NUMBER RIGIDITY IN POINT PROCESSES

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Abstract. In this paper, we survey some results on a notion known as number rigidity that arises in the study of point processes. Rigidity occurs when the number of points in a bounded set is determined by observing only the point pattern outside this set. We focus on the model of perturbed lattices, in which each point in $\mathbb{Z}^d$ is perturbed by a random variable to generate a point process on $\mathbb{R}^d$. Our main result is that in the transformed and perturbed one-dimensional integer lattice $\{|k|^\alpha + Y_k\}_{k\in\mathbb{Z}}$, the spacing parameter α plays an important role in the question of rigidity and exhibits a threshold phenomenon.

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1. Introduction

A point process on $\mathbb{R}^d$ is a random point pattern on $\mathbb{R}^d$. We can equivalently think of it as a random counting measure on $\mathbb{R}^d$ by counting the number of points within a given set. Although this paper is concerned with number rigidity, the broader idea here is spatial conditioning: if we are given a bounded subset B of $\mathbb{R}^d$ and only allowed to observe the point pattern outside of B, can we draw any conclusions about the point pattern inside B? Number rigidity says that the outside pattern determines the *number* of points within B. In the case where the number of points in the point patterns is finite, or where the points are distributed independently for disjoint sets, the question of rigidity is more or less direct. But,as we will see, in most instances establishing rigidity or non-rigidity becomes significantly more difficult.

An interesting example of a point process is the perturbed integer lattice. We start with a collection of independent, identically distributed (i.i.d.), $\mathbb{R}^d$-values random variables $\{x+Y_x\}_{x\in\mathbb{Z}^d}$, each associated with a vertex in $\mathbb{Z}^d$. By summing over their corresponding Dirac measures, we obtain a point process $\Pi = \sum_{x\in\mathbb{Z}^d} \delta_{x+Y_x}$

on $\mathbb{R}^d$. The intuitive idea for rigidity is that if the random variables have very large deviations from their centers, then the structure of perturbed lattice becomes less predictable, and we are less likely to have rigidity. Conversely, if the random variables tend to be centered at their corresponding vertices, then the structure of $\mathbb{Z}^d$ is well-preserved and allows for rigidity. The magnitude of the deviations can be reflected, for example, in the central moments of the random variables.

Building on this intuition, it seems plausible that the spacing between the random points also affects rigidity. To explore this idea, we will consider the model $\{|k|^\alpha + Y_k\}_{k\in\mathbb{Z}}$, where the Y_k are i.i.d. standard Guassian random variables. As we will see in Section 5, the resulting point process on $\mathbb{R}$ is rigid when $\alpha > 1/2$ and non-rigid when $\alpha \in (0, 1/2)$.

2. General Point Process Theory

In this section, we present a gentle, and mostly intuitive, introduction to the theory of point processes. We begin with a measurable space $(X, \mathcal{B})$, with X being a "nice" topological space and $\mathcal{B}$ being its Borel σ-algebra. The usual assumptions on X (e.g., local compactness, second countability) are there to prevent pathological examples. But for our purposes, it suffices to think of X in the most common cases, such as $\mathbb{R}^d$ or $\mathbb{C}$. It is often more convenient to understand a point process as a random countable subset of X with no limit points. Each elementary event is then a point pattern on X.

To make this somewhat more rigorous, we pair the measurable space $(X, \mathcal{B})$ with a probability space $(\Omega, \mathcal{F}, P)$. Throughout this paper, we will use $\mathbb{N}$ to denote the set of *nonnegative* integers $\{0, 1, 2, ...\}$. A point process Π is a mapping $\mathcal{B} \times \Omega \to \mathbb{N}$ satisfying the following conditions:

(1) for every $\omega \in \Omega$, the function $\Pi(\cdot, \omega) : \mathcal{B} \to \mathbb{N}$ is a locally finite measure on X;
(2) for every $B \in \mathcal{B}$, the function $\Pi(B, \cdot) : \Omega \to \mathbb{N}$ is a random variable taking values in the non-negative integers.

An easy way to make sense of this definition is to look at a point process from two perspectives. It will suffice to consider almost every $\omega \in \Omega$ as a countable subset of X without accumulation points. Then the map $\Pi(\cdot, \omega) : \Omega \to \mathbb{N}$ is nothing but the sum of the Dirac measures on X at each $a \in \omega$, i.e.,

$$\forall B \in \mathcal{B}, \Pi(B, \omega) = \sum_{a\in\omega} \delta_a \tag{2.1}$$

Alternatively, we could fix a measurable set B and restrict Π solely to a random measure on B counting the number of points, in which case the map $\Pi(B, \cdot)$ becomes a random variable on the space Ω. The latter will be the primary focus of this paper.

For notational convenience, we usually represent Π as a sum of Dirac measures,

$$\Pi = \sum_{k\in\mathbb{N}} \delta_{X_k}, \tag{2.2}$$

where $\{X_k\}_{k\in\mathbb{N}}$ are random elements of X. The important distinction here is that Π, as a sum of Dirac measures at these random elements, is different from directly summing these variables. In a point process, we draw a random countable subset S of X first, and then study the measure on X induced by the characteristic functions at the elements of S.

Having seen the definition of a point process, we will now look at some canonical examples to gain a better understanding.

Example 2.1. The Poisson point process Π on $\mathbb{R}^d$ with intensity parameter λ satisfies that for any disjoint bounded sets $B_1, ..., B_n \subset \mathbb{R}^d$, and any $k_1, ..., k_n$, we have

$$\mathbb{P}[\Pi(B_i) = k_i, 1 \leq i \leq n] = \prod_{i=1}^{n} e^{-\lambda\|B_i\|} \frac{(\lambda\|B_i\|)^{k_i}}{k_i!}, \tag{2.3}$$

where $\|\cdot\|$ denotes the Lebesgue measure. This formula shows that the Poisson point process Π is characterized by the conditions that 1) for any disjoint sets $B_1, ..., B_n$, the random variables $\Pi(B_1), ..., \Pi(B_n)$ are independent; 2) for each bounded B, $\Pi(B)$ follows a Poisson distribution with parameter $\lambda\|B\|$.

Example 2.2. The standard planar Gaussian Analytic Function (GAF) f on $\mathbb{C}$ is defined by a random power series expansion at zero,

$$f(z) = \sum_{k=0}^{\infty} \xi_k \frac{z^k}{\sqrt{k!}}, \tag{2.4}$$

where the ξ_k's are i.i.d. standard complex Gaussian random variables, i.e., $\xi_k \sim N(0, 1/2) + iN(0, 1/2)$ with the two components being independent. The zeros of $f(z)$ form a translation-invariant point process on $\mathbb{C}$ (see, e.g., Proposition 2.3.4 in [5]).

Example 2.3. The spectral distribution of random matrices is a point process on $\mathbb{R}$ or $\mathbb{C}$, depending on the setting being Hermitian or non-Hermitian. A much studied random matrix model is the Gaussian Unitary Ensemble (GUE). Consider a complex Wigner-type n-dimensional matrix H with the following properties:

(1) for $1 \leq i < j \leq n$, the H_{ij} are i.i.d. standard complex Gaussian;
(2) for $1 \leq i < j \leq n$, $H_{ji} = \overline{H_{ij}}$;
(3) for $1 \leq i \leq n$, the H_{ii} are i.i.d. standard real Gaussian, i.e., $H_{ii} \sim N(0, 1)$.

The distribution of this class of random matrices is invariant under unitary transformations, hence the name GUE. It can be shown (see, e.g., Theorem 4.3.8 in [5]) that the eigenvalues of H have joint density function

$$f(\lambda_1, ..., \lambda_n) = \frac{1}{Z_n} \exp\left(-\frac{1}{2} \sum_{i=1}^{n} \lambda_i^2\right) \prod_{i<j} |\lambda_i - \lambda_j|^2, \tag{2.5}$$

where Z_n is a normalizing constant depending on the dimension. However, in most other cases of point processes, we may not be able to obtain an explicit density function and will have to use different approaches.

3. Spatial Conditioning and Rigidity

One notion that arises in the study of point processes is spatial conditioning. We first fix a bounded measurable set $B \subset X$, and for simplicity, we will take X to be $\mathbb{R}^d$ or $\mathbb{C}$. For any point configuration $\omega \in \Omega$ on X, if we view it as a subset of X, then we can consider its intersections with B and B^C and obtain the decomposition

$$\omega_{\text{in}} \coloneqq \omega \cap B, \qquad \omega_{\text{out}} \coloneqq \omega \cap B^C. \tag{3.1}$$

Doing this for every element $\omega \in \Omega$ gives a decomposition of Ω into

$$\Omega_{\text{in}} := \{\omega \cap B | \omega \in \Omega\}, \qquad \Omega_{\text{out}} := \{\omega \cap B^C | \omega \in \Omega\}. \tag{3.2}$$

Then we can naturally restrict the point process Π to B and B^C by respectively letting

$$\Pi_{\text{in}}(A, \omega) := \Pi(A, \omega), A \subseteq B, A \in \mathcal{B} \tag{3.3}$$

$$\Pi_{\text{out}}(A, \omega) := \Pi(A, \omega), A \subseteq B^C, A \in \mathcal{B}. \tag{3.4}$$

We note here that the restrictions are not on the configurations Ω, but instead on the Borel σ-algebra $\mathcal{B}$ to B and B^C. Given such a decomposition of Π, when there is reasonable dependence among the random points in Π, it is natural to ask that if we could infer something about Π_{in} conditioned on Π_{out} having a certain configuration ω_{out}. Quite surprisingly, it can be demonstrated that in some cases, the number of points inside B is determined by the outside configuration. This phenomenon is known as number rigidity, and the formal definition is given as follows.

