ON THE BOCKSTEIN AND THE ADAMS SPECTRAL SEQUENCES

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ABSTRACT. In this paper, we present a theorem by May and Milgram regarding the Bockstein spectral sequence and the Adams spectral sequence in [7]. We begin by introducing cohomology operations and the two spectral sequences. Then we proceed to our main theorem and its proof.

CONTENTS

1. PRELIMINARIES

In this section, we introduce elements in the stable homotopy category in order to build up basic vocabularies for the rest of the paper.

Definition 1.1 (Spectra)**.** A spectrum E is a sequence of pointed spaces $(E_n)_{n\in\mathbb{N}}$ together with structure maps $\Sigma E_n \to E_{n+1}$. Maps between two spectra are defined level wise and have to commute with the structure maps.

Examples 1.2.

(1) The sphere spectrum $\mathbb{S}$ has S^n as its n-th space. It is constructed by applying suspension iteratively on S^0. The structure map $\Sigma S^n \to S^{n+1}$ is the canonical homeomorphism.
(2) For an abelian group A, the Eilenberg–MacLane spectrum HA has the Eilenberg–MacLane space $K(A,n)$ as its n-th space. The structure map is the adjoint to the homotopy equivalence $K(A,n) \to \Omega K(A,n+1)$.

In fact, it helps to think of a spectrum informally as a stabilized space. Indeed, it is the correct object to work with in the stable homotopy category. We can take homotopy groups of a spectrum X as

$$\pi_n(X) = \lim_{k\to\infty} \pi_{n+k}(X_k)\ .$$

Alternatively, we have

$$\pi_n(X) = [\Sigma^n \mathbb{S}, X]$$

where $(\Sigma^n \mathbb{S})$ is a spectrum with the t-th space $(\Sigma^n \mathbb{S})_t = \Sigma^n \mathbb{S}_t = S^{t+n}$. Given a spectrum E, we can take the E-cohomology of a spectrum X as

$$E^n(X) = [X, \Sigma^n E]$$

in positive degrees. We can also take the E-homology of X as

$$E_n(X) = \pi_n(E \wedge X) = [\Sigma^n \mathbb{S}, E \wedge X] \ .$$

Definition 1.3 (Cohomology operations)**.** Let E be a cohomology theory. For an integer k, the set of stable cohomology operations of degree k is the collection of natural transformations $E^0(-) \to E^k(-)$ that commute with suspension isomorphisms. We denote the set as $\mathcal{A}_E^k$. We claim that the set $\mathcal{A}_E^k$ forms a ring under addition and composition ([12, 3.2.1]). Summing over all k, we have a graded ring

$$\mathcal{A}_E = \bigoplus_k \mathcal{A}_E^k \ .$$

Now we introduce the Bockstein homomorphism as a crucial example of cohomology operations.

Example 1.4 (The Bockstein homomorphism)**.** The short exact sequence of coefficients

$$0 \to \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$$

induces the long exact sequence

$$\cdots \to H^n(X;\mathbb{Z}/p\mathbb{Z}) \xrightarrow{\partial_n} H^{n+1}(X;\mathbb{Z}) \xrightarrow{\times p} H^{n+1}(X;\mathbb{Z}) \xrightarrow{\pi_{n+1}} H^{n+1}(X;\mathbb{Z}/p\mathbb{Z}) \to \cdots$$

where ∂_n is the connecting homomorphism. We define the Bockstein homomorphism

$$\beta = \pi_{n+1}\partial_n : H^n(X;\mathbb{Z}/p\mathbb{Z}) \to H^{n+1}(X;\mathbb{Z}/p\mathbb{Z}) \ .$$

It increases the degree by 1. As a matter of fact, this homomorphism agrees with the connecting homomorphism associated with the short exact sequence of coefficients

$$0 \to \mathbb{Z}/p\mathbb{Z} \xrightarrow{\times p} \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0 \ .$$

This can be verified by carefully tracing each map. We also obtain the r-th order Bockstein homomorphism β_r as the connecting homomorphism of the short exact sequence of coefficients

$$0 \to \mathbb{Z}/p^r\mathbb{Z} \xrightarrow{\times p} \mathbb{Z}/p^{2r}\mathbb{Z} \to \mathbb{Z}/p^r\mathbb{Z} \to 0 \ .$$

When $p = 2$, the Bockstein homomorphism β is Sq^1 in the Steenrod operations. For each prime number p, the mod-p Steenrod algebra is a family of cohomology operations and it is essential in the calculation of various cohomology rings and stable homotopy theory, in particular the construction of the Adams spectral sequence. A detailed description of the Steenrod operations is in [11, 2].

2. The Bockstein and the Adams spectral sequences

We assume basic familiarity with spectral sequences. In this section, we introduce two important examples: the Bockstein spectral sequence and the Adams spectral sequence. They are the main subjects we study in this paper.

We follow the description of the Bockstein spectral sequence in [8, 24.2.3], and define it as a singly graded spectral sequence.

