TWO-DIMENSIONAL TOPOLOGICAL QUANTUM FIELD THEORIES AND FROBENIUS ALGEBRAS

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ABSTRACT. In this expository paper, we discuss topological quantum field theories (TQFTs) and the classification of two-dimensional topological quantum fields theories (2TQFTs). We define the category of cobordism classes, present the functorial definition of topological quantum field theories and discuss Frobenius algebras. Our main result is that there is a canonical equivalence of categories between the category $\mathbf{2TQFT}_{\Bbbk}$ of two-dimensional topological quantum field theories and the category $\mathbf{cFA}_{\Bbbk}$ of commutative Frobenius algbera.

CONTENTS

1. INTRODUCTION

Topological quantum field theories (TQFTs) are quantum field theories that are invariant under diffeomorphisms. They capture the global degrees of freedom in spacetime and can serve as prototype theories of quantum gravity. Apart from these physical motivations, mathematical interest in the TQFTs stems from their usefulness in producing topological invariants of manifolds.

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Although TQFTs in higher dimensions have been the most interesting and physically relevant, they are complicated obejcts to visualise and study. In dimension two, we can classify TQFTs by establishing a canonical equivalence between the category $\mathbf{2TQFT}_{\Bbbk}$ of two-dimensional TQFTs over a field $\Bbbk$ and the category $\mathbf{cFA}_{\Bbbk}$ of commutative Frobenius algebra over $\Bbbk$.

In this expository paper, we focus our attention on two-dimensional TQFTs. The paper is organised as follows. We first define cobordisms and the category of cobordism classes in Section 2. A discussion of the axiomatic formulation of n-dimensional TQFTs is then presented in Section 3 followed by a definition of an n-dimensional TQFT as a symmetric monoidal functor between the category of n-dimensional cobordism classes $\mathbf{nCob}$ and the category of vector spaces $\mathbf{Vect}_{\Bbbk}$. We also provide some physical intuitions behind the functorial definition of TQFTs in this section. We then explore the category of two-dimensional TQFTs in Section 4, listing its generators and relations. In Section 5, we define the Frobenius structure on a $\Bbbk$-algebra and the category of commutative Frobenius algebras $\mathbf{cFA}_{\Bbbk}$. Lastly in Section 6, we prove the the main classification theorem, that there is a canonical equivalence of categories $\mathbf{2TQFT}_{\Bbbk} = \mathbf{cFA}_{\Bbbk}$.

We assume that the readers have knowledge of the basic definitions in category theory and provide definitions where deemed appropriate. Throughout this paper, $\Bbbk$ denotes a field of characteristic zero. We use the convention of writing composition of functions and arrows from the left to the right. For example, the composition of two maps $f : X \to Y$ and $g : Y \to Z$ is written as $fg : X \to Z$. We also write the symbol of a function to the right of its argument, for example $f : X \to Y$ is given by $x \mapsto xf$.

2. Cobordisms

We first introduce the definition of cobordisms and the category of cobordism classes, which will be the foundation for our discussion of TQFTs.

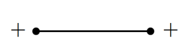
2.1. Basic Definitions.

Definition 2.1 (Oriented cobordism)**.** Let Σ_0 and Σ_1 be closed oriented $(n-1)$-manifolds. An *oriented cobordism* from Σ_0 to Σ_1 is an compact oriented n-dimensional manifold M together with smooth maps

$$\Sigma_0 \to M \leftarrow \Sigma_1$$

such that Σ_0 maps diffeomorphically (preserving orientation) onto the in-boundary of M, and Σ_1 maps diffeomorphically (preserving orientation) onto the out-boundary of M.

Example 2.2 (One-dimensional cobordisms)**.** A most fundamental cobordism is the closed interval. Take $I = [0, 1]$ with its standard orientation, with the boundary points 0 and 1 given standard orientation $+$:



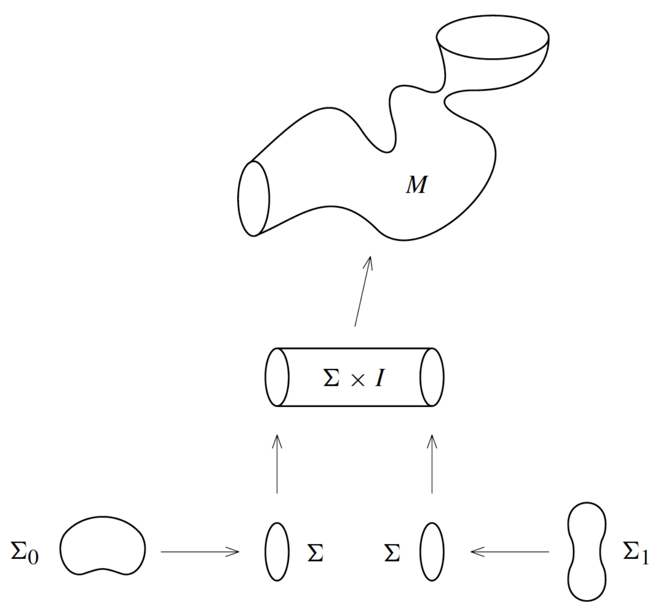
We can view 0 as the in-boubdary and 1 as the out-boundary so that I defines a cobordism from 0 to 1. If we take 0 with $+$ orientation and 1 with $-$ orientation, we obtain a cobordism from a two-point manifold to the empty manifold:

As illustrated by this example, a cobordism can be constructed from a nonempty manifold to the empty manifold - it is not a function.

Example 2.3 (Cylinders)**.** Take a closed oriented manifold Σ and cross it with the unit interval I with its standard orientation. The boundary of $\Sigma \times I$ consists of two copies of Σ: one which is the in-boundary $\Sigma \times \{0\}$ and another which is the out-boundary $\Sigma \times \{1\}$. This is a cobordism from Σ to Σ. This construction also gives a cobordism between any pair of $(n-1)$-manifolds Σ_0 and Σ_1 diffeomorphic to Σ,

$$\Sigma_0 \xrightarrow{\cong} \Sigma \xrightarrow{\cong} \Sigma \times \{0\} \subset \Sigma \times I$$
$$\Sigma_1 \xrightarrow{\cong} \Sigma \xrightarrow{\cong} \Sigma \times \{1\} \subset \Sigma \times I.$$

Any orientation-preserving diffeomorphism $\Sigma \times I \xrightarrow{\cong} M$ will also define a cobordism $M : \Sigma \to \Sigma$. Combining these two variations, we have a cobordism $M : \Sigma_0 \to \Sigma_1$ as shown in the following diagram [5].