Definition 3.1. A point process Π is *number rigid*, or *rigid*, if for every bounded measurable set $B \subset X$, there exists a deterministic measurable function $N : \Omega_{\text{out}} \to \mathbb{N}$ so it holds $P-$almost surely (a.s.) that

$$\left[\Pi_{\text{in}}(B) \middle| \Pi_{\text{out}}(B^C) = \sum_{a \in \omega_{\text{out}}} \delta_a\right] \equiv N(\omega_{\text{out}}). \tag{3.5}$$

Recall that P is the law of Π on Ω. The left-hand side is interpreted to denote the random variable $\Pi_{\text{in}}(B)$ on the restricted probability space $\Omega' = \{\lambda | \lambda \cap B^C = \omega_{\text{out}}\}$ for any given ω_{out}.

As we have remarked in the definition, any outside configuration ω_{out} results in a subspace of Ω consisting only of configurations that agree with ω_{out} on B^C. Number rigidity occurs if the random variable Π_{in} on this subspace is equal in an almost sure sense to a deterministic function taking values in the non-negative integers.

Having introduced the notion of number rigidity, it will be helpful to revisit some of the examples in Section 2 to see rigidity (or lack thereof) in some elementary applications. In Example 2.1 of the Poisson point process Π, since B and B^C are disjoint, the random variables $\Pi(B)$ and $\Pi(B^C)$ are independent. Therefore, for any outside configuration ω_{out}, we have

$$\left[\Pi(B) | \Pi(B^C) = \sum_{x \in \omega_{\text{out}}} \delta_x\right] = \Pi(B),$$

which cannot be deterministic if B has positive Lebesgue measure.

In Example 2.3, the spectral distribution of the GUE, considered as a point process on $\mathbb{R}$, is rigid. More generally, the spectral distribution for any finite-dimensional random matrix ensemble is rigid. This is because any n-dimensional matrix has n eigenvalues (counted with multiplicity). Consequently, given an outside configuration ω_{out}, the number of points inside B is determined by the simple relation

$$\Pi_{\text{in}}(B) = n - |\omega_{\text{out}}|.$$

Whether rigidity occurs in Example 2.2, on the other hand, is a highly non-trivial matter. Ghosh and Peres proved in [3] the following result.

Theorem 3.2. *(Informal statement) Let $f(z)$ be the standard planar Gaussian Analytic Function as defined in (2.4) and consider its zeros as a point process on $\mathbb{C}$. Let $\mathcal{D}$ denote a bounded open set on the complex plane. Then the point configuration on $\mathcal{D}^C$ determines, almost surely, the number of points in $\mathcal{D}$, the sum of points in $\mathcal{D}$, but "nothing more."*

This result goes beyond number rigidty and shows that spatial conditioning in the GAF zero process also determines the sum of points within $\mathcal{D}$. But perhaps even more interesting is the heuristic that besides the number and the sum, there is essentially nothing more we can conclude about the conditional distribution of points within $\mathcal{D}$. This idea is made precise by the mutual absolute continuity of measures. To see this, we first fix ω_{out} to be a possible configuration of points on $\mathcal{D}^C$. The first part of Theorem 3.2 tells us that there is $N(\omega_{\text{out}}) \in \mathbb{N}$ and $S(\omega_{\text{out}}) \in \mathbb{C}$ so that a.s., the number of points within $\mathcal{D}$ is $N(\omega_{\text{out}})$ and the sum of points is $S(\omega_{\text{out}})$. We take the set of all configurations on $\mathcal{D}$ with $N(\omega_{\text{out}})$ number of points and sum $S(\omega_{\text{out}})$ and embed them in $\mathbb{C}^{N(\omega_{\text{out}})}$ by considering all orderings to obtain

$$\Sigma_{N(\omega_{\text{out}})} = \left\{ \zeta \in \mathcal{D}^{N(\omega_{\text{out}})} \,\middle|\, \sum_{i=1}^{N(\omega_{\text{out}})} \zeta_i = S(\omega_{\text{out}}) \right\}.$$

This embedding allows us to define a measure $\rho(\omega_{\text{out}}, \cdot)$ on $\Sigma_{N(\omega_{\text{out}})}$ based on the conditional distribution of Π_{in} on $\mathcal{D}$ for each $\omega_{\text{out}} \in \Omega_{\text{out}}$. The last part of Theorem 3.2 means that the measure $\rho(\omega_{\text{out}}, \cdot)$ and the Lebesgue measure $\mathcal{L}$ on the set $\Sigma_{N(\omega_{\text{out}})}$ are mutually absolutely continuous. In particular, the measure $\rho(\omega_{\text{out}}, \cdot)$ is supported on all of $\Sigma_{N(\omega_{\text{out}})}$, which implies that almost any point configuration with $N(\omega_{\text{out}})$ points and sum $S(\omega_{\text{out}})$ is possible. This heuristic provides a glimpse at a notion related to rigidity, known as tolerance. We will discuss it in more detail in Section 4.

Besides these examples, another interesting setting to consider is perturbed lattices, or more general graphs. To begin with, we embed $\mathbb{Z}^d$ in $\mathbb{R}^d$, and assign a random variable Y_x in $\mathbb{R}^d$ to each $x \in \mathbb{Z}^d$. This gives us a field of random variables $\{x + Y_x | x \in \mathbb{Z}^d\}$, which in turn induces a point process on $\mathbb{R}^d$,

$$\Pi = \sum_{x \in \mathbb{Z}^d} \delta_{x+Y_x}. \tag{3.6}$$

It is possible study this model with very general assumptions, but even the simplest setting turns out to be interesting. For now, we will assume that the Y_x are i.i.d. Gaussian random variables with zero means in $\mathbb{R}^d$. It is shown in [4] that when $d = 1, 2$, the resulting point process is number rigid. However, when $d \geq 3$, the question of rigidity becomes more delicate as it relies on the variance of the Y_x.

Theorem 3.3. [7] *Consider the point process Π on $\mathbb{R}^d$ given in (3.6) with $d \geq 3$. Let Y_x be the i.i.d. Gaussian random variables $N(0, \sigma^2 I_d)$. There exist critical variances $\sigma(d)$, depending on the dimension, such that*

(1) *If $\sigma \in (0, \sigma(d))$, then Π is number rigid.*
(2) *If $\sigma \in (\sigma(d), \infty)$, then Π is non-rigid.*

As in a number of problems in probability, rigidity is another instance in which the dimension appears to play a role. An intuitive interpretation of Theorem 3.3 is that rigidity seems closely linked to how intense the perturbations are. Below the threshold $\sigma(d)$, the random variables $x + Y_x$ are, on average, not too deviated from their vertices $x \in \mathbb{Z}^d$. Therefore, the perturbed lattice stays close to the original structure $\mathbb{Z}^d$, allowing for rigidity. A supporting evidence for this intuition is that in [4], it is in fact proved that more generally,

(1) when $d = 1$, if the Y_x are i.i.d. random variables with bounded first moment, then Π is rigid;
(2) When $d = 2$, if the Y_x are i.i.d. random variables with bounded second moment, then Π is rigid.

Up until now, Theorem 3.2 is concerned with the GAF zero process, which can be shown to be translation invariant. Theorem 3.3 is, in a similar manner, periodic. That is, if we translate all the points along the edges in the lattice, this does not affect the setting of the problem. We can alternatively study rigidity in non-stationary structures. A result of this type is the following.

Theorem 3.4. [2] *Let Y_x be i.i.d. standard Gaussian random variables on $\mathbb{R}$. Consider the collection of random variables in $\mathbb{R}$ given by $\{\|x\|^\alpha + Y_x | x \in \mathbb{Z}^d\}$, and the induced point process $\Pi = \sum_{x\in\mathbb{Z}^d} \delta_{\|x\|^\alpha + Y_x}$ on $\mathbb{R}$. If $\alpha > d/2$, then Π is rigid.*

In this model, note that we are taking $\mathbb{Z}^d$ and mapping it into $\mathbb{R}$ via the function $\|\cdot\|^\alpha$. The parameter α determines the spacing of the transformed vertices $\|x\|^\alpha$ on $\mathbb{R}$. Therefore, the point process considered here is no longer stationary or translation invariant. We instead focus on the effect of spacing on rigidy. Theorem 3.4 echoes our interpretation for Theorem 3.3 in that when α is relatively large (specifically when $\alpha > d/2$), then the sufficient sparsity of $\|x\|^\alpha$ for $x \in \mathbb{Z}^d$ leads to rigidity.

4. Sufficient Conditions for Rigidity and Non-rigidity

At this point, it may seem that the problem concerning rigidity is either entirely trivial, or completely out-of-reach. However, in this section, we will first exposit a simple sufficient condition, given in [3], for proving rigidity. To that end, we need the definition of linear statistics.

Definition 4.1. Using the notation in (2.2) and viewing a point process Π on $\mathbb{R}^d$ as a collection $\{X_k\}_{k\in\mathbb{N}}$ of random elements of $\mathbb{R}^d$, the *linear statistic* corresponding to a function f on $\mathbb{R}^d$ is the random variable

$$\int f d[\Pi] := \sum_{k=0}^{\infty} f(X_k). \tag{4.1}$$

Theorem 4.2. *Let $\Pi = \sum_{k\in\mathbb{N}} \delta_{X_k}$ be a point process on $\mathbb{R}^d$. For any bounded measurable set $B \subset \mathbb{R}^d$, if there is a sequence of functions $(f_n)_{n\in\mathbb{N}}$ such that as $n \to \infty$,*

(1) *$f_n \to 1$ uniformly on B, and*
(2) *the variance of the linear statistics* $\textbf{\textit{Var}}[\int_{\mathbb{R}^d} f_n d[\Pi]] \to 0$,

then Π is number rigid.

