INTRODUCTION TO K-THEORY

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Abstract. In this paper we introduce topological K-theory, a generalized cohomology theory. We begin by introducing vector bundles and work towards Bott's periodicity theorem. Finally, some example calculations of $K(X)$ are made.

Contents

1. Introduction

In this paper we introduce topological K-theory, which is a generalized cohomology theory. The connections provided by K-theory between algebraic topology and other fields of mathematics make it a specially important cohomology theory. Additionally, extensions of K-theory have been found instrumental in characterizing phases of matter in Quantum Physics as well as in other physical problems.

2. Vector Bundles

2.1. **First Definitions and Examples.** We begin with a primer on vector bundles–the basic building blocks of Topological K-Theory–and their properties.

Definition 2.1. A vector bundle (p, E, X) over a field $\mathbb{K}$ (e.g. $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$) is a map $p : E \to X$ such that

(1) For any element $x \in X$, the fiber $p^{-1}(x)$ is homeomorphic to $\mathbb{K}^n$ for some positive integer n;
(2) There exists a collection $\{U_\alpha\}$ of open covers of X such that for each U_α there is a homeomorphism (often called a *trivialization*) $h_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{K}^n$ for some positive integer n, taking each fiber $p^{-1}(b)$ to $b \times \mathbb{K}^n$ by a vector space isomorphism;

(3) The homeomorphisms, which are called the transitions functions,
$$h_\alpha \circ h_\beta^{-1}|_{(U_\alpha \cap U_\beta)\times\mathbb{K}^n} : (U_\alpha \cap U_\beta) \times \mathbb{K}^n \to (U_\alpha \cap U_\beta) \times \mathbb{K}^n$$
are linear isomorphisms in each fiber $b \times \mathbb{K}^n$.

The space E is called the *total space*, but is often referred to as the vector bundle itself, and X is the *base space*. A vector bundle is said to have *dimension* n if the fiber over each point is homeomorphic to $\mathbb{R}^n$. If the base space is not connected, the dimension of the vector bundle is not necessarily well defined since fibers over different points might have different dimension.

As one would expect, vector bundles form a category, which we will denote by **Vect**. The objects in **Vect** are vector bundles and the morphisms from a vector bundle (p_1, E_1, X_1) to a second vector bundle (p_2, E_2, X_2) are pairs (f, g) with $f : X_1 \to X_2$ and $g : E_1 \to E_2$, making the following diagram commute:

$$\begin{tikzcd} E_1 \arrow[r, "g"] \arrow[d, "p_1"] & E_2 \arrow[d, "p_2"] \\ X_1 \arrow[r, "f"] & X_2 \end{tikzcd}$$

such that for a given fiber $p_1^{-1}(x)$, the map g restricted to the fiber $p_1^{-1}(x)$ is a linear map between vector spaces. A morphism (f, g) of vector bundles is an isomorphism if and only if f is a homeomorphism and g restricts to an isomorphism on each fiber. Given a vector bundle $p : E \to X$, $E_0 \subset E$ is a **subbundle** if $p|_{E_0} : E_0 \to X$ is a vector bundle.

We call two vector bundles over the same base space X *equivalent* if they are isomorphic by a morphism (id_X, g). In this paper, we are mainly interested in the subcategory of n-dimensional vector bundles. For a topological space X, we denote by $\mathcal{E}_n(X)$ the set of equivalence classes of n-dimensional vector bundles over X.

There is a very useful characterization of isomorphism of vector bundles, as indicated by the following lemma:

Lemma 2.2. *Consider a morphism of vector bundles $f : (p_1, E_1) \to (p_2, E_2)$ over the same base space X. Then, f is a bundle isomorphism if $f_x := f|_{p_1^{-1}(x)}$ is a linear isomorphism for each fiber $p_1^{-1}(x)$.*

Proof. We just need to show that there is a continuous inverse to f. This is the case because f is one-to-one and onto as f_x is a linear isomorphism and f takes each fiber in E_1 to the corresponding fiber in E_2. Continuity is only concerned with local matters of the space at hand, so we may restrict our attention to an open subset of B over which E_1, E_2 are trivial. Together with the local trivializations, we may then consider only continuous functions f from $U \times \mathbb{K}^n$ to $U \times \mathbb{K}^n$ sending $(x, v) \to (x, A_x \cdot v)$, where A_x belongs to the group of invertible linear transformations, $\mathrm{GL}_n(\mathbb{K})$. A_x is a matrix whose entries depend continuously on x as f is continuous. The inverse of A_x, A_x^{-1}, must also depend on x continuously as its entries can all be nicely written in terms of those of A_x. This gives a continuous inverse to f sending $(x, v) \to (x, A_x^{-1}(v))$. □

Example 2.3. A very natural vector bundle would be a projection of $X \times \mathbb{K}^n$ on to its first coordinate. Any n-dimensional vector bundle over X that is isomorphic to $X \times \mathbb{K}^n$ is called a *trivial* n-dimensional vector bundle. With this in mind, note that the trivialization condition in the definition of vector bundle is telling us that locally every vector bundle is trivial.

Example 2.4. Let X be a real n-dimensional manifold. Then, the **tangent bundle** TX is a vector bundle of X. For each $x \in X$, the fiber T_xX is the n-dimensional vector space lying tangent to the manifold.

Example 2.5. Let $E := [0,1] \times \mathbb{R}/\sim$, where $\sim$ is the equivalence given by the identification of $(0,t)$ with $(1,t)$. Quotienting the projection $I \times \mathbb{R} \to I$ by the equivalence above, we get a map $p : E \to S^1$. This is a vector bundle usually called the Mobius bundle.

Example 2.6. Recall that the n-dimensional real projective space $\mathbb{R}P^n$ is defined to be the set of lines through the origin in $\mathbb{R}^{n+1}$. Similarly, the n-dimensional complex projective space $\mathbb{C}P^n$ is defined to be the set of lines through the origin in $\mathbb{C}^{n+1}$. Over each of these spaces, we have the so-called **tautological line bundle**. Over $\mathbb{R}P^n$, the tautological line bundle γ_n^1 is the line bundle $p : E \to \mathbb{R}^{n+1}$ with total space,

$$E(\gamma_n^1) = \{(L, v) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} | v \in L\} \subset \mathbb{R}P^n \times \mathbb{R}^{n+1}$$

where $p : E \to \mathbb{R}^{n+1}$ is just the projection, sending $(L, v) \mapsto L$. An analogous definition is given for the tautological line bundle over $\mathbb{C}P^n$. We will see in Section 1.3 the importance of the tautological line bundles over projective spaces above.

Now that we have introduced the notion of isomorphism between n-dimensional vector bundles, one might wonder if there is an easy way to distinguish between two non-isomorphic vector bundles or determine when a vector bundle is trivial. And the answer is "yes" but for that we need to introduce the notion of a **section** on a vector bundle.

Definition 2.7. Let (p, E, X) be a n-dimensional vector bundle. A continous map $s : X \to E$ is a **section** if $p \circ s = id_X$. We can think of sections as a continuously varying family of vectors over X.