It is instructive to make an analogy between a cobordism and a movie showing how the object Σ_0 on the first frame of this movie evolves into the object Σ_1 on the last frame. Two "movies" can be equivalent when they meet the following criterion.

Definition 2.4 (Equivalent cobordisms)**.** Two oriented cobordisms from Σ_0 to Σ_1,

$$\begin{tikzcd} & M' & \\ \Sigma_0 \arrow[ur] \arrow[dr] & & \Sigma_1 \arrow[ul] \arrow[dl] \\ & M & \end{tikzcd}$$

are *equivalent* if there is an orientation-preserving diffeomorphism $\psi : M \xrightarrow{\cong} M'$ making the following diagram commute:

$$\begin{tikzcd} & M' & \\ \Sigma_0 \arrow[ur] \arrow[dr] & & \Sigma_1 \arrow[ul] \arrow[dl] \\ & M \arrow[uu, "\psi"', "\cong"] & \end{tikzcd}$$

Cobordisms equivalent in this sense are said to belong to the same *cobordism class*.

2.2. **The Category of Cobordism Classes.** We now assemble n-dimensional cobordisms into a category **nCob**. We identify the objects of **nCob** as $(n-1)$-dimensional closed oriented manifolds. Given two such objects Σ_0 and Σ_1, an arrow (or a morphism) from Σ_0 to Σ_1 is a diffeomorphism class of oriented cobordisms $M : \Sigma_0 \to \Sigma_1$, i.e. the cobordism classes defined in Definition 2.4. The identity arrow are just the cylinders (or the identity cobordisms).

The composition of arrows is given by gluing cobordism classes. We first consider the gluing of cylinders and then generalise it to general cobordisms.

Example 2.5 (Gluing of cylinders)**.** Consider two cobordisms $M_0 : \Sigma_0 \to \Sigma_1$ and $M_1 : \Sigma_1 \to \Sigma_2$ which are both equivalent to cylinders,

$$\phi_0 : M_0 \xrightarrow{\cong} \Sigma_1 \times [0,1]$$
$$\phi_1 : M_1 \xrightarrow{\cong} \Sigma_1 \times [1,2].$$

We can take $\phi := \phi_0 \amalg_{\Sigma_1} \phi_1 : M_0 \amalg_{\Sigma_1} M_1 \to S := \Sigma_1 \times [0,2]$ defined by gluing ϕ_0 and ϕ_1. We note that ϕ is a homeomorphism whose restrictions to M_0 and M_1 are diffeomorphisms and S has a smooth structure which agrees with $\Sigma_1 \times [0,1]$ and $\Sigma_1 \times [1,2]$. We then have a smooth structure on $M_0 M_1 : \Sigma_0 \to \Sigma_2$ via pullback along ϕ.

Heading towards constructing a category whose objects are the cobordism classes, we generalise Example 2.5. Consider two cobordisms $M_0 : \Sigma_0 \to \Sigma_1$ and $M_1 : \Sigma_1 \to \Sigma_2$. Take Morse functions $f_0 : M_0 \to [0,1]$ and $f_1 : M_1 \to [1,2]$ and consider the topological manifold $M_0 \amalg_{\Sigma_1} M_1$ with the induced continuous map $M_0 \amalg_{\Sigma_1} M_1 \to [0,2]$. Choose $\epsilon > 0$ so small that the two intervals $[1-\epsilon, 1]$ and $[1, 1+\epsilon]$ are regular for f_0 and f_1 respectively. Then the inverse image of these two intervals are diffeomorphic to cylinders. Within the interval $[1-\epsilon, 1+\epsilon]$, we are in the situation of Example 2.5 and we can take the smooth structure to be the one coming from the cylinder. This results in a composite cobordism M_0M_1 homeomorphic to $M_0 \amalg_{\Sigma_1} M_1$. The following result gives suitable uniqueness for smooth structures on M_0M_1.

Theorem 2.6. *Let Σ be an out-boundary of M_0 and an in-boundary of M_1, and consider the topological manifold $M_0M_1 := M_0 \amalg_\Sigma M_1$. Let α and β be two smooth structures on M_0M_1 which both induce the original structure on M_0 and M_1 (via pullpack along the inclusion maps). Then there is a diffeomorphism $\phi : (M_0M_1, \alpha) \xrightarrow{\cong} (M_0M_1, \beta)$ such that $\phi|_\Sigma = \mathrm{id}_\Sigma$.*

This means that the smooth struture on M_0M_1 is unique up to diffeomorphism and that the construction above is well-defined. For more details on the proof of Theorem 2.6, see Section 1.3 in [5].

So far we have shown that given two specific cobordisms $M_0 : \Sigma_0 \to \Sigma_1$ and $M_1 : \Sigma_1 \to \Sigma_2$, there is a well-defined diffeomorphism class of cobordisms $M_0M_1 : \Sigma_0 \to \Sigma_2$. It remains to show that the resultant cobordism class of M_0M_1 does not depend on our initial choice of M_0 and M_1. Suppose we have two diffeomorphism $\psi_0 : M_0 \to M_0'$ and $\psi_1 : M_1 \to M_1'$,

$$\begin{tikzcd}
 & M_0' & & M_1' & \\
\Sigma_0 \arrow[ur] \arrow[dr] & & \Sigma_1 \arrow[ul] \arrow[dl] \arrow[ur] \arrow[dr] & & \Sigma_2 \arrow[ul] \arrow[dl] \\
 & M_0 \arrow[uu, "\psi_0", "\cong"'] & & M_1 \arrow[uu, "\psi_1", "\cong"'] &
\end{tikzcd}$$

then there is a gluing M_0M_1 and a gluing $M_0'M_1'$. The two diffeomorphisms, ψ_0 and ψ_1, also glue in the category of continuous maps. Hence we have a homeomorphism $\psi : M_0M_1 \xrightarrow{\cong} M_0'M_1'$

$$\begin{tikzcd}
 & M_0'M_1' & \\
\Sigma_0 \arrow[ur] \arrow[dr] & & \Sigma_2 \arrow[ul] \arrow[dl] \\
 & M_0M_1 \arrow[uu, "\psi", "\cong"'] &
\end{tikzcd}$$

We can use ψ to define smooth structure on $M_0'M_1'$, and, by Theorem 2.6, $M_0'M_1'$ is in the same diffeomorphic class as M_0M_1. Therefore, gluing is a well-defined composition for cobordism classes. The associativity of this composition follows from the construction.