Proof. Note first that we want to compute

$$\int_B 1 d[\Pi].$$

We can write it as the sum of the following three terms:

$$\int_B 1d[\Pi] - \int_B f_n d[\Pi] = \int_B (1-f_n)d[\Pi], \tag{4.2}$$

$$\mathbf{E}\left[\int_{\mathbb{R}^d} f_n d[\Pi]\right] - \int_{B^C} f_n d[\Pi], \tag{4.3}$$

$$\int_{\mathbb{R}^d} f_n d[\Pi] - \mathbf{E}\left[\int_{\mathbb{R}^d} f_n d[\Pi]\right]. \tag{4.4}$$

Since B is a bounded measurable set, almost surely we have $\Pi(B,\cdot) < \infty$. By uniform convergence, the distance between f_n and 1 on B is bounded above uniformly for all large n. Therefore, by dominated convergence, almost surely we have

$$\lim_{n\to\infty} \int_B (1-f_n)d[\Pi] = \int_B \lim_{n\to\infty} (1-f_n)d[\Pi] = 0,$$

so (4.2) tends to zero almost surely. Now using Chebyshev's inequality,

$$\mathbf{P}\left(\left|\int_{\mathbb{R}^d} f_n d[\Pi] - \mathbf{E}\left[\int_{\mathbb{R}^d} f_n d[\Pi]\right]\right| > \epsilon\right) \leq \frac{\mathbf{Var}\left[\int_{\mathbb{R}^d} f_n d[\Pi]\right]}{\epsilon^2} \to 0$$

for all $\epsilon > 0$, so (4.4) converges to zero in probability. Therefore, there is a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ so that (4.4) converges to zero almost surely. It follows that along this subsequence $(f_{n_k})_{k\in\mathbb{N}}$, (4.2) and (4.4) both vanish as $k\to\infty$. Finally, we observe that (4.3) is a measurable function for all n with respect to Ω_{out} because the second term is an integral only on B^C. Since the pointwise limit of measurable functions is again measurable, we conclude that the number of points inside B,

$$\int_B 1d[\Pi] = \lim_{k\to\infty} \mathbf{E}\left[\int_{\mathbb{R}^d} f_{n_k} d[\Pi]\right] - \int_{B^C} f_{n_k} d[\Pi],$$

is a measurable function with respect to outside configurations Ω_{out}. □

When the point process takes place in $\mathbb{R}$ or $\mathbb{C}$ and is bounded below, a good choice for the sequence of functions $(f_n)_{n\in\mathbb{N}}$ is to let $f_n = e^{-x/n}$, since the uniform convergence is easily seen for all bounded sets. The advantage of Theorem 4.2 as a criterion for rigidity is that the variances of the linear statistics are reasonably easy to compute or estimate, provided that the sequence f_n is sufficiently simple and that we have explicit formulas or estimates for the density function of the point process.

It is natural to ask the question of whether Theorem 4.2 is necessary for rigidity. In an attempt to answer this, we look at the specific example of using the semigroup method to study random Schrödinger operators, as discussed in [2]. If Π is the eigenvalue point process of the random Schrödinger operator H, and $f_n(\cdot) = e^{-(\cdot)/n}$ is the sequence of exponential functions defined for linear operators, then we can write

$$\int f_n d[\Pi] = \mathrm{Tr}[e^{-H/n}].$$

Therefore, the sufficient condition in Theorem 4.2 amounts (equivalently) to showing that $\lim_{t\to 0} \mathbf{Var}[\mathrm{Tr}[e^{-tH}]] = 0$. However, [2] provides specific examples of random Schrödinger operators that are number rigid, but for which

$$\liminf_{t\to 0} \mathbf{Var}[\mathrm{Tr}[e^{-tH}]] > 0. \tag{4.5}$$

This means that the vanishing of $\mathbf{Var}[\int e^{-(\cdot)/n} d[\Pi]]$ is not necessary for rigidity. Therefore, to prove rigidity in certain problems, one would need to try a different sequence of functions or use an altogether different sufficient condition. However, the hardness of the former is that the resulting computations may be overly difficult or even impossible. On the other hand, the question of whether Theorem 4.2 is a necessary condition for rigidity is unclear: we do not know if there exists a rigid point process so that the variance condition fails for every choice of f_n.

Our above discussion now motivates an approach for proving non-rigidity. Proposed in [4], a theory known as insertion and deletion tolerance turns out to be useful in proving non-rigidity, which is something we have already touched on in the discussion the GAF zero process. The idea stems from the natural question of examining the effect of adding or removing a point in a point process. For instance, if Π is the point process generated by the collection of random variables $\{X_k\}_{k\in\mathbb{N}}$, what happens when we delete the random variable Y_0 and look at the resulting point process Π_0? If Π_0 is "indistinguishable" from Π, then we can conclude that Π is not rigid because removing a point certainly decreases the number of points in some set with positive probability. Similarly, if adding a point X_∞ results in a point process that is almost the same as the original, then Π is also non-rigid. To make these ideas more precise, we introduce the following definitions.

Definition 4.3. Let $\Pi = \sum_{k\in\mathbb{N}} \delta_{X_k}$ be a point process on $\mathbb{R}^d$ and consider the point process $\Pi_i := \sum_{k\in\mathbb{N}-\{i\}} \delta_{X_k}$ with X_i removed. We say that Π is X_i-*deletion intolerant* if the laws of Π_i and Π are mutually absolutely continuous. And Π is *deletion-intolerant* if Π is X_i-deletion intolerant for all $i \in \mathbb{N}$.

Definition 4.4. Let Π be a point process as in Definition 4.3. Let U be a Borel-measurable subset of $\mathbb{R}^d$ with positive and finite Lebesgue measure, and let X_∞ be the random variable that is distributed on U uniformly and independently of Π. Define $\Pi_U := \sum_{k\in\mathbb{N}\cup\{\infty\}} \delta_{X_k}$. We say that Π is U-*insertion tolerant* if the laws of Π and Π_U are mutually absolutely continuous. And Π is *insertion-tolerant* if it is U-insertion tolerant for all U Borel and with positive, finite Lebesgue measure.

It seems intuitively clear that both deletion-intolerance and insertion-intolerance would provide a sufficient condition for non-rigidity, but it is more nuanced. In the setting of perturbed lattices, we have the following result from [7]:

Theorem 4.5. *Consider the collection of random variables $\{x + Y_x\}_{x\in\mathbb{Z}^d}$ and its associated point process Π on $\mathbb{R}^d$. If the Y_x are i.i.d. and have an everywhere positive density, then both deletion-intolerance and insertion-intolerance imply non-rigidity.*

Proof. Suppose that for a given $x \in \mathbb{Z}^d$, the point processes $\Pi_x = \sum_{k\in\mathbb{Z}^d-\{x\}} \delta_{k+Y_k}$ and Π are mutually absolutely continuous but Π is rigid. Let B be the unit ball in $\mathbb{R}^d$. For any outside configuration ω_{out}, by rigidity the number of points inside $\Pi_{\text{in}}(B) = N(\omega_{\text{out}})$ is determined almost surely. Since Π is mutually absolutely continuous with Π_x, the number of points in $(\Pi_x)_{\text{in}}(B) = N(\omega_{\text{out}})$ is also determined almost surely. But on the event $E = \{x+Y_x \in B\}$, Π_x and Π have the same outside configuration but Π has one more point inside B than does Π_x. Since Y_x has an everywhere positive density, the event E happens with positive probability, giving a contradiction. The proof for the insertion-tolerance case is similar. $\square$

5. First Problem

In this section, we study the setting of a transformed and perturbed one-dimensional integer lattice. Let $\{Y_k\}_{k\in\mathbb{Z}}$ be i.i.d. standard Gaussian random variables. Consider the collection of random variables $\{|k|^\alpha + Y_k | k \in \mathbb{Z}\}$, where $\alpha \in (0,\infty)$ is a parameter of spacing, and the resulting point process $\Pi = \sum_{k\in\mathbb{Z}} \delta_{|k|^\alpha+Y_k}$. We will show that this point process demonstrates a phase transition at $\alpha = 1/2$ regarding rigidity. The precise statement is given as follows.

Theorem 5.1. *Consider the point process* Π *on* $\mathbb{R}$ *as above.*

1. *If* $\alpha \in (1/2,\infty)$*, then* Π *is rigid.*
2. *If* $\alpha \in (0,1/2)$*, then* Π *is non-rigid.*

The first part of Theorem 5.1 is a direct consequence of the main result in [2], though we will exposit the proof below. The essence is using the sufficient condition for rigidity stated in Theorem 4.2. The proof of the second part of Theorem 5.2 applies the notion of deletion-tolerance introduced in Section 4 and builds on an approach in [7] using convergence of martingales.

We will now prove the first part of Theorem 5.1. Fix $\alpha > 1/2$. Consider the collection of functions $f_t(x) = e^{-tx}$ with $t > 0$, which converges uniformly to 1 on all bounded sets. The variance of the linear statistics is given by

$$\mathbf{Var}\left[\int f_t d[\Pi]\right] = \mathbf{Var}\left[\sum_{k\in\mathbb{Z}} e^{-t(|k|^\alpha+Y_k)}\right]. \tag{5.1}$$

Since the Y_k are independent, we want to be able to exchange the order of the sum and the variance. But since the sum is countably infinite, this requires some work. Applying Corollary A.2.1 to the sequence $X_n = \sum_{k\in\mathbb{Z},|k|=n} e^{-t(|k|^\alpha+Y_k)}$, it suffices to check that the following two conditions are satisfied.

$$\text{For each } t>0, \sum_{k\in\mathbb{Z}} \mathbf{E}\left[e^{-t(|k|^\alpha+Y_k)}\right] < \infty. \tag{5.2}$$

$$\sum_{k\in\mathbb{Z}} \mathbf{Var}\left[e^{-t(|k|^\alpha+Y_k)}\right] \to 0 \text{ as } t \to 0. \tag{5.3}$$

Since tY_k and $-tY_k$ have the same distribution and the e^{tY_k}'s are lognormal with mean $e^{t^2/2}$, we have

$$\sum_{k\in\mathbb{Z}} \mathbf{E}\left[e^{-t(|k|^\alpha+Y_k)}\right] = \sum_{k\in\mathbb{Z}} e^{-t|k|^\alpha}\mathbf{E}[e^{-tY_k}] \le 2\sum_{k\in\mathbb{N}} e^{-tk^\alpha}\mathbf{E}[e^{tY_k}] = 2e^{t^2/2}\sum_{k\in\mathbb{N}} e^{-tk^\alpha},$$

