz)**.** *Let $M$ be a spin manifold of dimension disivible by 4 which admits a positive scalar curvature metric. Then $\widehat{A}(M) = 0$*

To prove this theorem, we need to introduce Dirac operators. If $n = 2k$ then $Cl_n \otimes \mathbb{C}$ is the algebra of $2^k \times 2^k$ complex matrices and so it naturally acts on the spinor module $\Delta = \mathbb{C}^{2^k}$; in particular, $Spin(n) \subseteq Cl_{2k}$ acts on $\Delta$.

**Definition 5.4.** If $M$ is a $n$-dimensional spin manifold then starting with the principal $Spin(n)$ bundle $P_{Spin}(TM) \to M$ we can produce the associated fiber bundle

$$\Delta \to P_{Spin}(M) \times_{Spin(n)} \Delta \to M \tag{5.5}$$

This is called the complex spinor bundle; we shall denote its total space by $S$ and its smooth sections by $\Gamma(S)$.

The action $\mathbb{R}^n \otimes \Delta \subseteq Cl_n \otimes \mathbb{C} \otimes \Delta \to \Delta$ induces a map of bundles

$$TM \otimes S \to S$$

We shall use this map to define Dirac operators:

**Definition 5.6.** The Dirac operator $D : \Gamma(S) \to \Gamma(S)$ is given by:

$$D(\psi)(x) = \Sigma_i e_i \cdot \nabla_{e_i}(\psi) \tag{5.7}$$

where $\{e_1, e_2, \ldots\}$ is an orthonormal basis of $T_x M$ and $\nabla_{e_i}(\psi) \in S_x$ is the covariant derivative of $\psi$ in the direction of $e_i$.

We can define a $\mathbb{Z}/2$ grading on $\Delta$ by setting $\Delta^+$ to consist of points fixed under multiplication by the volume element $\omega_{\mathbb{C}} = i^k e_1 \cdots e_n \in Cl_n \otimes \mathbb{C}$, and $\Delta^-$ to consist of points $p$ such that $\omega_{\mathbb{C}} p = -p$. This induces a grading on the spinor bundle $S = S^+ \oplus S^-$ and it can be shown that the Dirac operator $D$ takes $\Gamma(S^+)$ to $\Gamma(S^-)$ and vice-versa. We denote the restrictions of $D$ on $S^+, S^-$ by $D^+, D^-$ respectively; it turns out that $D^-$ is the adjoint operator of $D^+$ ([1]).

To relate Dirac operators to $\widehat{A}(M)$ we shall need the notion of the index of an elliptic differential operator such as $D$ (or $D^+, D^-$).

**Definition 5.8.** The index of an elliptic differential operator $D$ is

$$index(D) = dimKer(D) - dimCoker(D) \tag{5.9}$$

It turns out that if $M$ is a spin manifold of dimension divisible by 4,

$$index(D^+) = \widehat{A}(M) \tag{5.10}$$

This is a consequence of the Atiyah-Singer index theorem, which asserts that the analytical index defined above is equal to a certain topological invariant called the topological index; one can then compute directly that the topological index of $D^+$ is $\widehat{A}(M)$.

We will need one more ingredient for the proof of Theorem 5.3, the Lichnerowicz–Weitzenböck formula:

$$D^2\psi = \nabla^*\nabla\psi + \frac{1}{4}Sc \cdot \psi \tag{5.11}$$

Here $\nabla : \Gamma(S) \to \Gamma(S \otimes T^*M)$ is a connection on the spinor bundle $S$ induced by lifting the Levi-Cevita connection of $TM$ to $P_{Spin(M)}$ ($T^*M$ is the cotangent bundle). We use $\nabla^* : \Gamma(S \otimes T^*M) \to \Gamma(S)$ to denote the adjoint to $\nabla$ and $Sc$ to denote the scalar curvature function on $M$.

