# CLIFFORD ALGEBRAS AND BOTT PERIODICITY

YUQIN KEWANG

Abstract. In this paper we will observe the eight-fold periodicity in Clifford algebras and use it to explain the real Bott periodicity in topological $K$-theory through the Atiyah-Bott-Shapiro map [4]. We will also discuss the equivalence between $KO$-orientability and Spin-structures, the vector fields on spheres problem, and the problem of normed division algebras over $\mathbb{R}$ as applications.

Contents



## 1. Introduction

The Bott periodicity theorem is essential in algebraic topology because it determines how topological $K$-theories are realized as generalized cohomology theories. Hence, people came up with many different proofs of Bott periodicity theorem to

*Date*: October 14, 2022.

better understand it. However, the hardest thing to explain is why the eight-fold periodicity in real topological $K$-theory arises.

In 1963, Atiyah-Bott-Shapiro [4] discovered that the eight-fold periodicity in Clifford modules coincides with the periodicity in real $K$-theory, and this striking coincidence can be further developed into a ring isomorphism between the Clifford modules and the real $K$-theory of a point since both sides are equipped with nice ring structures. This is an exciting moment that Clifford algebras step into the area of topology. Through this ring isomorphism, the algebraic structure of real $K$-theory is captured by the Clifford modules and the eight-fold periodicity hence arises from Clifford algebras in the sense of Morita equivalence.

The goal of this paper is to explicate the construction of the *Atiyah-Bott-Shapiro map* (abbreviated as ABS map) which is a ring homomorphism connecting the Clifford modules with real topological $K$-theory and prove that it becomes a ring isomorphism in a special case, following [4]. In this paper, we fill in the gaps in the proofs in [4] and try to clarify the ideas behind the technical details. Also, the paper [4] restricted the discussion of the real $K$-theory Thom isomorphism only for vector bundles of dimension of a multiple of 8 and it has already been remarked by Karoubi [18] that such restriction is unnecessary. We generalize the result to an arbitrary dimension and include a complete proof of the equivalence between $KO$-orientability and Spin-structures.

We first observe the eight-periodicity in Clifford algebras and Clifford modules (Table 1, Table 2, Table 3) in Section 2. Further, we will construct the ring structures of the Clifford modules (Definition 2.27) and give an explicit computation of the rings (Theorem 2.33, Theorem 2.34). Then, we aim to construct the ABS map

$$\alpha_P : A_k \to \bigoplus_{k\geq 0} \widetilde{KO}(T(V)) \tag{1.1}$$

where $P \to X$ is a principal $\mathrm{Spin}(k)$-bundle, $X$ is a based finite CW complex, and $V = P \times_{\mathrm{Spin}(k)} \mathbb{R}^k$ is the associated vector bundle of $P$. Roughly speaking, for every $\mathbb{Z}/2$-graded Clifford module $M = M^0 \oplus M^1$, we associate a vector bundle $E = P \times_{\mathrm{Spin}(k)} M = E^0 \oplus E^1$ over $X$ with it which is also $\mathbb{Z}/2$-graded. Pulling back through $\pi : D(V) \to X$ gives us two bundles $\pi^*E^1$ and $\pi^*E^0$ over $D(V)$ which coincide on $S(V)$, and a morphism $\pi^*E^1 \to \pi^*E^0$ by multiplication by elements in the Clifford algebra. Since the morphism restricts to an isomorphism on $S(V)$, the difference bundle construction, called *Euler characteristic* in [4], gives us an element in $KO(D(V), S(V)) = \widetilde{KO}(T(V))$.

The main theme of Section 3 is to introduce this difference bundle construction (Theorem 3.5). To further show that the ABS map is multiplicative, we generalize our bundles $\pi^*E^1 \to \pi^*E^0$ to *sequence of bundles* (Definitions 3.2, Definition 3.3, Definition 3.8). This generalization mainly works to ease calculations but brings no new mathematics. The multiplicative property of the Euler characteristic is stated in Proposition 3.12 and a computation of the product is summarized in Proposition 3.16. These provide essential data to ensure the ABS map to be multiplicative.

In particular, composing the restriction map $i_x^* : \widetilde{KO}(T(V)) \to \widetilde{KO}(T(V_x)) = \widetilde{KO}(S^k) = KO^{-k}(\mathrm{pt})$ with the ABS map gives us a map $\alpha : A_k \to KO^{-k}(\mathrm{pt})$. This induces a ring isomorphism

$$\alpha : A_* \to \bigoplus_{k\geq 0} KO^{-k}(\mathrm{pt})$$

where $A_*$ is the graded ring of Clifford modules, called the *Atiyah-Bott-Shapiro isomorphism*. The main goal of Section 4 is to prove the ring isomorphism. The idea is that as we have computed the ring structure of Clifford modules in Section 2,

combining this data with the ring structure of the real $K$-theory of a point, which is well-known, it suffices to check that the ABS isomorphism sends generators to generators, so it will naturally become a ring isomorphism. Although we mainly focus on the real $K$-theory in this paper, the parallel results carry out for the complex case as well.

Proving the ABS isomorphism is the climax of this paper. However, the significance of the ABS map is more than this. In Section 5, we will first use the ABS map to prove that the $KO$-orientability of a vector bundle is equivalent to that it admits a Spin-structure. The ABS map also plays a role in the computation of the real $K$-theory of real stunted projective spaces. This computation plays a very important role in Adams' solution to the upper bound of *the vector fields on spheres problem*. Since the construction of the maximal number of vector fields also comes from Clifford algebras, we will introduce this problem briefly in Section 5.2 and outline how Clifford algebras and the ABS map are involved there. Finally, as an interesting corollary of the vector fields on spheres problem which also connects to Clifford algebras closely, we will introduce a solution to the classification of normed division algebras over $\mathbb{R}$ in Section 5.3. The main goal of introducing the vector fields on spheres problem and the normed division real algebras problem is to show how Clifford algebras appear as key ingredients in many other problems in mathematics.

The paper [4] inspires people to look for proofs of Bott periodicity using Clifford algebras. However, it is not easy to directly connect the topological $K$-theory with Clifford algebras; the construction of the ABS map is rather technical and not computational-friendly. Hence, the strategy for using Clifford algebras to prove Bott periodicity is to find alternative models for topological $K$-theory which are closely related to Clifford algebras, so we can use Clifford algebras to prove the Bott periodicity in those models first; then show that the Bott periodicity in the alternative models imply the original Bott periodicity. The first attempt is given by Wood [28] in 1966 using $\mathbb{Z}/2$-graded Banach algebras to realize the periodicity in Clifford algebras. In 1968, Karoubi [18] introduced the $K$-theories of Banach categories which are additive categories endowed with extra topological structures, generalizing Wood's result. This new $K$-theory directly inherits the periodicity of Clifford algebras and is equivalent to the topological one. In 1969, Atiyah-Singer [5] gave a different proof by studying the action of some particular Fredholm operators on $\mathbb{Z}/2$-graded Hilbert spaces acted by Clifford algebras. The Fredholm operators are connected to topological $K$-theory by the Atiyah-Jänich theorem [3, Theorem A1] that the real $K$-theory of a compact space is represented by the space of all Fredholm operators on an infinite-dimensional real Hilbert space.

## 2. Clifford algebras and Clifford modules

This section aims to introduce the Clifford modules which appear on the left hand side of the ABS map, construct the ring structure, and finally compute the ring (Theorem 2.33, Theorem 2.34). To set up the foundation, we need to first understand the Clifford algebras.

### 2.1. **Clifford algebras.**

We will first define the general Clifford algebras in Section 2.1.1, and move to some particular Clifford algebras (denoted as $C_k$) we are most interested in, the ones derived from Euclidean spaces $\mathbb{R}^n$, in Section 2.1.2. A complete computation of $C_k$ will be given in Section 2.1.3.

2.1.1. *Basic definitions.* We denote $k$ as a field. Let $E$ be a $k$-vector space and $Q : E \to k$ be a quadratic form over $E$.

**Definitions 2.1.** (1) The *tensor algebra over* $E$ is

$$T(E) = \bigoplus_{i=0}^{\infty} T^i E$$

where for $i > 0$, $T^i E = \underbrace{E \otimes E \otimes \cdots \otimes E}_{k \text{ times}}$ and $T^0 E = k$.

(2) The *Clifford algebra of* $Q$ is defined by the quotient

$$C(Q) = T(E)/(x \otimes x - Q(x) \mid x \in E).$$

where $(-)$ refers to the ideal generated by the elements in the brackets.

We embed the vector space $E$ into $C(Q)$ through the composition $i_Q : E \hookrightarrow T(E) \twoheadrightarrow C(Q)$ which is injective.

**Example 2.2.** When $Q = 0$, the Clifford algebra $C(Q)$ is the exterior algebra $\Lambda(E)$.

**Proposition 2.3** (The universal property of Clifford algebras)**.** *For any $k$-algebra $A$ with a linear homomorphism $\phi : E \to A$ such that*

$$\phi(x)^2 = Q(x) \quad \text{in } A, \tag{2.4}$$

*there exists a unique $k$-algebra homomorphism $\tilde{\phi} : C(Q) \to A$ such that $\tilde{\phi} \circ i_Q = \phi$.*

Define $F^q T(E) = \sum_{i \le q} T^i(E)$ and $F^q C(Q) = F^q T(E)/(x \otimes x - Q(x) \mid x \in E)$. Notice that $F^q C(Q)$ gives a filtration over $C(Q)$. Moreover, denote $GC(Q)$ as the associated graded algebra of $C(Q)$ with respect to this filtration, and we have $GC(Q) \cong \Lambda(E)$. Hence, $\dim_k C(Q) = 2^{\dim_k E}$. Let $e_1, \cdots, e_n$ be a basis of $i_Q(E)$. Then

$$\{e_{i_1} e_{i_2} \cdots e_{i_k} \mid 1 \le i_1 < \cdots < i_k \le n\} \bigcup \{1\}$$

forms a basis for $C(Q)$.

Let $\pi : T(E) \to C(Q)$ be the quotient map. Define the $\mathbb{Z}/2$-grading on $C(Q)$ by $C^0(Q) = \pi\left(\sum_{i=0}^{\infty} T^{2i}(E)\right)$ and $C^1(Q) = \pi\left(\sum_{i=0}^{\infty} T^{2i+1}(E)\right)$. It is worth noticing that we equip $C(Q)$ with the $\mathbb{Z}/2$-grading while $T(E)$ is a $\mathbb{Z}$-graded algebra. This is because $(x \otimes x - Q(x) \mid x \in E)$ is not homogeneous when $Q \neq 0$.

2.1.2. *The algebras $C_k$.* From now on, we fix base field to be $\mathbb{R}$ and $E = \mathbb{R}^k$ where $k \ge 1$ is an integer. Consider the quadratic form $Q_k : \mathbb{R}^k \to \mathbb{R}$,

$$(x_1, \cdots, x_k) \mapsto -\sum_{i=1}^{k} x_i^2.$$

We denote the Clifford algebra of $Q_k$ as $C_k$. In particular, we identify $\mathbb{R}^k$ with its embedding image $i_Q(\mathbb{R}^k)$ in $C_k$ and $\mathbb{R}$ with $\mathbb{R} \cdot 1$ in $C_k$. For $k = 0$, we define $C_0 = \mathbb{R}$. For each $C_k$, its corresponding complex version is defined by the complexification, i.e. $C_k \otimes_{\mathbb{R}} \mathbb{C}$.

Now we want to develop an alternative description of $C_k$ using generators and relations, which turns out to be more useful. Let $e_i = (0, \cdots, 1, \cdots, 0)$ be the unit vector in $\mathbb{R}^k$ with 1 in the $i$-th position and 0 in the others. Then $C_k$ is the universal algebra over $\mathbb{R}$ generated by $\{e_1, \cdots, e_k\}$ subject to the relation

$$\begin{cases} e_i^2 = -1, \\ e_i e_j + e_j e_i = 0, \quad i \neq j. \end{cases}$$

This follows directly from Lemma 2.6 and the following definition of graded tensor product where $k$ can be taken as an arbitrary field.

**Definition 2.5.** $A = \bigoplus_{\alpha=0,1} A^\alpha$, $B = \bigoplus_{\beta=0,1} B^\beta$ are $\mathbb{Z}/2$-graded $k$-algebras. The *graded tensor product* $A\hat{\otimes}_k B$ of $A$ and $B$ over $k$ is

$$\bigoplus_{i=0,1} \bigoplus_{\alpha+\beta\equiv i \pmod 2} A^\alpha \otimes B^\beta$$

with the multiplication given by

$$(u \otimes x)(y \otimes v) = (-1)^{\alpha\beta} uy \otimes xv$$

where $x \in B^\beta, y \in A^\alpha, u \in A, v \in B$.