Note that every vector bundle has a canonical section, the **zero section**, which assigns to each point in the base space the zero vector in each fiber. By definition of vector bundles, one can see that the image of the zero section is a subspace of the total space which projects homeomorphically onto the base space. In addition, note that one can distinguish non-isomorphic vector bundles by looking at the complement of the zero section since any vector bundle isomorphism $f : E_1 \to E_2$ must take the zero section of E_1 onto the zero section of E_2, so that the complement of the zero sections in E_1 and E_2 must be homeomorphic.

Given the zero section, one might wonder if every vector bundle admits a section whose values are all nonzero, and the answer is no. For example, if one considers the tangent bundle of the n-sphere S^n, one can show that TS^n has a nonvanishing section if and only if n is odd. This tells us that the tangent bundles of S^{2n} are far from being trivial since a trivial bundle has a nonvanishing section, and an isomorphism between vector bundles takes nonvanishing sections to nonvanishing sections. The following lemma establishes a stronger characterization as to when an n-dimensional vector bundle is trivial:

Lemma 2.8. *Let (p, E, X) be an n-dimensional vector bundle. Then $p : E \to B$ is isomorphic to the trivial bundle if and only if it has n sections $s_1, \cdots, s_n$ such that the vectors $s_1(x), \cdots, s_n(x)$ are linearly independent in each fiber $p^{-1}(x)$.*

Proof. If the bundle has n sections s_i such that the vectors $s_i(b)$ are independent in each section, we construct the following map:

$$h : B \times \mathbb{R}^n$$
$$h(b, t_1, \ldots, t_n) = \sum_i s_i(b) t_i$$

This map is a linear isomorphism in each fiber. Since its composition with a local trivialization $p^{-1}(U) \to U \times \mathbb{R}^n$ is continuous, the map h is itself continuous. This gives us an isomorphism between bundles since a continuous map between two bundles is an isomorphism whenever we also have a linear isomorphism between corresponding fibers by Lemma 2.2. The other direction follows from the fact that the trivial bundle does indeed have such sections and that an isomorphism between vector bundles takes linearly independent sections to linearly independent sections. □

The tautological line bundles over projective spaces are not trivial. In particular, we can show that one cannot find a nonvanishing section for them while the trivial bundle has nonvanishing sections. Consider the case where $n = 1$, the real projective space $\mathbb{R}P^1$ is isomorphic to S^1 with antipodal points identified with each other. Take a point $x \in S^1$ and let $s(x) = v$ be a radial vector - that is, in the line defined by x and its antipodal point $\overline{x}$. Going around the circle, the vectors $s(x)$ will have to go from pointing away from the center of S^1 to pointing towards the center such that $s(y) = 0$ for some $y \in S^1$ - as $s(\overline{x}) = v$ and the map must be continuous. A similar visualization and argument can be made for higher dimensions.

2.2. **Operations on Vector Bundles.** Now that we understand what vector bundles are, we can study all the operations we can perform on them. Given that the fibers of a vector bundle are finite dimensional vector spaces, it turns out there are very natural extensions of some linear algebra operations to vector bundles. Here, we will briefly mention how direct sums, tensor products and inner products show up. We will see that the tensor product and the direct sum operations *almost* produce a ring structure on the set of equivalence classes of vector bundles over a given space X, which we can complete with a Grothendieck construction, leading us to the K-rings of K-theory (see Section 3).

Linear Algebra. Here are some canonical operations from vector spaces that we can perform on vector bundles by applying the operations fiber-wise:

(1) The **direct sum or Whitney sum operation**: Given two vector bundles $p_1 : E_1 \to X$, $p_2 : E_2 \to X$ over the same base space X, the Whitney sum $E_1 \oplus E_2$ is defined (as a set) to be $E_1 \oplus E_2 := \{(v_1, v_2) \in E_1 \times E_2 | p_1(v_1) = p_2(v_2)\}$. Note that this definition where corresponding fibers are summed gives us a pretty natural projection map $p : E_1 \oplus E_2 \to X$, and $p^{-1}(x) = p_1^{-1}(x) \oplus p_2^{-1}(x)$ so that the name is reasonable.

(2) The **tensor product operation**: Given two vector bundles $p_1 : E_1 \to X$, $p_2 : E_2 \to X$ over the same base space X, then their tensor product $E_1 \otimes E_2$ is defined (as a set) to be $E_1 \otimes E_2 := \{\sqcup p_1^{-1}(x) \otimes p_2^{-1}(x) : x \in B\}$.
(3) The **exterior power operation**: Given a vector bundle $p : E \to X$, then the k-exterior power $\Lambda^k(E)$ is defined to be a vector bundle whose fibers are $\Lambda^k(F_x)$, where F_x are fibers of E.
(4) The **Hom-bundle operation**: Given two vector bundles $p_1 : F \to X$ and $p_2 : E \to X$, then the Hom-bundle $Hom(E, F)$ is defined such that the fiber over a point $x \in X$ is the set of linear maps from the fiber E_x to the fiber F_x.
(5) The **dual operation:** Given a vector bundle $E \to X$, then the dual vector bundle E^* is a vector bundle whose fiber over $x \in X$ is the dual space of the fiber of E over x, that is the space of linear forms over the vector space E_x.

We note that the set $\boldsymbol{Vect}(X)$ then forms a semi-ring under the Whitney sum and the tensor product described above, with $\mathbb{R} \times X$ being the multiplicative identity and $0 \times X$ the additive one.

Another important concept in linear algebra is the notion of an **inner product**, which allows us to define the notion of *orthogonality*. In turn, for instance, this allows us to ask for the complement of a subspace $V_1 \subset V$, such that $V_1 \oplus V_1^{\perp} = V$. In the context of vector bundles, we will now see that, under certain conditions, we can find a complement to subbundles.

Definition 2.9. An *inner product* for a vector bundle E is a map $\langle -, - \rangle : E \oplus E \to \mathbb{K}$ which restricts to an inner product over each fiber.

Proposition 2.10. *If X is paracompact, then any vector bundle $p : E \to B$ over X admits an inner product.*

Proof. We first provide the general idea of the proof. The inner product we have to go work with is the one in $\mathbb{R}^n$, so we want to take that to the relevant space and show that we get a nice inner product defined on each fiber. To take $\mathbb{R}^n$'s inner product to each fiber of E, we may relate E to $\mathbb{R}^n$ through the local trivializations and the vector bundle itself. So, we have $h_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ and with it we can pullback the standard inner product in $\mathbb{R}^n$ to one on $p^{-1}(U_\alpha)$, $\langle \cdot, \cdot \rangle_\alpha$. Because B is said to be paracompact, we know we have, for each cover, a subordinate partition of unity, $\{\varphi_\beta\}$. Then, without any complications as we have only finitely many φ_β non-zero near any point in B, we may stitch the different $\langle \cdot, \cdot \rangle_\alpha$ by setting the desired inner product to be:

$$\langle v, w \rangle := \sum_\beta \varphi_\beta(p(v)) \langle v, w \rangle_{\alpha(\beta)}$$

where $U_{\alpha(\beta)}$ contains the support of $\{\varphi_\beta\}$. □

Once we have an inner product on vector bundles, we can now talk about orthogonal complement.