In conclusion, we have assembled n-dimensional cobordisms into a category **nCob** with the following data:

(1) objects: $(n-1)$-dimensional closed oriented manifolds;
(2) arrows: diffeomorphism classes of n-dimensional manifolds;
(3) identity arrows: cylinders;
(4) composition of arrows: gluing of cobordisms.

2.3. **nCob is Symmetric Monoidal.** The category **nCob** admits a monoidal structure which is realised via the disjoint union construction. Given two cobordisms $M : \Sigma_o \to \Sigma_1$ and $M' : \Sigma_o' \to \Sigma_1'$, their disjoint union $M \amalg M'$ is a cobordism from $\Sigma_0 \amalg \Sigma_0'$ to $\Sigma_1 \amalg \Sigma_1'$. M and M' can be thought of as a particular choice of representatives of cobordism classes. Taking other representatives will only produce disjoint unions that are diffeomorphic to $M \amalg M'$. Hence the disjoint union of cobordism classes is well-defined.We also have the empty cobordism $\emptyset_n : \emptyset_{n-1} \to \emptyset_{n-1}$. This means that the triple $(\mathbf{nCob}, \amalg, \emptyset_n)$ is a monoidal category, which we will write as $(\mathbf{nCob}, \amalg, \emptyset)$ for simplicity.

Furthermore, the monoidal category $(\mathbf{nCob}, \amalg, \emptyset)$ admits a symmetric structure. We first recall the definition of a symmetric strict monoidal category.

Definition 2.7 (Symmetric strict monoidal category)**.** A strict monoidal category $(\mathbf{V}, \Box, I)$ is called a *symmetric strict monoidal category* if for each pair of objects X, Y there is a given *twist map*

$$\tau_{X,Y} : X \Box Y \to Y \Box X$$

subjected to the following three axioms:

(1) the maps are natural, that is, for every arrow in $\mathbf{V} \times \mathbf{V}$ (i.e. for every pair of arrows $f : X \to X'$ and $g : Y \to Y'$), the diagram

$$\begin{tikzcd} X \Box Y \arrow[r, "\tau_{X,Y}"] \arrow[d, "f \Box g"'] & Y \Box X \arrow[d, "g \Box f"] \\ X' \Box Y' \arrow[r, "\tau_{X',Y'}"'] & Y' \Box X' \end{tikzcd}$$

commutes. This means that the collection of twist maps assemble into a natural transformation τ, going from

$$\mathbf{V} \times \mathbf{V} \xrightarrow{\mu} \mathbf{V} \quad \text{to} \quad \mathbf{V} \times \mathbf{V} \xrightarrow{\text{twist}} \mathbf{V} \times \mathbf{V} \xrightarrow{\mu} \mathbf{V}.$$

(2) for every triple of objects X, Y, Z, the following two disgrams commute:

$$\begin{tikzcd} X \Box Y \Box Z \arrow[rr, "\tau_{X,Y \Box Z}"] \arrow[dr, "\tau_{X,Y} \Box \mathrm{id}_Z"'] & & Y \Box Z \Box X \\ & Y \Box X \Box Z \arrow[ur, "\mathrm{id}_Y \Box \tau_{X,Z}"'] & \end{tikzcd}$$

$$\begin{tikzcd} X \Box Y \Box Z \arrow[rr, "\tau_{X \Box Y,Z}"] \arrow[dr, "\mathrm{id}_X \Box \tau_{Y,Z}"'] & & Z \Box X \Box Y \\ & X \Box Z \Box Y \arrow[ur, "\tau_{X,Z} \Box \mathrm{id}_Y"'] & \end{tikzcd}$$

(3) $\tau_{X,Y} \tau_{Y,X} = \mathrm{id}_{X \Box Y}$.

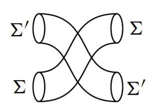
Note that the symmetric strict monoidal categories are just the permutative categories. General symmetric monoidal categories are defined by relaxing the commutation requirement on the diagrams from commuting "on the nose" to commuting up to isomorphisms. An example of a symmetric (non-strict) monoidal category which is important for the later part of the paper is the vector space.

Example 2.8 (Vector spaces)**.** For every pair of vector spaces V, V' there is a canonical twist map

$$\sigma_{V,V'} : V \otimes V' \to V' \otimes V$$

which exchange the order of factors in the tensor product. $(\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk, \sigma)$ is a symmetric monoidal category.

The symmetric structure of $(\mathbf{nCob}, \amalg, \emptyset)$ is realised via the twist cobordism, induced by the twist diffeomorphism $T_{\Sigma,\Sigma'} : \Sigma \amalg \Sigma' \xrightarrow{\cong} \Sigma' \amalg \Sigma$. We represent it graphically as



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We now check that the three axioms in Definition 2.7 are satisfied by the twist cobordism. Conditions (2) and (3) are easy to check by considering the twist diffeomorphisms. The naturality condition (1) is given by the commutativity of the following diagram for two cobordisms $M : \Sigma_0 \to \Sigma_1$ and $M' : \Sigma_0' \to \Sigma_1'$:

$$\begin{tikzcd}
\Sigma_0 \amalg \Sigma_0' \arrow[r, "M \amalg M'"] \arrow[d, "T_{\Sigma_0,\Sigma_0'}"'] & \Sigma_1 \amalg \Sigma_1' \arrow[d, "T_{\Sigma_1,\Sigma_1'}"] \\
\Sigma_0' \amalg \Sigma_0 \arrow[r, "M' \amalg M"'] & \Sigma_1' \amalg \Sigma_1
\end{tikzcd}$$

Therefore, $(\mathbf{nCob}, \amalg, \emptyset, T)$ is a symmetric monoidal category.