**Lemma 2.6.** *As $\mathbb{R}$-algebras, $C_1 \cong \mathbb{C}$ and*

$$C_k \cong \underbrace{\mathbb{C}\hat{\otimes}\cdots\hat{\otimes}\mathbb{C}}_{k \text{ times}}.$$

To prove Lemma 2.6 we need the following lemma which is stated below without proof.

**Lemma 2.7.** *Let $E = E_1 \oplus E_2$ be an orthogonal decomposition of $E$ with respect to $Q$. Let $Q_i = Q|_{E_i}, i = 1, 2$. Then there is an isomorphism*

$$C(Q) \cong C(Q_1)\hat{\otimes}_k C(Q_2).$$

*Proof of Lemma 2.6.* The isomorphism $C_1 \cong \mathbb{C}$ follows from $\varphi : E = \mathbb{R} \to \mathbb{C}, x \mapsto ix$ which extends to $\tilde{\varphi} : C_1 \to \mathbb{C}$ by applying the universal property (2.4) of $C_1$. Observe that $\tilde{\varphi}$ maps generators to generators as an $\mathbb{R}$-vector space homomorphism; hence it is an isomorphism. $C_k \cong C_1 \hat{\otimes}\cdots\hat{\otimes}C_1$ follows by applying Lemma 2.7 repeatedly. □

2.1.3. *Computing $C_k$.* This section focuses to compute the algebras $C_k$ (Table 1). This is important because it lays a foundation to our calculations of the Clifford modules in Section 2.2.

In the following, let $F = \mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$ and write $F(n)$ for the $F$-algebra of $n \times n$ matrices over $F$. We will assume the following facts about matrix algebras.

**Facts 2.8.** *(1) $F(n) \cong \mathbb{R}(n) \otimes_{\mathbb{R}} F$;*
*(2) $\mathbb{R}(n) \otimes \mathbb{R}(m) \cong \mathbb{R}(nm)$;*
*(3) $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$;*
*(4) $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}(2)$;*
*(5) $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{R}(4)$.*

We first introduce a new family of Clifford algebras which mirror our $C_k$'s. They are used as an intermediary in the computation.

**Definition 2.9.** $C'_k$ is defined as the universal algebra over $\mathbb{R}$ generated by $\{1, e_1, \cdots, e_k\}$ subject to the relation

$$\begin{cases} e_i^2 = 1 \\ e_i e_j + e_j e_i = 0 \quad i \neq j. \end{cases}$$

Indeed, $C'_k = C(-Q_k)$ is a Clifford algebra.

**Proposition 2.10.** *There are isomorphisms*

$$C_k \otimes_{\mathbb{R}} C'_2 \cong C'_{k+2}, \qquad C'_k \otimes_{\mathbb{R}} C_2 \cong C_{k+2}.$$

*Proof.* We denote the generators of $C_k$ by $\{1, e_1, \cdots e_k\}$ and the generators of $C'_k$ by $\{1, e'_1, \cdots e'_k\}$. To show that $C_k \otimes_{\mathbb{R}} C'_2 \cong C'_{k+2}$, consider the map

$$\psi : \mathbb{R}^{k+2} \to C_k \otimes C'_2$$
$$e'_i \mapsto \begin{cases} e_{i-2} \otimes e'_1 e'_2, & 3 \leq i \leq k+2; \\ 1 \otimes e'_i, & 1 \leq i \leq 2. \end{cases}$$

Since $\psi$ satisfies the universal property (2.4) of $C'_{k+2}$, it can be extended as $\psi : C'_{k+2} \to C_k \otimes C'_2$ and is a vector space isomorphism because it sends basis to basis. Then $\psi$ is an $\mathbb{R}$-algebra isomorphism. $C'_k \otimes_{\mathbb{R}} C_2 \cong C_{k+2}$ follows similarly. □

**Observation 2.11.** (1) $C_1 \cong \mathbb{C}$;
(2) $C_2 \cong \mathbb{H}$;
(3) $C'_1 \cong \mathbb{R} \oplus \mathbb{R}$;
(4) $C'_2 \cong \mathbb{R}(2)$.

*Proof.* (1) and (2) follow from Lemma 2.6. (3) is given by $C'_1 \cong \mathbb{R}\frac{1+e'_1}{2} \oplus \mathbb{R}\frac{1-e'_1}{2}$ and one checks that it is a direct sum of rings. The isomorphism in (4) is determined by $e'_1 \mapsto \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $e'_2 \mapsto \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$. □

Now, using Facts 2.8, Proposition 2.10 and Observation 2.11, an inductive calculation gives us the following table of Clifford algebras.

| $k$ | $C_k$ | $C'_k$ | $C_k \otimes_{\mathbb{R}} \mathbb{C} = C'_k \otimes_{\mathbb{R}} \mathbb{C}$ |
|---|---|---|---|
| 1 | $\mathbb{C}$ | $\mathbb{R} \oplus \mathbb{R}$ | $\mathbb{C} \oplus \mathbb{C}$ |
| 2 | $\mathbb{H}$ | $\mathbb{R}(2)$ | $\mathbb{C}(2)$ |
| 3 | $\mathbb{H} \oplus \mathbb{H}$ | $\mathbb{C}(2)$ | $\mathbb{C}(2) \oplus \mathbb{C}(2)$ |
| 4 | $\mathbb{H}(2)$ | $\mathbb{H}(2)$ | $\mathbb{C}(4)$ |
| 5 | $\mathbb{C}(4)$ | $\mathbb{H}(2) \oplus \mathbb{H}(2)$ | $\mathbb{C}(4) \oplus \mathbb{C}(4)$ |
| 6 | $\mathbb{R}(8)$ | $\mathbb{H}(4)$ | $\mathbb{C}(8)$ |
| 7 | $\mathbb{R}(8) \oplus \mathbb{R}(8)$ | $\mathbb{C}(8)$ | $\mathbb{C}(8) \oplus \mathbb{C}(8)$ |
| 8 | $\mathbb{R}(16)$ | $\mathbb{R}(16)$ | $\mathbb{C}(16)$ |

TABLE 1.

Notice that Proposition 2.10 implies

$$C_{k+4} \cong C'_{k+2} \otimes C_2 \cong C_k \otimes C'_2 \otimes C_2 \cong C_k \otimes C_4$$

and hence

$$C_{k+8} \cong C_k \otimes C_8 \cong C_k \otimes \mathbb{R}(16). \tag{2.12}$$

If $C_k \cong F(n)$, then $C_{k+8} \cong F(16n)$, so $C_k$ and $C_{k+8}$ have the same coefficient ring but of different dimensions. This guarantees the eight-fold periodicity of Clifford modules which will be discussed in detail in Section 2.2. For the complex case, we observe that

$$C_{k+2} \otimes_{\mathbb{R}} \mathbb{C} \cong (C_k \otimes_{\mathbb{R}} \mathbb{C}) \otimes \mathbb{R}(2)$$

leads to a two-fold periodicity.

2.2. **Clifford modules.** We now introduce the left hand side objects in the ABS map, the ring $A_*$, and calculate its ring structure.

In Section 2.2.1 we define the Clifford modules $M(C_k)$ and calculate them (Table 2). Then, in Section 2.2.2, we define $A_k$ as a cokernel of $M(C_k)$, and derive the calculations of $A_k$ based on $M(C_k)$ (Table 3). From the computations of $A_k$, we observe that they coincide with $KO^{-k}(\mathrm{pt})$ and share the same eight-fold periodicity. In Section 2.2.3, we construct the ring structure on $A_* = \sum_k A_k$, and give a computation of the ring $A_*$ (Theorem 2.33, Theorem 2.34).

2.2.1. *$M(C_k)$ and $M^c(C_k)$.*

**Definitions 2.13.** We define $M(C_k)$ to be the free abelian group generated by the isomorphism classes of irreducible $\mathbb{Z}/2$-graded $C_k$-modules. Similarly, we define $N(C_k)$ to be the free abelian group generated by the isomorphism classes of irreducible (ungraded) $C_k$-modules. Usually we call $M(C_k)$ and $N(C_k)$ the *Clifford modules*.

We denote the corresponding free abelian groups with respect to the complex Clifford algebras $C_k \otimes_{\mathbb{R}} \mathbb{C}$ by $M^c(C_k)$ and $N^c(C_k)$ and call them the *complex Clifford modules*.

The reason that we introduce $N(C_k)$ after $M(C_k)$ is because it forgets about the grading structure on $C_k$-modules and hence is easier to compute. Indeed, the computation of $N(C_k)$ is closely connected to the computation of $M(C_k)$ according to the following result. Recall that $C_k = C_k^0 \oplus C_k^1$ is given a $\mathbb{Z}/2$-grading.

**Proposition 2.14.** *(1) Let $M = M^0 \oplus M^1$ be a $\mathbb{Z}/2\mathbb{Z}$-graded $C_k$-module. Naturally, $M^0$ is a $C_k^0$-module. Then there is an isomorphism*

$$\begin{aligned} M(C_k) &\cong N(C_k^0) \\ M = M^0 \oplus M^1 &\mapsto M^0 \\ C_k \otimes_{C_k^0} M &\leftarrow\!\shortmid M. \end{aligned}$$

*For the complex case, we also have $M^c(C_k) \cong N^c(C_k^0)$.* [4, Proposition 5.3]

*(2) Define*

$$\begin{aligned} \phi : \mathbb{R}^k &\to C_{k+1}^0 \\ e_i &\mapsto e_i e_{k+1} \end{aligned}$$

*where $\{e_1, \cdots, e_k\}$ generates $C_k$. Since $\phi$ satisfies the universal property (2.4) of $C_k$, it extends to an isomorphism $\phi : C_k \xrightarrow{\sim} C_{k+1}^0$.* [4, Proposition 5.4]

An important algebraic tool we need is the following result on representations of artinian rings.

**Theorem 2.15** ([16, Theorem 4.4])**.** *Let $R$ be a semi-simple aritinian ring and $R = R_1 \oplus \cdots \oplus R_s$ be the direct sum decomposition of the simple components of $R$. Let $I_i$ be a minimal left ideal in $R_i$. Then $\{I_1, \cdots, I_s\}$ is a set of representatives of the isomorphism classes of irreducible $R$-modules. Moreover, any $R$-module is completely reducible.*

**Definition 2.16.** Let $a_k$ (resp. $a_k^c$) be the $\mathbb{R}$-dimension (resp. $\mathbb{C}$-dimension) of $M^0$ when $M = M^0 \oplus M^1$ is an irreducible $\mathbb{Z}/2\mathbb{Z}$-graded module for $C_k$ (resp. $C_k \otimes_{\mathbb{R}} \mathbb{C}$).

We now start to compute $M(C_k)$ and $M^c(C_k)$. In Table 1, notice that $C_k$ is always isomorphic to the direct sum of rings of finite dimensional matrices over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ which are all division rings. According to the structure theorem for semi-primitive aritinian rings [16, page 203], $C_k$ is semi-simple and aritinian. Then Theorem 2.15 tells us that when $k \neq 3, 7$, $C_k$ has only one (ungraded) irreducible module up to isomorphism, the $n$-tuples of elements in $F$ ($F = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$); and when $k = 3, 7$, $C_k$ has exactly two (ungraded) irreducible modules (up to isomorphism), each inherited from the summand. In summary,

$$N(C_k) \cong \begin{cases} \mathbb{Z}, & k \neq 3, 7 \\ \mathbb{Z} \oplus \mathbb{Z}, & k = 3, 7. \end{cases}$$

On the other hand, using Proposition 2.14 we get

$$M(C_k) \cong N(C_k^0) \cong N(C_{k-1}).$$

Hence,

$$M(C_k) \cong \begin{cases} \mathbb{Z}, & k \neq 4, 8 \\ \mathbb{Z} \oplus \mathbb{Z}, & k = 4, 8. \end{cases}$$

The calculation of $M^c(C_k)$ is similar. Now we are able to write down the following table of $M(C_k)$, $M^c(C_k)$, $a_k$ and $a_k^c$.

| $k$ | $C_k$ | $M(C_k)$ | $a_k$ | $M^c(C_k)$ | $a_k^c$ |
|---|---|---|---|---|---|
| 1 | $\mathbb{C}$ | $\mathbb{Z}$ | 1 | $\mathbb{Z}$ | 1 |
| 2 | $\mathbb{H}$ | $\mathbb{Z}$ | 2 | $\mathbb{Z} \oplus \mathbb{Z}$ | 1 |
| 3 | $\mathbb{H} \oplus \mathbb{H}$ | $\mathbb{Z}$ | 4 | $\mathbb{Z}$ | 2 |
| 4 | $\mathbb{H}(2)$ | $\mathbb{Z} \oplus \mathbb{Z}$ | 4 | $\mathbb{Z} \oplus \mathbb{Z}$ | 2 |
| 5 | $\mathbb{C}(4)$ | $\mathbb{Z}$ | 8 | $\mathbb{Z}$ | 4 |
| 6 | $\mathbb{R}(8)$ | $\mathbb{Z}$ | 8 | $\mathbb{Z} \oplus \mathbb{Z}$ | 4 |
| 7 | $\mathbb{R}(8) \oplus \mathbb{R}(8)$ | $\mathbb{Z}$ | 8 | $\mathbb{Z}$ | 8 |
| 8 | $\mathbb{R}(16)$ | $\mathbb{Z} \oplus \mathbb{Z}$ | 8 | $\mathbb{Z} \oplus \mathbb{Z}$ | 8 |

TABLE 2.