Definition 2.1 (Bockstein spectral sequence)**.** Let C be a degree-wise free $\mathbb{Z}$-chain complex. Tensoring the short exact sequences of coefficients

$$0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$$

with C, we can form the following exact couple

$$\begin{tikzcd} H_*(C;\mathbb{Z}) \arrow[rr, "i=\times p"] & & H_*(C;\mathbb{Z}) \arrow[dl, "j=mod\ p"] \\ & H_*(C;\mathbb{Z}/p\mathbb{Z}) \arrow[ul, "k"] & \end{tikzcd}$$

and we thus obtain the mod-p Bockstein spectral sequence E^r_n with the following properties.

(1) We have $E^1_n = H_n(C;\mathbb{Z}/p\mathbb{Z})$ and differentials
$$\beta_1 = jk : H_n(C;\mathbb{Z}/p\mathbb{Z}) \to H_{n-1}(C;\mathbb{Z}/p\mathbb{Z}) \ .$$
Note that the Bockstein β is the homological version of Example 1.4.

(2) To calculate the higher pages explicitly, we use
$$0 \to p^{r-1}H_n(C;\mathbb{Z}) \otimes \mathbb{Z}/p\mathbb{Z} \to E^r_n \to \mathrm{Tor}(p^{r-1}H_{n-1}(C;\mathbb{Z}), \mathbb{Z}/p\mathbb{Z}) \to 0 \ . \tag{2.2}$$
Observe that when $r = 1$, this is precisely the universal coefficient theorem.

(3) We assume that the homology of C is finitely generated in each degree from now on, and we have convergence as follows:
$$E^1_n C \Rightarrow E^\infty_n C = (H_n(C;\mathbb{Z})/\text{torsion}) \otimes \mathbb{Z}/p\mathbb{Z} \ .$$

In addition, knowing E^r for all r and E^∞, we have a complete description of the $\mathbb{Z}/p^r\mathbb{Z}$ components and the $\mathbb{Z}$ components in $H_*(C;\mathbb{Z})$, as described in [8, 24.2.3]. We shall exploit this fact extensively in our main theorem. Using the Bockstein spectral sequence, we can gather information on the integral homology of a space or a spectrum given information about its mod-p homology.

We demonstrate a Bockstein spectral sequence calculation using the Moore spectrum $\mathbb{S}/p^s$.

Example 2.3 (The Bockstein spectral sequence for $\mathbb{S}/p^s$)**.** For a prime number p, we have a map

$$\mathbb{S} \xrightarrow{p^s} \mathbb{S}$$

inducing multiplication by p^s in the 0-th homotopy. The Moore spectrum $\mathbb{S}/p^s$ is constructed as the cofiber of p^s. It has only one non-trivial homology group $H_0(\mathbb{S}/p^s;\mathbb{Z}) = \mathbb{Z}/p^s\mathbb{Z}$. Using Formula 2.2, we can easily calculate the mod-p Bockstein spectral sequence, since its homology groups are very simple. To summarize the result, we have $E^r_0\mathbb{S}/p^s = E^r_1\mathbb{S}/p^s = \mathbb{Z}/p\mathbb{Z}$ for $1 \le r \le s+1$ and $E^r_n\mathbb{S}/p^s = 0$ for all other r, n as graphed below in 2.3.

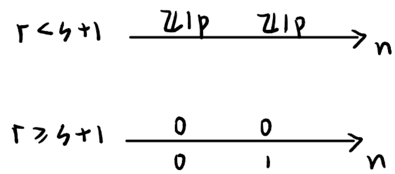

FIGURE 1. The Bockstein spectral sequence of $\mathbb{S}/p^s\mathbb{S}$.

The integral homology of $\mathbb{S}/p^s$ has no free parts, so it makes sense that every spot in the spectral sequence vanishes eventually. Moreover, we can make some observations on the differentials. Since nothing changes in the E^n-pages in the Bockstein spectral sequence for $1 \leq n < s+1$, the first $(s-1)$ Bockstein homomorphisms have to be trivial. After the E_s-page, the whole Bockstein spectral sequence collapses, so we know at page s the Bockstein homomorphism is an isomorphism.

We now come to the Adams spectral sequence. Since its construction is rather complicated, we only sketch out what is needed to understand the rest of this paper. A complete construction of the Adams spectral sequence can be found in [12, 4.3] and [10, 5].

We use the Adams spectral sequence to compute the homotopy classes of maps between two spectra X and Y. Here we take $X = \mathbb{S}$ and restrict our attention on computing $\pi_*(Y)$.