*Proof.* (Of Theorem 5.3) Let $\langle -, - \rangle$ be an inner product on each fiber of the spinor bundle. We define the inner product of sections in $\Gamma(S)$ by:

$$\langle \psi, \phi \rangle = \int_M \langle \psi(x), \phi(x) \rangle dvol(x) \tag{5.12}$$

Now if $\psi$ is the kernel of $D$, then

(5.13)

$$0 = \langle D^2\psi, \psi \rangle = \langle \nabla^*\nabla\psi + \frac{1}{4}Sc \cdot \psi, \psi \rangle = |\nabla\psi|^2 + \frac{1}{4}\langle Sc \cdot \psi, \psi \rangle \geq \frac{1}{4}\langle Sc \cdot \psi, \psi \rangle$$

Since $Sc > 0$ by assumption, this implies that $\psi = 0$ i.e. $Ker(D) = 0$. Thus $Ker(D^+) = Ker(D^-) = 0$ and because $D^-$ is adjoint to $D^+$, $Coker(D^+) = Ker(D^-) = 0$. By definition,

$$index(D^+) = dimKer(D^+) - dimCoker(D^+) = 0$$

and the equality with the topological index gives $\hat{A}(M) = 0$. □

## 6. The Clifford index

Theorem 5.3 can be generalized to manifolds of dimension not divisible by 4 by replacing the $\widehat{A}$ genus with another invariant, called the Clifford index.

**Definition 6.1.** If $M$ is an $n$-dimensional spin manifold we define the $Cl_n$-linear spinor bundle as

$$Cl_n \to Spin(M) \times_{Spin(n)} Cl_n \to M$$

The Dirac operator $D$ can be defined just like for the complex spinor bundle, and now commutes with the Clifford action. In particular, $Ker(D)$ is a $\mathbb{Z}/2$ graded Clifford module. To make it independent of the choice of Riemannian metric on $M$, we consider it as an element of

$$Ker(D) \in M_n/M_{n+1} = KO_n \tag{6.2}$$

(recall the identification found at the end of section 2). This is the Clifford index $\alpha(M) \in KO_n$.

**Theorem 6.3** (Hitchin)**.** *Let $M$ be a spin manifold of dimension $n$. If $M$ admits a positive scalar curvature metric, then $\alpha(M) = 0$*

This is proven similarly to Lichnerowicz's theorem ([12]).

It turns out that, under certain assumptions, the converse is true:

**Theorem 6.4** (Stolz)**.** *Let $M$ be a simply connected, closed, spin manifold of dimension $n \geq 5$. Then $M$ admits a metric with positive scalar curvature metric if and only if $\alpha(M) = 0$.*

In [2], Rosenberg found explicit generators for $Ker(\alpha)$ in dimensions less than 24, that turn out to be cobordism classes of total spaces of $\mathbb{HP}^2$ bundles. This is enough to prove Stolz's theorem in low dimensions by fact 4.1 and the Gromov-Lawson Theorem 3.2. We don't have such explicit generators in higher dimensions, however the same argument can still be applied due to the following result:

**Theorem 6.5** (Stolz)**.** *Let $M$ be a simply connected, closed spin manifold with $\alpha(M) = 0$. Then $M$ is spin cobordant to the total space of a fiber bundle with the fiber $\mathbb{HP}^2$.*

We will explain how this theorem is proved in the remaining two sections of the paper.

## 7. The Spin case

To prove theorem 6.5, we will construct a map $\psi$ so that $Im(\psi) \subset \Omega_*^{Spin}$ is the subgroup represented by total spaces of $\mathbb{HP}^2$ bundles. We first describe a bundle that will be helpful in defining $\psi$.

Let the group of isometries of $\mathbb{HP}^2$ be $G$ and its isotropy subgroup be $H$. Then $\mathbb{HP}^2 \cong G/H$ and the inclusion map $H \to G$ induces a map of classifying spaces $BH \to BG$. This is a fiber bundle with fiber $G/H = \mathbb{HP}^2$ and for any $\mathbb{HP}^2$ bundle $\tilde{N} \to N$, there is some map $f : N \to BG$ such that $f^*(BH) = \tilde{N}$. Then we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{HP}^2 & & \mathbb{HP}^2 \\ \downarrow & & \downarrow \\ \tilde{N} & \xrightarrow{f^*} & BH \\ \downarrow & & \downarrow \\ N & \xrightarrow{f} & BG \end{array}$$

and the subgroup of spaces represented by total spaces of $\mathbb{HP}^2$ bundles of dimension $n > 8$ is given by the image of

$$\psi : \Omega_{n-8}^{Spin}(BG) \to \Omega_n^{Spin} \tag{7.1}$$

which maps the cobordism class of $f : N \to BG$ to the cobordism class of the pullback $\tilde{N}$. From this, we see the theorem is equivalent to the exactness of

$$\Omega_{n-8}^{Spin}(BG) \xrightarrow{\psi} \Omega_n^{Spin} \xrightarrow{\alpha} KO_n \tag{7.2}$$