Moreover, in (2.12) we have $C_{k+8} \cong C_k \otimes_{\mathbb{R}} \mathbb{R}(16)$, so

$$\begin{aligned} M(C_{k+8}) &\cong M(C_k) \\ M^c(C_{k+2}) &\cong M^c(C_k), \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} a_{k+8} &= 16a_k \\ a_{k+2}^c &= 2a_k^c. \end{aligned} \tag{2.18}$$

Theorem 2.15 tells us that every $\mathbb{Z}/2$-graded $C_k$-module is completely reducible; hence the Grothendieck group of $\mathbb{Z}/2$-graded $C_k$-modules coincide with $M(C_k)$. For this reason, we may also call $M(C_k)$ the Grothendieck group of $\mathbb{Z}/2$-graded $C_k$-modules.

(2.17) is the eight-fold periodicity (resp. two-fold periodicity) in Clifford modules (resp. complex Clifford modules), guaranteed by (2.12). Now we want to explain this periodicity from the perspective of Morita equivalence.

**Definition 2.19.** We say that two rings $R$ and $S$ are *Morita equivalent* if there exist bimodules ${}_RM_S$ and ${}_SN_R$ such that ${}_RM \otimes_S N_R \simeq {}_RR_R$ and ${}_SN \otimes_R M_S \simeq {}_SS_S$.

For any ring $R$, $R$ is Morita equivalent to $R(n)$ if we consider the bimodules ${}_RR^n_{R(n)}$ and ${}_{R(n)}R^n_R$. In particular, since $C_{k+8} \cong C_k \otimes_{\mathbb{R}} \mathbb{R}(16)$, $C_k$ and $C_{k+8}$ are Morita equivalent.

Moreover, the rings $R$ and $S$ are Morita equivalent if and only if the category of left $R$-modules is equivalent to the category of left $S$-modules [16, Morita I, page 167, and Morita II, page 178], and this implies the group isomorphism between the Grothendieck groups of left $R$-modules and left $S$-modules. Hence, the

Morita equivalence between $C_k$ and $C_{k+8}$ implies the group isomorphism between $M(C_k)$ and $M(C_{k+8})$, and then reveals the eight-fold periodicity. More details about Morita equivalence can be found in [16, Section 3.12].

The isomorphism in (2.17) can be explicitly written down as the multiplication by an isomorphism class of irreducible modules of $C_8$, as will be further explained in Section 2.2.3.

2.2.2. *$A_k$ and $A_k^c$.* Let $i : C_k \to C_{k+1}$ be the inclusion. It induces a group homomorphism

$$i^* : M(C_{k+1}) \to M(C_k).$$

**Definition 2.20.** We define

$$A_k := \operatorname{coker} i^* = M(C_k)/i^* M(C_{k+1}).$$

Similarly, in the complex case we define

$$A_k^c := M^c(C_k)/i^* M^c(C_{k+1}).$$

By abusing notations, we also call $A_k$ and $A_*$ the *Clifford modules*.

Now we want to calculate $A_k$ and $A_k^c$. The calculation of $A_k$ when $k \neq 4n$ ($n \in \mathbb{Z}_{\geq 0}$) (resp. $A_k^c$ when $k \neq 2n$) is simple because there is only one irreducible $\mathbb{Z}/2\mathbb{Z}$-graded module for $C_k$ (resp. $C_k \otimes_{\mathbb{R}} \mathbb{C}$) up to isomorphism. The results are listed in the following table.

| $k$ | $C_k$ | $M(C_k)$ | $A_k$ | $a_k$ | $M^c(C_k)$ | $A_k^c$ | $a_k^c$ |
|---|---|---|---|---|---|---|---|
| 1 | $\mathbb{C}$ | $\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | 1 | $\mathbb{Z}$ | 0 | 1 |
| 2 | $\mathbb{H}$ | $\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | 2 | $\mathbb{Z} \oplus \mathbb{Z}$ | $\mathbb{Z}$ | 1 |
| 3 | $\mathbb{H} \oplus \mathbb{H}$ | $\mathbb{Z}$ | 0 | 4 | $\mathbb{Z}$ | 0 | 2 |
| 4 | $\mathbb{H}(2)$ | $\mathbb{Z} \oplus \mathbb{Z}$ | $\mathbb{Z}$ | 4 | $\mathbb{Z} \oplus \mathbb{Z}$ | $\mathbb{Z}$ | 2 |
| 5 | $\mathbb{C}(4)$ | $\mathbb{Z}$ | 0 | 8 | $\mathbb{Z}$ | 0 | 4 |
| 6 | $\mathbb{R}(8)$ | $\mathbb{Z}$ | 0 | 8 | $\mathbb{Z} \oplus \mathbb{Z}$ | $\mathbb{Z}$ | 4 |
| 7 | $\mathbb{R}(8) \oplus \mathbb{R}(8)$ | $\mathbb{Z}$ | 0 | 8 | $\mathbb{Z}$ | 0 | 8 |
| 8 | $\mathbb{R}(16)$ | $\mathbb{Z} \oplus \mathbb{Z}$ | $\mathbb{Z}$ | 8 | $\mathbb{Z} \oplus \mathbb{Z}$ | $\mathbb{Z}$ | 8 |

TABLE 3.

(2.17) tells us that

$$\begin{aligned} A_{k+8} &\cong A_k \\ A_{k+2}^c &\cong A_k^c. \end{aligned} \tag{2.21}$$

It remains to prove that $A_{4n} \cong \mathbb{Z}$ and $A_{2n}^c \cong \mathbb{Z}$. We will use the following $*$ operation on $M(C_k)$ and $M^c(C_k)$ as an important tool.

**Definition 2.22.** Let $M = M^0 \oplus M^1$ be a $C_k$-module (resp. $C_k \otimes \mathbb{C}$-module) with its $\mathbb{Z}/2\mathbb{Z}$-grading. Define

$$M^* = M^1 \oplus M^0;$$

that is, $M^*$ is the module obtained from $M$ by interchanging its grading decomposition. Then $M^*$ is again a $\mathbb{Z}/2\mathbb{Z}$-graded $C_k$-module (resp. $C_k \otimes \mathbb{C}$-module). This induces an involutive operation on $M(C_k)$ (resp. $M^c(C_k)$), called the $*$ *operation*.

**Proposition 2.23.** *Let $k = 4n$. If $M$ and $N$ are the two non-isomorphic irreducible $C_k$-modules, then*

$$M \cong N^* \quad \text{and} \quad N \cong M^*.$$

*Proof.* First we reduce this problem to ungraded modules. Let $\lambda \in \mathrm{Aut}(C_k)$ and construct a $\mathbb{Z}/2\mathbb{Z}$-graded $C_k$-module $M^\lambda$ from $M$ in the following way: the underlying set of $M^\lambda$ is the same as $M$, but the action of $C_k$ on $M^\lambda$ is defined as

$$x \cdot m = \lambda(x)m, \qquad x \in C_k.$$

In particular, consider the automorphisms

$$\begin{aligned} \alpha : C_{k-1} &\to C_{k-1} \\ e_i &\mapsto -e_i \end{aligned} \qquad\qquad \begin{aligned} \beta : C_k &\to C_k \\ x &\mapsto e_k x e_k^{-1} \end{aligned}$$

where $\{e_1, \cdots, e_k\}$ generates $C_k$. We only stated the action of $\alpha$ on generators; its detailed construction will be later explained in **??**. One checks that $\alpha$ and $\beta$ fit into the commutative diagram

$$\begin{tikzcd} C_{k-1} \arrow[r, "\phi"] \arrow[d, "\alpha"'] & C_k^0 \arrow[d, "\beta"] \\ C_{k-1} \arrow[r, "\phi"'] & C_k^0 \end{tikzcd} \tag{2.24}$$

where $\phi$ is defined in Proposition 2.14 (2). Consider the map

$$\begin{aligned} e_k \cdot - : M^* &\to M^\beta \\ m &\mapsto e_k \cdot m \end{aligned}$$

which is a $\mathbb{Z}/2\mathbb{Z}$-graded $C_k$-module homomorphism. Since multiplication by $e_k$ induces a $C_k$-module isomorphism $M^0 \cong M^1$, $e_k \cdot -$ is an isomorphism. Hence, using (2.24) we have the following commutative diagram

$$\begin{tikzcd} M(C_k) \arrow[r, "\sim"] \arrow[d, "*"'] & N(C_k^0) \arrow[r, "\sim"] \arrow[d, "e_k\cdot -"', "\sim"] & N(C_{k-1}) \arrow[d, "-^\alpha"] \\ M(C_k) \arrow[d, "e_k\cdot -"', "\sim"] & N(C_k^0) \arrow[r, "\sim"] & N(C_{k-1}) \\ M(C_k) \arrow[ur, "\sim"] & & \end{tikzcd}$$

from which we see that the $*$ operation

$$\begin{aligned} * : M(C_k) &\to M(C_k) \\ M &\mapsto M^* \end{aligned}$$

corresponds to

$$\begin{aligned} -^\alpha : N(C_{k-1}) &\to N(C_{k-1}) \\ N &\mapsto N^\alpha. \end{aligned}$$

Therefore we have reduced the problem of the $*$ operation on $M(C_k)$ to the problem of $-^\alpha$ operation on $N(C_{k-1})$.

Then, it suffices to show that the operation $-^\alpha$ switches the two irreducible modules of $C_{k-1}$. From Table 1 we see that when $k = 4n$, $C_{k-1} \cong C_{k-2} \otimes C_{k-2}$ and $C_{k-2}$ is simple. Recall that a ring $R$ can be expressed as a finite direct product of simple rings if and only if $1 \in R$ can be written as a sum of orthogonal centrally primitive idempotents [20, page 327]. Hence we seek the central elements in $C_{4n-1}$.

Indeed, the center of $C_{4n-1}$ is spanned by 1 and $\omega = e_1e_2\cdots e_{4n-1}$. Further, $\omega^2 = 1$. Thus, there are two orthogonal centrally primitive idempotents $(1 \pm \omega)/2$, and $C_{k-1} = C_{k-1}(1+\omega)/2 \oplus C_{k-1}(1-\omega)/2$ is a decomposition of its simple components. By Theorem 2.15,

$$N_1 = C_{k-1}\frac{1+\omega}{2} \quad \text{and} \quad N_2 = C_{k-1}\frac{1-\omega}{2}$$

represent the two non-isomorphic classes of irreducible modules of $C_{k-1}$. Moreover, notice that there is a $C_{k-1}$-module isomorphism $N_i^\alpha \cong \alpha(N_i)$ where $i = 1, 2$. Since $\alpha(\omega) = -\omega$, $\alpha$ switches $N_1$ and $N_2$ and hence and $-^\alpha$ switches $N_1$ and $N_2$. □

**Corollary 2.25.** $A_{4n} \cong \mathbb{Z}$.

*Proof.* Let $x, y$ be the isomorphism classes of the two distinct irreducible modules in $M(C_{4n})$ and $z$ be the class of the irreducible module in $M(C_{4n+1})$. Then $z^* = z$ and Proposition 2.23 tells us that $x^* = y$ and $y^* = x$. Let $i : C_{4n} \to C_{4n+1}$ be the inclusion. Then $(i^*z)^* = i^*z^* = i^*z$. Counting the dimension, the only possibility is $i^*z = x + y$. Hence

$$A_{4n} = M(C_{4n})/i^*M(C_{4n+1}) = \mathbb{Z}x \oplus \mathbb{Z}y/\mathbb{Z}(x+y) \cong \mathbb{Z}.$$

□

The proof of $A^c_{2n} \cong \mathbb{Z}$ follows similarly.

2.2.3. *The ring structure of Clifford modules.*

**Notation 2.26.** We write $M_* = \bigoplus_{k=0}^\infty M(C_k)$ and $A_* = \bigoplus_{k=0}^\infty A_k$; $M^c_*$ and $A^c_*$ are the corresponding complex versions.

In this section we will define the ring structures of $M_*$ and $A_*$, and their complex versions. Moreover, we will give an explicit calculation of the rings $A_*$ and $A^c_*$ in Theorem 2.33 and Theorem 2.34 respectively, and use the calculation to describe the complexification map in Proposition 2.35.

In Definition 2.5 we have introduced the graded tensor product of two graded algebras. Similarly, we give the following definition of the graded tensor product of two graded modules.