3. Topological Quantum Field Theories

3.1. **Mathematical Formulation.** We now present an axiomatic definition of TQFTs modified from Atiyah's original formulation [1, 2] and then link it to $(\mathbf{nCob}, \amalg, \emptyset, T)$ discussed in Section 2.3.

Definition 3.1 (Topological quantum field theories)**.** An *n-dimensional topological quantum field theroy* (TQFT) is a rule $\mathcal{A}$ which to each closed oriented $(n-1)$-dimensional manifold Σ associates a vector space $\Sigma\mathcal{A}$, and to each oriented cobordism $M : \Sigma_0 \to \Sigma_1$ associated a linear map $M\mathcal{A}$ from $\Sigma_0\mathcal{A}$ to $\Sigma_1\mathcal{A}$. The rule $\mathcal{A}$ satisfies the following five axioms.

(1) Two equivalent cobordisms must have the same image:
$$M \cong M' \implies M\mathcal{A} \cong M'\mathcal{A}.$$
(2) The cylinder $\Sigma \times I$, thought of as a cobordism from Σ to itself, must be sent to the identity map of $\Sigma\mathcal{A}$.
(3) Given a composition $M = M'M''$ then
$$M\mathcal{A} = (M'\mathcal{A})(M''\mathcal{A}).$$
(4) Disjoint union goes to tensor product: if $\Sigma = \Sigma' \amalg \Sigma''$ then $\Sigma\mathcal{A} = \Sigma'\mathcal{A} \otimes \Sigma''\mathcal{A}$; if $M : \Sigma_0 \to \Sigma_1$ is the disjoint union of $M' : \Sigma_0' \to \Sigma_1'$ and $M'' : \Sigma_0'' \to \Sigma_1''$ then $M\mathcal{A} = M'\mathcal{A} \otimes M''\mathcal{A}$.
(5) The empty manifold $\Sigma = \emptyset$ must be sent to the ground field $\Bbbk$. The empty cobordism is sent to the identity map of $\Bbbk$.

Readers familiar with topology should note that $\Sigma\mathcal{A}$ is the vector space associated with Σ after applying rule $\mathcal{A}$, not the suspension of $\mathcal{A}$.

To understand the relationship between **nCob** and TQFTs, we recall the definition of symmetric monoidal functors.

Definition 3.2 (Symmetric monoidal functors)**.** Given two symmetric monoidal categories $(\mathbf{V}, \Box, I, \tau)$ and $(\mathbf{V}', \Box', I', \tau')$, a *symmetric monoidal functor* $F : \mathbf{V} \to \mathbf{V}'$ preserves the symmetric structure, namely that for every pair of objects X, Y in $\mathbf{V}$ we have

$$\tau_{X,Y} F = \tau'_{XF,YF}.$$

Comparing this categorical definition with Definition 3.1, we arrive at the functorial definition of a TQFT.

Definition 3.3 (Functorial definition of topological quantum field theories)**.** An *n-dimensional topological quantum field theory* is a symmetric monoidal functor from $(\mathbf{nCob}, \amalg, \emptyset, T)$ to $(\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk, \sigma)$, where σ is defined as in Example 2.8.

The morphisms between TQFTs defined this way are the monoidal natural transformation.

Definition 3.4 (Monoidal natural transformations)**.** Let $(\mathbf{V}, \Box, I)$ and $(\mathbf{V}', \Box', I')$ be two monoidal categories and let

$$\mathbf{V} \overset{G}{\underset{F}{\rightrightarrows}} \mathbf{V}'$$

be two monoidal functors. A natural transformation $u : F \to G$ is called a *monoidal natural transformation* if for every two objects X, Y in $\mathbf{V}$ we have

$$u_X \Box' u_Y = u_{X \Box Y},$$

and $u_I = \mathrm{id}_{I'}$.

We can therefore assemble n-dimensional TQFTs into a category $\mathbf{nTQFT}_{\Bbbk}$.

3.2. **Physical Intuition.** At this point, let us make a brief detour into physics and appreciate the functorial definition presented above.

Take a cobordism $M : \Sigma_0 \to \Sigma_1$ and recall the movie analogy of cobordisms mentioned in Section 2.1. We can consider the $(n-1)$-dimensional manifolds Σ_0 and Σ_1 as $(n-1)$dimensional spaces. The n-dimensional manifold M shows how the space Σ_0 changes into the space Σ_1, that is, it captures the time dimension of the "movie". Putting this together, cobordisms represent spacetimes which is the main object studied in general relativity.

In the light of the functorial definition, a TQFT sends each $(n-1)$-dimensional space to a vector space and sends cobordisms between $(n-1)$ dimensional spaces to operators. The composition of cobordisms is translated into composition of operators and the identity cobordisms become the identity operator. We recover the main ingredients of quantum mechanics as well.

In summary, the functorial definition of TQFT shows a way to translate physics about spacetime to physics about quantum states - a prototype theory for quantum gravity! For a more detailed treatment of the physical intuitions behind the functorial definition, see [3] and [4].

4. The Category of Two-dimensional Cobordisms **2Cob**

Returning from the physics detour, we now focus on the category of two-dimensional cobordisms **2Cob**. Though for applications in physics, the $(1 + 1)$-dimensional spacetime represented by **2Cob** might not be interesting, it is easier to visualise and better studied because two-dimensional manifolds are completely classified. We first examine the objects of **2Cob**.

Definition 4.1 (Skeleton of a category)**.** A *skeleton* of a category is a full subcategory comprising exactly one object from each isomorphism class.

We observe that every closed oriented 1-manifold is diffeomorphic to a finite disjoint union of circles. Combined with the following proposition, we have that two obejcts of **2Cob** are in the same isomorphism class of **2Cob** if and only if they have the same number of connected components.

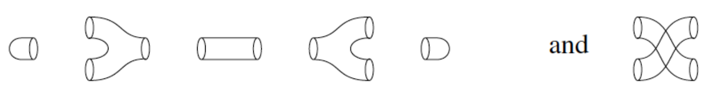
Proposition 4.2. *Two closed oriented 1-manifolds Σ_0 and Σ_1 are diffeomorphic if and only if there is an invertible cobordism between them.*

For 1-manifolds, up to homotopy, the only diffeomorphisms are the permutations of its connected components. Hence the only invertible 2-cobordisms are the permutation cobordisms.