Definition 2.4 (The Adams spectral sequence)**.** For a connective spectrum X (meaning $\pi_i(X) = 0$ for $i < 0$), we have the Adams spectral sequence $E_r^{s,t}$ such that

(1) The differentials are defined as
$$d_r : E_r^{s,t}X \to E_r^{s+r,t+r-1}X \ .$$
(2) We usually start with its E_2-page:
$$E_2^{s,t}X = \mathrm{Ext}_{\mathcal{A}_p}(H^*(X), H^*(\mathbb{S})) = \mathrm{Ext}_{\mathcal{A}_p}(H^*(X), \mathbb{Z}/p\mathbb{Z})$$
where $\mathcal{A}_p$ is the mod-p Steenrod algebra for a prime p.
(3) Convergence:
$$E_2^{s,t}X \Rightarrow \pi_{t-s}(X_p^\wedge) \ .$$
The spectrum $X_p^\wedge$ here is called the p-completion of X. It is equipped with a map $X \to X_p^\wedge$ which induces the p-completion of the groups $\pi_*(X)$. If X is of finite type, then
$$\pi_{t-s}(X_p^\wedge) = \pi_{t-s}(X) \otimes \mathbb{Z}_p$$
where $\mathbb{Z}_p$ represent the p-adic integers.

Since the total degree for the Adams spectral sequence is $t-s$, it is conventional that we use $t-s$ as the horizontal axis and s as the vertical axis when we graph the spectral sequence. In this way, elements of the same total degree can be easily found along one vertical stem.

Example 2.5. We have encountered the Moore spectrum $\mathbb{S}/p^s\mathbb{S}$ as an example of the Bockstein spectral sequence, and now we want to see the E_2-page of the Adams

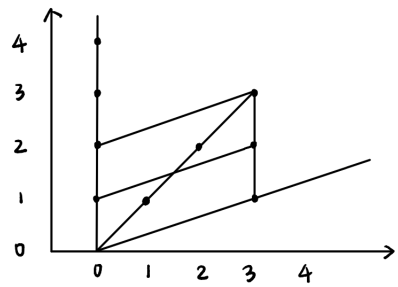

FIGURE 2. E_2-page of the Adams spectral sequence of $\mathbb{S}$ from [5].

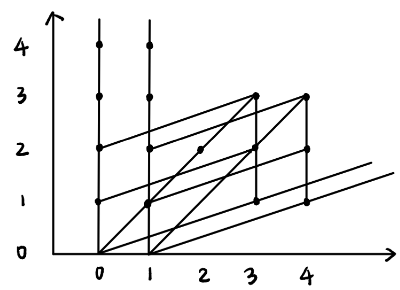

FIGURE 3. E_2-page of the Adams spectral sequence of $\mathbb{S}/p^s\mathbb{S}$.

spectral sequence of $\mathbb{S}/p^s\mathbb{S}$. Below is the lower part of the E_2-page of the Adams spectral sequence of $\mathbb{S}$ (Figure 2) and $\mathbb{S}/p^s\mathbb{S}$ (Figure 3).

The E_2-page of $\mathbb{S}/p^s\mathbb{S}$ consists of two copies of the E_2-page of the Adams spectral sequence of the sphere spectrum $\mathbb{S}$ that is off by one degree horizontally. That's no surprise, since we saw in Example 2.3 that the Moore spectrum $\mathbb{S}/p^s\mathbb{S}$ is constructed via two copies of $\mathbb{S}$. Observe that at $t - s = 0$ and 1, there are two vertical stems that seem to be going very high up. Each one of them is called a spike and it actually has infinite height. We will see its precise definition shortly in Definition 3.2.

3. Main Results

In this section we introduce our main theorem on the correspondence between the Bockstein spectral sequence and the Adams spectral sequence. We first state an important theorem by Adams and Liulevicius, and we prove a few important lemmas. Then we glue all the pieces together and prove the main theorem. We also revisit the Moore spectrum to illustrate the correspondence.

3.1. **Background.**

We begin by introducing some preliminary definitions and results on the Adams spectral sequence. Fix a prime p, and from now on we write $\mathcal{A}$ for $\mathcal{A}_p$ for simplicity.

Definition 3.1 (Infinite cycles)**.** There is a canonical generator a_0 of $E_2^{1,1}\mathbb{S}$. It is an infinite cycle such that if $x \in E_\infty^{s,t}$ and if $y \in F^s\pi_{t-s}X$ projects to x, then

$py \in F^{s+1}\pi_{t-s}X$ projects to a_0x. Here F^n is the decreasing filtration of π_*X given by the construction of the Adams spectral sequence.

Definition 3.2 (Spikes)**.** Let M be an $\mathcal{A}$-module and a_0 be the infinite cycle defined in Definition 3.1. If $x \in E_2M$ is not of the form a_0x' for any x' and if $a_0^ix \neq 0$ for all i, we say that x generates the spike $\{a_0^ix\}$. Using $(t-s)$ as the horizontal axis and s as the vertical axis, the elements in a spike form a vertical stem of an infinite height.