**Definition 2.27.** Let $M, N$ be $\mathbb{Z}/2\mathbb{Z}$-graded modules of $C_k$ and $C_l$ respectively. We define $M\hat{\otimes}N$, called the *graded tensor product* of $M$ and $N$, as a $C_k\hat{\otimes}C_l$-module in the following way:

(1) The underlying group has the grading

$$(M\hat{\otimes}N)^0 = M^0 \otimes N^0 \oplus M^1 \otimes N^1,$$
$$(M\hat{\otimes}N)^1 = M^1 \otimes N^0 \oplus M^0 \otimes N^1;$$

(2) The action of $C_k\hat{\otimes}C_l$ on $M\hat{\otimes}N$ is given by

$$(x \otimes y) \cdot (m \otimes n) = (-1)^{qi}(x \cdot m) \otimes (y \cdot n), \qquad y \in C^q_l, m \in M^i \quad (q, i = 0, 1).$$

Now we define the ring structure on $M_*$. Consider the isomorphism

$$\phi_{k,l} : C_{k+l} \to C_k\hat{\otimes}C_l \tag{2.28}$$

which is defined as the extension of the map

$$e_i \mapsto \begin{cases} e_i \otimes 1 & 1 \le i \le k, \\ 1 \otimes e_{i-k} & k+1 \le i \le k+l \end{cases}$$

where $\{e_1, \cdots, e_{k+l}\}$ generates $C_{k+l}$. This induces the following composition

$$\begin{aligned} M(C_k) \times M(C_l) &\to M(C_k \hat{\otimes} C_l) \to M(C_{k+l}) \\ (M, N) &\mapsto \quad M \hat{\otimes} N \quad \mapsto \phi^*_{k,l}(M \hat{\otimes} N) \end{aligned}$$

which is bilinear. Hence, we have the pairing

$$M(C_k) \otimes M(C_l) \to M(C_{k+l}).$$

which induces the $\mathbb{Z}$-graded ring structure over $M_*$. We denote the product by $(u, v) \mapsto u \cdot v$ for $u \in M(C_k), v \in M(C_l)$. The following proposition summarizes the basic properties of this ring structure.

**Proposition 2.29** ([4, Proposition 6.2])**.** *Let $u \in M(C_k), v \in M(C_l)$. We have the following properties:*

*(1) the multiplication $u \cdot v$ is associative;*
*(2) $(u \cdot v)^* = u \cdot v^*$;*
*(3)* $u \cdot v = \begin{cases} v \cdot u, & \text{if } kl \text{ is even,} \\ (v \cdot u)^*, & \text{if } kl \text{ is odd;} \end{cases}$
*(4) If $i : C_{k-1} \to C_k$ is the inclusion, then the induced map*

$$i^* : M(C_k) \to M(C_{k-1})$$

*is the restriction homomorphism, and*

$$u \cdot i^* v = i^*(u \cdot v)$$

*for $k \geq 1$.*

In particular, from Proposition 2.29 (4) we know that $\operatorname{im} i^*$ is an ideal of $M_*$. Hence $A_* = \operatorname{coker} i^*$ naturally inherits the ring structure from $M_*$.

Next we show that the eight-fold periodicity in (2.17) can be realized by multiplying an isomorphism class of irreducible graded modules of $C_8$.

**Corollary 2.30.** *Let $\lambda \in M(C_8)$ be one isomorphism class of the irreducible graded modules of $C_8$. Then multiplication by $\lambda$ induces an isomorphism*

$$M(C_k) \xrightarrow{\sim} M(C_{k+8})$$

*and*

$$A_k \xrightarrow{\sim} A_{k+8},$$

*where $k \geq 0$.*

*Proof.* We first show that multiplying by $\lambda$ induces isomorphism $M(C_k) \xrightarrow{\sim} M(C_{k+8})$.

When $k \neq 4n$, let $x$ be the class of the irreducible module of $C_k$. Then

$$\dim(\lambda \cdot x)^0 = \dim \lambda^0 \cdot \dim x^0 + \dim \lambda^1 \cdot \dim x^1 = 2a_8 \cdot a_k = 16a_k = a_{k+8}$$

where the last equality comes from (2.18). Therefore, $\lambda \cdot x$ corresponds to the class of the irreducible module of $C_{k+8}$, so $\lambda \cdot - : M(C_k) \xrightarrow{\sim} M(C_{k+8})$ is an isomorphism.

When $k = 4n$, let $x$ and $y$ be classes of the two irreducible graded modules of $C_k$. Proposition 2.23 tells us that $x^* = y$. Again, by comparing dimensions, $\lambda \cdot x$ corresponds to one class of the irreducible graded modules of $C_{k+8}$. Since

$$\lambda \cdot y = \lambda \cdot x^* = (\lambda \cdot x)^*,$$

where the last equality comes from Proposition 2.29, (2), $\lambda \cdot y$ corresponds to the other class of irreducible graded modules of $C_{k+8}$. Hence $\lambda \cdot - : M(C_k) \xrightarrow{\sim} M(C_{k+8})$ is an isomorphism.

The isomorphism $A_k \xrightarrow{\sim} A_{k+8}$ given by multiplication by $\lambda$ follows directly from $M_*$. □

When $k = 4n$, recall that in the proof of Proposition 2.23 we showed the two generators of $N(C_{k-1})$ are represented by

$$N_1 = C_{k-1}\frac{1+\omega}{2} \quad \text{and} \quad N_2 = C_{k-1}\frac{1-\omega}{2}.$$

Using Proposition 2.14, let $M_1$ and $M_2$ be two irreducible graded $C_k$-modules representing the two generators of $M(C_k)$. Then,

$$M_1^0 \cong C_k^0\frac{1+\omega}{2} \quad \text{and} \quad M_2^0 = C_k^0\frac{1-\omega}{2}.$$

Observe that $\omega = e_1 \cdots e_{4n}$ acts as 1 on $M_1^0$ and acts as $-1$ on $M_2^0$ since $\omega^2 = 1$. Hence, the behavior of $\omega$ on the zero-degree of the irreducible graded $C_k$-modules distinguishes the two generators.

**Definition 2.31.** Let $M$ be a $\mathbb{Z}/2\mathbb{Z}$-graded module of $C_{4n}$. For any $\varepsilon \in \mathbb{Z}$, we say that $M$ is an *$\varepsilon$-module* if $\omega$ acts as $\varepsilon$ on $M^0$. This definition also applies for the graded $C_{2n} \otimes_{\mathbb{R}} \mathbb{C}$-modules.

**Lemma 2.32.** *(1) If $M$ is an $\varepsilon$-module for $C_{4n}$, then $M^*$ is a $(-\varepsilon)$-module;*
*(2) If $M$ is an $\varepsilon$-module and $M'$ an $\varepsilon'$-module for $C_{4n}$, then $M\hat{\otimes}M'$ is an $\varepsilon\varepsilon'$-module for $C_{8n}$.*

*Proof.* Notice that every element in $M^1$ can be expressed in the form $e_i x$ where $e_i$ is a generator of $C_k$ and $x \in M^0$. Since $\omega e_i x = -e_i \omega x = -e_i(\varepsilon x) = -\varepsilon(e_i x)$, $\omega$ acts as $-\varepsilon$ on $M^1$; hence $M^*$ is a $(-\varepsilon)$-module. The second part follows directly from (1). □

Now we compute the ring structure of $A_*$.

**Theorem 2.33.** *$A_*$ is the graded-commutative ring generated by a unit $1 \in A_0, \xi \in A_1, \mu \in A_4, \lambda \in A_8$ with relations:*

$$\begin{aligned} 2\xi &= 0 \\ \xi^3 &= 0 \\ \xi\mu &= 0 \\ \mu^2 &= 4\lambda. \end{aligned}$$

*Proof.* $2\xi = 0$ because $A_1 \cong \mathbb{Z}/2$. Since $\dim(\xi^2)^0 = \dim \xi^0 \cdot \dim \xi^0 + \dim \xi^1 \cdot \dim \xi^1 = 2$, $\xi^2$ is nonzero; hence it generates $A_2$. $\xi^3 = 0$ and $\xi\mu = 0$ because the groups they lie in are both zero. Hence it remains to show that $\mu^2 = 4\lambda$.

Let $\mu$ be the generator of $A_4$ and $\lambda$ be the generator of $A_8$. We know that $\mu^2 = a\lambda$ for some integer $a$. To determine the coefficient $a$, we need to go back to $M(C_*)$ where the ring structure is explicitly defined and work on particular irreducible graded $C_*$-modules. Let $M$ be an irreducible 1-module for $C_4$. Consider $\omega = e_1e_2e_3e_4$. By Lemma 2.32 (2), $M\hat{\otimes}M$ is an 1-module for $C_8$. Now let $W$ be an irreducible 1-module for $C_8$. Then we know that $M\hat{\otimes}M \cong aW$. Since $\dim M\hat{\otimes}M = 64$ and $\dim W = 16$, it follows that $a = 4$. □

A parallel result holds for the complex case where two generators of $M^c(C_{2l})$ correspond to $\pm i^l$-modules respectively.

**Theorem 2.34.** *There is a ring isomorphism*

$$A_*^c \cong \mathbb{Z}[\mu^c].$$

Now, knowing the ring structures of both $A_*$ and $A_*^c$, we will study the relation between these two rings, as summarized by the following result.

**Proposition 2.35.** *There is a ring homomorphism*

$$c : A_* \to A_*^c$$

*given by the complexification* $M \mapsto M \otimes_{\mathbb{R}} \mathbb{C}$*, under which* $\xi \mapsto 0, \mu \mapsto 2(\mu^c)^2$ *and* $\lambda \mapsto (\mu^c)^4$*.*

*Proof.* $c(\xi) = 0$ because there is no nonzero 2-torsion element in $A_1^c$. Again, to determine the coefficients $a, b$ where $c(\mu) = a(\mu^c)^2$ and $c(\lambda) = b(\mu^c)^4$, we need to go back to $M(C_*)$ and $M^c(C_*)$. Let $M$ be an irreducible 1-module for $C_4$. Then $M \otimes_{\mathbb{R}} \mathbb{C}$ is a complex $(-1)$-module for $C_4 \otimes_{\mathbb{R}} \mathbb{C}$. Let $N$ be an $i$-module for $C_2 \otimes_{\mathbb{R}} \mathbb{C}$. Hence $M \otimes_{\mathbb{R}} \mathbb{C} = aN\hat{\otimes}N$. Since $\dim M \otimes_{\mathbb{R}} \mathbb{C} = 2a_4 = 8$ and $\dim(N\hat{\otimes}N) = 2a_4^c = 4$, we see that $a = 2$. Then, $c(\lambda) = c(\mu^2/4) = (\mu^c)^4$. □

## 3. The difference bundle construction

This section focuses to introduce the difference bundle construction which is the last step in the construction of the ABS map. The result applies equally to real and complex vector bundles, and we will just refer to vector bundles.

We first introduce our main objects, the sequences of bundles over $(X, Y)$ for $Y$ a subcomplex of a based finite CW complex $X$ (Section 3, Definition 3.3) in Section 3.1. Then, in Section 3.2, we will introduce the difference bundle construction (Theorem 3.5) which connects $L_n(X, Y)$ with topological $K$-theory. In Section 3.3, we will construct a product structure over $L_n(X, Y)$ (Proposition 3.12) and give an explicit computation of the product of two elements (Proposition 3.16). In these, Many proofs of the results are omitted in this section for clarity and those who are interested can find the detailed proofs in Appendix B.

**Convention 3.1.** In this section, all base spaces of the vector bundles are assumed to be based finite CW complexes.

### 3.1. Sequences of bundles.

**Definitions 3.2.** Let $Y \subseteq X$ be based finite CW complexes. For each positive integer $n$,

- consider $\mathscr{C}_n(X, Y)$ consisting of sequences
$$E = (0 \to E_n \xrightarrow{\sigma_n} E_{n-1} \xrightarrow{\sigma_{n-1}} \cdots \to E_1 \xrightarrow{\sigma_1} E_0 \to 0)$$
where the $E_i$'s are vector bundles over $X$, $\sigma_i$'s are homomorphisms of vector bundles defined over $Y$, and the sequence is exact over $Y$. There is a natural levelwise direct sum operation on the sequences $E$ which endows $\mathscr{C}_n(X, Y)$ with an abelian semi-group structure;
- a *homomorphism* $E \to E'$ in $\mathscr{C}_n(X, Y)$ consists of levelwise isomorphisms of vector bundles $E_i \to E_i'$ on $X$ such that the squares are commutative over $Y$;
- an *elementary sequence* in $\mathscr{C}_n(X, Y)$ is a sequence of the form $0 \to E_i \xrightarrow{\mathrm{id}} E_i \to 0$; that is, $E_i = E_{i-1}$ with $\sigma_i = \mathrm{id}$ for some $i$, and $E_j = 0$ for $j \neq i, i-1$.