Proposition 2.11. *If E is a vector bundle over a paracompact base space X and $E_0 \subset E$ is a subbundle, then there is a complement subbundle $E_0^{\perp} \subset E$ such that $E_0 \oplus E_0^{\perp} \approx E$.*

Proof. If we show that $E_0^\perp \to B$ is a vector bundle, we will then easily have that $E_0 \oplus E_0^\perp \simeq E$. We now show that the local trivialization condition is fulfilled by $E_0^\perp$, but since this is only a question about the local behavior, we may take E to be $B \times \mathbb{R}^n$. We claim that the restriction of the local trivialization h of E to $E_0^\perp$ is the local trivialization for $E_0^\perp$. We will find a set of n independent vectors locally for all $b \in U$. If E_0 is m-dimensional, it has as many independent sections, $\{s_i\}_1^m$. We extend this to a set of n independent vectors by first picking $n-m$ independent vectors $s_{m+1}, \dots, s_n$ in the fiber $p^{-1}(b_0)$. Since the determinant is a continuous function, we may take these same n vectors for all nearby fibers and they will still be independent. We may turn these independent vectors into orthogonal ones with Gram-Schmidt, obtaining orthogonal s_i' such that $\{s_i'\}_1^m$ are a basis for E_0. This allows us to define a trivialization h sending E_0 to $U \times \mathbb{R}^m$ and $E_0^\perp$ to $U \times \mathbb{R}^{n-m}$, which gives us that the restriction of h to $E_0^\perp$ is what we were after. □

Given the proposition above, one might wonder if given an n-dimensional vector bundle $p : E \to X$, one can provide an embedding of E into a trivial bundle. The answer is "yes" provided we ask that the base space x is compact (here, paracompact is not enough).

Lemma 2.12. *Given a vector bundle $E \to X$ over a compact Hausdorff base space X, there exists a vector bundle $E' \to X$ such that $E \oplus E'$ is the trivial bundle.*

Proof. If we show that an arbitrary vector bundle $p : E \to B$ is a subbundle of a trivial vector bundle we are done by Proposition 2.11. We will find, using Urysohn's lemma, a finite cover of the base space B and use the functions given to us by Urysohn's lemma to construct projections of the trivializations h_x to build an isomorphism between E and a subbundle of $B \times \mathbb{R}^N$.

At each point x, there is a neighborhood U_x over which E is trivial. Urysohn's guarantees the existence of a function $\phi_x : B \to [0, 1]$ that is non-zero at x and zero outside U_x. The set $\{\phi_x^{-1}((0, 1]) : x \in B\}$ is a cover of B, since B is compact, we have a finite cover $\{\phi_i^{-1}((0, 1])\}_i$.[1] Finally, we may construct coordinate maps

$$g_i : E \to \mathbb{R}^n$$
$$g_i(v) := \phi_i(p(v)) \cdot \pi_i \circ h_i(v)$$

and let $g := (g_1, \dots, g_n) : E \to \mathbb{R}^N$, where $\pi_i : B \times \mathbb{R}^n \to \mathbb{R}^n$ is the projection on the coordinate. Finally, consider $f := (p, g) : E \to B \times \mathbb{R}^N$; g is a linear injection on each fiber and the image of f is a subbundle of $B \times \mathbb{R}^N$. The result follows. □

Pullback. Besides vector space operations, we can perform other categorical constructions on vector bundles - here, we will review the notion of *pullback* on vector bundles. As we will see below, this is a very important constructions since it allows us to treat $\mathcal{E}_n(-)$, as defined in the previous section, as a set-value contravariant functor from the category of topological spaces. The idea is very simple. Suppose we are given a vector bundle $p : E \to B$ and a map $f : A \to B$, where A is any topological space. Then, it is somewhat natural to wonder if we can build a vector bundle over A from the one over B.

Given two topological spaces B and A, a continuous map $f : A \to B$, and a vector bundle $p : E \to B$, the pullback of (p, E, B) along f, denoted as (f^*p, f^*E, A), is the vector bundle fitting into the following commutative diagram:

[1] B is assumed compact and Hausdorff, so it's normal.

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d, "f^*p"] & E \arrow[d, "p"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

Such an object can be constructed quite naturally. And in fact, it has the universal property, as illustrated by the following proposition.

Proposition 2.13. *Given a vector bundle $p : E \to B$ and a map $f : A \to B$, there exists a vector bundle $f^*p : f^*E \to A$ with a map $f^* : E \to E$ taking the fibers of f^*E isomorphically onto the corresponding fibers of E. Therefore, there exists a function $f^* : \mathcal{E}_n(B) \to \mathcal{E}_n(A)$ taking the isomorphism class of E to that of f^*E.*

Proof. Let $f^*E := \{(a, v) \in A \times E : f(a) = p(v)\}$, f^*p and f^* be projections on to the first and second coordinates, respectively. The following diagram commutes,

$$\begin{tikzcd} f^*E \arrow[r, "f^*"] \arrow[d, "f^*p"] & E \arrow[d, "p"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

We will now show that $f^*p : f^*E \to A$ as constructed above is a vector bundle. First, note that because $p : E \to B$ is a vector bundle, $\mathbb{1} \times p : A \times E \to A \times B$ is easily seen to be a vector bundle and so is its restriction to $\{(a, f(a)) \in A \times B : a \in A\}$. Lastly, the projection $(a, f(a)) \to a$ is a homeomorphism and so, because f^*p factors as a map from $f^*E \to \{(a, f(a)) \in A \times B : a \in A\} \to A$ by $(a, v) \to (a, f(a)) = (a, p(v)) \to a$, f^*p is a vector bundle over A.

Uniqueness follows by simply noting that if $p' : E' \to A$ is a vector bundle satisfying the proposition with the map $f' : E' \to E$, the natural map $v' \to (p'(v'), f'(v'))$ takes the fibers of this arbitrary E' to their corresponding fibers in the vector bundle f^*E by a vector space isomorphism. Then, the result follows by invoking Lemma 2.2. □

Thus, we can think of $\mathcal{E}_n(-)$ as a contravariant set-valued functor from the category of topological spaces. The vector bundle f^*E constructed above is often referred to as the *pullback bundle*. Note that we could similarly define a set-valued functor $\mathcal{E}(-)$, giving us all vector bundles over a given topological space.

Given the pullback construction and the vector space operations presented above, one might wonder how they interact with one another. As one would hope, the pullback construction behaves as expected with respect to composition of functions, direct sum and tensor product:

$$(fg)^*(E) \simeq g^*(f^*(E))$$
$$1^*(E) \simeq E$$
$$f^*(E_1 \oplus E_2) \simeq f^*(E_1) \oplus f^*(E_2)$$
$$f^*(E_1 \otimes E_2) \simeq f^*(E_1) \otimes f^*(E_2)$$

One important result is that pullbacks by homotopic maps are isomorphic.