Now observe that for each fixed $t > 0$, since

$$\lim_{k\to\infty} \frac{e^{-tk^\alpha}}{k^{-2}} = \lim_{k\to\infty} \frac{k^2}{e^{tk^\alpha}} = \lim_{n\to\infty} \frac{n^{2/\alpha}}{e^{tn}} = 0,$$

by the comparison test we conclude that (5.2) holds. Similarly, we can estimate (5.3):

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \mathbf{Var}\left[e^{-t(|k|^\alpha+Y_k)}\right] &= \sum_{k\in\mathbb{Z}} e^{-2t|k|^\alpha}\mathbf{Var}\left[e^{-tY_k}\right] \le 2\sum_{k\in\mathbb{N}} e^{-2tk^\alpha}\mathbf{Var}[e^{tY_k}] \\ &= \left(2e^{t^2}\right)\cdot\left(\frac{e^{t^2}-1}{t^{1/\alpha}}\right)\cdot\left(t^{1/\alpha}\sum_{k\in t^{1/\alpha}\mathbb{N}} e^{-2k^\alpha}\right),\end{aligned}$$

where we have used that the variance of the e^{tY_k}'s is $(e^{t^2}-1)e^{t^2}$. Note that the first term clearly converges to 2 as $t \to 0$. Applying L'Hôpital's rule to the second term gives us

$$\lim_{t\to 0} \frac{e^{t^2}-1}{t^{1/\alpha}} = 2 \lim_{t\to 0} t^{2-1/\alpha} e^{t^2} = 0,$$

provided that $\alpha > 1/2$. Via a Riemann sum and a change of variables, the third term can be written as

$$\begin{aligned} \lim_{t\to 0} t^{1/\alpha} \sum_{k\in t^{1/\alpha}\mathbb{N}} e^{-2k^\alpha} &= \int_0^\infty e^{-2x^\alpha}\,dx \\ &= \frac{1}{\alpha}\cdot\left(\frac{1}{2}\right)^{1/\alpha}\cdot\int_0^\infty y^{1/\alpha-1}e^{-y}\,dy \\ &= \frac{1}{\alpha}\cdot\left(\frac{1}{2}\right)^{1/\alpha}\cdot\Gamma\left(\frac{1}{\alpha}\right) < \infty \end{aligned}$$

Therefore, we conclude that variance of the linear statistics in (5.1) vanishes as $t \to 0$ and that Π is rigid if $\alpha \in (1/2, \infty)$.

We move on to the second part of Theorem 5.1. Our general strategy is to remove the random variable at the origin and show that the law of the resulting point process is mutually absolutely conitnuous with that of Π. Recall from Section 4 that this means that Π is 0-deletion tolerant. As a necessity for the proof, we will take a slight detour right now to introduce a notion known as pushforward measure.

Definition 5.2. Given measurable spaces $(X_1, \mathcal{B}_1)$, $(X_2, \mathcal{B}_2)$, a measurable function $f : X_1 \to X_2$, and a measure $\mu : \mathcal{B}_1 \to [0,\infty]$ on X_1, there is a *pushforward* measure $\mu_f : \mathcal{B}_2 \to [0,\infty]$ on X_2 defined by

$$\mu_f(B) := \mu(f^{-1}(B)). \tag{5.4}$$

The crucial reason that this defines a measure on X_2 is that the inverse images of disjoint sets under a function are disjoint. Indeed, for any $\{B_i\}_{i\in\mathbb{N}} \subset \mathcal{B}_1$, we always have

$$\mu_f\left(\cup_{i\in\mathbb{N}} B_i\right) = \mu\left(f^{-1}\left(\cup_{i\in\mathbb{N}} B_i\right)\right) = \mu\left(\cup_{i\in\mathbb{N}} f^{-1}(B_i)\right) = \sum_{i\in\mathbb{N}} \mu(f^{-1}(B_i)) = \sum_{i\in\mathbb{N}} \mu_f(B_i).$$

One important property of pushforward measures is that they preserve mutual absolute continuity.

Theorem 5.3. *Given measurable spaces $(X_1, \mathcal{B}_1)$, $(X_2, \mathcal{B}_2)$, measurable functions $f : X_1 \to X_2$, and measures μ and ν on X_1, if μ and ν are mutually absolutely continuous, then so are the pushforward measures μ_f and ν_f.*

Proof. Note that

$$\mu_f(B) = 0 \Leftrightarrow \mu(f^{-1}(B)) = 0 \Leftrightarrow \nu(f^{-1}(B)) = 0 \Leftrightarrow \nu_f(B).$$

□

To see the significance of using pushforward measures, suppose a point process Π in $\mathbb{R}^d$ is given in the form of a collection of random variables $\{X_i\}_{i\in\mathbb{N}}$. Although we can label these random variables using $\mathbb{N}$, this information is lost when we look at a point configuration ω on $\mathbb{R}^d$: we do not know which point in ω is "coming from" which random variable. A way to interpret this is that we are taking a random sequence in $(\mathbb{R}^d)^{\mathbb{N}}$ given by $(X_0, X_1, ...)$, forgetting the order of its coordinates, and only focusing on the random Dirac measures on $\mathbb{R}^d$ at these points. Formally, this procedure is a map Φ from $(\mathbb{R}^d)^I$ (with I being an arbitrary countable index set) to the space of measures on $\mathbb{R}^d$ given by

$$\Phi((a_i)_{i\in I}) = \sum_{i\in I} \delta_{a_i}. \tag{5.5}$$

Using this map as a pushforward map, we can study point processes of the form $\Pi = \sum_{i\in I} \delta_{X_i}$ for a collection of random variables $\{X_i\}_{i\in I}$ by looking at the random vector $(X_i)_{i\in I}$. More specifically, if we want to show that the laws of the two point processes $\Pi_1 = \sum_{i\in I_1} \delta_{X_i}$ and $\Pi_2 = \sum_{i\in I_2} \delta_{X_2}$ are mutually absolutely continuous, it suffices, by Theorem 5.3, to show that the laws of the two random vectors $(X_i)_{i\in I_1}$ and $(X_i)_{i\in I_2}$ are mutually absolutely continuous. There are certainly measurability issues here we have to take care of. Below we summarize the main assumptions that need to be checked, and we defer the proofs of these to Appendix B:

Lemma 5.4. *(Informal statement)*

(1) *The inverse image of the set of locally finite counting measures on $\mathbb{R}^d$ under Φ is a measurable subset of $(\mathbb{R}^d)^I$ with the product Borel σ-algebra.*
(2) *Denote this inverse image by $\mathbb{L}$. In the context of Theorem 5.1, the random vector takes values in $\mathbb{L}$ almost surely.*
(3) *The map Φ is measurable from $\mathbb{L}$ to the space of locally finite counting measures on $\mathbb{R}^d$.*

To see how this pushforward map Φ plays a role in the proof, recall that in the setting of our specific problem, we can write the point process as $\Pi = \sum_{k\in\mathbb{Z}} \delta_{|k|^\alpha+Y_k}$, where the Y_k are i.i.d. standard Gaussians. To prove that Π is non-rigid when $\alpha < 1/2$, we show that it is 0-deletion tolerant. This means that we remove the random variable at the origin in Π to obtain a new point process $\Pi_0 = \sum_{k\in\mathbb{Z}-\{0\}} \delta_{|k|^\alpha+Y_k}$ and prove that the laws of Π and Π_0 are mutually absolutely continuous. Via the pushforward map given in (5.5), note that Π is generated by the random vector

$$(|k|^\alpha + Y_k)_{k\in\mathbb{Z}}. \tag{5.6}$$

taking values in $\mathbb{R}^{\mathbb{Z}}$. We would also like to find a random vector in $\mathbb{R}^{\mathbb{Z}}$ that generates the point process Π_0 via the map in (5.5). But since Π_0 does not have an $\mathbb{R}$-valued random variable indexed by 0, we would seemingly have a null zeroth coordinate if we were to take the same random vector as (5.6). This can be resolved by shifting each coordinate of (5.6) with $k \geq 1$ to the left by one. That is, we consider

$$(\hat{Y}_k)_{k\in\mathbb{Z}}, \hat{Y}_k = \begin{cases} |k|^\alpha + Y_k \text{ if } k \in \mathbb{Z}, k < 0 \\ |k+1|^\alpha + Y_{k+1} \text{ if } k \in \mathbb{Z}, k \geq 0. \end{cases} \tag{5.7}$$

It is worth noting that (5.7) indeed induces the point process Π because the pushforward map in (5.5) ignores the indexing. At this point, we can finally appreciate the introduction of pushforward measures to the proof of Theorem 5.1, since the proof of non-rigidity amounts to showing that the laws of the random vectors in (5.6) and (5.7) on $\mathbb{R}^{\mathbb{Z}}$ are mutually absolutely continuous.

To simplify the problem even further, we introduce another pushforward map from $\mathbb{R}^{\mathbb{Z}}$ to $\mathbb{R}^{\mathbb{Z}}$ in order to center the collection of random variables $\{|k|^\alpha + Y_k\}_{k\in\mathbb{Z}}$.

Consider the map

$$\Psi((a_i)_{i\in\mathbb{Z}}) = (a_i + |i|^\alpha)_{i\in\mathbb{Z}}. \tag{5.8}$$

Given that the domain and the codomain $\mathbb{R}^{\mathbb{Z}}$ are both equipped with the product Borel σ-algebra, it can be verified that Ψ is measurable because the inverse image (which is nothing more than a translation at each coordinate) of any cylinder set in the generator is again a cylinder set. Then applying the map Ψ to the random vector

$$(Y_k)_{k\in\mathbb{Z}}, \tag{5.9}$$

we obtain the random vector in (5.6). Similarly, applying Ψ to the random vector

$$(Z_k)_{k\in\mathbb{Z}}, Z_k = \begin{cases} Y_k \text{ if } k\in\mathbb{Z}, k<0 \\ |k+1|^\alpha - |k|^\alpha + Y_{k+1} \text{ if } k\in\mathbb{Z}, k\geq 0, \end{cases} \tag{5.10}$$

we obtain the random vector in (5.7). Therefore, by using the composition of pushforward maps $\Phi\circ\Psi$, we have reduced the proof of non-rigidity to showing that the laws of the random vectors in (5.9) and (5.10) on $\mathbb{R}^{\mathbb{Z}}$ are mutually absolutely continuous.