The elementary sequences determine an equivalence relation $\sim$ in $\mathscr{C}_n(X, Y)$ in the following sense: $E \sim E'$ if there exist elementary sequences $P^i, Q^j \in \mathscr{C}_n(X, Y)$ such that $E \oplus P^1 \oplus \cdots \oplus P^r \cong E' \oplus Q^1 \oplus \cdots \oplus Q^s$. In other words, we regard sequences $E$ and $E'$ to be equivalent if they are isomorphic after taking direct sums with elementary sequences.

**Definition 3.3.** Define $L_n(X, Y) = \mathscr{C}_n(X, Y)/\sim$ where $\sim$ is the equivalence relation defined above. $L_n(X, Y)$ inherits the abelian semi-group structure from $\mathscr{C}_n(X, Y)$ given by direct sums. If $Y = \varnothing$, we write $L_n(X) = L_n(X, \varnothing)$.

$L_n(X,Y)$ are the main objects that we work with for the difference bundle construction. Observe that $L_n(X,Y)$ defines a functor

$$L_n : \text{The category of pairs of based finite CW complexes} \to \mathbf{SemiAb}.$$

Consider the inclusion map

$$\mathscr{C}_n(X,Y) \hookrightarrow \mathscr{C}_{n+1}(X,Y)$$
$$E = (0 \to E_n \to \cdots \to E_0 \to 0) \mapsto (0 \to 0 \to E_n \to \cdots \to E_0 \to 0);$$

the image of $E$ in $\mathscr{C}_{n+1}(X,Y)$ is obtained by adding $E_{n+1} = 0$. This gives us a sequence of inclusions

$$\mathscr{C}_1(X,Y) \hookrightarrow \mathscr{C}_2(X,Y) \hookrightarrow \cdots \hookrightarrow \mathscr{C}_n(X,Y) \hookrightarrow \cdots$$

which passes to

$$L_1(X,Y) \to L_2(X,Y) \to \cdots \to L_n(X,Y) \to \cdots .$$

These inclusions are abelian semi-group homomorphisms.

**Proposition 3.4.** *The abelian semi-group homomorphisms $j_n : L_1(X,Y) \to L_n(X,Y)$ are isomorphisms for $1 \le n < \infty$.*

Now, since all the $L_n(X,Y)$'s are isomorphic, why should we define so many of them instead of just focusing on a single one, for example, $L_1(X,Y)$? Indeed, when we involve $\mathscr{C}_n(X,Y)$'s and $L_n(X,Y)$'s in our construction of the ABS map, we are just using the case when $n = 1$. However, the higher-degree semi-groups make it easier to describe the *product structure*

$$L_n(X,Y) \otimes L_m(X',Y') \to L_{n+m}(X \times X', X \times Y' \cup Y \times X')$$

which will be discussed in Section 3.3, which allows the ABS map to naturally become a group homomorphism. One can definitely use the isomorphism $L_n(X,Y) \cong L_1(X,Y)$ to pass this product structure to a multiplication on $L_1(X,Y)$, but this is not computationally friendly because the extension $\sigma'_{n+1} : E_{n+1} \to E_n$, though has been proved to exist, is not constructed explicitly.

3.2. **Euler characteristics.** Following is our main result for the difference bundle construction.

**Theorem 3.5.** *There exists a unique natural transformation between the functor $L_1$ and the (complex or real) $K$-theory $\chi : L_1 \to K$ which for $Y = \varnothing$ is given by $\chi(E) = E_0 - E_1$. Moreover, $\chi$ is an isomorphism.*

Here we abused the notation of $K$ to denote either complex or real $K$-theory, depending on whether we are considering complex or real vector bundles for $\mathscr{C}_n$ and $L_n$. For any integer $1 \le n < \infty$, we can use the isomorphism $j_n : L_1 \xrightarrow{\sim} L_n$ to define

$$\chi : L_n \xrightarrow[\sim]{j_n^{-1}} L_1 \xrightarrow{\chi} K$$

which we still denote as $\chi$, and it is an isomorphism as well. Moreover, from our construction of $j_n^{-1} : L_n \to L_1$ in the proof of Proposition 3.4, we know that when $Y = \varnothing$, $\chi(E) = \sum_{i=0}^{n} (-1)^i E_i$.

**Definition 3.6.** We call the map $\chi$ *the Euler characteristic.*

Notice that $\chi(E) = E_0 - E_1$ for $E \in L_1(X,\varnothing)$ coincides with the geometric description of $K(X)$ that every element can be written as a difference of two vector bundles over $X$. This shows how $L_n(X,Y)$ can be interpreted as an alternative description of the relative $K$-group $K(X,Y)$. However, the product structure on $L_n(X,Y)$ (which will be introduced in Section 3.3) is easier to compute than $K(X,Y)$.

We will only introduce the construction of the Euler characteristic $\chi : L_1 \to K$ here. The proof that such Euler characteristic is unique and a natural isomorphism is contained in Section B.2. Consider a pair of CW complexes $(X, Y)$. Let $X_i := X \times \{i\}$, where $i = 0, 1$. We first construct a new space $A$ from $(X, Y)$ by gluing $X_0$ and $X_1$ along $Y$

$$A = X_0 \cup_Y X_1 = X_0 \sqcup X_1/(y, 0) \sim (y, 1), y \in Y.$$

Then we have the natural retractions

$$\pi_i : A \to X_i, \quad (a, i) \mapsto a$$

for $i = 0, 1$ which induces a map on $K$-groups $\pi_i^* : K(X_i) \to K(A)$. Recall that in $K$-theory we have the following exact sequence

$$\begin{tikzcd} K^{n-1}(A) \arrow[r, "j_i^*"] & K^{n-1}(X_i) \arrow[l, bend right, "\pi_i^*"'] \arrow[r] & K^0(A, X_i) \arrow[r, "\rho_i^*"] & K^0(A) \arrow[r, "j_i^*"] & K^0(X_i) \arrow[l, bend right, "\pi_i^*"'] \arrow[r] & K^1(A, X_i) \end{tikzcd}$$

where $n = 2$ for real $K$-theory and $n = 8$ for complex $K$-theory. Since we have $j_i^* \circ \pi_i^* = \mathrm{id}$, $j_i^*$ is naturally surjective. Hence, by exactness, the maps $K^0(X_i) \to K^1(A, X_i)$ and $K^{n-1}(X_i) \to K^0(A, X_i)$ are zero maps, so we get a split exact sequence

$$\begin{tikzcd} 0 \arrow[r] & K(A, X_i) \arrow[r, "\rho_i^*"] & K(A) \arrow[r, "j_i^*"] & K(X_i) \arrow[l, bend right, "\pi_i^*"'] \arrow[r] & 0. \end{tikzcd}$$

Also, if we identify $X$ with $X_i$, we have natural inclusions $\phi_i : (X, Y) \to (A, X_{i+1})$ where we regard $i \in \mathbb{Z}/2\mathbb{Z}$. This induces an isomorphism on $K$-groups $\phi_i^* : K(A, X_{i+1}) \xrightarrow{\sim} K(X, Y)$ because $A/X_{i+1} \simeq X/Y$.

Now, consider a sequence $E = (0 \to E_1 \xrightarrow{\sigma} E_0 \to 0) \in \mathscr{C}_1(X, Y)$ where by definition $\sigma : E_1 \to E_0$ is an isomorphism of vector bundles over $Y$. Construct a vector bundle $F \to A$ by

$$F := E_0 \sqcup E_1/(e_1, y) \sim (\sigma(e_1), y), \quad y \in Y, e_1 \in E_1|_{\{y\}}.$$

Then the isomorphism class of $F$ depends only on the isomorphism class of $E$ in $\mathscr{C}_1(X, Y)$; that is, if there exists another vector bundle $F' \to A$ which is constructed from $E' = (0 \to E_1' \xrightarrow{\sigma'} E_0' \to 0)$ such that $F \cong F'$ as vector bundles over $A$, the restriction of the isomorphism over $X_0$ and $X_1$ gives rise to the isomorphism $E \cong E'$, and vice versa.

Let $F_i := \pi_i^*(E_i) \in K(A)$. By definition, we have $F|_{X_i} = F_i|_{X_i} = E_i$, which implies that $(F - F_i)|_{X_i} = 0$ where the minus sign means taking fiberwise orthogonal complement. Hence $F - F_1 \in \ker j_i^* = \operatorname{im} \rho_1^*$. $\rho_1^*$ being injective ensures that $\rho_1^{*-1}(F - F_i)$ is well-defined, so we may define

$$\chi(E) = (\phi_0^* \circ \rho_1^{*-1})(F - F_1) \in K(X, Y).$$

This gives us a map

$$\begin{aligned} \chi : \mathscr{C}_1(X, Y) &\to K(X, Y) \\ E &\mapsto (\phi_0^* \circ \rho_1^{*-1})(F - F_1). \end{aligned} \tag{3.7}$$

Observe that $\chi$ is additive, i.e. $\chi(E \oplus E') \cong \chi(E) \oplus \chi(E')$. Moreover, if $E$ is an elementary sequence, we have $\chi(E) = 0$. Indeed, $E$ being elementary implies that $E_1 = E_0$ and $\sigma = \mathrm{id}$. Since $F_1 = \pi_1^*(E_1)$, we have $F = F_1$ and $F - F_1 = 0$. Notice that $\rho_1^*$ is injective and $\phi_0^*$ is bijective, so $\chi(E) = 0$. Therefore, (3.7) factors through $L_1(X, Y)$:

$$\chi : L_1(X, Y) \to K(X, Y).$$

Our construction of $\chi$ is natural.

The last step is to check what $\chi$ looks like when $Y = \varnothing$. Indeed, we have

$$\begin{aligned} A &= X_0 \sqcup X_1 \\ F &= E_0 \times \{0\} \sqcup E_1 \times \{1\} \\ F_i &= E_i \times \{0\} \sqcup E_i \times \{1\} \\ F - F_1 &= (E_0 - E_1) \times \{0\} \sqcup \{0\} \times \{1\} \end{aligned}$$

so $\chi(E) = (\phi_0^* \circ \rho_1^{*-1})(F - F_1) = E_0 - E_1$, as we expected.

3.3. **Products.** Now we aim to construct the product structure for $L_n$'s and prove the multiplicative property of the Euler characteristic under this product structure (Proposition 3.12). Although we remark at the end of Section 3.1 that it is difficult to use the isomorphism $L_1 \cong L_n$ to compute the product of $E, E' \in L_1(X, Y)$ in $L_1(X, Y)$, we still find an element $F \in L_1(X, Y)$ such that $\chi(F) = \chi(EE') = \chi(E)\chi(E')$ (Proposition 3.16).

**Definition 3.8.** We denote $\mathscr{D}_n(X, Y)$ as the set of sequences $E = (0 \to E_n \xrightarrow{\sigma_n} E_{n-1} \xrightarrow{\sigma_{n-1}} \cdots \to E_1 \xrightarrow{\sigma_1} E_0 \to 0)$ where the $E_i$'s are vector bundles over $X$, $\sigma_i$'s are homomorphisms of vector bundles defined over $X$, $\sigma_{i-1}\sigma_i = 0$, and the sequence is exact over $Y$.

By restricting the homomorphisms $\sigma_i$ to $Y$, we get a sequence in $\mathscr{C}_n(X, Y)$. This gives us an inclusion map $\varphi : \mathscr{D}_n(X, Y) \to \mathscr{C}_n(X, Y)$.

**Definition 3.9.** Two sequences $E, E' \in \mathscr{C}_n(X, Y)$ (resp. $\mathscr{D}_n(X, Y)$) are *homotopic* if there exists an element $F \in \mathscr{C}_n(X \times I, Y \times I)$ (resp. $\mathscr{D}_n(X \times I, Y \times I)$) such that $E \cong F|_{X \times \{0\}}$ and $E' \cong F|_{X \times \{1\}}$.

The homotopy relation behaves well under our natural quotient map $\mathscr{C}_n(X, Y) \to L_n(X, Y)$ in the following sense.

**Proposition 3.10.** *Homotopic elements in $\mathscr{C}_n(X, Y)$ define the same elements in $L_n(X, Y)$. Hence, we have a well-defined induced homomorphism $\mathscr{C}_n(X, Y)/\simeq \to L_n(X, Y)$ where $\simeq$ is the homotopy relation.*

*Proof.* This follows from the isomorphism $\chi : L_n(X, Y) \xrightarrow{\sim} K(X, Y)$ in Theorem 3.5 by noticing that the $K$-theory $K(X, Y)$ is homotopy-invariant. □

**Proposition 3.11.** *The induced map*

$$\varphi : \mathscr{D}_n(X, Y)/\simeq \to \mathscr{C}_n(X, Y)/\simeq,$$

*where $\simeq$ is the homotopy relation, is a bijection.*

The main result of the product structure on $L_n(X, Y)$ is summarized as follows.