From this, we obtain the skeleton of **2Cob** as follows. Let $\mathbf{0}$ denote the empty 1-manifold; let $\mathbf{1}$ denote a given circle Σ, and let $\mathbf{n}$ denote the disjoint union of n copies of Σ. Then the full subcategory $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, ...\}$ is a skeleton of **2Cob**. From here onwards, we abuse notions and denote this skeleton as **2Cob**.

We would like to now characterise **2Cob** by looking at the generators and relations of this symmetric monoidal category. It is a classical result that **2Cob** is completely generated by the cobordisms listed in the following proposition.

Proposition 4.3 (Generators of **2Cob**)**.** *The symmetric monoidal category* **2Cob** *is generated under composition (serial connection) and disjoint union (parallel conncetion) by the following six cobordisms:*



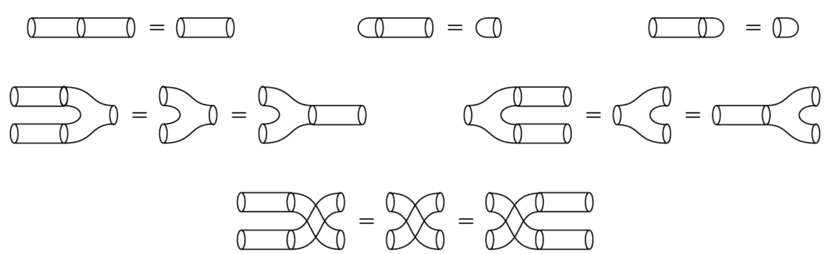
In each cobordism shown, the in-boundaries are drawn on the left and the out-boundaries are drawn on the right.

Instead of giving a proof for Proposition 4.3, we would like to bring in some physical intuitions again. The six generators above can be viewed as depicting the processes of the creation of a particle, the merging of two particles, no change, the splitting of a particle into two, the annihilation of a particle and the exchange of two particles respectively. This is in analogy to the Feynman diagrams. For a detailed proof of the proposition, see Section 1.4 in [5].

We also list the relations of **2Cob** in the following proposition.

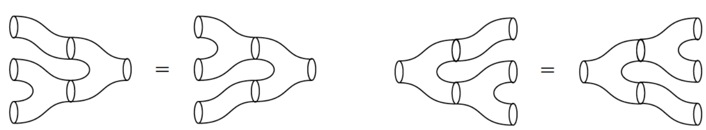
Proposition 4.4 (Relations of **2Cob**)**.** *The relations of* **2Cob** *are the following.*

(1) Identity relations:



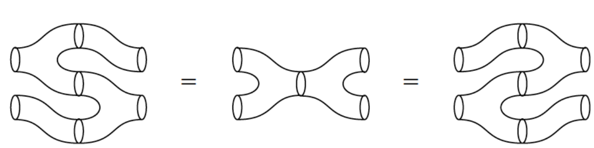
(2) Unit and counit:

(3) Associativity and coassociativity:

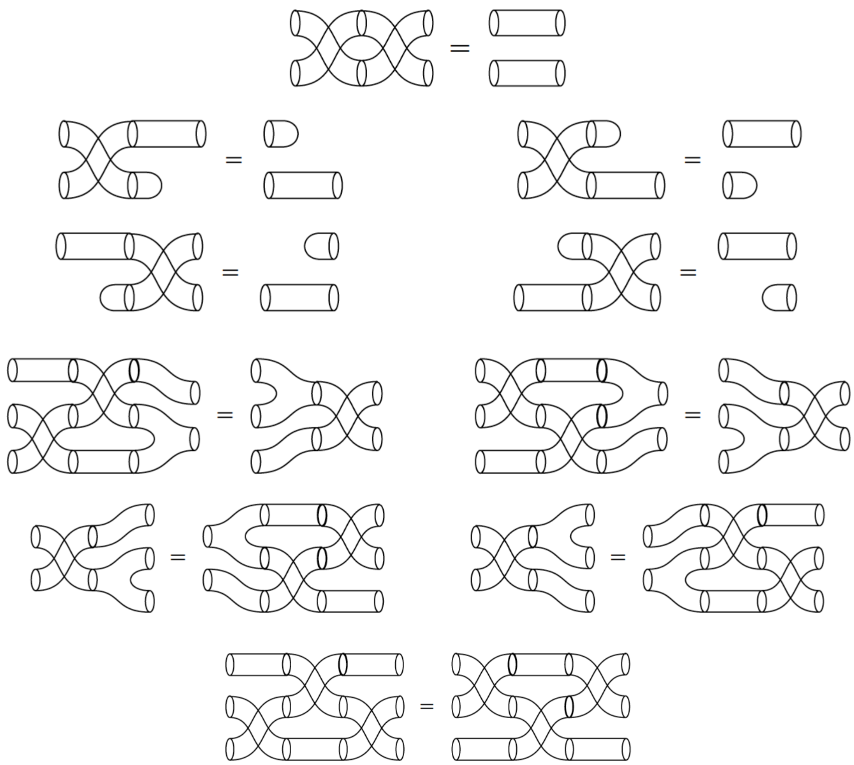


(4) Commutativity and cocommutativity:

(5) The Frobenius relation:

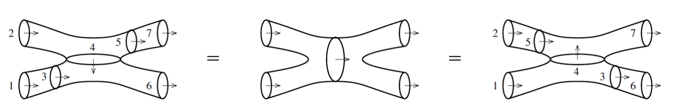


(6) Relations involving the twist cobordism:

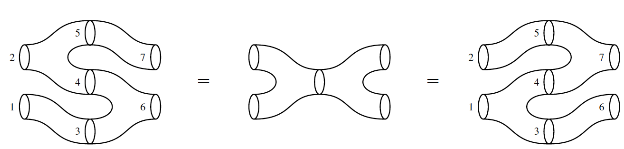


With the physical intuition we mentioned following Proposition 4.3, it is relatively straight-forward to see why these relations should hold. The Frobenius relation (5) is probably the only one that needs more explanation. We therefore prove it here.