Note that if our random vectors were finite-dimensional, say $(Y_k)_{|k|\leq n}$ and $(Z_k)_{|k|\leq n}$, then their laws would certainly be mutually absolutely continuous because they would both be mutually absolutely continuous with the Lebesgue measure on $\mathbb{R}^{\{|k|\leq n\}}$. Moreover, their Radon-Nikodym derivative would be given by the quotient of the joint Gaussian density functions. Because of this simplicity in the finite-dimensional cases, it will be convenient if there is a criterion for the mutual absolute continuity of the random vectors in (5.9) and (5.10) based on the two sequences of truncations,

$$\{(Y_k)_{|k|\leq n}\}_{n\in\mathbb{N}}, \{(Z_k)_{|k|\leq n}\}_{n\in\mathbb{N}}. \tag{5.11}$$

In order to formalize this strategy, we use a martingale argument provided in [6] (see Theorem 12.32). The reader may refer to Appendix C for a brief overview of the results that lead up to this theorem.

Theorem 5.5. *Let μ, ν be two probability measures on a space with σ-algebra $\mathcal{F}$. Assume that $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is a filtration such that $\mathcal{F}_n \nearrow \mathcal{F}$, i.e., the union of the $\mathcal{F}_n$ generates $\mathcal{F}$, and denote $\mu_n = \mu_{|\mathcal{F}_n}$ and $\nu_n = \nu_{|\mathcal{F}_n}$. Suppose $\mu_n \ll \nu_n$ for all n and let*

$$X_n = \frac{d\mu_n}{d\nu_n}.$$

Then $(X_n)_{n\in\mathbb{N}}$ is a nonnegative martingale and ν-almost surely convergent. Moreover, denote

$$X = \limsup_{n\to\infty} X_n.$$

Then for any $A\in\mathcal{F}$ we have

$$\mu(A) = \int_A X\,d\nu + \mu(A\cap\{X=\infty\}). \tag{5.12}$$

In particular,

1. *If $\mu(\{X=\infty\})=0$, then $\mu\ll\nu$.*
2. *If $\nu(\{X=0\})=0$, then $\nu\ll\mu$.*

Proof. First note that for each $n \geq 0$ and $A \in \mathcal{F}_n$ we have

$$\begin{aligned}\int_A \mathbf{E}[X_{n+1}|\mathcal{F}_n]d\nu &= \int_A X_{n+1}d\nu = \int_A \frac{d\mu_{n+1}}{d\nu_{n+1}}d\nu_{n+1} = \mu_{n+1}(A)\\ &= \mu(A) = \int_A \frac{d\mu_n}{d\nu_n}d\nu_n = \int_A X_n d\nu.\end{aligned}$$

So $\mathbf{E}[X_{n+1}|\mathcal{F}_n] = X_n$ ν-a.s. and $(X_n)_{n\in\mathbb{N}}$ is a nonnegative martingale (w.r.t. $(\{\mathcal{F}_n\},\nu)$). By nonnegativity we have $\mathbf{E}[|X_n|] = \mathbf{E}[X_n] = \mathbf{E}[X_0]$ for each n and $\sup_{n\in\mathbb{N}}\mathbf{E}[|X_n|] = \mathbf{E}[X_0] < \infty$. Since $(X_n)_{n\in\mathbb{N}}$ is a uniformly L^1-bounded martingale, we can apply Theorem C.6 to see that it converges ν-almost surely. To show (5.12), we write it in the following equivalent form:

$$\mu(A\cap\{X<\infty\}) = \int_A X d\nu \text{ for all } A \in \mathcal{F}. \tag{5.13}$$

For $A \in \mathcal{F}_k$ and $n > k$ we have $\mu(A) = \int_A X_n d\nu$. It follows from Fatou's lemma that

$$\int_A X d\nu = \int_A \liminf_{n\to\infty} X_n d\nu \leq \liminf_{n\to\infty}\int_A X_n d\nu = \mu(A). \tag{5.14}$$

Since $\mathcal{F}$ is generated by $\{\mathcal{F}_k\}_{k\in\mathbb{N}}$, the inequality in (5.14) holds for every $A \in \mathcal{F}$. Therefore, for all $A \in \mathcal{F}$ we have

$$\mu(A\cap\{X<\infty\}) \geq \int_{A\cap\{X<\infty\}} X d\nu = \int_A X d\nu. \tag{5.15}$$

On the other hand, for $A \in \mathcal{F}_k$ and $n > k$ we also have

$$\mu(A\cap\{X_n<M\}) = \int_{A\cap\{X_n<M\}} X_n d\nu \leq \int_A X_n \wedge M d\nu.$$

It follows from bounded convergence that

$$\begin{aligned}\mu(A\cap\{X<M\}) &= \mu(\cup_{m=0}^{\infty}(A\cap\{\sup_{l\geq m} X_l < M\})) = \lim_{n\to\infty}\mu(A\cap\{\sup_{l\geq n} X_l<M\})\\ &\leq \lim_{n\to\infty}\mu(A\cap\{X_n<M\}) \leq \lim_{n\to\infty}\int_A X_n\wedge M d\nu\\ &= \int_A \lim_{n\to\infty} X_n\wedge M d\nu = \int_A X\wedge M d\nu.\end{aligned}$$

Then by monotone convergence we have

$$\begin{aligned}\mu(A\cap\{X<\infty\}) &= \lim_{M\to\infty}\mu(A\cap\{X<M\}) \leq \lim_{M\to\infty}\int_A X\wedge M d\nu\\ &= \int_A \lim_{M\to\infty} X\wedge M d\nu = \int_A X d\nu.\end{aligned}$$

Since this inequality holds for any k and $A \in \mathcal{F}_k$, we conclude that it must also hold for every $A \in \mathcal{F}$. Combining this inequality with (5.15) gives (5.13). If $\mu(\{X=\infty\}) = 0$, then $\mu(A) = \int_A X d\nu$ and thus $\mu \ll \nu$. If $\nu(\{X=0\}) = 0$, then X is ν-almost surely positive because it is nonnegative. So $\mu(A) = 0$ implies that $\int_A X d\nu = 0$, which in turn implies that $\nu(A) = 0$. □

In order to apply Theorem 5.5 to our specific problem, let μ be the distribution of $(Y_k)_{k\in\mathbb{Z}}$ on $\mathbb{R}^{\mathbb{Z}}$, and the let ν be the distribution of $(Z_k)_{k\in\mathbb{Z}}$ on $\mathbb{R}^{\mathbb{Z}}$, where the notations are consistent with (5.9) and (5.10). We equip $\mathbb{R}^{\mathbb{Z}}$ with the product Borel

σ-algebra. Recall that our goal is to let μ_n and ν_n be the distributions of the truncations $(Y_k)_{|k|\leq n}$ and $(Z_k)_{|k|\leq n}$, respectively. The natural way to do this is to consider the filtration

$$\mathcal{F}_n = \prod_{|k|\leq n} \mathcal{B}_k \times \prod_{|k|>n} \mathbb{R}.$$

First note that since each cylinder set only has finitely many coordinates distinct from $\mathbb{R}$, the filtration $\{\mathcal{F}_n\}$ contains all the cylinder sets and therefore generates $\mathcal{F}$, the product Borel σ-algebra on $\mathbb{R}^{\mathbb{Z}}$. Moreoever, since the usual Borel σ-algebra on $\mathbb{R}$ is only given on the index set $\{|k| \leq n\}$, the restriction of μ (respectively ν) to $\mathcal{F}_n$ is determined only by the truncation $(Y_k)_{|k|\leq n}$ (resp. $(Z_k)_{|k|\leq n}$). Since the Y_k are independent and so are the Z_k, this allows us to explicitly compute the Radon-Nikodym derivatives:

$$\frac{d\mu_n}{d\nu_n} = \frac{\prod_{|k|\leq n} \frac{1}{\sqrt{2\pi}} e^{-x_k^2/2}}{\prod_{-n\leq k\leq -1} \frac{1}{\sqrt{2\pi}} e^{-x_k^2/2} \cdot \prod_{0\leq k\leq n} \frac{1}{\sqrt{2\pi}} e^{-(x_k-(k+1)^\alpha+k^\alpha)^2/2}}. \tag{5.16}$$

Writing $\mathbf{x}_n = (x_0, ..., x_n)$ and $\mathbf{v}_n = (v_k)_{0\leq k\leq n} = ((k+1)^\alpha - k^\alpha)_{0\leq k\leq n}$, we can simplify (5.16) to obtain

$$\begin{aligned} \frac{d\mu_n}{d\nu_n} &= \frac{\frac{1}{(2\pi)^{(n+1)/2}} \exp(-\|\mathbf{x}_n\|_2^2/2)}{\frac{1}{(2\pi)^{(n+1)/2}} \exp(-\|\mathbf{x}_n - \mathbf{v}_n\|_2^2/2)} = \exp\left(\frac{-\|\mathbf{x}_n\|_2^2 + \|\mathbf{x}_n - \mathbf{v}_n\|_2^2}{2}\right) \\ &= \exp\left(-\langle \mathbf{x}_n, \mathbf{v}_n\rangle + \frac{\|\mathbf{v}_n\|_2^2}{2}\right) = \exp\left(-\langle \mathbf{x}_n, \mathbf{v}_n\rangle\right) \cdot \exp\left(\frac{\|\mathbf{v}_n\|_2^2}{2}\right). \end{aligned}$$

This immediately implies the following.