**Proposition 3.12** (Douady)**.** *There is a natural product*

$$L_n(X, Y) \otimes L_m(X', Y') \to L_{n+m}(X \times X', X \times Y' \cup Y \times X')$$

*which is induced by the tensor product of vector bundles; we also denote this product as $\otimes$. Moreover, the Euler characteristic is multiplicative with respect to this product, i.e. $\chi(E \otimes E') = \chi(E)\chi(E')$.*

**Notation 3.13.** Let $V$ and $W$ be vector bundles (resp. principal $G$-bundles) over based spaces $X$ and $Y$ respectively. Let $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ be the projections to the first and second coordinate. We denote

$$\begin{aligned} V \boxplus W &= p_1^* V \oplus p_2^* W, \\ V \boxtimes W &= p_1^* V \otimes p_2^* W \end{aligned}$$

as the direct sum and tensor product operations for vector bundles (resp. principal $G$-bundles) over two different base spaces. Then, $V \boxplus W$ and $V \boxtimes W$ become vector bundles (resp. principal $G$-bundles) over $X \times Y$.

For homomorphisms $f : V \to V'$ and $g : W \to W'$, the induced homomorphisms are denoted as

$$f \boxplus g = p_1^* f \oplus p_2^* g : V \boxplus V' \to W \boxplus W',$$
$$f \boxtimes g = p_1^* f \otimes p_2^* g : V \boxtimes V' \to W \boxtimes W'.$$

*Proof of Proposition 3.12.* We first define the product structure over $\mathscr{D}_n$:

$$\mathscr{D}_n(X,Y) \otimes \mathscr{D}_m(X',Y') \to \mathscr{D}_{n+m}(X \times X', X \times Y' \cup Y \times X'). \tag{3.14}$$

Given $E \in \mathscr{D}_n(X,Y), F \in \mathscr{D}_m(X',Y')$ such that

$$E = (0 \to E_n \xrightarrow{\sigma_n} \cdots \xrightarrow{\sigma_1} E_0 \to 0),$$
$$F = (0 \to F_m \xrightarrow{\tau_m} \cdots \xrightarrow{\tau_1} F_0 \to 0),$$

we define the product $E \otimes F$ by

$$(E \otimes F)_i = \sum_{k=0}^{i} E_k \boxtimes F_{i-k} \tag{3.15}$$

with the vector bundle homomorphisms defined as

$$\sigma_i : (E_0 \boxtimes F_i) \oplus \cdots \oplus (E_i \boxtimes F_0) \to (E_0 \boxtimes F_{i-1}) \oplus \cdots \oplus (E_{i-1} \boxtimes F_0)$$
$$(a_{0,i}, a_{1,i-1}, \cdots, a_{i,0}) \mapsto ((\mathrm{id} \boxtimes \tau_i)(a_{0,i}) + (\sigma_1 \boxtimes \mathrm{id})(a_{1,i-1}), \cdots).$$

This gives an element in $\mathscr{D}_{n+m}(X \times X', X \times Y' \cup Y \times X')$; further, it is additive and compatible with homotopies. Hence, using Proposition 3.10 and Proposition 3.11, this product is passed to to $L_n$:

$$L_n(X,Y) \otimes L_m(X',Y') \to L_{n+m}(X \times X', X \times Y' \cup Y \times X').$$

The multiplicative property of Euler characteristic with respect to this product can be rephrased by the commutativity of the following diagram

$$\begin{tikzcd}
L_n(X,Y) \otimes L_n(X',Y') \arrow[r] \arrow[d, "\chi\otimes\chi"'] & L_{n+m}(X \times X', X \times Y' \cup Y \times X') \arrow[d, "\chi"] \\
K(X,Y) \otimes K(X',Y') \arrow[r] & K(X \times X', X \times Y' \cup Y \times X')
\end{tikzcd}$$

where the bottom horizontal arrow is the external product for relative $K$-theory.

For $Y = Y' = \varnothing$, since $\chi(E) = \sum_i (-1)^i E_i$, we can see from (3.15) that $\chi(E \otimes E') = \chi(E)\chi(E')$. The general case follows from the naturality of $\chi$. □

The following computation will become a key process in showing that $\chi_V$ is multiplicative (Proposition 4.13) so that the ABS map is a group homomorphism.

**Proposition 3.16.** *Let $E = (0 \to E_1 \xrightarrow{\sigma} E_0 \to 0) \in \mathscr{D}_1(X,Y)$ and $E' = (0 \to E_1' \xrightarrow{\sigma'} E_0' \to 0) \in \mathscr{D}_1(X',Y')$ where all the bundles are equipped with metrics. Let $F = (0 \to F_1 \xrightarrow{\tau} F_0 \to 0) \in \mathscr{D}_1(X \times X', X \times Y' \cup Y \times X')$ be defined by*

$$F_1 = E_0 \boxtimes E_1' \oplus E_1 \boxtimes E_0'$$
$$F_0 = E_0 \boxtimes E_0' \oplus E_1 \boxtimes E_1'$$
$$\tau = \begin{pmatrix} 1 \boxtimes \sigma' & \sigma \boxtimes 1 \\ \sigma^* \boxtimes 1 & -1 \boxtimes \sigma'^* \end{pmatrix}$$

*where $\sigma^*$ and $\sigma'^*$ denote the adjoints of $\sigma$ and $\sigma'$ respectively. Then, $\chi(F) = \chi(E)\chi(E')$.*

## 4. The main theorem

The main goal for this section is to construct the Atiyah-Bott-Shapiro map and and prove that it becomes a ring isomorphism when restricting to a point. For a based finite CW complex $X$, we will construct two maps

$$\chi_V : A(V) \to \widetilde{KO}(T(V)),$$

and

$$\beta_P : A_k \to A(V),$$

where $P$ is a principal $\mathrm{Spin}(k)$-bundle over $X$, $V$ is a vector bundle over $X$ associated with $P$, $T(V)$ is the Thom space of $V$, and $A(V)$ is the bundle analogy of the Clifford module $A_k$ (Definitions 4.5), in Section 4.3 and Section 4.4 respectively. Their composition gives the ABS map

$$\alpha_P : A_k \xrightarrow{\beta_P} A(V) \xrightarrow{\chi_V} \widetilde{KO}(T(V))$$

which gives rise to the desired group isomorphism $\alpha : A_k \to \widetilde{KO}(S^k)$ for each dimension $k$. This extends to the ABS isomorphism (Theorem 4.23)

$$\alpha : A_* \to \bigoplus_{k\geq 0} KO^{-k}(\mathrm{pt})$$

which is a ring isomorphism. Similarly, we have the corresponding complex ring isomorphism

$$\alpha^c : A^c_* \to \bigoplus_{k\geq 0} K^{-k}(\mathrm{pt}).$$

4.1. **A prelude: Pin groups and Spin groups.** Before we start our constructions of $\chi_V$ and $\beta_P$, we first introduce Pin groups and Spin groups which are Lie groups defined from Clifford algebras. We will encounter them in the averaging argument (4.14) when we prove the multiplicative property of $\chi_V$ and in the principal $\mathrm{Spin}(k)$-bundle $P$ when we construct $\beta_P$. Recall that we use $k$ to denote a field, $E$ a $k$-vector space and $Q$ a quadratic form over $E$.

**Notations 4.1.**
- Recall that we use $k$ to denote a field, $E$ a $k$-vector space and $Q$ a quadratic form over $E$.
- Let $R$ be an arbitrary ring. We write $R^\times$ for the set of units in $R$.

We first introduce some operations over general Clifford algebras $C(Q)$ which will later be used to define Clifford groups. Consider

$$-^t : T^k(E) \to T^k(E), \quad x_1 \otimes \cdots \otimes x_k \mapsto x_k \otimes \cdots \otimes x_1$$

which factors through $C(Q)$ because the ideal $(x \otimes x - Q(x) \mid x \in E)$ is invariant under this map, so we get $-^t : C(Q) \to C(Q)$ and it is identity on $E$. Also, we have

$$\alpha : E \to C(Q), \quad x \mapsto -i_Q(x)$$

where $i_Q : E \to C(Q)$ is the embedding. Since $\alpha$ satisfies the universal property (2.4) of $C(Q)$, it extends to a homomorphism $\alpha : C(Q) \to C(Q)$. Composing these two maps together gives us

$$\bar{\cdot} : C(Q) \to C(Q), \quad x \mapsto \bar{x} = \alpha(x^t)$$

which defines a *norm map* on $C_k$

$$N : C_k \to C_k, \quad x \mapsto x \cdot \bar{x}.$$

This definition is compatible with the norm on $\mathbb{R}^n$ in the sense that for $x \in \mathbb{R}^k$, $N(x) = -x^2 = -Q_k(x) = \sum_{i=1}^k x_i^2$.

**Definitions 4.2.** For $k \geq 1$, we define

- $\Gamma_k = \{x \in C_k^\times \mid \text{for any } y \in \mathbb{R}^k, \alpha(x)yx^{-1} \in \mathbb{R}^k\}$;
- $\mathrm{Pin}(k) = \ker(N : \Gamma_k \to \mathbb{R}^\times) = \{x \in \Gamma_k \mid N(x) = 1\}$ and its complex version $\mathrm{Pin}^c(k) = \mathrm{Pin}(k) \times_{\mathbb{Z}/2} \mathrm{U}(1)$;
- $\mathrm{Spin}(k) = \mathrm{Pin}(k) \cap C_k^0$ and its complex version $\mathrm{Spin}^c(k) = \mathrm{Spin}(k) \times_{\mathbb{Z}/2} \mathrm{U}(1)$.

**Example 4.3.** When $k = 1$, $\mathrm{Pin}(1) = \mathbb{Z}/4$ and $\mathrm{Spin}(1) = \mathbb{Z}/2$.

It worths noticing that $\mathrm{Pin}(k)$ and $\mathrm{Spin}(k)$ are double covers of Lie groups $\mathrm{O}(k)$ and $\mathrm{SO}(k)$ respectively in the sense of the following short exact sequences

$$1 \to \mathbb{Z}/2 \to \mathrm{Pin}(k) \xrightarrow{\rho} \mathrm{O}(k) \to 1$$

$$1 \to \mathbb{Z}/2 \to \mathrm{Spin}(k) \xrightarrow{\rho} \mathrm{SO}(k) \to 1$$

[4, Theorem 3.11], so they inherit the Lie group structures from the later two. Similarly, we also have short exact sequences for the complex ones

$$1 \to \mathrm{U}(1) \to \mathrm{Pin}^c(k) \to \mathrm{O}(k) \to 1$$

$$1 \to \mathbb{Z}/2 \to \mathrm{Spin}^c(k) \to \mathrm{SO}(k) \times \mathrm{U}(1) \to 1.$$

4.2. **Clifford bundles.** The left hand side of the ABS map is purely algebraic while the right hand side is purely topological. Hence, we need a bridge to connect the algebra and the topology - the Clifford bundles, which are the fiber-bundle analogies of Clifford algebras and Clifford modules.

**Convention 4.4.** For notational convenience we will consider real vector bundles and the real $K$-theory in the following discussions. The complex case is entirely parallel.

Let $X$ be a based finite CW complex and $V$ be a real Euclidean vector bundle over $X$, i.e. there is a positive definite inner product on the fiber $V_x$ of $V$ for every point $x \in X$, which we denote as $\langle - \mid - \rangle_x$ as it depends on $x$ continuously. Now, the fiber $V_x$ becomes a vector space equipped with a positive quadratic form $Q_x$ which is equivalent to the inner product $\langle - \mid - \rangle_x$ on $V_x$. Hence, we are able to form bundles of Clifford algebras, Pin groups and Spin groups, Clifford modules, etc.

**Definitions 4.5.**

(1) The *Clifford bundle* $C(V)$ of $V$ is defined as a bundle of algebras whose fiber at $x$ is the Clifford algebra $C(-Q_x)$.

(2) Similarly, we can define the bundles of groups $\mathrm{Pin}(V)$ and $\mathrm{Spin}(V)$ whose fibers at $x$ are groups $\mathrm{Pin}(V_x)$ and $\mathrm{Spin}(V_x)$ respectively.

(3) Let $E = E^0 \oplus E^1$ be a $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle over $X$. $E$ is called a *$\mathbb{Z}/2\mathbb{Z}$-graded $C(V)$-module* if we have vector bundle homomorphisms

$$V \otimes_{\mathbb{R}} E^0 \to E^1, \qquad V \otimes_{\mathbb{R}} E^1 \to E^0 \tag{4.6}$$

denoted by $v \otimes e \mapsto v(e)$, such that

$$v(v(e)) = -Q_x(v)e, \qquad v \in V_x. \tag{4.7}$$

The action of $V$ on $E$ (4.6) naturally extends to an action of $C(V)$ on $E$ by the universal property Proposition 2.3.