However, we would like to instead work with an exact sequence involving homotopy groups. To do this, we must first define spectra. An $\Omega$-spectrum $T$ is a sequence of spaces $\{T_q\}$ and maps $s_q : \Sigma T_q \to T_{q+1}$ where we can also take the adjoint of $s_q$ to form a weak homotopy equivalence $o_q : T_q \to \Omega T_{q+1}$. Here $\Omega X$ denotes the loop space of $X$. The homotopy groups of a spectrum $T$ are defined as the colimit:

$$\pi_m(T) = \lim_{\to}(\ldots \to \pi_{m+k}(T_k) \to \pi_{m+k+1}(T_{k+1}) \to \ldots)$$

Where the maps $\pi_{m+k}(T_k) \to \pi_{m+k+1}(T_{k+1})$ are given by the composition

$$\pi_{m+k}(T_k) \xrightarrow{\Sigma_*} \pi_{m+k}(\Sigma T_k) \xrightarrow{(s_k)_*} \pi_{m+k+1}(T_{k+1})$$

The homology of a spectrum can be defined similarly.

It is possible to construct a spectrum $MG$ associated to a group $G$ which satisfies $\pi_*(MG) \cong \Omega_*^G$. This isomorphism is known as the Pontryagin-Thom construction and the spectrum is an example of a Thom spectrum. For more information on how Thom spectra are defined and the Pontryagin-Thom construction, we refer the reader to [6].

We will be particularly interested in $MSpin$. Using the Pontryagin-Thom construction referenced above, it turns out that we can get an exact sequence that is isomorphic to 7.2:

$$\pi_n(MSpin \wedge \Sigma^8 BG_+) \xrightarrow{T_*} \pi_n(MSpin) \xrightarrow{D_*} \pi_n(ko)$$

where $ko$ is the connective real $K$-theory spectrum, which has $\pi_n(ko)$ trivial for $n < 0$ and $\pi_n(ko) \cong KO_n$ for $n \geq 0$, $D$ is the orientation class defined in [3]. The definition of $T$ is more complicated and can be seen in [5].

It can be shown that the composition

$$MSpin \wedge \Sigma^8 BG_+ \xrightarrow{T} MSpin \xrightarrow{D} ko$$

is nullhomotopic. Then $T$ can be factored as

$$MSpin \wedge \Sigma^8 BG_+ \xrightarrow{\widehat{T}} \widehat{MSpin} \longrightarrow MSpin$$

where $\widehat{MSpin}$ is the homotopy fiber of $D : MSpin \to ko$. Then to prove exactness of the sequence on homotopy groups, it is enough to show that $\widehat{T}_*$ is surjective. It turns out that it is convenient to localize. This means that for $\mathbb{Z}_{(p)} \cong \{\frac{a}{b} \mid b$ is prime to $p\}$ we will show that

$$\widehat{T}_* : \pi_n(MSpin \wedge \Sigma^8 BG_+) \otimes \mathbb{Z}_{(p)} \to \pi_n(\widehat{MSpin}) \otimes \mathbb{Z}_{(p)}$$

is surjective for $p = 2$ and $p \neq 2$ separately. We will focus on the proof of surjectivity localized at 2 due to Stolz [5]. The proof of surjectivity localized away from 2 is due to Kreck and Stolz [7].