Definition 3.3 ($\mathcal{A}_0$-free)**.** For $\mathcal{A}_0 = E\{\beta\} \subset \mathcal{A}$ where $E\{\beta\}$ denotes the exterior algebra generated by the cohomological Bockstein homomorphism β, we say that an $\mathcal{A}$-module M is $\mathcal{A}_0$-free if and only if the Margolis homology $H(M,\beta) = 0$. Here we treat the left β-action on $\mathcal{A}$ (therefore on M) as differentials, and so it makes sense to take the homology.

Now we state a vanishing theorem by Adams and Liulevicius. It gives a range in which $E_2^{*,*}$ in the Adams spectral sequence for a spectrum vanishes. This theorem comes handy for us to construct the correspondence between the Bockstein and the Adams spectral sequence.

Theorem 3.4 (Adams–Liulevicius)**.** *Let M be an $(m+1)$-connected $\mathcal{A}_0$-free $\mathcal{A}$-module; i.e. as a graded module M_* is trivial below degree $(m+2)$. Then we have*

$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(M, \mathbb{Z}/p\mathbb{Z}) = 0$$

for $s \geq 1$ and $t - s \leq m + f(s)$, where $f(s) = 2(p-1)$ if $p > 2$ and

$$f(s) = \begin{cases} 8k+1, & \text{if } s = 4k \\ 8k+2, & \text{if } s = 4k+1 \\ 8k+3, & \text{if } s = 4k+2 \\ 8k+5, & \text{if } s = 4k+3 \end{cases}$$

if $p = 2$.

The proof of Theorem 3.4 is due to Adams in [1] and Liulevicius in [6].

Proposition 3.5 (Change of ring)**.** *For a Hopf algebra A over a field k and a normal Hopf subalgebra $B \subset A$, we have*

$$\mathrm{Ext}_A(B,k) \cong \mathrm{Ext}_M(k,k) \text{ where } B = A \otimes_M k\ .$$

We use Proposition 3.5 to compute the E_2-pages of various Adams spectral sequences. The detailed proof can be found in [2, VI.4.13], .

3.2. Calculation: the Adams spectral sequence for HA.

The Eilenberg–MacLane spectrum HA is categorized by $\pi_0(HA) = A$ and $\pi_n(HA) = 0$ for all other n. Now we calculate the Adams spectral sequence of several Eilenberg–MacLane spectra. This result is essential to the proof of our main result in Theorem 3.8.

Theorem 3.6.

(1) $E_2H\mathbb{Z}/p\mathbb{Z} = E_\infty H\mathbb{Z}/p\mathbb{Z}$ has a class $\mathbb{Z}/p\mathbb{Z}$ in bidegree $(0,0)$ and 0 anywhere else.

(2) $E_2H\mathbb{Z} = E_\infty H\mathbb{Z}$ is a spike generated by $y \in E_2^{0,0}H\mathbb{Z}$.

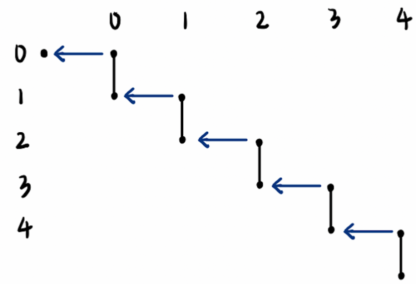

FIGURE 4. The free $E\{\beta\}$-resolution of $\mathbb{Z}/p\mathbb{Z}$.

(3) *For $r \geq 2$, $E_2 H\mathbb{Z}/p^r\mathbb{Z}$ is the sum of a spike generated by $y \in E_2^{0,0} H\mathbb{Z}/p^r\mathbb{Z}$ and a spike generated by $z \in E_2^{1,0} H\mathbb{Z}/p^r\mathbb{Z}$ with $d_r(a_0^i z) = a_0^{i+r} y$, and $E_\infty H\mathbb{Z}/p^r\mathbb{Z}$ has a basis $\{a_0^i y \mid 0 \leq i < r\}$.*

Proof. Unless noted otherwise, all (co)homology is computed with mod-p coefficients. We have the following classical results:

$$\begin{aligned} H^*(H\mathbb{Z}/p\mathbb{Z}) &= \mathcal{A} \cdot \iota \\ H^*(H\mathbb{Z}) &= (\mathcal{A}/\mathcal{A}\beta) \cdot \iota \\ H^*(H\mathbb{Z}/p^r\mathbb{Z}) &= (\mathcal{A}/\mathcal{A}\beta) \cdot \iota \oplus (\mathcal{A}/\mathcal{A}\beta) \cdot \beta_r \iota \end{aligned}$$

where ι denotes the fundamental class and β_r denotes the r-th cohomological Bockstein. For those interested, the proof for $p = 2$ can be found in [13, §2]. We know

$$E_2^{s,t} X = \mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(X), \mathbb{Z}/p\mathbb{Z}) \ ,$$

so we can use the results above to calculate the E_2-page of each Adams spectral sequence.