(4) We define $M(V)$ to be the Grothendieck group of $\mathbb{Z}/2\mathbb{Z}$-graded $C(V)$-modules and $A(V) = \mathrm{coker}(M(V \oplus 1) \to M(V))$. These are the bundle analogies of the Clifford modules $M_k$ and $A_k$.

Comparing these definitions with those of the algebraic concepts in Section 2, we may regard the original algebraic ones as the special cases when $X = \mathrm{pt}$.

4.3. **Constructing $\chi_V$.** In this section construct the map

$$\chi_V : A(V) \to \widetilde{KO}(T(V)),$$

where $T(V) = D(V)/S(V)$ is the *Thom space* of the Euclidean vector bundle $V \to X$, $D(V)$ is the unit disk bundle of $V$ and $S(V)$ as the unit sphere bundle of $V$. Recall that we assume $X$ to be a based finite CW complex, so it is compact. Hence, $T(V)$ is the one-point compactification of $V$. Without loss of generality, we may assume that $(D(V), S(V))$ is a CW pair.

The idea of constructing $\chi_V$ is to first associate a complex to every element in $M(V)$ and apply the Euler characteristic to get an element in real $K$-theory; then pass it to $A(V)$.

4.3.1. *First step.* Let $E = E^0 \oplus E^1$ be a $\mathbb{Z}/2\mathbb{Z}$-graded $C(V)$-module. Then $E$ determines an element in $M(V)$. Now we associate a complex to $E$.

Let $\pi : D(V) \to X$ be the projection map. Then we have two pullback bundles $\pi^*E^1 \to D(V)$ and $\pi^*E^0 \to D(V)$. Define a bundle homomorphism

$$\begin{aligned} \sigma(E) : \pi^*E^1 &\to \pi^*E^0, \\ (e, v, x) &\mapsto (-v(e), v, x) \end{aligned} \tag{4.8}$$

where $v \in D(V)_x, e \in E^1_x, x \in X$. Then restricting $\sigma(E)$ to $S(E)$ gives an isomorphism because from (4.7) we know that $(e, v, x) \mapsto (v(e), v, x)$ is its inverse when $v \in S(V)_x$. Hence, the complex

$$0 \to \pi^*E^1 \xrightarrow{\sigma(E)} \pi^*E^0 \to 0 \tag{4.9}$$

is an element in $\mathscr{D}_1(D(V), S(V))$. Applying the composition of maps

$$\mathscr{D}_1(D(V), S(V)) \to \mathscr{C}_1(D(V), S(V)) \to L_1(D(V), S(V)) \xrightarrow{\chi} KO(D(V), S(V))$$

to (4.9), we obtain an element in $KO(D(V), S(V)) = \widetilde{KO}(T(V))$. This defines a map

$$\chi_V : M(V) \to \widetilde{KO}(T(V)) \tag{4.10}$$

and it is further a group homomorphism.

4.3.2. *Second step.* The next step to reduce (4.10) to $A(V)$.

If $E \in \mathrm{im}(M(V \oplus 1) \to M(V))$, then $E$ has the structure of $C(V \oplus 1)$-module extending the $C(V)$-module structure. Repeating the procedure in the paragraph above by replacing $V$ with $V \oplus 1$, we extend the bundle isomorphism $\sigma(E)|_{S(V)}$ from $S(V)$ to $S^+(V \oplus 1)$, the upper hemisphere of $S(V \oplus 1)$. Then, as a sequence over $S^+(V \oplus 1)$,

$$0 \to \pi^*E^1 \xrightarrow{\sigma(E)} \pi^*E^0 \to 0$$

is isomorphic to an elementary sequence in $\mathscr{C}_1(S^+(V \oplus 1), S(V))$; hence it equals to zero in $L_1(S^+(V \oplus 1), S(V))$. Applying the homeomorphism $(D(V), S(V)) \cong (S^+(V \oplus 1), S(V))$, $E$ is mapped to zero in $L_1(D(V), S(V))$. This implies that $\chi_V(E) = 0 \in \widetilde{KO}(T(V))$. Therefore, the map (4.10) is reduced to

$$\chi_V : A(V) \to \widetilde{KO}(T(V)), \tag{4.11}$$

as we desire.

4.3.3. *Multiplicative properties of $\chi_V$.* Now we want to discuss the multiplicative properties of (4.11), which is a key ingredient of making the ABS isomorphism into a ring homomorphism. Let $V, W$ be Euclidean bundles over $X, Y$ respectively. Then $V \boxplus W$ is an Euclidean bundle over $X \times Y$ and $(V \boxplus W)_{(x,y)} = V_x \oplus W_y$.

Recall that there is a natural homeomorphism

$$T(V) \wedge T(W) \cong T(V \boxplus W)$$

where $\wedge$ is the smash product of two based spaces [2, Lemma 2.3]. This induces a homomorphism

$$\widetilde{KO}(T(V)) \otimes \widetilde{KO}(T(W)) \to \widetilde{KO}(T(V \boxplus W)) \tag{4.12}$$

where for $a \in \widetilde{KO}(T(V)), b \in \widetilde{KO}(T(W))$, we denote the image of $a \otimes b$ simply as $ab$.

**Proposition 4.13.** *The following diagram commutes*

$$\begin{tikzcd} A(V) \otimes A(W) \arrow[r, "\hat{\boxtimes}"] \arrow[d, "\chi_V \otimes \chi_W"] & A(V \boxplus W) \arrow[d, "\chi_{V \boxplus W}"] \\ \widetilde{KO}(T(V)) \otimes \widetilde{KO}(T(W)) \arrow[r] & \widetilde{KO}(T(V \boxplus W)) \end{tikzcd}$$

*where for $E \in A(V), F \in A(W)$, $E \hat{\boxtimes} F = p_1^* E \hat{\otimes} p_2^* F$ is induced by the graded tensor product of Clifford modules, and the bottom horizontal arrow is the map (4.12). Thus,*

$$\chi_{V \boxplus W}(E \hat{\boxtimes} F) = \chi_V(E)\chi_W(F)$$

*for $E \in A(V)$ and $F \in A(W)$.*

*Proof.* First notice that

$$\begin{aligned} \chi_V(E)\chi_W(F) \in \widetilde{KO}(T(V \boxtimes W)) &= \widetilde{KO}(T(V) \wedge T(W)) \\ &= KO(D(V) \times D(W), D(V) \times S(W) \cup S(V) \times D(W)). \end{aligned}$$

As we did in Section 4.3.1, we can associate $E$ with $0 \to \pi^* E^1 \xrightarrow{\sigma(E)} \pi^* E^0 \to 0$ which is an element in $\mathscr{D}_1(D(V), S(V))$, and same for $F$. Apply Proposition 3.16 and it follows that $\chi_V(E)\chi_W(F) = \chi(G)$ where $G = (0 \to G_1 \xrightarrow{\tau} G_0 \to 0)$ is an element in $\mathscr{D}_1(D(V) \times D(W), D(V) \times S(W) \cup S(V) \times D(W))$ defined by

$$\begin{aligned} G_1 &= \pi^* E^0 \boxtimes \pi^* F^1 \oplus \pi^* E^1 \boxtimes \pi^* F^0 \\ G_0 &= \pi^* E^0 \boxtimes \pi^* F^0 \oplus \pi^* E^1 \boxtimes \pi^* F^1 \\ \tau &= \begin{pmatrix} 1 \boxtimes \sigma(F) & \sigma(E) \boxtimes 1 \\ \sigma(E)^* \boxtimes 1 & -1 \boxtimes \sigma(F)^* \end{pmatrix}. \end{aligned}$$

Next, we calculate $\sigma(E)^*$ and $\sigma(F)^*$. Recall that $E = E^0 \oplus E^1$ is a $\mathbb{Z}/2\mathbb{Z}$-graded $C(V)$-module. Analogous to Spin groups, we have $\mathrm{Spin}(V) = \mathrm{Pin}(V) \cap C(V)^0$; hence $E^0$ is a $\mathrm{Spin}(V)$-module. Consider the norm on $E^0$ which is induced by the inner product and denote it as $\|-\|$. We may assume that this norm is invariant under $\mathrm{Spin}(V)$; otherwise we apply an averaging argument by considering the new norm

$$\|e\|' = \int_{v \in \mathrm{Spin}(V)} \|v(e)\| d\mu \tag{4.14}$$

where $e \in E_x, v \in \mathrm{Spin}(V)_x, x \in X, d\mu$ is the Haar measure. This norm can be extended to a norm on $E$ which is invariant under $\mathrm{Pin}(V)$ and such that $E^0$ and

$E^1$ are orthogonal complements. There is also a norm on $V$ induced by the inner product which, by abusing notation, we still denote as $\|-\|$. Now, for $v \in V_x, v \neq 0$, we have $v/\|v\| \in \mathrm{Pin}(V_x)$. Hence,

$$\|\frac{v}{\|v\|}(e)\| = \frac{\|v(e)\|}{v} = \|e\|$$

implies that $\|v(e)\| = \|v\|\|e\|$. Then,

$$\|v(e)\|^2 = \langle v(e), v(e)\rangle = \langle v^* v(e), e\rangle, \tag{4.15}$$

while on the other hand,

$$\|v(e)\|^2 = \|v\|^2\|e\|^2 = \|v\|^2\langle e, e\rangle = \langle \|v\|^2 e, e\rangle = \langle (-v)(v(e)), e\rangle. \tag{4.16}$$

Comparing (4.15) and (4.16), we conclude that $v^* = -v$. Therefore, we rewrite

$$\tau = \begin{pmatrix} 1 \boxtimes \sigma(F) & \sigma(E) \boxtimes 1 \\ -\sigma(E) \boxtimes 1 & 1 \boxtimes \sigma(F) \end{pmatrix}$$

where we used $\sigma(E)^* = -\sigma(E), \sigma(F)^* = -\sigma(F)$. For any $(v, w) \in D(V) \times D(W)$, by definition (4.8) we know that

$$\tau|_{(v,w)} = \begin{pmatrix} 1 \boxtimes -w & -v \boxtimes 1 \\ v \boxtimes 1 & 1 \boxtimes -w \end{pmatrix}.$$

On the other hand, notice that $E\hat{\boxtimes}F$ is a $C(V)\hat{\boxtimes}C(W)$-module. Parallel to the isomorphism $C_{k+l} \cong C_k\hat{\otimes}C_l$ (2.28), we have $C(V \boxplus W) \cong C(V)\hat{\boxtimes}C(W)$; hence, $E\hat{\boxtimes}F$ is a $C(V\boxplus W)$-module. Consider its associated complex $0 \to \pi^*(E\hat{\otimes}F)^1 \xrightarrow{\sigma(E\hat{\otimes}F)} \pi^*(E\hat{\otimes}F)^0 \to 0$ which lies in $\mathscr{D}_1(D(V\boxplus W), S(V\boxplus W))$. Then, for any $v+w \in V\boxplus W$, we have

$$\sigma(E\hat{\otimes}F)|_{v+w} = \begin{pmatrix} 1 \boxtimes w & v \boxtimes 1 \\ v \boxtimes 1 & -1 \boxtimes w \end{pmatrix}.$$

Therefore,

$$\tau = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \sigma(E\hat{\otimes}F) \tag{4.17}$$

Let $D'(V \boxplus W)$ be the disk bundle of radius 2 and let

$$S'(V \boxplus W) = \overline{D'(V \boxplus W) - D(V \boxplus W)}.$$

Then we have two homotopy equivalences induced by inclusions

$$i : (D(V \boxplus W), S(V \boxplus W)) \to (D'(V \boxplus W), S'(V \boxplus W))$$
$$j : (D(V) \times D(W), D(V) \times S(W) \cup S(V) \times D(W)) \to (D'(V \boxplus W), S'(V \boxplus W)).$$

Comparing $G$ with $0 \to \pi^*(E\hat{\otimes}F)^1 \xrightarrow{\sigma(E\hat{\otimes}F)} \pi^*(E\hat{\otimes}F)^0 \to 0$, from (4.17) we see that they define the same element in real $K$-theory under the isomorphism

$$KO(D(V) \times D(W), D(V) \times S(W) \cup S(V) \times D(W)) \\ \cong KO(D'(V \boxplus W), S'(V \boxplus W)) \cong KO(D(V \boxplus W)$$

which is induced by $i$ and $j$. Therefore, $\chi(G) = \chi_{V\boxplus W}(E\hat{\boxtimes}F)$. □

4.4. **Constructing $\beta_P$.** Now we start to construct the other map

$$\beta_P : A_k \to A(V)$$

where $P$ is a fixed principal $\mathrm{Spin}(k)$-bundle over $X$, and $V$ is a vector bundle over $X$ constructed using $P$.