Theorem 2.14. *If $f_0, f_1 : A \to B$ are homotopic maps and A is compact Hausdorff, then we have that $f_0^*(E)$ is isomorphic to $f_1^*(E)$, where E is a vector bundle over B.*

This theorem is an immediate consequence of the following technical proposition

Proposition 2.15. *If X is paracompact, given a vector bundle $E \to X \times I$, its restrictions over $X \times \{0\}$ and $X \times \{1\}$ are isomorphic.*

For the proof of the theorem, just take the homotopy between f_0 and f_1 given by $F : A \times I \to B$ and note that the restrictions of $F^*(E)$ over $A \times \{0\}$ and $A \times \{1\}$ are $f_0^*(E)$ and $f_1^*(E)$, respectively. See [2].

An important result then follows easily:

Corollary 2.16. *Let f be a homotopy equivalence $f : A \to B$ of paracompact spaces, then we have a bijection $f^* : Vect^n(B) \to Vect^n(A)$. In particular, every vector bundle over a contractible space is trivial.*

Proof. If g is a homotopy inverse of f, then it's clear we have $f^* g^* = 1$ and $g^* f^* = 1$. So, g^* is the inverse of f^*, and we have the desired bijection. $\square$

2.3. **Classifying Vector Bundles.** In general, classifying all vector bundles over a give base space is difficult. In this subsection we will first work our way through the classification of bundles over S^k, one of the more tractable cases, to later arrive at the main result we prove of vector bundles: for paracompact base spaces, we have a bijective correspondence between the isomorphism classes of vector bundles and homotopy classes of maps between the base space, X, and the Grassmanian manifold G_n, a space we will later precisely define.

Universal Bundle. It turns out that all n-dimensional vector bundles can be constructed as pullbacks of one special vector bundle, the *universal bundle*, $E_n \to G_n$, where we define:

Definition 2.17. The Grassmanian manifold $G_n(\mathbb{C}^k)$ is, as a set, the set of n-dimensional subspaces in $\mathbb{R}^k$. Its topology is obtained as the quotient topology obtained from the natural surjective map $V_n(\mathbb{C}^k) \to G_n(\mathbb{C}^k)$, where $V_n(\mathbb{C}^k)$ is called the Stiefel manifold and is given by n-tuples of orthogonal vectors in $\mathbb{C}^k$.[2]

Definition 2.18. There is a canonical n-dimensional bundle of $G_n(\mathbb{C}^k)$ given by $E_n(\mathbb{C}^k) := \{(l, v) \in G_n(\mathbb{C}^k) \times \mathbb{C}^k | v \in l\}$

We topologize $G_n(\mathbb{C}^\infty) := \bigcup_k G_n(\mathbb{R}^k)$ and $E_n(\mathbb{C}^\infty) := \bigcup_k E_n(\mathbb{C}^k)$ with the weak topology, open subsets of $G_n(\mathbb{C}^\infty)$ and $E_n(\mathbb{C}^\infty)$ should intersect with each $G_n(\mathbb{C}^k)$ and $E_n(\mathbb{C}^k)$, respectively, in an open set. Note that the inclusion $\mathbb{C}^k \subset \mathbb{C}^{k+1}$ gives us the inclusions $G_n(\mathbb{C}^k) \subset G_n(\mathbb{C}^{k+1}) \subset \ldots$ and $E_n(\mathbb{C}^k) \subset E_n(\mathbb{C}^{k+1})$.

Before we prove the general classification mentioned above, we state the following lemma:

Lemma 2.19. *The projection $\rho : E_n(\mathbb{C}^k) \to G_n(\mathbb{C}^k)$, $p(l, v) = v$, is a vector bundle, for finite and infinite k.*

We won't prove the above lemma, but it's worth giving a general idea of the proof. We focus on the finite dimensional case. Pick $l \in G_n(\mathbb{C}^k)$ and let π_l be the

[2]Note that the Stiefel manifold inherits as a subspace the topology of the product of n copies of S^{k-1}. The surjection sends the n-tuple of orthogonal vectors to the subspace spanned by them.

projection onto l, then let U_l be the set of elements of $G_n(\mathbb{C}^k)$ whose projection onto l is n-dimensional,

$$U_l := \{l' \in G_n(\mathbb{C}^k) : \pi_l(l') \text{ is n-dimensional}\}$$

The outline of the proof is as follows: show that U_l is open and that the map $h : p^{-1}(U_l) \to U_l \times \mathbb{C}^n$ given by $h(l', v) = (l', \pi_l(v))$ is a local trivialization of $E_n(\mathbb{C}^k)$.

We simplify notation and write G_n and E_n for $G_n(\mathbb{C}^\infty)$ and $E_n(\mathbb{C}^\infty)$. $G_n(\mathbb{C}^\infty)$ is also denoted by $\mathrm{BU}(n)$ and called the classifying space of the unitary group $U(n)$. We now prove the classification theorem:

Theorem 2.20. *For paracompact X, the map $[X, G_n] \to Vect^n(X), [f] \to f^*(E_n)$ is a bijection.*

Proof. Let $p : E \to X$. We first show injectivity of the map $[X, G_n] \to \mathrm{Vect}^n(X)$. If there were two maps $f_0, f_1 : X \to G_n$ such that $E \approx f_0^*(E) \approx f_1^*(E)$, we would get two maps $g_0, g_1 : E \to \mathbb{R}^\infty$:

$$g_i := \pi \circ \widetilde{f_i}$$

where $\pi : E_n \to \mathbb{R}^\infty$, $\pi(l, v) = v$. We have that g_0, g_1 are linear injections on each fiber as π and $\widetilde{f}$ are both linear injections on each fiber. We now show that f_0, f_1 are homotopic by arguing that g_0, g_1 are homotopic. The result will then follow. Note that the injective map $L_t : \mathbb{R}^\infty \to \mathbb{R}^\infty$ given by

$$L_t(x_1, \dots) = (1 - t)(x_1, x_2, \dots) + t(x_1, 0, x_2, 0, \dots)$$

can move the image of g_0 into the odd coordinates. We may similarly move the image of g_1 into the even coordinates. Then, we have a homotopy $g_t = (1 - t)g_0' + tg_1'$, where g_i' are just the g_i's with their coordinates moved into the odd/even coordinates as discussed above. Finally, we have a homotopy between f_0 and f_1: $f_t(x) = g_t(p^{-1}(x))$.

Now we show the map to be surjective. For this, we use that given an open cover $\{U_\alpha\}$ of X, we can find a partition of unity supported in an countable open cover $\{V_k\}$ such that each V_k is a disjoint union of open sets each contained in some U_α.