(1) There is a canonical identification of $\mathcal{A} \cdot \iota$ with $\mathcal{A}$. As

$$\mathcal{A} = \mathcal{A} \ \otimes_{\mathbb{Z}/p\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \ ,$$

by the Proposition 3.5, we know

$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(H\mathbb{Z}/p\mathbb{Z}), \mathbb{Z}/p\mathbb{Z}) = \mathrm{Ext}_{\mathbb{Z}/p\mathbb{Z}}^{s,t}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \ .$$

The free $\mathbb{Z}/p\mathbb{Z}$-resolution of $\mathbb{Z}/p\mathbb{Z}$ is concentrated in degree 0. Therefore, by taking the Hom of the resolution into $\mathbb{Z}/p\mathbb{Z}$, we again get a chain complex concentrated in degree 0. After taking homology, we know that $\mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(H\mathbb{Z}/p\mathbb{Z}), \mathbb{Z}/p\mathbb{Z})$ is $\mathbb{Z}/p\mathbb{Z}$ at $(0,0)$ and 0 everywhere else, which gives us $E_2 H\mathbb{Z}/p\mathbb{Z}$. Since there are no differentials, the single copy of $\mathbb{Z}/p\mathbb{Z}$ persists to the E_∞-page.

(2) We observe

$$\mathcal{A}/\mathcal{A}\beta = \mathcal{A} \ \otimes_{E\{\beta\}} \mathbb{Z}/p\mathbb{Z} \ ,$$

as $(a\beta, r) = (a, \beta r) = 0$. Again by Proposition 3.5, we know

$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(H\mathbb{Z}), \mathbb{Z}/p\mathbb{Z}) = \mathrm{Ext}_{E\{\beta\}}^{s,t}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \ .$$

We draw the $E\{\beta\}$-resolution of $\mathbb{Z}/p\mathbb{Z}$ as in Figure 4 where each vertical bar represents $E\{\beta\}$ and the dots represent $\mathbb{Z}/p\mathbb{Z}$.

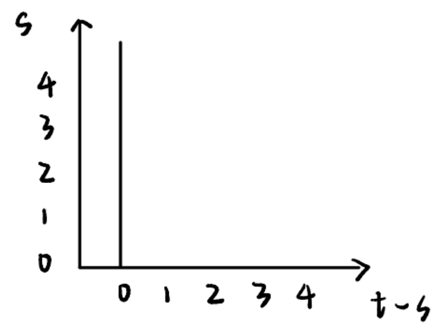

FIGURE 5. E_n $(n \geq 2)$ and E_∞-page of the Adams spectral sequence of $H\mathbb{Z}$

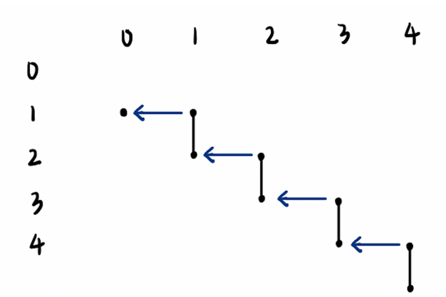

FIGURE 6. The free $E\{\beta\}$-resolution of $\mathbb{Z}/p\mathbb{Z}$ with the degree shifted.

Therefore, for each non-negative integer r, at $(s,t) = (r,r)$, we have a copy of $\mathbb{Z}/p\mathbb{Z}$. As there are no differentials, after taking the homology we get $\mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(H\mathbb{Z}), \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$ at (r,r). We see that they form a spike generated in $E_2^{0,0}$. Since there are no differentials, the spike persists to E_∞. Therefore, we can draw the spectral sequence as in Figure 5.

(3) We can compute the Ext of each summand of the cohomology ring separately and then add them up, and we already did half of the work in the last calculation. Now we only need to calculate the $(\mathcal{A}/\mathcal{A}\beta)\cdot\beta_r\iota$ part. Note that this is almost identical as the last part. The only difference here is that $\beta_r\iota$ is of degree 1, as the cohomological Bockstein homomorphism increases the degree by 1. So we start our resolution at $(0,1)$, and we have a similar $E\{\beta\}$-resolution of $\mathbb{Z}/p\mathbb{Z}$ as in Figure 6.

Similarly, after taking the homology we get $\mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(H\mathbb{Z}), \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$ at $(r, r+1)$. We see that they form a spike generated in $E_2^{0,1} H\mathbb{Z}/\mathbb{Z}_{p^r}$. To summarize, $E_2 H\mathbb{Z}/p^r\mathbb{Z}$ is generated a spike at $y \in E_2^{0,0}$ and a spike at $z \in E_2^{0,1}$, and the differentials follow directly from how they are defined. Since $\pi_0(H\mathbb{Z}/p^r\mathbb{Z}) = \mathbb{Z}/p^r\mathbb{Z}$ and $\pi_1(H\mathbb{Z}/p^r\mathbb{Z}) = 0$, the spike generated at $E_2^{0,1}$ does not persist to E_∞ and must vanish after the E_r-page. Therefore, we know that the lower part of the stem $\{a_0^i y \mid 0 \leq i < r\}$ is untouched by

d_r and thus persists to E_∞. Therefore, we can graph the spectral sequence as in Figure 7.