Lemma 5.6. *Let $\mathbf{v} := ((k+1)^\alpha - k^\alpha)_{k\in\mathbb{N}}$. If $\|\mathbf{v}\|_2^2 < \infty$, then*

$$X := \limsup_{n\to\infty} \frac{d\mu_n}{d\nu_n} = \exp\left(\frac{\|\mathbf{v}\|_2^2}{2}\right) \cdot \limsup_{n\to\infty} \exp\left(-\langle \mathbf{x}_n, \mathbf{v}_n\rangle\right). \tag{5.17}$$

In view of Theorem 5.5, if the RHS of (5.17) does not go to infinity under μ or go to zero under ν, then we have mutual absolute continuity. In the following lemma, we show the stronger result that under both laws, the RHS of (5.17) is almost surely convergent to a nonzero, positive limit in $\mathbb{R}$.

Lemma 5.7. *If $\|\mathbf{v}\|_2^2 < \infty$, then $\mu(\{0 < X < \infty\}) = \nu(\{0 < X < \infty\}) = 1$.*

Proof. By Lemma 5.6 and the properties of $\exp(-(\cdot))$, it suffices to show that the series

$$\langle \mathbf{x}_n, \mathbf{v}_n\rangle = \sum_{k=0}^{n} v_k x_k$$

is almost surely convergent under both μ and ν. In the case of μ, we have the infinite series $\sum_{k=0}^\infty v_k Y_k$, where the Y_k are i.i.d. standard Gaussians. Note that

$$\sum_{k=0}^\infty \mathbf{E}[v_k Y_k] = \sum_{k=0}^\infty v_k \mathbf{E}[Y_k] = 0$$

and

$$\sum_{k=0}^\infty \mathbf{Var}[v_k Y_k] = \sum_{k=0}^\infty v_k^2 \mathbf{Var}[Y_k] = \|\mathbf{v}\|_2^2 < \infty.$$

Therefore, by Theorem A.2 the series $\sum_{k=0}^\infty v_k Y_k$ is almost surely convergent in $\mathbb{R}$. Under the law of ν, we have the series

$$\sum_{k=0}^\infty v_k Z_k = \sum_{k=0}^\infty v_k(v_k + Y_{k+1}) = \|\mathbf{v}\|_2^2 + \sum_{k=0}^\infty v_k Y_{k+1},$$

where the Y_{k+1} are again i.i.d. standard Gaussians. By the same argument, we know that $\sum_{k=0}^\infty v_k Z_k$ is also almost surely convergent in $\mathbb{R}$. □

From these two lemmas we can deduce that if $\mathbf{v}$ is square summable, then μ and ν are mutually absolutely continuous and we can conclude non-rigidity. We use the following lemma to finish the proof of part 2 of Theorem 5.1.

Lemma 5.8. *If $\alpha \in (0, 1/2)$, then $\mathbf{v} = ((k+1)^\alpha - k^\alpha)_{k\in\mathbb{N}}$ is square-summable.*

Proof. By the mean value theorem, for each $k \in \mathbb{N}$, there exists $u_k \in [k, k+1]$ so that $\alpha u_k^{\alpha-1} = (k+1)^\alpha - k^\alpha$. Since $\alpha \in (0, 1/2)$, it follows that $2\alpha - 2 < -1$ and therefore

$$\sum_{k=0}^\infty ((k+1)^\alpha - k^\alpha)^2 = \sum_{k=0}^\infty \alpha^2 u_k^{2\alpha-2} \le \alpha^2 \sum_{k=0}^\infty k^{2\alpha-2} = \alpha^2 \zeta(2\alpha - 2) < \infty.$$

□

6. Acknowledgements

This paper is written as a project report for the UChicago Math REU 2021. I extend my sincerest gratitude to my mentor, Prof. Pierre Yves Gaudreau Lamarre, for suggesting the topic, as well as for his constant help and support throughout the time I have known him. I am grateful to Prof. Peter May for organizing the REU, year in and year out. It is also a pleasure to thank Prof. Gregory Lawler for supervising the probability sub-program and for his much helpful advice. Lastly, I enjoyed many wonderful discussions I had with some friends in the REU.

Appendix A. Convergence of Series of Random Variables

In this section, we exposit a few standard results in probability theory concerning the convergence of infinite series of random variables. These theorems are required for some technical parts of proofs in Section 5. Given an infinite sequence of random variables $(X_i)_{i\in\mathbb{N}}$, we denote the partial sums as $S_n = \sum_{i=0}^n X_i$ and say that the series $\sum_{i=0}^\infty X_i$ converges almost surely if the partial sums $(S_n)_{n\in\mathbb{N}}$ converge almost surely as $n \to \infty$. The first result, known as Kolmogorov's inequality, is useful for establishing series convergence for independent random variables.

Theorem A.1. (Kolmogorov's inequality) *Let $X_0, ..., X_n$ be independent random variables defined on a common probability space. Assume, in addition, that for every $i = 0, ..., n$, we have $\mathbf{E}[X_i] = 0$ and $\mathbf{Var}[X_i] < \infty$. Then for each $\epsilon > 0$, it holds that*

$$\mathbf{P}\left(\max_{0\le k\le n} |S_k| > \epsilon\right) \le \frac{\mathbf{Var}[S_n]}{\epsilon^2} = \frac{\sum_{i=0}^n \mathbf{E}[X_i^2]}{\epsilon^2}. \tag{A.1}$$

Proof. For a fixed $n \in \mathbb{N}$ and $\epsilon > 0$, note that we can express the event $\{\max_{0\le k\le n} |S_k| > \epsilon\}$ as a disjoint union of the events

$$E_k = \{|S_k| > \epsilon, |S_i| \le \epsilon \text{ for all } i < k\}$$

with k ranging from 0 to n. Each event E_k is interpreted to mean that the minimum i such that $|S_i| > \epsilon$ is k. By Chebyshev's inequality we have

$$\mathbf{P}(E_k) \leq \frac{\mathbf{E}[S_k^2 \mathbf{1}_{E_k}]}{\epsilon^2} \leq \frac{\mathbf{E}[(S_k^2 + (S_n - S_k)^2) \cdot \mathbf{1}_{E_k}]}{\epsilon^2}. \tag{A.2}$$

Note that $S_k \mathbf{1}_{E_k}$ only depends on $X_0, ..., X_k$, while $S_n - S_k$ only depends on $X_{k+1}, ..., X_n$. Therefore, by independence we have

$$\mathbf{E}[S_k \mathbf{1}_{E_k} \cdot (S_n - S_k)] = \mathbf{E}[S_k \mathbf{1}_{E_k}] \cdot \mathbf{E}[S_n - S_k] = 0.$$

Combining this with (A.2) gives

$$\mathbf{P}(E_k) \leq \frac{\mathbf{E}[(S_k^2 + (S_n - S_k)^2 + 2S_k(S_n - S_k)) \cdot \mathbf{1}_{E_k}]}{\epsilon^2} = \frac{\mathbf{E}[S_n^2 \mathbf{1}_{E_k}]}{\epsilon^2}.$$

Summing over $k = 0, ..., n$ we obtain

$$\mathbf{P}\left(\max_{0 \leq k \leq n} |S_k| > \epsilon\right) = \sum_{k=0}^{n} \mathbf{P}(E_k) \leq \frac{\mathbf{E}[S_n^2 \mathbf{1}_{\max |S_k| > \epsilon}]}{\epsilon^2} \leq \frac{\mathbf{E}[S_n^2]}{\epsilon^2} = \frac{\sum_{i=0}^n \mathbf{E}[X_i^2]}{\epsilon^2}.$$

□

This maximal inequality provides a tool for establishing the Kolmogorov's two-series theorem, a criterion for the almost sure convergence of random series.

Theorem A.2. (Kolmogorov's two series theorem) *Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of independent random variables with expected values $\mathbf{E}[X_i] = \mu_i$ and variances $\mathbf{Var}[X_i] = \sigma_i^2$. Assume that $\sum_{i \in \mathbb{N}} \mu_i$ and $\sum_{i \in \mathbb{N}} \sigma_i^2$ are both convergent. Then $\sum_{i \in \mathbb{N}} X_i$ converges almost surely and in L^2.*

Proof. Since $\sum_{i \in \mathbb{N}} \mu_i$ is convergent, we may assume WLOG that $\mu_i = 0$ for all i, and the general result follows from considering $\sum_{i \in \mathbb{N}} (X_i - \mu_i) + \sum_{i \in \mathbb{N}} \mu_i$. Again denote $S_n = \sum_{i=0}^n X_i$. For fixed $N, m \in \mathbb{N}$, by Theorem 6.1 we have the estimate

$$\mathbf{P}(\max_{N \leq n \leq N+m} |S_n - S_N| > \epsilon) \leq \epsilon^{-2} \sum_{i=N}^{N+m} \mathbf{Var}[S_{N+m} - S_N] = \epsilon^{-2} \sum_{i=N}^{N+m} \sigma_i^2 \leq \epsilon^{-2} \sum_{i=N}^{\infty} \sigma_i^2.$$

Letting $m \to \infty$ we obtain

$$\mathbf{P}(\sup_{n \geq N} |S_n - S_N| > \epsilon) \leq \epsilon^{-2} \sum_{i=N}^{\infty} \sigma_i^2.$$

It follows from the convergence of $\sum_{i \in \mathbb{N}} \sigma_i^2$ that $\sum_{i=N}^{\infty} \sigma_i^2$ vanishes as $N \to \infty$. Therefore,

$$\lim_{N \to \infty} \mathbf{P}(\sup_{n \geq N} |S_n - S_N| > \epsilon) = 0,$$

which implies that $\limsup_n S_n - \liminf_n S_n \leq \epsilon$ almost surely. Taking ϵ along a countable sequence that goes to zero gives the almost sure convergence of $\sum_{i \in \mathbb{N}} X_i$. To prove convergence in L^2, we first denote the almost sure limit of S_n by S. An application of Fatou's Lemma yields

$$\begin{aligned}\mathbf{E}[(S_n - S)^2] &= \mathbf{E}[(S_n - \lim_{k \to \infty} S_k)^2] = \mathbf{E}[\lim_{k \to \infty} (S_n - S_k)^2] \\ &\leq \liminf_{k \to \infty} \mathbf{E}[(S_n - S_k)^2] = \liminf_{k \to \infty} \sum_{i=n}^{k} \sigma_i^2 \leq \sum_{i=n}^{\infty} \sigma_i^2.\end{aligned}$$