Given an n-dimensional vector bundle and a cover $\{U_\alpha\}$ of X such that p is trivial over each U_α, by the lemma above, we get a countable open cover $\{U_i\}$ and a partition ϕ_i supported in each U_i. Take the trivialization $p^{-1}(U_i) \to U_i \times \mathbb{R}^n$ and project the image into $\mathbb{R}^n$; let this map be g. Because only finitely many of the ϕ_i are nonzero around a given point and $(\phi_i p)g$ is zero outside $p^{-1}(U)_i$, we may then construct a map $g : E \to \mathbb{R}^\infty$ where the coordinates are given by $(\phi_i p)g$ - a linear injection on each fiber. □

We don't concern ourselves with them here, but there are similar results to Proposition 3.9 and Theorem 2.20 in the case of real bundles.

3. $K(X)$ and $\widetilde{K}(X)$ rings

In this section we first define and present the basic properties of topological K-theory and then introduce some tools, in special Bott's Periodicity Theorem, that allow us to calculate the K-theory of certain spaces. We will showcase the applicability of Bott's theorem, as well as that of some of the other basic properties of the K rings, by calculating the K-theory of some example spaces. We only consider connected spaces and usually ones that also are compact Hausdorff, but we will later note that defining topological K-theory for non-compact spaces works just fine.

3.1. **Definitions and First Results.** Recall that in the previous section we noted that $\textbf{\textit{Vect}}(X)$ forms a semi-ring under the direct sum and tensor product. Now, we define the K-theory of the space X as the completion of this semi-ring through the Grothendieck construction, the same used to get the integers $\mathbb{Z}$ from the natural numbers $\mathbb{N}$.

Definition 3.1. The K-theory $K(X)$ of X is defined to be the ring constructed from $\textbf{\textit{Vect}}(X)$ through the Grothendieck construction.

Proposition 3.2. *Any element of $K(X)$ can be written as the difference between two bundles where one is a trivial bundle, $E - \varepsilon^n$.*

Proof. This is the case because, starting with an element $E - E'$, where we have the formal difference of bundles from the construction of $K(X)$, we can by Lemma 2.12 find a third bundle E" such that $E" \oplus E' \approx \epsilon^n$. Then, $(E - E') + (E" - E") = (E \oplus E") - (E' \oplus E") = E^* - \epsilon^n$, where $E^* = (E \oplus E")$ is a bundle. □

Over pointed spaces, we may consider the map $\textbf{\textit{Vect}}(X) \to \mathbb{Z}$ sending a vector bundle to the dimension of the fiber above the base point. Bearing in mind how the addition and multiplication of vector bundles are defined fiber-wise, we see that this map is an homomorphism between the semi-ring of vector bundle isomorphism classes, $\textbf{\textit{Vect}}(X)$, and their dimensions, $\mathbb{N} \subset \mathbb{Z}$. Furthermore, it gives us an induced homomorphism of rings $K(X) \to \mathbb{Z}$.

Definition 3.3. The reduced K-theory of a space X, $\widetilde{K}(X)$, is defined to be the kernel of the homomorphism $K(X) \to \mathbb{Z}$.

From the above, we get that, being an ideal of $K(X)$, $\widetilde{K}(X)$ is a ring without identity and that $K(X) \simeq \widetilde{K}(X) \oplus \mathbb{Z}$.

Now, we briefly address the representability of K-theory. We do this by connecting the classification of the vector bundles in the previous section with K-theory. Theorem 2.20 established that the n-dimensional vector bundles, those in $\mathcal{E}_n(X)$, are classified by the classes of maps from X into the classifying space: in the complex case, $\mathcal{E}_n(X) \simeq [X, BU(n)]$. If we consider the colimit BU of $BU(n)$ with the union topology, because a map from a compact space X into BU, together with its homotopies, has all of its image contained in one of the $BU(n)$, we get that $[X, BU] \simeq \operatorname{colim}[X, BU(n)]$. Now, we consider the colimit of the left handside of Theorem 2.20, take the colimit of $\mathcal{E}_n(X)$ taken over the map sending a bundle to the Whitney sum of itself with the one-dimensional trivial bundle. To represent K-theory we now work towards relating the colimit of $\mathcal{E}_n(X)$ with the reduced K ring, $\widetilde{K}$. Consider for a moment the following equivalence relation

Definition 3.4. We set two vector bundles to be under our reduced equivalence $E_1 \sim E_2$ if there are trivial bundles of dimensions n, m such that $E_1 \oplus \varepsilon^n \simeq E_2 \oplus \varepsilon^m$. $\mathcal{E}(X)$ denotes the set of classes under this equivalence relation.

Then, as the above notation might suggest, we have that the set of classes under this equivalence relation is the colimit of $\mathcal{E}_n(X)$, which is indeed the case when dealing with connected spaces as we are. If $E_1 \oplus \varepsilon^{q-n} \simeq E_2 \oplus \varepsilon^{q-m}$, noting that the colimit map sends $E_1 \in \mathcal{E}_n(X)$ to $E_1 \oplus \varepsilon^{q-n} \in \mathcal{E}_q(X)$, and so on successively, and $E_2 \in \mathcal{E}_m(X)$ to $E_2 \oplus \varepsilon^{q-m} \in \mathcal{E}_q(X)$, we see that the colimit coincides with the reduced class. So that we have:

$$\mathcal{E}(X) \simeq [X, BU]$$

This gives us a classification for K-theory as we have the following relation between K-theory and $\mathcal{E}$:

Proposition 3.5. *There is a natural isomorphism $\mathcal{E}(X) \to \widetilde{K}(X)$.*

And we also have that

Proposition 3.6. *There is an isomorphism $K(X) \simeq [X, BU \times \mathbb{Z}]$.*

Proof. Using that every element of $K(X)$ can be written as $[E] - \varepsilon^q$, where $[E]$ is the class of an n-dimensional vector bundle classified by the map $f : X \to BU$, consider the map sending $[E] - \varepsilon^q$ to $f \oplus (n - q)$.[3] □

Up to this point and in later parts of this section we only consider compact spaces, but we will now briefly consider the non-compact case. We just showed that $K(X)$ is represented by classes of functions from X into $BU \times \mathbb{Z}$: $K(X) \simeq [X \to BU \times Z]$. Noting that, for non-compact spaces, classes of such functions are likewise well-defined, we may define the K-theory of a non-compact space X as follows:

$$K(X) := [X \to BU \times \mathbb{Z}]$$

From its construction, the K-theory of X compact is a ring. Although we may hope $K(X)$ of non-compact X is a ring as well, it's not obvious this is the case. While we don't prove it, this is indeed the case.

We now show that the rings $K(X)$ and $\widetilde{K}(X)$ are homotopy invariant.

Proposition 3.7. *Let X, Y be compact Hausdorff spaces and $f : X \to Y$ a homotopy equivalence, then we have an isomorphism:*

$$F : \widetilde{K}(X) \to \widetilde{K}(Y)$$

Proof. Since f is a homotopy equivalence, there exists a continuous function $g : Y \to X$ and homotopies

$$g \circ f \sim 1_X$$
$$f \circ g \sim 1_Y$$

Then, Theorem 2.14 gives us that g^* is the inverse of f^* and so we have a bijection between the isomorphism classes of vector bundles over X and those over Y. The desired result then follows. □

As a final remark in this section, we note that $K(X)$ and $\widetilde{K}(X)$ are functors over X. Via pullbacks, a map $f : X \to Y$ induces a map $K(Y) \to K(X)$ sending $E_1 - E_2$ to $f^*(E_1) - f^*(E_2)$, which is a ring homomorphism. The usual equalities $f^* \circ g^* = (g \circ f)^*$ and $1^* = 1$ also can easily be seen to hold by considering the corresponding properties of pullbacks.