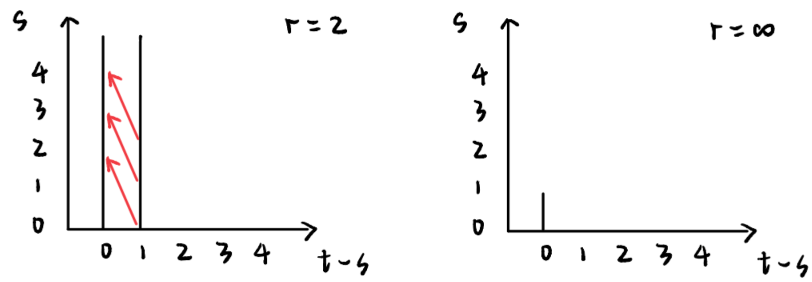

FIGURE 7. E_n ($n \geq 2$, left) and E_∞-page (right) of the Adams spectral sequence of $H\mathbb{Z}$

□

Proposition 3.7 (Spectral sequence of a wedge sum)**.** *Let X and Y be spectra, then*

$$E_r(X \vee Y) = E_r X \oplus E_r Y \ .$$

Proof. This follows directly from the construction of the cohomological spectral sequences and the Eilenberg–Steenrod axioms for cohomology. □

3.3. **Statement of the main theorem.** Back in Section 2, we looked at the Bockstein and the Adams spectral sequence of the Moore spectrum $\mathbb{S}/p^s\mathbb{S}$. Recall that in the Bockstein spectral sequence (Figure 2.3) there are copies of $\mathbb{Z}/p\mathbb{Z}$ at $n = 0$ and $n = 1$ in the lower pages, and that everything becomes trivial after page $(s+1)$. Also recall that in the Adams spectral sequence (Figure 2.5) there are two spikes at the $t - s = 0, 1$, and we deduced that the spikes vanish in higher pages. We see a correspondence here between classes in the Bockstein spectral sequence and spikes in the Adams spectral sequence of $\mathbb{S}/p^s\mathbb{S}$. It seems like a class at E_n^r in the Bockstein spectral sequence corresponds to a spike at $E_r^{s,t}$ in the Adams spectral sequence where $s - t = 0$. In fact, this correspondence is no accident, which brings us to the main result of this paper. This result is due to J. P. May and R. J. Milgram in [7].

Theorem 3.8 (May–Milgram)**.** *Let X be an $(m-1)$-connected spectrum with finite type integral homology. For $r \geq 1$, let C_r be a basis for the r-th page $E^r X$ of the mod-p homological Bockstein spectral sequence of X. We also choose C_r such that*

$$C_r = D_r \cup \beta_r D_r \cup C_{r+1} \ ,$$

where $D_r, \beta_r D_r$ and C_{r+1} (the set of cycles under β_r that is projected to the chosen basis of $E^{r+1}X$) are disjoint linearly independent subsets of $E^r X$. Then we have

(i) For $r \geq 2$ and $r = \infty$, the set of spikes in $E_r X$ of the Adams spectral sequence is in one-to-one correspondence with the elements in C_r: if $c \in C_r$ has degree q, it corresponds to a spike generated by $\gamma \in E_r^{s,t} X$ where $q = t - s$. Additionally, we conclude $f(s) + m \leq q$.

(ii) *For $d \in D_r$, if $\delta \in E_r^{s,t}X$ generates d and $\epsilon \in E_r^{s,t}X$ generates $\beta_r d$, then we have*
$$d_r(a_0^i \delta) = a_0^{i+r+s-u}\epsilon$$
provided $m + f(i+s) \geq t - s$.

3.4. **Proof of the main theorem.** The proof involves concrete constructions of objects and maps. In order to present it in a clear and a coherent way, we break it down to three parts here. We first use our given spectrum X to construct a spectrum Y that is a wedge sum of Eilenberg–MacLane spectra.

Construction 3.9. We first construct a map $\phi_i : X \to K(H_i(X;\mathbb{Z}), i)$ in the following way. Ignoring the torsion groups not a power of p, we write $H_i(X;\mathbb{Z})$ as the sum of cyclic groups C_{ij}. By the universal coefficient theorem, we have
$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{i-1}(X;\mathbb{Z}), C_{ij}) \to H^i(X;C_{ij}) \xrightarrow{h} \mathrm{Hom}_{\mathbb{Z}}(H_i(X;\mathbb{Z}), C_{ij}) \to 0 \ .$$
Since the map h is surjective, we can find a class $u_j \in H^i(X;C_{ij})$ that maps to the projection map in $\mathrm{Hom}_{\mathbb{Z}}(H_i(X;\mathbb{Z}), C_{ij})$. By the representability of integral cohomology, u_j represents a map
$$\phi_{ij} : X \to K(C_{ij}, i) \ .$$
Taking the sum of these maps over j, we have
$$\phi_i = \sum_j \phi_{ij} : X \to \bigvee_j K(C_{ij}, i) = K(H_i(X;\mathbb{Z}), i) \ .$$
Again taking the sum of these maps over i, we have our desired map
$$\phi = \sum_i \phi_i : X \to \bigvee_i K(H_i(X;\mathbb{Z}), i) \ .$$