Proof. (Proof for the Frobenius relation) The second cobordism in relation (5) can be cut in three ways [5]:



where the direction of an arrow indicates whether the given boundary is an in-boundary or an out-boundary. Drawing the picture as follows proves the Frobenius relation.



□

We remark that the set of relations we present above are sufficient but not minimal. It is shown that the Frobenius relation (5) together with the unit and counit relation (2) imply the associativity and coassociativity relation (3). For a detailed proof on the sufficiency of these relations, see Section 1.4 in [5].

5. Frobenius Algebras

Having examined the structure of **2Cob**, we move on to discuss Frobenius algebras. The motivation for this move will become evident as we proceed to discuss the graphical calculus for Frobenius algebras.

5.1. **Basic Definitions.** We first recall some algebraic definitions to introduce the various symbols we will need later.

Definition 5.1 ($\Bbbk$-algebras)**.** A $\Bbbk$-*algebra* is a $\Bbbk$-vector space A together with two $\Bbbk$-linear maps

$$\mu : A \otimes A \to A, \ \eta : \Bbbk \to A,$$

called the multiplication and unit map respectively, such that the following three disgrams commute:

$$\begin{tikzcd}
& A\otimes A\otimes A \arrow[dl, "\mu\otimes\mathrm{id}_A"'] \arrow[dr, "\mathrm{id}_A\otimes\mu"] & \\
A\otimes A \arrow[dr, "\mu"'] & & A\otimes A \arrow[dl, "\mu"] \\
& A &
\end{tikzcd}$$

$$\begin{tikzcd}
\Bbbk\otimes A \arrow[r, "\eta\otimes\mathrm{id}_A"] \arrow[dr] & A\otimes A \arrow[d, "\mu"] & A\otimes A \arrow[d, "\mu"'] & A\otimes\Bbbk \arrow[l, "\mathrm{id}_A\otimes\eta"'] \arrow[dl] \\
& A & A &
\end{tikzcd}$$

Definition 5.2 (Frobenius structrue)**.** Given a finite-dimensional $\Bbbk$-algebra A, a *Frobenius structure* is given by any one of the following equivalent structures:

(1) a linear functional $\epsilon : A \to \Bbbk$, called the *Frobenius form*, whose nullspace contains no nontrivial left ideals;
(2) an associative nondegenerate pairing $\beta : A \otimes A \to \Bbbk$, called the *Frobenius pairing*;
(3) a left A-isomorphism $A \xrightarrow{\cong} A^*$;
(4) a right A-isomorphism $A \xrightarrow{\cong} A^*$.

Definition 5.3 (Symmetric Frobenius algebras)**.** A Frobenius algebra A is called a *symmetric Frobenius algebra* if one (and hence all) of the following equivalent conditions holds.

(1) The Frobenius form $\epsilon : A \to \Bbbk$ is central, i.e. $(ab)\epsilon = (ba)\epsilon$ for all $a, b \in A$
(2) The pairing $< | >$ is symmetric, i.e. $< a|b >=< b|a >$ for all $a, b \in A$.
(3) The left A-isomorphism $A \xrightarrow{\cong} A^*$ is also right-linear.
(4) The right A-isomorphism $A \xrightarrow{\cong} A^*$ is also left-linear.

Definition 5.4 (Coalgebras)**.** A *coalgebra* over $\Bbbk$ is a vector space A together with two $\Bbbk$-linear maps

$$\delta : A \to A \otimes A,\ \epsilon : A \to \Bbbk$$

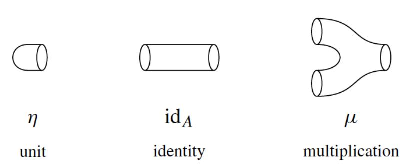
such that the following three disgrams commute:

$$\begin{tikzcd}
& A \otimes A \otimes A & \\
A \otimes A \arrow[ur, "\delta\otimes\mathrm{id}_A"] & & A \otimes A \arrow[ul, "\mathrm{id}_A\otimes\delta"'] \\
& A \arrow[ul, "\delta"] \arrow[ur, "\delta"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\Bbbk \otimes A & A \otimes A \arrow[l, "\epsilon\otimes\mathrm{id}_A"'] \\
& A \arrow[ul] \arrow[u, "\delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes A \arrow[r, "\mathrm{id}_A\otimes\epsilon"] & A \otimes \Bbbk \\
A \arrow[u, "\delta"] \arrow[ur] &
\end{tikzcd}$$

The map δ is called *comultiplication* and ϵ is called *counit*. The axioms expressed in the disgrams are called the *coassociativity* and the *counit condition*.

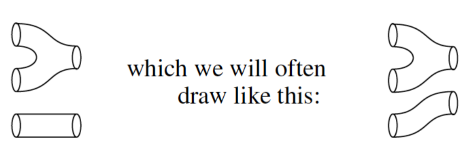
5.2. **The Graphical Calculus.** We now introduce the graphical calculus to state two important results about Frobenius algebra. The graphical calculus provides a dictionary that translates statements about Frobenius algebra into graphical representations. First, the maps that define a $\Bbbk$-algebra are:



.

Throughout the rest of this paper, these symbols have the status of formal mathematical symbols. The symbol corresponding to each $\Bbbk$-linear map $\phi : A^m \to A^n$ has m boundaries on the left: one for each factor of A in the source, ordered such that the first factor in the tensor prodcut corresponds to the bottom input hole and the last factor corresponds to the top input hole. The n boundaries on the right corresponds to the target A^n with similar convention for ordering. We also note that A^0 is just $\Bbbk$.

The tensor product of two maps is drawn as the disjoint union of two symbols. For example, the map $\mathrm{id}_A \otimes \mu$ is drawn as:

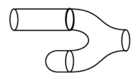


.