[3] We are using connectedness here when saying that E is an n-dimensional bundle.

3.2. **Bott's Periodicity Theorem.** Calculating the K-theory of arbitrary spaces is in general hard, but there are certain tools that allow us to calculate the K-theory of a given space in terms of that of another we already know. In particular, the *Fundamental Product Theorem* gives us a tool with which to calculate $K(X \times S^2)$ in terms of $K(X)$. To motivate it, consider the line bundle over S^2, we call it H, it satisfies the following relation: $(H \otimes H) \oplus 1 \approx H \oplus H$. Before we move towards proving Bott's Periodicity theorem, we now introduce *clutching constructions* which will be important in the setup of Bott's periodicity theorem.

Clutching Construction. Because we are interested in the K-theory of $X \times S^2$, we now show how to construct vector bundles over base spaces $X \times S^k$ through the so-called *Clutching Construction.*

Definition 3.8. Consider the sphere S^k as the union of its two hemispheres, D^k_- and D^k_+, intersecting in the equator S^{k-1}. Now, given a vector bundle $p : E \to X$, and an automorphism $f : E \times S^k \to E \times S^k$ of the product vector bundle $p \times \mathbf{1} : E \times S^k \to X \times S^k$, we can construct an n-dimensional bundle $E_f \to S^k$ by letting E_f to be the quotient

$$X \times D^k_- \times \mathbb{R}^n \sqcup X \times D^k_+ \times \mathbb{R}^n / \sim$$

where we identify $(x, y, v) \in X \times \partial D^k_- \times \mathbb{R}^n$ with $(x, y, f(x, y)(v)) \in X \times \partial D^k_- \times \mathbb{R}^n$, f is called the clutching function.

Note that if we have homotopic clutching functions $f, g : S^{k-1} \to \mathrm{GL}_n(\mathbb{C})$, the bundles E_f and E_g are isomorphic. Let $H : I \times S^k \to \mathrm{GL}_n(\mathbb{C})$ be a homotopy between f and g. Then, we may use the clutching construction to get a vector bundle $E_H \to I \times S^{k-1}$. This vector bundle E_H restricts to E_f and E_g at $\{0\} \times S^k$ and $\{1\} \times S^k$. By Proposition 2.15, we are done. This shows that the map $\Phi : [S^{k-1}, \mathrm{GL}_n(\mathbb{C})] \to \mathrm{Vect}^n_{\mathbb{C}}(S^k)$, $f \to E_f$ is well-defined. In fact, we have a bijection.

Proposition 3.9. *The map* $\Phi : [S^{k-1}, GL_n(\mathbb{C})] \to Vect^n_{\mathbb{C}}(S^k)$, $f \to E_f$, *is a bijection.*

Now we prove that the line bundle over S^2 satisfies $(H \otimes H) \oplus 1 \approx H \oplus H$.

Lemma 3.10. *The line bundle over* S^2, H, *satisfies* $(H \otimes H) \oplus 1 \approx H \oplus H$.

Proof. The clutching construction for each of the two sides of the isomorphism is given by:

$$f_{H \oplus H} : z \to \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$$

and

$$f_{(H \times H) \oplus 1} : z \to \begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix}$$

Now, we rely on the fact that the $\mathrm{GL}(\mathbb{C}, n)$ is a path connected space to say that there exists a continuous map $\alpha : [0, 1] \to \mathrm{GL}(\mathbb{C}, 2)$ such that $\alpha(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\alpha(1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; that is, a path from the identity matrix to the matrix that swaps entries. Then, we can construct a homotopy between $f_{H\oplus H}$ and $f_{(H\otimes H)\oplus 1}$ as follows:

$$g : [0,1] \times S^1 \to \mathrm{GL}(2,\mathbb{C})$$
$$(t,z) \to \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \alpha(t) \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \alpha(t)$$

To finish we just use the bijection in Projection 3.9. □

Writing the relation above in $K(S^2)$ we naturally have $H^2 + 1 = 2H$ and so $(H-1)^2 = 0$. Now, this gives us a ring homomorphism $\mathbb{Z}[H]/(H-1)^2 \to K(S^2)$ between the quotient polynomial ring generated by H quotiented out by the ideal $(H-1)^2$. We then have a ring homomorphism between $K(X) \otimes \mathbb{Z}[H]/(H-1)^2 \to K(X) \otimes K(S^2)$. Since we are interested in the K-theory of $X \times S^2$, we compose this homomorphism with the natural external product, $K(X) \otimes K(Y) \to K(X \times Y)$ constructed out of the projection maps from $X \times Y$ to X and Y. This external product will be denoted by the usual symbol, $*$. The Fundamental Product Theorem states that this map is an isomorphism, thus allowing us to relate $K(X \times S^2)$ with $K(X)$.

Theorem 3.11. *If X is a compact Hausdorff space, the homomorphism $\mu : K(X) \otimes Z[H]/(H-1)^2 \to K(X \times S^2)$ is a ring isomorphism.*

A complete proof of the Fundamental Product Theorem would be long and isn't key to the main focus of this work. As such, we restrict ourselves to proving the following lemma that is an important step in the theorem's proof.

Lemma 3.12. *Every vector bundle $p : E' \to X \times S^2$ is isomorphic to $[E, f]$ for some vector bundle E over X and a clutching function f.*

Proof. Denoting by $[E, f]$ the vector bundle obtained from the clutching construction, we want to show that $E' \approx [E, f]$ for some f and E that we shall construct. If we see the sphere S^2 as the union of two disks glued at their boundaries, $S^2 = D_0 \cup D_\infty$, define E_α as the restriction of p over $X \times D_\alpha$, $\alpha \in \{0, \infty\}$. We claim that the total space E that we are after is the restriction of E' over $X \times \{1\}$. Now, recall that disks are contractible, then we have that the projection of $X \times D_\alpha$ to $X \times \{1\}$ is homotopic to the identity map, $\pi \approx \mathrm{Id}$.