We will use the mod 2 Adams spectral sequence. For a CW spectrum $X$, it converges to $\pi_{t-s}(X) \otimes \mathbb{Z}_2$ and the $E_2$ page is given by:

$$E_2^{s,t} = Ext_{A_*}^{s,t}(H^*(X;\mathbb{Z}/2), \mathbb{Z}/2) \tag{7.3}$$

where $A_*$ is the dual Steenrod algebra given by the homology of the Eilenberg-MacLane spectrum $H\mathbb{Z}/2$ where here and for the remainder of the paper, we are taking homology with $\mathbb{Z}/2$ coefficients. The spectrum is formed by the set of Eilenberg-MacLane spaces

$$\{K(\mathbb{Z}/2, 0), K(\mathbb{Z}/2, 1), K(\mathbb{Z}/2, 2), \ldots\}.$$

In our case, we will be taking $X$ as $MSpin \wedge BG_+$ or $\widehat{MSpin}$. Then the Adams spectral sequence will converge to $\pi_*(MSpin \wedge BG_+) \otimes \mathbb{Z}_2$ and $\pi_*(\widehat{MSpin}) \otimes \mathbb{Z}_2$ respectively.

It can be shown that

**Theorem 7.4** (Stolz)**.** *The induced map on the $E_2$ page*

$$\widehat{T}_* : Ext_{A_*}^{s,t}(\mathbb{Z}/2, H_*(MSpin \wedge \Sigma^8 BG_+)) \to Ext_{A_*}^{s,t}(\mathbb{Z}/2, H_*(\widehat{MSpin}))$$

*is a surjection for all $s, t$.*

However, to get a surjection on homotopy groups we would need a surjection on the $E_\infty$ terms. Then we need the following result which we will prove in the remainder of this paper.

**Theorem 7.5** (Stolz)**.** *The Adams spectral sequence converging to $\pi_*(MSpin \wedge \Sigma^8 BG_+) \otimes \mathbb{Z}_2$ collapses.*

To understand why the spectral sequence collapses, we need to know the structure of $H_*(MSpin \wedge \Sigma^8 BG_+)$ and $H_*(\widehat{MSpin})$ as $A_*$ comodules so that we are able to work with the $Ext$ groups in the $E_2$ page. So we will first establish some results on the $A_*$-comodule structure of $MSpin$-module spectra. Here an $MSpin$-module spectrum $Y$ is a spectrum equipped with a multiplication map $\mu : MSpin \wedge Y \to Y$.

It is important to know that $H_*(MSpin)$ has a subalgebra that is mapped isomorphically into $H_*(ko)$. We have an isomorphism $A_* \cong \mathbb{Z}/2[\zeta_1, \zeta_2, ...]$ where $\zeta_i$ has degree $2^{i-1}$. Then

$$H_* ko \cong \mathbb{Z}/2[\zeta_1^4, \zeta_2^2, \zeta_3, \zeta_4]$$

which is proved in section 4 of [5]. We will use this subalgebra to explain how we can obtain a simplified $A_*$-comodule structure for any $MSpin$ module spectrum. Namely, we will show that any $MSpin$ module spectrum has homology isomorphic to $A_* \Box_{A(1)_*} M$ where

$$A(1)_* = A_*/(\zeta_1^4, \zeta_2^2, \zeta_3, ...)$$

and $M$ is an $A(1)_*$ comodule and $\Box$ is the cotensor product which is analogous to the tensor product of modules. More specifically, if $M, N$ are left and right $A_*$-comodules respectively, then $M \Box_{A_*} N$ is defined to be the kernel of the map

$$\phi = \phi_{M,N} : M \otimes N \to M \otimes A_* \otimes N$$

given by

$$\phi(m \otimes n) = \Delta_1(m) \otimes n - m \otimes \Delta_2(n)$$

where $\Delta_1 : M \to M \otimes A_*, \Delta_2 : N \to N \otimes A_*$ are the comodule structure maps. Now let $Y$ be an $MSpin$ module spectrum with multiplication map $\mu$. Note the induced map on homology $\mu_*$ makes $H_*(Y)$ a module over $H_*(MSpin)$. It is a fact that

$$R := \mathbb{Z}/2[\zeta_1^4, \zeta_2^2, \zeta_3, \zeta_4] \cong A_* \square_{A(1)_*} \mathbb{Z}/2$$

If we restrict $\mu_*$ to $R \otimes H_*(Y)$, then we can see $H_*Y$ as a module over $R$. Now let $\overline{H_*(Y)} = \mathbb{Z}/2 \otimes_R H_*(Y)$ be the $R$-indecomposable quotient of $H_*(Y)$. It turns out we have an isomorphism

$$\phi_Y : H_*(Y) \to A_* \square_{A(1)_*} \overline{H_*(Y)} \tag{7.6}$$

which is our desired comodule structure [5].