The composition of maps is reperesented by joining the output holes of the first symbol with the input holes of the second symbol. For example, the composition map

$$\Bbbk \otimes A \xrightarrow{\eta \otimes \mathrm{id}_A} A \otimes A \xrightarrow{\mu} A$$

is drawn as



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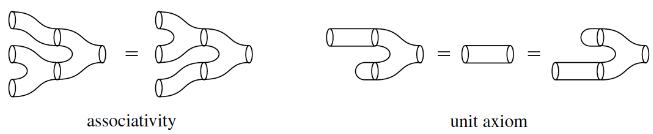
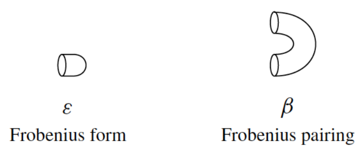
The axioms of $\Bbbk$-algebra in Definition 5.1 are now expressed as



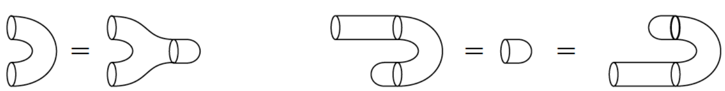
.

We proceed to express the Frobenius structure in Definition 5.2 in the graphical dictionary. For our purpose, the Frobenius form $\epsilon : A \to \Bbbk$ and Frobenius pairing $\beta : A \otimes A \to \Bbbk$ are the most helpful. They are depicted as



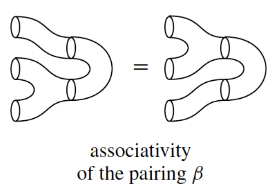
.

The relations between ϵ and β are depicted as



.

The associativity of Frobenius pairing is depicted as

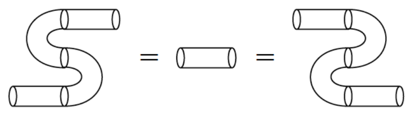


.

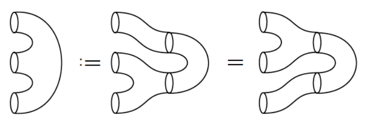
The nondegeneracy condition on the Frobenius pairing requires the existence of the following symbol

,

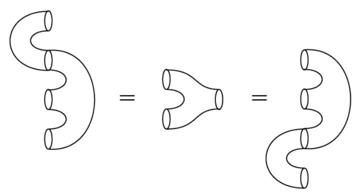
such that the following relation (the snake relation) holds



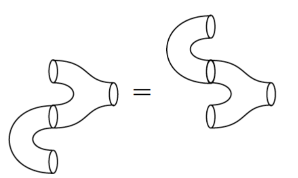
So far we have obtained graphical representations for all the structures of a Frobenius algebra, but let us explore these graphics a bit further. We can depict the three-point function $\phi : A \otimes A \otimes A \to \Bbbk$ given by $\phi := (\mu \otimes \mathrm{id}_A)\beta = (\mathrm{id}_A \otimes \mu)\beta$ by combining the symbols above:



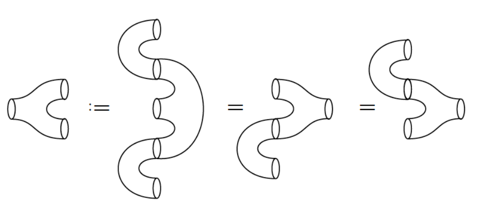
and we have the following relation for the three-point function:



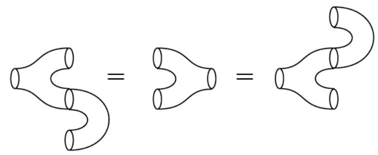
This also gives us the following relation:



We can now define the symbol for the comultiplication $\delta : A \to A \otimes A$:



Conversely, using β and the snake relation, we have the following relation: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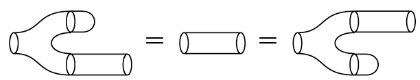
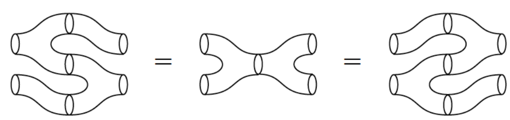
This gives us the following lemmas.

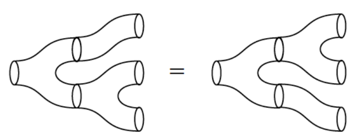
Lemma 5.5. *The Frobenius form ϵ is counit for δ:*


.

Lemma 5.6. *The comultiplication δ satisfies the following relation, called the Frobenius relation:*


.

Lemma 5.7. *The comultiplication is coassociative:*


.

Lemmas 5.5 through 5.7 imply two main results of this section.

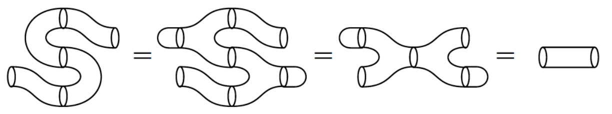
Proposition 5.8. *Given a Frobenius algebra (A, ϵ), there exists a unique comultiplication whose counit is ϵ and which satisfies the Frobenius relation, and this comultiplication is coassociative.*

Proposition 5.9. *Let A denote a vector space equipped with a multiplication map $\mu : A \otimes A \to A$, with unit $\eta : \Bbbk \to A$, a comultiplication $\delta : A \to A \otimes A$, with counit $\epsilon : A \to \Bbbk$ and suppose that the Frobenius relation holds. Then*

(1) the vector space A is of finite dimension;
(2) the multiplicaiton μ is associative, and thus A is a finite-dimensional $\Bbbk$-algebra. The comultiplication is coassociative;
(3) the counit ϵ is a Frobenius form, and (A, ϵ) is a Frobenius algebra.

With the help of the graphical calculus, these propositions can be proved by inserting units, counits and relations at suitable locations. We show the proof for Proposition 5.9 (1) as an example below. Interested readers might consult Sections 2.2 and 2.3 in [5] for proofs of the rest of the propositions.

Proof. (Proof for Proposition 5.9 (1)) Let $\beta := \mu\epsilon$. We will show that β is nondegenerate, i.e. establish the snake relation. We put unit and counit on the left-hand part of the Frobenius relation and use the unit axiom and Lemma 5.5:

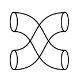


This proves the left hand side of the snake relation. Similarly, we can prove the right hand side of the snake relation. Therefore β is nondegenerate. The finite dimensionality of A follows with the help of the following lemma.