Therefore, the pullbacks are isomorphic, by Theorem 2.14, and we have that $E \times D_\alpha \approx \pi^*(E) \approx \mathrm{Id}^*(E) = E_\alpha$. Therefore, there are isomorphisms $h_\alpha : E_\alpha \to E \times D_\alpha$. Finally, we show that $f = h_0 \circ h_\infty^{-1}$ is a clutching function. □

Exact Sequences and Bott's. The Fundamental Product Theorem lends itself pretty easily to the proof of another important theorem, *Bott's Periodicity Theorem*:

Theorem 3.13. *In compact Hausdorff spaces, the homomorphism of groups $\beta : \widetilde{K}(X) \to \widetilde{K}(\Sigma^2 X)$, $\beta(x) = (H-1)*x$ is an isomorphism. In terms of classifying spaces, this means:*

$$BU \times Z \simeq \Omega^2(BU \times Z).$$

The Fundamental Product Theorem relates the K-theory group of $S^2 \times X$ with that of X and so, since we want to prove a result about $\widetilde{K}(\Sigma^2 X)$, we will first relate $K(X)$ and the relative K group $K(X, A)$, $A \subset X$ - do not forget that $\Sigma^2 X \simeq S^2 \wedge X$ and that $X \wedge Y = (X \times Y)/(X \vee Y)$. Then, we will prove a relation between the

K-theory groups of $X \times Y$ with those of the smash product and wedge sum of X and Y. In order to do the above, we now introduce the reader to a very important exact sequence in K-theory which we will use widely from here on.

Proposition 3.14. *For a compact Hausdorff space X and a closed subspace $A \subset X$, the following is an exact sequence that can be extended infinitely to the left.*

$$\cdots \longrightarrow \widetilde{K}(\Sigma(X/A)) \longrightarrow \widetilde{K}(\Sigma X) \longrightarrow \widetilde{K}(\Sigma A) \longrightarrow \widetilde{K}(X/A) \longrightarrow \widetilde{K}(X) \longrightarrow \widetilde{K}(A)$$

Proof. To prove this we will refer to the long cofibration sequence

$$A \hookrightarrow X \hookrightarrow X \cup Cone(A) \hookrightarrow (X \cup Cone(A)) \cup Cone(X)$$
$$\hookrightarrow ((X \cup Cone(A)) \cup Cone(X)) \cup Cone(X \cup Cone(A)) \hookrightarrow \cdots$$

obtained by attaching a cone on the space two steps back to the left of the sequence. Note now that by collapsing the outermost cone in the three rightmost elements written out above, the collapsed spaces will be, from left to right, X/A, SA, SX - where SX is the suspension of X. Then, since this canonical quotient is a homotopy equivalence, we are done by the homotopy invariance of the K rings in Proposition 3.7. ☐

In fact, this result is general, requiring only the suspension axiom and the short sequence $0 \to A \to X \to X/A \to 0$. See [1]. The propositions that follow are also general, true for any representable functor, and are due completely to homotopy theory.

Lemma 3.15. *If A is contractible, the quotient map $q : X \to X/A$ induces a bijection, for all dimensions n, $q^* : Vect^n(X/A) \to Vect^n(X)$.*

Here we note that topological K-Theory satisfies the axioms of a generalized cohomology theory and Proposition 3.14 essentially establishes that the axiom requiring that the functor, here that of $K(X)$, take homotopy cofiber sequences into exact ones holds.

We now will work towards a relation between $K(X \times Y)$ and $K(X \wedge Y)$. For that, we will first prove the splitting of the K-theory group of X into that of $A \subset X$ and the relative K theory of X, A. Then, we will make extensive use of Proposition 3.14 to obtain a splitting of $K(X \times Y)$ in terms of $K(X \wedge Y)$.

Proposition 3.16. *Let X be a compact Hausdorff space and $A \subset X$ a closed subset, such that there is a rectraction $r : X \to A$ such that its composition with the inclusion $i : A \to X$ is the identity function on A. Then, there is a splitting of the K-theory group of X as follows:*

$$\widetilde{K}(X) \simeq \widetilde{K}(A) \oplus K(X, A)$$

The same equality with the reduced groups of X and A instead of the full K-theory group also holds, if we are dealing with pointed spaces.

Proof. The long exact sequence gives us that

$$\widetilde{K}(\Sigma X) \xrightarrow{\Sigma i^*} \widetilde{K}(\Sigma A) \xrightarrow{f} \widetilde{K}(X/A) \longrightarrow \widetilde{K}(X) \xrightarrow{i^*} \widetilde{K}(A)$$

We have that $i^* \circ r^* \simeq 1_A^*$, and so i^* is surjective. Similarly, its suspension is also surjective. Therefore, we have that the sequence

$$0 \longrightarrow \widetilde{K}(X/A) \longrightarrow \widetilde{K}(X) \xrightarrow{i^*} \widetilde{K}(A) \longrightarrow 0$$

is exact. We have that $\ker(f) = \mathrm{Im}(\Sigma i^*) = \widetilde{K}(\Sigma A)$ and so the leftmost map from 0 to $\widetilde{K}(X/A)$ is justified. Similarly, the map sending all of $\widetilde{K}(A)$ to 0 has as kernel the image of i^*. The result then follows. $\square$

Proposition 3.17. *Let (X, x) and (Y, y) be pointed compact Hausdorff spaces. There is an isomorphism $\widetilde{K}(X \vee Y) \simeq \widetilde{K}(X) \oplus \widetilde{K}(Y)$.*

Proof. We will again make use of the long exact sequences to reach the desired result. First, note that $X \subset X \wedge Y$ is a closed subset, as its complement, $Y \setminus \{y\}$ is open because Y is Hausdorff. Second, by the definition of the wedge sum, $X \wedge Y/X \simeq Y$. Then, we may reason as in the proof of Proposition 3.16 to obtain the following split exact sequence

$$0 \to \widetilde{K}(X) \to \widetilde{K}(X \wedge Y) \to \widetilde{K}(Y) \to 0$$

$\square$

Proposition 3.18. *For any two compact Hausdorff pointed spaces, X and Y, we have an isomorphism of groups:*

$$\widetilde{K}(X \times Y) \simeq \widetilde{K}(X \wedge Y) \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y)$$

Proof. We start with the long sequence from Proposition 3.14

$$\widetilde{K}(\Sigma(X \times Y)) \xrightarrow{\Sigma i^*} \widetilde{K}((\Sigma X) \vee (\Sigma Y)) \longrightarrow \widetilde{K}(X \wedge Y) \longrightarrow \widetilde{K}(X \times Y) \xrightarrow{i^*} \widetilde{K}(X \vee Y)$$

Then, we use the result from Proposition 3.17 to rewrite the above as:

$$\widetilde{K}(\Sigma(X \times Y)) \xrightarrow{\Sigma i^*} \widetilde{K}(\Sigma X) \oplus \widetilde{K}(\Sigma Y) \longrightarrow \widetilde{K}(X \wedge Y) \longrightarrow \widetilde{K}(X \times Y) \xrightarrow{i^*} \widetilde{K}(X) \oplus \widetilde{K}(Y)$$

Setting $p_1 : X \times Y \to X$, $p_2 : X \times Y \to Y$ to be the projections onto the first and second components of $X \times Y$, the map $\widetilde{K}(X \times Y) \longrightarrow \widetilde{K}(X) \oplus \widetilde{K}(Y)$ has a section given by:

$$\widetilde{K}(X) \oplus \widetilde{K}(Y) \to \widetilde{K}(X \times Y)$$
$$(a, b) \to p_1^*(a) + p_2^*(b)$$

This gives us the short split exact sequence

$$0 \to \widetilde{K}(X \wedge Y) \longrightarrow \widetilde{K}(X \times Y) \longrightarrow \widetilde{K}(X) \oplus \widetilde{K}(Y) \to 0$$

from which the result follows. $\square$

With the above results, we are now able to prove Bott's Periodicity Theorem.