## 8. Showing the Adams Spectral Sequence Collapses

We are now ready to compute the $E_2$ term of the Adams spectral sequence

$$E_2^{s,t} = Ext_{A_*}^{s,t}(\mathbb{Z}/2, H_*(MSpin \wedge BG_+))$$

Since $MSpin \wedge BG_+$ can be considered as an $MSpin$ module spectrum we have by the results in the previous section:

$$H_*(MSpin \wedge BG_+) \cong A_* \square_{A(1)_*} \overline{H_*(MSpin \wedge BG_+)} = A_* \square_{A(1)_*} (\overline{H_*(MSpin)} \otimes H_*(BG_+))$$

Then by the change of rings isomorphisms, we have

$$Ext_{A_*}^{s,t}(\mathbb{Z}/2, H_*(MSpin \wedge BG_+)) \cong Ext_{A(1)_*}^{s,t}(\mathbb{Z}/2, \overline{H_*(MSpin)} \otimes H_*(BG_+))$$

so we only need to find the structure of $\overline{H_*(MSpin)} \otimes H_*BG_+$ as an $A(1)_*$ comodule. This turns out to be difficult but there is a result of Margolis [11] which says any free $A_*$ comodule that is a direct summand of $\overline{H_*(MSpin)} \otimes H_*BG_+$ splits off as some Eilenberg MacLane spectrum. Then we only need to find a stably equivalent comodule structure $N$. That is an $N$ such that $N \oplus F' \cong \overline{H_*(MSpin)} \otimes H_*BG_+ \oplus F'$ where $F$ and $F'$ are free $A_*$ comodules.

Particularly, Anderson, Brown, and Peterson found the direct summands for a stably equivalent comodule for $H_*(MSpin)$ in [9] and Stolz found direct summands for a stably equivalent comodule for $H_*(BG_+)$ in [5]. The desired Ext groups for these summands have all been determined in [10]. They turn out to have a periodicity $Ext^{r,s} \cong Ext^{r+4,s+12}$ and so we have a periodicity of $r - s$ degree 8 in our Ext groups. As the $E_2$ page we are considering is formed by direct sums of copies of these Ext groups, then we also have this periodicity for the $E_2$ page as well.

By inspection of the Ext groups of each of the summands, we have for degree reasons that nonzero differentials can occur only in $r - s$ degree 1 or 2 mod 8. However, by the multiplicativity of the spectral sequence the elements from which the differentials originate are formed by multiplication of some $h_1 \in Ext^{1,2}$ with elements with $r - s$ degree 0 mod 8 from which only trivial differentials originate. Then all the differentials are trivial and the spectral sequence collapses.

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## References

[1] Blaine Lawson and Marie-Louise. Spin Geometry. Princeton University Press. 1990
[2] Jonathan Rosenberg. Manifolds of Positive Scalar Curvature: A Progress Report. 2006
[3] Michael Atiyah, Raoul Bott, and Arnold Shapiro. Clifford Modules. 1963
[4] Stephan Stolz. Manifolds of Positive Scalar Curvature. 2006
[5] Stephan Stolz. Simply Connected Manifolds of Positive Scalar Curvature. Annals of Mathematics, 1992.
[6] Robert Stong. Notes on Cobordism Theory. Princeton University Press and Tokyo University Press, 1968
[7] Mathias Kreck and Stephan Stolz. $\mathbb{HP}^2$-bundles and elliptic cohomology. Acta Math. 171 (1993), 231-261
[8] Mikhael Gromov and Blaine Lawson. The Classification of simply Connected Manifolds of Positive Scalar Curvature. Annals of Mathematics, 111 (1980), 423-434
[9] D. W. Anderson, E. H. Brown, F. P. Peterson, The structure of the spin cobordism ring, Ann. of Math 86 (1967), 271-298
[10] J. F. Adams and S. Priddy, Uniqueness of BSO, Math. Proc. Cambridge Phil. Soc. 80 (1978), 475-509
[11] H. R. Margolis, Eilenberg-MacLane spectra, Proc. A.M.S. 43 (1974), 409-415
[12] N. Hitchin, Harmonic Spinors, Advances in Mathematics, 14 (1974), 1-55