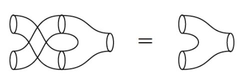
Lemma 5.10. *The pairing $\beta : V \times W \to \Bbbk$ is degenerate in W if and only if W is finite-dimensional and the induced map $\beta_{\text{left}} : W \to V^*$ is injective. Nondegeneracy in V is equivalent to finite dimensionality of V and injectivity of $V \to W^*$.*

□

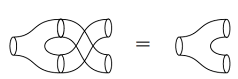
5.3. **Commutativity and Cocommutativity.** To express commutativity and cocommutativity in the graphical langauge, we introduce again the symbol for the twist map of a vector space (see Example 2.8):



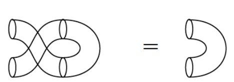
Let A be a commutative algebra with multiplication depicted in Section 5.2. Then we can picture the axiom of being a commutative algebra as



For a coalgebra A with comultiplication depicted as in Section 5.2, we can picture the axiom of being cocommutative as



If (A, β) is a Frobenius algebra, then we can picture of the condition of being a symmetric Frobenius algebra as



5.4. **The Category of Frobenius Algebras.** We would like to now assemble Frobenius algebras into a category and give it the monoidal structure, like what we did for two-dimensional cobordisms.

Definition 5.11 (The category of Frobenius algebra)**.** A *Frobenius algebra homomorphism* $\phi : (A, \epsilon) \to (A', \epsilon')$ between two Frobenius algebras is an algebra homomorphism which is at the same time a coalgebra homomorphism. $\mathbf{FA}_{\Bbbk}$ is the category of Frobenius algebra over $\Bbbk$ and Frobenius algebra homomorphisms. $\mathbf{cFA}_{\Bbbk}$ is the full subcategory of commutative Frobenius algebras.

Lemma 5.12. *A Frobenius algebra homomorphism $\phi : A \to A'$ is always invertible, so that the category $\mathbf{FA}_{\Bbbk}$ is a groupoid (and so is $\mathbf{cFA}_{\Bbbk}$).*

To see the monoidal structure in $\mathbf{FA}_{\Bbbk}$, it is natural to consider the tensor product of two Frobnius algebras.

Proposition 5.13. *Given two algebras A and A', consider their tensor product $A \otimes A'$ as vector spaces. Then component-wise multiplication makes $A \otimes A'$ into an algebra:*

$$(A \otimes A') \otimes (A \otimes A') \to A \otimes A'$$
$$(x \otimes x') \otimes (y \otimes y') \mapsto xy \otimes x'y'.$$

We remark that in order for A to only interact with A and A' to only interact with A' when juxtaposing two multiplications, the twist map play a central role in defining the new multiplication. The associavity and unit axioms for being an algebra are easily checked using the graphical calculus and the relations for twist maps. Similarly, we observe the following proposition.

Proposition 5.14. *The tensor product of two coalgebras is a coalgebra.*

These two propositions imply that the tensor product of two Frobenius algebras is a Frobenius algebra. Therefore, we have the following result.

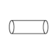
Proposition 5.15. $(\mathbf{FA}_{\Bbbk}, \otimes, \Bbbk)$ *is monoidal;* $(\mathbf{cFA}_{\Bbbk}, \otimes, \Bbbk)$ *is symmetric monoidal.*

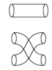
6. Classification of 2D Topological Quantum Field Theory

Recall the functorial definition of TQFT (Definition 3.3). A two-dimensional TQFT is a symmetric monoidal functor from the symmetric monoidal category $(\mathbf{2Cob}, \amalg, \emptyset, T)$ to $(\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk, \sigma)$. Since a monoidal functor is completely determined by its values on the generators of the source category, to specify a symmetric monoidal functor $\mathcal{A} : \mathbf{2Cob} \to \mathbf{Vect}_{\Bbbk}$, we specify a vector space A as the image of $\mathbf{1} \in \mathbf{2Cob}$ and a linear map for each of the generators. The symmetric condition on $\mathcal{A}$ demands that the image of the twist cobordism must be the usual twist map for the tensor product of vector spaces σ.

Fixing vector space A, we construct the following correspondences

$$\mathbf{2Cob} \longrightarrow \mathbf{Vect}_{\Bbbk}$$
$$\mathbf{1} \longmapsto A$$
$$\mathbf{n} \longmapsto A^n$$

 $\longmapsto [\mathrm{id}_A : A \to A]$

 $\longmapsto [\sigma : A^2 \to A^2].$

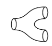
The images of the generators of **2Cob** are

$$\textit{\textbf{2Cob}} \longrightarrow \textit{\textbf{Vect}}_{\Bbbk}$$

 $\longmapsto [\eta : \Bbbk \to A]$

 $\longmapsto [\mu : A^2 \to A]$

 $\longmapsto [\varepsilon : A \to \Bbbk]$

 $\longmapsto [\delta : A \to A^2].$

In conclusion, given a two-dimensional TQFT $\mathcal{A}$, the image vector space $A = \mathbf{1}\mathcal{A}$ is a commutative Frobenius algebra. Conversely, given a commutative Frobenius algebra (A, ϵ), we can construct $\mathcal{A}$ using the above correspondences.

The correspondences are also true for arrows. As seen in Section 3.1, the arrows in $\mathbf{2TQFT}_{\Bbbk}$ are the monoidal natural transformations. Given two TQFTs $\mathcal{A}$ and $\mathcal{B}$ with $A = \mathbf{1}\mathcal{A}$ and $B = \mathbf{1}\mathcal{B}$, a natural transformation u between them consists of linear maps $A^n \to B^n$ for each $n \in \mathbb{N}$. For u to be a monoidal natural transformation, $A^n \to B^n$ must be completely determined by the map $A^1 \to B^1$. The naturality of u requires that $A^n \to B^n$ are all compatible with arrows in **2Cob**, which follows from the fact that $A \to B$ is a $\Bbbk$-algebra homomorphism and coalgebra homomorphism. Conversely, given a Frobenius algebra homomorphism between two commutative Frobenius algebras, we can construct a monoidal natural transformation between the TQFTs by applying the above correspondences in the reverse direction.

We arrive at the main theorem of this paper.

Theorem 6.1. *There is a canonical equivalence of categories between the category of two-dimensional TQFTs and the category of commutative Frobenius algebras*

$$\mathbf{2TQFT}_{\Bbbk} \cong \mathbf{cFA}_{\Bbbk}.$$

Acknowledgements

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