Theorem 3.13. *In compact Hausdorff spaces, the homomorphism of groups $\beta : \widetilde{K}(X) \to \widetilde{K}(\Sigma^2 X)$, $\beta(x) = (H-1)*x$ is an isomorphism. In terms of classifying spaces, this means:*

$$BU \times Z \simeq \Omega^2(BU \times Z).$$

Proof. We start with the isomorphism of Theorem 3.11 and recall Proposition 3.18 to write:

$$K(S^2) \otimes K(X) \longrightarrow K(S^2 \times X)$$
$$\left(\widetilde{K}(S^2) \oplus \mathbb{Z}\right) \otimes \left(\widetilde{K}(X) \oplus \mathbb{Z}\right) \simeq \widetilde{K}(S^2 \times X) \oplus \mathbb{Z}$$
$$\simeq \widetilde{K}(S^2 \wedge X) \oplus \widetilde{K}(S^2) \oplus \widetilde{K}(X) \oplus \mathbb{Z}$$
$$\Longrightarrow \widetilde{K}(S^2) \otimes \widetilde{K}(X) \simeq \widetilde{K}(S^2 \wedge X) \simeq \widetilde{K}(\Sigma^2 X)$$

Now, it suffices to note that the map β is the composition of this (reduced) external product isomorphism with the map $\widetilde{K}(X) \to \widetilde{K}(S^2) \otimes \widetilde{K}(X)$ sending $a \to (H-1) \otimes a$, which is also isomorphic as $\widetilde{K}(S^2)$ is infinite cyclic generated by $H-1$.

□

3.3. **Calculating $K(X)$ for Example X's.** We will now use some of the tools developed above to calculate the K-theory of some example spaces.

3.3.1. *S^n.* Using Bott's periodicity theorem we have

$$\begin{aligned}\widetilde{K}(S^{2n+1}) &= \widetilde{K}(S^2 \wedge S^{2n-1}) = \widetilde{K}(\Sigma^2 S^{2n-1}) = \widetilde{K}(\Sigma^2 S^{2n-1}) \\ &= \widetilde{K}(\Sigma^2 S^{2n-1}) \because \widetilde{K}(X) = \widetilde{K}(\Sigma^2 X) \\ &\implies \widetilde{K}(S^{2n+1}) = \widetilde{K}(S^1)\end{aligned}$$

and similarly,

$$\begin{aligned}\widetilde{K}(S^{2n}) &= \widetilde{K}(S^2 \wedge S^{2n-2}) = \widetilde{K}(\Sigma^2 S^{2n-2}) = \widetilde{K}(\Sigma^2 S^{2n-2}) \\ &= \widetilde{K}(\Sigma^2 S^{2n-2}) \because \widetilde{K}(X) = \widetilde{K}(\Sigma^2 X) \\ &\implies \widetilde{K}(S^{2n}) = \widetilde{K}(S^0)\end{aligned}$$

Finally, we know that every complex bundle on S^1 is trivial so that $K(S^1) \simeq \mathbb{Z}$, and $\widetilde{K}(S^1) = 0$ as $K(X) \simeq \widetilde{K}(X) \oplus \mathbb{Z}$. On the other hand, S^0 is just two points. Note now that we have, using $*$ for a point, that the pointed space of a point, $*_+$ is the same as the space of two points, and that our development of the reduced group gives us that

$$\widetilde{K}(*_+) = \widetilde{K}(* \cup *) = K(*) \implies K(* \cup *) = \widetilde{K}(* \cup *) \oplus \mathbb{Z} = K(*) \oplus \mathbb{Z} = \mathbb{Z} \oplus \mathbb{Z}$$

A vector bundle over one point is simply the assignment of one vector space to the point. An isomorphism between such vector bundles will just consist of linear bijective maps. Finite dimensional vector spaces are isomorphic exactly when they have the same dimension and so the monoid of isomorphism classes of vector bundles over a point is just the natural numbers, whose Grothendieck group - and thus $K(*)$ - is the additive group of the integers.

3.3.2. *Torus.* To calculate the K-theory of the simple torus, we use Proposition 3.18 to get:

$$\begin{aligned}&\widetilde{K}(T^2) \simeq \widetilde{K}(T^1 \wedge S^1) \oplus \widetilde{K}(T^1) \oplus \widetilde{K}(S^1) \\ \implies &\widetilde{K}(T^2) = \widetilde{K}(S^1 \wedge S^1) \oplus \widetilde{K}(S^1) \oplus \widetilde{K}(S^1) = \widetilde{K}(S^2) = \mathbb{Z} \oplus \mathbb{Z}\end{aligned}$$

3.3.3. *CW Complex by Atiyah.* Now we will calculate the K-theory of a special cell complex, given as an example by [3]. We say a space X is a cell complex if there is a filtration by closed sets $X_{-1} \subset X_0 \subset X_l \subset \cdots \subset X_n = X$ such that each $X_k - X_{k-1}$ is a disjoint union of open k-cel1s, and $X_{-1} = \emptyset$.

Proposition 3.19. *If X is a cell complex such that $X_{2n+1} = X_{2n}$ for all n, then we have that*

$$\begin{aligned}K^0(X) &= 0 \\ K^1(X) &= \textit{free abelian group with generators} \\ &\quad \textit{in correspondence with the cells of } X\end{aligned}$$

Proof. First, note that X_{2n}/X_{2n-2} is the union of $2n$-spheres with one point in common. Then, using Bott's Periodicity Theorem, we get

$$K^1(X_{2n}, X_{2n-2}) = 0$$
$$K^0(X_{2n}, X_{2n-2}) = \mathbb{Z}^k$$

where k_{2n} is the number of $2n$-spheres. Then, we may write

$$\begin{aligned} K^0(X_{2n}) &= K^0(X_{2n}, X_{2n-2}) \oplus K^0(X_{2n-2}) \\ &= K^0(X_{2n}, X_{2n-2}) \oplus \cdots \oplus K^0(X_0) \\ &= \mathbb{Z}^{k_{2n}} \oplus \mathbb{Z}^{k_{2n-2}} \oplus \cdots \oplus (\mathbb{Z} \oplus \mathbb{Z}) \end{aligned}$$

$$\begin{aligned} K^1(X_{2n}) &= K^1(X_{2n}, X_{2n-2}) \oplus K^1(X_{2n-2}) \\ &= K^1(X_{2n}, X_{2n-2}) \oplus \cdots \oplus K^1(X_0) \\ &= 0 \end{aligned}$$

where we used $K^0(*) = \mathbb{Z}$. By induction the desired result is obtained. □

We note that this example is not without some consequence as the classifying space BU(n), for instance, is one space that only has cells in even dimensions and the above tells us both that calculating $K^0(\mathrm{BU}(n))$ is enough and what to expect of $K^0(\mathrm{BU}(n))$.

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