For simplicity of notations, we write
$$Y = \bigvee_i K(H_i(X;\mathbb{Z}), i) \ .$$
Passing ϕ to homology, we now want to calculate the quotient of the map
$$\phi_* : H_*(X;\mathbb{Z}) \to H_*(Y;\mathbb{Z}) \ .$$

Lemma 3.10. *With the map ϕ_* in Construction 3.9, we build a short exact sequence*
$$0 \to H_*(X;\mathbb{Z}) \xrightarrow{\phi_*} H_*(Y;\mathbb{Z}) \to M_* \to 0 \ .$$
We claim that
$$M_* = \sum_q H_q(\bigvee_{i \leq q-2} K(H_i(X;\mathbb{Z}), i); \mathbb{Z}/p\mathbb{Z})) \ .$$

Proof. This is not difficult to show, but we need to carefully trace down what each map does at each level of homology. We first observe that ϕ_i induces an isomorphism on the i-th homology
$$\phi_i : H_i(X) \to H_i(K(H_i(X;\mathbb{Z}), i); \mathbb{Z})$$
as it is the wedge sum of projection maps on each component. Then we have a monomorphism

(3.11)
$$\phi_* : H_*(X;\mathbb{Z}) \to H_*(K(H_i(X;\mathbb{Z}), i); \mathbb{Z}) \hookrightarrow H_*(\bigvee_i K(H_i(X;\mathbb{Z}), i)) = H_*(Y;\mathbb{Z}) \ .$$

We also observe that ϕ_i induces an epimorphism on the $(i+1)$-th homology, since the Hurewicz theorem gives $H_{i+1}(K(H_i(X;\mathbb{Z}),i);\mathbb{Z})=0$.

Everything we have discussed so far takes place in integral homology, and now we move toward the mod-p homology. By the universal coefficient theorem, we have

$$H_i(X;\mathbb{Z}/p\mathbb{Z}) = H_i(X;\mathbb{Z})\otimes\mathbb{Z}/p\mathbb{Z}\oplus \mathrm{Tor}(H_{i-1}(X;\mathbb{Z}),\mathbb{Z}/p\mathbb{Z})$$

mapping to

$$H_i(K(H_i(X;\mathbb{Z}),i);\mathbb{Z}/p\mathbb{Z}) = H_i(X;\mathbb{Z})\otimes\mathbb{Z}/p\mathbb{Z}\ .$$

Note that we have

$$H_i(K(H_{i-1}(X;\mathbb{Z}),i-1);\mathbb{Z}/p\mathbb{Z}) = \mathrm{Tor}(H_{i-1}(X),\mathbb{Z}/p\mathbb{Z}),$$

so we still have a monomorphism

$$\phi_* : H_*(X;\mathbb{Z}/p\mathbb{Z})\to H_*(Y;\mathbb{Z}/p\mathbb{Z})\ .$$

It is similar in the $(i+1)$-th homology. We have

$$H_{i+1}(X;\mathbb{Z}/p\mathbb{Z}) = H_{i+1}(X;\mathbb{Z})\otimes\mathbb{Z}/p\mathbb{Z}\oplus \mathrm{Tor}(H_i(X;\mathbb{Z}),\mathbb{Z}/p\mathbb{Z})$$

mapping to

$$H_{i+1}(K(H_i(X;\mathbb{Z}),i);\mathbb{Z}/p\mathbb{Z}) = \mathrm{Tor}(H_i(X;\mathbb{Z}),\mathbb{Z}/p\mathbb{Z})\ .$$

We also have

$$H_{i+1}(K(H_i(X;\mathbb{Z}),i);\mathbb{Z}/p\mathbb{Z}) = \mathrm{Tor}(H_{i-1}(X),\mathbb{Z}/p\mathbb{Z}).$$

Moreover, for $i>q$, note that

$$\phi_{i*} : H_q(X;\mathbb{Z}/p\mathbb{Z})\to H_q(K(H_i(X;\mathbb{Z}),i);\mathbb{Z}/p\mathbb{Z})$$

is 0 by the Hurewicz Theorem. Given we just showed that ϕ_{q*} and ϕ_{q-1*} do not contribute to the cofiber at degree-q homology, we conclude that the only maps that contribute to the cofiber at the degree-q homology are ϕ_i for $i\le q-2$. Thus, we have

$$M_q = H_q(\bigvee_{i\le q-2} K(H_i(X;\mathbb{Z}),i);\mathbb{Z}/p\mathbb{Z}))\ .$$

□

We are now going to figure out the relationship between E_2X and E_2Y using what we've figured out about M_* and Theorem 3.4.