Since the RHS vanishes as $n \to \infty$, this shows L^2 convergence. □

Corollary A.2.1. Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of independent random variables. Using the same notations as Theorem A.2, if $\sum_{i\in\mathbb{N}}\mu_i$ and $\sum_{i\in\mathbb{N}}\sigma_i^2$ both converge, then

$$\mathbf{Var}\left[\sum_{i\in\mathbb{N}} X_i\right] = \sum_{i\in\mathbb{N}}\sigma_i^2. \tag{A.3}$$

Proof. WLOG assume $\mu_i = 0$ for all $i\in\mathbb{N}$. From Theorem A.2 we know that the partial sums S_n converge in L^2 to $S=\sum_{i\in\mathbb{N}} X_i$. Since L^2 convergence implies L^1 convergence, we have

$$|\mathbf{E}[S_n]-\mathbf{E}[S]| \le \mathbf{E}[|S_n - S|] \to 0,$$

and thus $\mathbf{E}[S]=0$. Again by convergence in L^2(-norm) we conclude that

$$\mathbf{Var}[S] = \mathbf{E}[S^2] = \lim_{n\to\infty}\mathbf{E}[S_n^2] = \sum_{i\in\mathbb{N}}\sigma_i^2.$$

□

We conclude this section with Kolmogorov's three-series theorem, which provides a sufficient condition for the convergence of a random series based on three non-random series.

Theorem A.3. *Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of independent random variables. The random series $\sum_{i\in\mathbb{N}} X_i$ converges almost surely if for some $A>0$ and $Y_i := X_i\mathbf{1}_{|X_i|\le A}$, the following three conditions hold:*

(1) $\sum_{i\in\mathbb{N}}\mathbf{P}(|X_i|\ge A)$ *converges;*
(2) $\sum_{i\in\mathbb{N}}\mathbf{E}[Y_i]$ *converges;*
(3) $\sum_{i\in\mathbb{N}}\mathbf{Var}[Y_i]$ *converges.*

Proof. By the Borel-Cantelli lemma, the first condition implies that almost surely, $X_i = Y_i$ for large i. Therefore, $\sum_{i\in\mathbb{N}} X_i$ converges if and only if $\sum_{i\in\mathbb{N}} Y_i$ converges. Applying Theorem 6.2 gives the almost sure convergence of $\sum_{i\in\mathbb{N}} Y_i$, hence the almost sure convergence of $\sum_{i\in\mathbb{N}} X_i$. □

Appendix B. Revisiting Lemma 5.4

Recall the setting in which we are given a collection of $\mathbb{R}^d$-valued random variables $\{X_i\}_{i\in I}$, where I is an arbitrary countable set of indices. This induces a point process $\Pi=\sum_{i\in I}\delta_{X_i}$ on $\mathbb{R}^d$. Lemma 5.4 deals with the measurability of the map $\Phi((a_i)_{i\in I}) = \sum_{i\in I}\delta_{a_i}$ from $(\mathbb{R}^d)^I$ to the space of counting measures on $\mathbb{R}^d$. Let $\mathcal{N}^{\#}_{\mathbb{R}^d}$ denote the subset of locally finite counting measures on $\mathbb{R}^d$, and let $\mathbb{L}\subset(\mathbb{R}^d)^I$ be the inverse image of $\mathcal{N}^{\#}_{\mathbb{R}^d}$ under the map Φ. Our first concern is whether $\mathbb{L}$ is a measurable subset of $(\mathbb{R}^d)^I$ equipped with the product Borel σ-algebra.

For each $(k_1,\dots,k_d)\in\mathbb{Z}^d$, let $B(k_1,\dots,k_d) = [k_1,k_1+1)\times\cdots\times[k_d,k_d+1)$. Note that we can write $\mathbb{L}$ as the countable intersection

$$\mathbb{L} = \bigcap_{B(k_1,\dots,k_d),(k_1,\dots,k_d)\in\mathbb{Z}^d}\left\{a\in(\mathbb{R}_d)^I : \Phi(a)(B(k_1,\dots,k_d))<\infty\right\}.$$

For each $(k_1,\dots,k_d)$, we then have

$$\left\{a\in(\mathbb{R}_d)^I : \Phi(a)(B(k_1,\dots,k_d))<\infty\right\} = \bigcup_{l\in\mathbb{N}}\left\{a\in(\mathbb{R}_d)^I : \Phi(a)(B(k_1,\dots,k_d))<l\right\}.$$

For each $l \in \mathbb{N}$, we can write

$$\begin{aligned}\left\{a \in (\mathbb{R}_d)^I : \Phi(a)(B(k_1, ..., k_d)) < l\right\} &= \{a \in (\mathbb{R}^d)^I : \Phi(a)(B(k_1, ..., k_d)) \geq l\}^C \\ &= \bigcup_{S \subset I, |S|=l} \left(\prod_{i \in S} B(k_1, ..., k_d) \times \prod_{i \notin S} \mathbb{R}^d \right)\end{aligned}$$

as a countable union of cylinder sets. This verifies that $\mathbb{L}$ is indeed measurable in the product Borel σ-algebra on $(\mathbb{R}^d)^I$.

The next thing to check is that Φ is measurable from $\mathbb{L}$ to $\mathcal{N}^{\#}_{\mathbb{R}^d}$. We assume the fact that $\mathcal{N}^{\#}_{\mathbb{R}^d}$ is a Polish space, i.e., a complete, separable metric space (see, e.g., Proposition 9.1.IV-(iii) in [1]). The notion of convergence in $\mathcal{N}^{\#}_{\mathbb{R}^d}$ is weak convergence of measures, and in $(\mathbb{R}^d)^I$ it is pointwise convergence. Since I is countable, we can pick $\{I_n\}$ so that $|I_n| = n$ and $I_n \uparrow I$. Consider the map

$$\Phi_n((a_i)_{i \in I}) = \sum_{i \in I_n} \delta_{a_i}.$$

Then Φ is the pointwise limit of Φ_n. Since the pointwise limit of a sequence of measurable functions into a metric space ($\mathcal{N}^{\#}_{\mathbb{R}^d}$) is always measurable (see, e.g., Section 8.16 in [8]), it suffices to show that each Φ_n is measurable. In fact, we will show that Φ_n is continuous. Let $a^n \to a$ in $(\mathbb{R}^d)^I$. Then $a_i^n \to a_i$ for each $i \in I_n$. It follows that for every continuous f,

$$\int f d \left(\sum_{i \in I_n} \delta_{a_i^n} \right) = \sum_{i \in I_n} f(a_i^n) \to \sum_{i \in I_n} f(a_i) = \int f d \left(\sum_{i \in I_n} \delta_{a_i} \right).$$

So each Φ_n is continuous and hence measurable.

Lastly, we would like to establish that in the setting of Theorem 5.1, the law of the point process $\Pi = \sum_{k \in \mathbb{Z}} \delta_{|k|^\alpha + Y_k}$ (where Y_k are i.i.d. standard Gaussians) has support in $\mathbb{L}$. Let $B \subset \mathbb{R}^d$ be bounded and measurable, then there exists some M so that $B \subseteq [-M, M]$. Since $\alpha > 0$, there exists $N \in \mathbb{N}$ so that $N^\alpha > M$. Then for every k with $|k| \geq N$, we have

$$\begin{aligned}\mathbf{P}(|k|^\alpha + Y_k \leq M) &= \mathbf{P}(Y_k \leq M - |k|^\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{M-|k|^\alpha} e^{-x^2/2} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{|k|^\alpha - M}^{\infty} e^{-x^2/2} dx \leq \frac{1}{\sqrt{2\pi}} \int_{|k|^\alpha - M}^{\infty} \frac{x}{|k|^\alpha - M} e^{-x^2/2} dx \\ &= \frac{1}{(|k|^\alpha - M)\sqrt{2\pi}} \int_{|k|^\alpha - M}^{\infty} \frac{\partial}{\partial x} (-e^{-x^2/2}) dx \\ &= \frac{1}{(|k|^\alpha - M)\sqrt{2\pi}} \cdot \exp\left(-\frac{(|k|^\alpha - M)^2}{2} \right) \\ &\leq C \cdot \exp\left(-\frac{(|k|^\alpha - M)^2}{2} \right),\end{aligned}$$

where we may take $C = \frac{1}{(N^\alpha - M)\sqrt{2\pi}}$. It follows that

$$\sum_{k\in\mathbb{Z}} \mathbf{P}(|k|^\alpha + Y_k \le M) = \sum_{|k|<N} \mathbf{P}(|k|^\alpha + Y_k \le M) + \sum_{|k|\ge N} \mathbf{P}(|k|^\alpha + Y_k \le M)$$
$$\le 2N + 2C \cdot \sum_{k\ge N} \exp\left(-\frac{(k^\alpha - M)^2}{2}\right)$$

Since M is a constant and $\exp(-\frac{(k^\alpha-M)^2}{2})$ decays faster than k^{-p} for any $p>1$, by a comparison test we know that the above sum is finite. Therefore, we conclude by the Borel-Cantelli lemma that $\Pi((-\infty, M]) < \infty$ almost surely and consequently, $\Pi(B)$ almost surely.

Appendix C. A Martingale Convergence Theorem

This section provides a brief introduction to martingales and some useful tools for studying their convergence properties. The primary reference for this material is [9].

Definition C.1. We begin with a *filtered space* $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbf{P})$, where $(\Omega, \mathcal{F}, \mathbf{P})$ is a usual probability space, and $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is a *filtration.* That is,

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}$$

is an increasing family of σ-algebras in $\mathcal{F}$. We denote the σ-algebra generated by this filtration by

$$\mathcal{F}_\infty = \sigma\left(\bigcup_{n\in\mathbb{N}} \mathcal{F}_n\right) \subseteq \mathcal{F}.$$

Definition C.2. A process $(X_n)_{n\in\mathbb{N}}$ is a *martingale* with respect to $(\{\mathcal{F}_n\}, \mathbf{P})$ if the following three conditions hold:

1. For every $n \in \mathbb{N}$, the random variable X_n is $\mathcal{F}_n$-measurable.
2. For every $n \in \mathbb{N}$, $\mathbf{E}[|X_n|] < \infty$.
3. For every $n \ge 1$, $\mathbf{E}[X_n|\mathcal{F}_{n-1}] = X_{n-1}$ a.s.