Lemma 3.12. *In the Adams spectral sequence, we have*

$$E_2^{s,t}X \cong E_2^{s,t}Y$$

when $s\ge 2$ and $t-s\le m+f(s-1)$.

Proof. Taking the dual of map (3.11), we have

$$0\to M\to H^*(Y;\mathbb{Z}/p)\to H^*(X;\mathbb{Z}/p)\to 0 \tag{3.13}$$

where

$$M=\sum_q H^q(\bigvee_{i\le q-2} K(H_i(X;\mathbb{Z}),i);\mathbb{Z}/p\mathbb{Z}))$$

is the dual of M_*. We claim that $H(\mathcal{A}/\mathcal{A}\beta,\beta)=0$ by the properties of the Steenrod Algebra, so M is $\mathcal{A}_0$-free given the cohomology groups described in Theorem 3.6. We also observe that M is $(m+1)$-connected, since X is by hypothesis $(m-1)$-connected.

Equation (3.13) induces the long exact sequence

$$\cdots \to \mathrm{Ext}_{\mathcal{A}}^{s-1,t}(M;\mathbb{Z}/p\mathbb{Z}) \to E_2^{s,t}X \to E_2^{s,t}Y \to \mathrm{Ext}_{\mathcal{A}}^{s,t}(M;\mathbb{Z}/p\mathbb{Z}) \to \cdots$$

Applying Theorem 3.4 to M, we see that

$$E_2^{s,t}X \to E_2^{s,t}\bigvee_i K(H_i(X;\mathbb{Z}),i)$$

is an epimorphism when $s \geq 1$ and $t-s \leq m+f(s)$, and an isomorphism when $s \geq 2$ and $t-s \leq m+f(s-1)$. □

Now we are ready to paste all the pieces together and prove Theorem 3.8.

Proof of Theorem 3.8. We want to show that there is a correspondence between basis C_r in the Bockstein spectral sequence and spikes in the Adams spectral sequence for a spectrum X. For part (i), we start from the Adams spectral sequence. By Lemma 3.12, we have

$$\begin{aligned} E_2^{s,t}X &= E_2^{s,t}(\bigvee_i K(H_i(X;\mathbb{Z}),i)) \\ &= \bigoplus_i E_2^{s,t}K(H_i(X;\mathbb{Z}),i) \\ &= \bigoplus_{i,j} E_2^{s,t}K(C_{ij},i) \end{aligned}$$

within the range of the isomorphism. Therefore, by Theorem 3.6, within the range we have complete knowledge of $E_2^{s,t}X$ in terms of spikes, as all the C_{ij}'s are either $\mathbb{Z}$ or $\mathbb{Z}/p^r\mathbb{Z}$.

Now we make our way toward the Bockstein spectral sequence. We know that $H_*(X;\mathbb{Z})$ is a direct sum of cyclic groups. In fact, the generators of those subgroups in $H_*(X;\mathbb{Z})$ of order p^r are represented by elements of $\beta_r D_r$, and the generators of those subgroups in $H_*(X;\mathbb{Z})$ of infinite order are represented by elements of C_∞ ([8, 24.2.3]).

We have shown that there is a correspondence between basis C_r in the Bockstein spectral sequence and spikes in the Adams spectral sequence for a spectrum X. Now we proceed to observe how degrees work.

In the beginning of Section 3.3 we used the Moore spectrum $\mathbb{S}/p^s\mathbb{S}$ to demonstrate this correspondence. More concretely, let $c \in \beta_r D_r$ represent the generator of $C_{qj} = \mathbb{Z}/p^r\mathbb{Z}$ in $H_q(X)$. Therefore, the class c corresponds to a spike in $E_r^{s,t}(K(C_{qj},q))$. We know that it contributes toward $\pi_q(X)$; thus, the spike must be generated by some element in $E_r^{s,t}$ with $t-s=q$. In addition, we showed that any spike in $E_2^{s,t}K(C_{qj},q)$ is generated at $s=0$ in Theorem 3.6. Since the spike in $E_2^{s,t}K(C_{qj},q) \subset E_2^{s,t}Y$ is isomorphic with a spike in $E_2^{s,t}X$ within the range of isomorphism, the spike in $E_2^{s,t}X$ corresponding to c must be generated below the range of isomorphism. Therefore, we have

$$f(s)+m \leq q = t-s\ .$$

Note that it's the same story if we take a class $c' \in C_\infty$ representing the generator of $C_{kj} = \mathbb{Z}$ in $H_k(X)$. It then corresponds to a spike in $E_\infty^{s,t}(K(C_{kj},k))$ where $t-s=k$.

Part (ii) of the theorem follows directly from part (iii) Theorem 3.6, given that $a_0^i\delta$ is within the range of the isomorphism. □

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