Similarly, the process $(X_n)_{n\in\mathbb{N}}$ is a *supermartingale* if condition (iii) is replaced by

$$\forall n \ge 1,\ \mathbf{E}[X_n|\mathcal{F}_{n-1}] \le X_{n-1}\ a.s.$$

The process $(X_n)_{n\in\mathbb{N}}$ is a *submartingale* if condition (iii) is replaced by

$$\forall n \ge 1,\ \mathbf{E}[X_n|\mathcal{F}_{n-1}] \ge X_{n-1}\ a.s.$$

Definition C.3. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbf{P})$ be a filtered space. A process $(X_n)_{n\in\mathbb{N}}$ is said to be *predictable* if X_{n+1} is measurable with respect to $\mathcal{F}_n$ for each n.

Now we exposit an inequality, known as Doob's Upcrossing Inequality, that will be useful for establishing the convergence of L^1-bounded supermartingales (and hence martingales).

Definition C.4. Let $(X_n)_{n\in\mathbb{N}}$ be a process on the filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbf{P})$. Let $a, b \in \mathbb{R}$ with $a < b$. The *number of upcrossings of* $[a,b]$ *made by* $n \to X_n(\omega)$ *by time* N, denoted by $U_N[a,b](\omega)$, is defined to be the largest k in $\mathbb{N}$ such that we can find

$$0 \le s_1 < t_1 < \cdots < s_k < t_k \le N,\ \text{with}\ X_{s_i}(\omega) < a,\ X_{t_i}(\omega) > b \text{ for } 1 \le i \le k.$$

Theorem C.5. *Let $(X_n)_{n\in\mathbb{N}}$ be a supermartingale and $U_N[a,b]$ be the number of upcrossings as in Definition C.4. Then we have the inequality*

$$(b-a)\mathbf{E}[U_N[a,b]] \le \mathbf{E}[(X_N-a)^-], \tag{C.1}$$

where

$$(X_N-a)^- := \begin{cases} a-X_N & \textit{if } X_N \le a, \\ 0 & \textit{otherwise.} \end{cases}.$$

Proof. Define the sequence $(C_n)_{n\ge 1}$ inductively as follows:

$$C_1(\omega) = \begin{cases} 1 & \text{if } X_0(\omega) < a, \\ 0 & \text{otherwise.} \end{cases}$$

Inductively,

$$C_n(\omega) = \begin{cases} 1 & \text{if } C_{n-1}(\omega) = 1 \text{ and } X_{n-1}(\omega) \le b \\ 1 & \text{if } C_{n-1}(\omega) = 0 \text{ and } X_{n-1}(\omega) < a, \\ 0 & \text{otherwise.} \end{cases}$$

To gain a sense of this definition, we begin with the case when $X_0(\omega) < a$ and $C_1(\omega) = 1$. Then $C_n(\omega)$ remains 1 until the first time X_n crosses above b, at which point $C_n(\omega)$ becomes 0. It then remains 0 until X_n crosses below a, at which point $C_n(\omega)$ returns to 1. If we begin with $X_0(\omega) \ge a$ and $C_1(\omega) = 0$, then $C_n(\omega)$ remains 0 until X_n crosses below a, at which point $C_n(\omega)$ becomes 1 and proceeds as in the first case. Now we define a new process

$$Y_n := (C\cdot X)_n := \sum_{1\le k\le n} C_k(X_k - X_{k-1}).$$

The claim is that

$$Y_N(\omega) \ge (b-a)U_N[a,b] - (X_N(\omega)-a)^-. \tag{C.2}$$

Denote $U_N[a,b] = k$. Then by definition there are $0 \le s_1 < t_1 < \cdots < s_k < t_k \le N$ so that $X_{s_i} < a < b < X_{t_i}$ for all $1 \le i \le k$. Let $m_1 = \min\{n : X_n < a\}$ and for $2 \le i \le k+1$ let $m_i = \min\{n : n \ge t_{i-1}, X_n < a\}$. For each $1 \le i \le k$, let $l_i = \min\{n : n \ge s_i, X_n > b\}$. Then by the definition of C_n we know that for $n \le N$, $C_n(\omega) = 1$ if and only if $m_i + 1 \le n \le l_i$ for $i \le k$ or $n \ge m_{k+1} + 1$. It follows that we can write

$$\begin{aligned} Y_n(\omega) &= \sum_{i=1}^{N} C_i(\omega)(X_i(\omega) - X_{i-1}(\omega)) \\ &= \sum_{j=1}^{k}\left[\sum_{i=m_j+1}^{l_j} C_i(\omega)(X_i(\omega) - X_{i-1}(\omega))\right] + \sum_{i=m_{k+1}+1}^{N} C_i(\omega)(X_i(\omega) - X_{i-1}(\omega)) \\ &= \sum_{j=1}^{k}(X_{l_j}(\omega) - X_{m_j}(\omega)) + X_N(\omega) - X_{m_{k+1}}(\omega), \\ &\ge (b-a)U_N[a,b] + X_N(\omega) - X_{m_{k+1}}(\omega) \end{aligned}$$

where in the last line we set $X_N(\omega) - X_{m_{k+1}}(\omega) = 0$ if $m_{k+1} \ge N$. Note that if $X_N(\omega) \le a$, then

$$X_N(\omega) - X_{m_{k+1}}(\omega) > X_N(\omega) - a = -(X_N(\omega)-a)^-$$

because $X_{m_{k+1}}(\omega) < a$. If $X_N(\omega) > a$, then

$$X_N(\omega) - X_{m_{k+1}}(\omega) > X_N(\omega) - a > 0 = -(X_N(\omega) - a)^-.$$

This proves (C.2). Now note that by definition C_n is predictable. Therefore, we have

$$\mathbf{E}[Y_n - Y_{n-1}|\mathcal{F}_{n-1}] = \mathbf{E}[C_n(X_n - X_{n-1})|\mathcal{F}_{n-1}] = C_n\mathbf{E}[(X_n - X_{n-1})|\mathcal{F}_{n-1}] \leq 0$$

because $(X_n)_{n\in\mathbb{N}}$ is a supermartingale and C_n is nonnegative. This shows that $(Y_n)_{n\in\mathbb{N}}$ is a supermartingale as well. By definition Y_0 is null and thus $\mathbf{E}[Y_0] = 0$. Inductively, by the tower law we obtain

$$\mathbf{E}[Y_n] = \mathbf{E}[\mathbf{E}[Y_n|\mathcal{F}_{n-1}]] \leq \mathbf{E}[Y_{n-1}].$$

This implies that $\mathbf{E}[Y_n] \leq 0$ for all n. We conclude the proof of the theorem by taking expectations of both sides of (C.2). □

Now we are ready to prove Doob's martingale convergence theorem.

Theorem C.6. *Let $(X_n)_{n\in\mathbb{N}}$ be a uniformly L^1-bounded supermartingale, i.e.,*

$$\sup_{n\in\mathbb{N}} \mathbf{E}[|X_n|] < \infty.$$

Then almost surely, $X_\infty \coloneqq \lim_{n\to\infty} X_n$ exists and is in L^1.

Proof. Suppose $X_\infty = \lim_{n\to\infty} X_n$ exists. Then by Fatou's lemma we have

$$\mathbf{E}[|X_\infty|] = \mathbf{E}[\liminf_{n\to\infty} |X_n|] \leq \liminf_{n\to\infty} \mathbf{E}[|X_n|] \leq \sup_{n\in\mathbb{N}} \mathbf{E}[|X_n|] < \infty.$$

To prove the first assertion, we define the event

$$\begin{aligned}\Lambda &\coloneqq \{\omega : X_n(\omega) \text{ does not converge in } \mathbb{R}\} \\ &= \{\omega : \liminf_{n\to} X_n(\omega) < \limsup_{n\to\infty} X_n(\omega)\} \\ &= \bigcup_{a,b\in\mathbb{Q}, a<b} \{\omega : \liminf_{n\to} X_n(\omega) < a < b < \limsup_{n\to\infty} X_n(\omega)\}.\end{aligned}$$

Since Λ is the union of countably many events, it suffices to show that $\Lambda_{a,b} \coloneqq \{\omega : \liminf_{n\to} X_n(\omega) < a < b < \limsup_{n\to\infty} X_n(\omega)\}$ for each $a, b \in \mathbb{Q}$ has probability zero. Note that $\omega \in \Lambda_{a,b}$ only if the number of upcrossings of $[a, b]$ made by $n \to X_n(\omega)$ is infinite. Define $U_\infty[a, b] = \lim_{N\to\infty} U_N[a, b]$. Then $\omega \in \Lambda_{a,b}$ only if $U_\infty[a, b](\omega) = \infty$. Therefore, it suffices to show that $\mathbf{P}(U_\infty[a, b] = \infty) = 0$. By Theorem C.5 we have

$$(b - a)\mathbf{E}[U_N[a, b]] \leq \mathbf{E}[(X_N - a)^-] \leq \sup_{n\in\mathbb{N}} \mathbf{E}[|X_N|] + |a| < \infty,$$

which implies that $\mathbf{E}[U_N[a, b]] < \infty$. Since this bound is uniform in N and $U_N[a, b] \uparrow U_\infty[a, b]$, by monotone convergence we have

$$\mathbf{E}[U_\infty[a, b]] = \mathbf{E}[\lim_{N\to\infty} U_N[a, b]] = \lim_{N\to\infty} \mathbf{E}[U_N(a, b)] < \infty.$$

In particular, $\mathbf{P}(U_\infty[a, b] = \infty) = 0$, as desired. □

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