4.4.1. *The construction.*

**Definition 4.18.** Let $G$ be a topological group, $X$ be a right $G$-space and $Y$ be a left $G$-space. The *balanced product* of $X$ and $Y$ over $G$ is

$$X \times_G Y := X \times Y/(xg, y) \sim (x, gy),$$

where $x \in X, y \in Y, g \in G$. Equivalently, we can regard $Y$ as a right $G$-space by $y \cdot g := g^{-1}y$, so $G$ acts on the right on $X \times Y$ by $(x, y) \cdot g = (xg, g^{-1}y)$; thus, $X \times_G Y = (X \times Y)/G$.

Choose a principal $\mathrm{Spin}(k)$-bundle $P$ over $X$ and define $V = P \times_{\mathrm{Spin}(k)} \mathbb{R}^k$ by the balanced product. Then $V$ is a vector bundle over $X$ of dimension $k$. If $M$ is a graded $C_k$-module, equip $M$ with the discrete topology; then $P \times_{\mathrm{Spin}(k)} M$ is naturally a graded $C(V)$-module. Further, if $M \in A_k$, then $P \times_{\mathrm{Spin}(k)} M \in A(V)$. Thus, for each degree $k$ and principal $\mathrm{Spin}(k)$-bundle $P$ over $X$, we obtain a map

$$\begin{aligned} \beta_P : A_k &\to A(V) \\ M &\mapsto P \times_{\mathrm{Spin}(k)} M. \end{aligned} \tag{4.19}$$

It naturally extends to a group homomorphism.

4.4.2. *Multiplicative property of $\beta_P$.* Similar as $\chi_V$, we discuss the multiplicative properties of $\beta_P$ so that the ABS isomorphism becomes a ring homomorphism. Let $P, P'$ be principal $\mathrm{Spin}(k), \mathrm{Spin}(l)$-bundles over $X$ and $X'$ respectively. Then $P \boxplus P'$ is a principal $\mathrm{Spin}(k) \times \mathrm{Spin}(l)$ bundle over $X \times X'$. Notice that there is a group homomorphism

$$\mathrm{Spin}(k) \times \mathrm{Spin}(l) \to \mathrm{Spin}(k + l)$$

which is induced by the graded tensor product $C_k \times C_l \to C_k \hat{\otimes} C_l = C_{k+l}$ (2.28). Then $\mathrm{Spin}(k + l)$ becomes a left $\mathrm{Spin}(k) \times \mathrm{Spin}(l)$ space. Define

$$P'' = (P \boxplus P') \times_{\mathrm{Spin}(k)\times\mathrm{Spin}(l)} \mathrm{Spin}(k + l).$$

This is a principal $\mathrm{Spin}(k + l)$-bundle over $X \times X'$.

Define $V = P \times_{\mathrm{Spin}(k)} \mathbb{R}^k$, $V' = P' \times_{\mathrm{Spin}(l)} \mathbb{R}^l$, $V'' = P'' \times_{\mathrm{Spin}(k+l)} \mathbb{R}^{k+l}$. Then we have the following multiplicative property which follows directly from the construction.

**Proposition 4.20.** *For $a \in A_k, b \in A_l$, we have*

$$\beta_{P''}(ab) = \beta_P(a)\beta_{P'}(b).$$

*where the product on the right hand side is induced by $\hat{\boxtimes}$.*

4.5. **The main theorem.** Now we are able to first define the Atiyah-Bott-Shapiro map and then prove that it is a ring isomorphism.

**Definition 4.21.** For each degree $k$, the *Atiyah-Bott-Shapiro map* is defined to be the composition

$$\alpha_P : A_k \xrightarrow{\beta_P} A(V) \xrightarrow{\chi_V} \widetilde{KO}(T(V))$$

where $P$ is a principal $\mathrm{Spin}(k)$-bundle over $X$ and $V = P \times_{\mathrm{Spin}(k)} \mathbb{R}^k$ is the $k$-dimensional vector bundle over $X$ associated with $P$.

Then, $\alpha_P$ naturally inherits the multiplicative property from $\chi_V$ and $\beta_P$.

**Proposition 4.22.**

$$\alpha_{P''}(ab) = \alpha_P(a)\alpha_{P'}(b)$$

*where the notations are the same as in Proposition 4.20.*

Let $V \to X$ be a $k$-dimensional vector bundle. For any point $x \in X$, we have an inclusion $i_x : T(V_x) \to T(V)$ which induces a map on real $K$-theory

$$i_x^* : \widetilde{KO}(T(V)) \to \widetilde{KO}(T(V_x)).$$

Fix a dimension $k$. For a principal $\mathrm{Spin}(k)$-bundle $P$ and $V = P \times_{\mathrm{Spin}(k)} \mathbb{R}^k$, consider the composition

$$\alpha : A_k \xrightarrow{\alpha_P} \widetilde{KO}(T(V)) \xrightarrow{i_x^*} \widetilde{KO}(T(V_x)) = \widetilde{KO}(S^k)$$

which is independent of the choice of principal $\mathrm{Spin}(k)$-bundle $P$ as $P$ becomes the trivial $\mathrm{Spin}(k)$-bundle when we restrict to $x$. Summing over all nonnegative integers $k$, we get the *Atiyah-Bott-Shapiro isomorphism*

$$\alpha : A_* \to \bigoplus_{k\geq 0} KO^{-k}(\mathrm{pt})$$

where we used the fact that $\widetilde{KO}(S^k) = \widetilde{KO}(S^k \wedge S^0) = \widetilde{KO}^{-k}(S^0) = KO^{-k}(\mathrm{pt})$. The ABS isomorphism $\alpha$ is naturally a ring homomorphism due to the multiplicative property of $\alpha_P$.

**Theorem 4.23** (Main theorem)**.** *The ABS isomorphism*

$$\alpha : A_* \to \bigoplus_{k\geq 0} KO^{-k}(\mathrm{pt})$$

*is a ring isomorphism.*

*Proof.* As we already know that $\alpha$ is a ring homomorphism and are clear about the ring structures on both sides, it suffices to check that $\alpha$ sends generators to generators. The ring structures of $\bigoplus_{k\geq 0} K^{-k}(\mathrm{pt})$ and $\bigoplus_{k\geq 0} KO^{-k}(\mathrm{pt})$ are summarized as follows:

(1) $\bigoplus_{k\geq 0} K^{-k}(\mathrm{pt}) = \mathbb{Z}[x]$ where $x \in K^{-2}(\mathrm{pt})$ corresponds to the canonical line bundle over $S^2 = \mathbb{C}P^1$;
(2) $\bigoplus_{k\geq 0} KO^{-k}(\mathrm{pt}) = \mathbb{Z}[a, z, y]/(2a, a^3, az, z^2 - 4y)$ where $a \in KO^{-1}(\mathrm{pt})$, $z \in KO^{-4}(\mathrm{pt})$, $y \in KO^{-8}(\mathrm{pt})$, and $a$ corresponds to the canonical line bundle over $S^1 = \mathbb{R}P^1$;
(3) $y \mapsto x^4$ under the complexification map $\bigoplus_{k\geq 0} KO^{-k}(\mathrm{pt}) \to \bigoplus_{k\geq 0} K^{-k}(\mathrm{pt})$.

The first eight groups of $\bigoplus_{k\geq 0} KO^{-k}(\mathrm{pt})$ are listed in the following table.

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $KO^{-k}(\mathrm{pt})$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}$ |

Table 4.

Recall that in Theorem 2.33 and Theorem 2.34 we have calculated the ring structures of $A_*$ and $A_*^c$ respectively. Since the ring structure of the complex case is simpler, we will first prove that the complex analog of $\alpha$,

$$\alpha^c : A_*^c \to \bigoplus_{k\geq 0} K^{-k}(\mathrm{pt}),$$

is a ring isomorphism.

In Theorem 2.34 we have shown that $A_*^c = \mathbb{Z}[\mu^c]$ where $\mu^c$ is the generator of $A_2^c$, so it suffices to show that $\alpha^c(\mu^c) = x$ where the class $x \in K^{-2}(\mathrm{pt})$ is represented by the canonical line bundle over $S^2 = \mathbb{C}P^1$. First notice that regarding $\alpha$ as the composition $i_x^* \circ \alpha_P$ for an arbitrary $X$ is equivalent to taking $X = \mathrm{pt}$ when we construct $\alpha_P$, where $P$ becomes the trivial Spin(2)-bundle $\mathrm{pt} \times \mathrm{Spin}(2)$. We will take the latter approach. Recall that the class $\mu^c$ is represented by $\mathbb{C}^2$, so $\beta_P(\mu^c) = (\mathrm{pt} \times \mathrm{Spin}(2)) \times_{\mathrm{Spin}(2)} \mathbb{C}^2 = \mathrm{pt} \times \mathbb{C}^2$ is the trivial bundle. It is regarded as a $C(V)$-module where $V = \mathrm{pt} \times \mathbb{R}^2$ is the 2-dimensional real vector bundle. Then, $D(V) = \mathrm{pt} \times D^2$ and $S(V) = \mathrm{pt} \times S^1$, and $\mathrm{pt} \times \mathbb{C}^2$ is associated to the following element in $\mathscr{D}_1(\mathrm{pt} \times D^2, \mathrm{pt} \times S^1)$

$$0 \to \mathrm{pt} \times \mathbb{C} \times D^2 \xrightarrow{\sigma} \mathrm{pt} \times \mathbb{C} \times D^2 \to 0$$

where $\sigma(\mathrm{pt}, z, t) = (\mathrm{pt}, -tz, t)$. Applying the Euler characteristic to the sequence above, we get an element in $\widetilde{KO}(S^2) = KO^{-2}(\mathrm{pt})$ represented by

$$\mathbb{C} \times D^2 \sqcup \mathbb{C} \times D^2/(z,t) \sim (-tz, t), t \in \partial D^2 = S^1$$

which is a 1-dimensional complex vector bundle over $S^2$. Its clutching function is given by

$$f : S^1 \to \mathbb{C}^\times, \quad t \mapsto -t.$$

Notice that the clutching function of the canonical line bundle over $S^2$ is given by $g : S^1 \to \mathbb{C}^\times, t \mapsto t$, and $f \simeq g$ by the following homotopy

$$F : S^1 \times [0,1] \to \mathbb{C}^\times, \quad (t,s) \mapsto t \cdot e^{is\pi}.$$

Hence, $f$ and $g$ determine the same element in $\widetilde{KO}(S^2)$, so $\alpha^c(\mu^c) = x$.

The map $\alpha^c$ helps to prove that $\alpha$ is an isomorphism. By Theorem 2.33, it suffices to calculate the image of $\xi, \mu$ and $\lambda$ under $\alpha$. Consider the commutative diagram

$$\begin{tikzcd} A_* \arrow[r, "\alpha"] \arrow[d, "c"] & \bigoplus_{k\geq 0} KO^{-k}(\mathrm{pt}) \arrow[d, "c"] \\ A_*^c \arrow[r, "\alpha^c"] & \bigoplus_{k\geq 0} K^{-k}(\mathrm{pt}) \end{tikzcd}$$

where the vertical arrows are the complexification maps denoted by $c$. By Proposition 2.35, we know that $c(\lambda) = (\mu^c)^4$, so

$$c \circ \alpha(\lambda) = \alpha^c \circ c(\lambda) = \alpha^c((\mu^c)^4) = x^4 = c(y).$$

Thus, $\alpha(\lambda) - y \in \ker c$. On the other hand, since $\alpha(\lambda) - y \in KO^{-8}(\mathrm{pt})$ and $KO^{-8}(\mathrm{pt}) \cap \ker c = 0$, we have $\alpha(\lambda) = y$.

Assume that $\alpha(\mu) = kz$ where $k$ is a positive integer. Then

$$c \circ \alpha(\mu^2) = c(k^2 z^2) = c(4k^2 y) = 4k^2 x^4,$$
$$\alpha^c \circ c(\mu^2) = \alpha^c(4(\mu^c)^4) = 4x^4.$$

This implies that $k^2 = 1$ and hence we may take $\alpha(\mu) = z$.

Similar to our discussion in the complex case, $\alpha(\xi)$ is the canonical line bundle over $\mathbb{R}P^1 = S^1$, so $\alpha(\xi) = a$. This completes our proof that $\alpha$ is a ring isomorphism as we have shown that it sends generators to generators. □

## 5. Applications

5.1. **$KO$-orientability and Spin-structures.** Recall that $V \to X$ is a $k$-dimensional vector bundle. For any point $x \in X$, we have a group homomorphism

$$i_x^* : \widetilde{KO}^k(T(V)) \to \widetilde{KO}^k(T(V_x)) = \widetilde{KO}^k(S^k) = KO(\mathrm{pt}) = \mathbb{Z}.$$

**Definitions 5.1.** (1) $V$ is called *KO-orientable* if there exists a class $\mu_V \in \widetilde{KO}^k(T(V))$ such that for any point $x \in X$, $i_x^*(\mu_V)$ is a generator. In that case, such a choice of $\mu_V$ is called a *KO-orientation* (or *Thom class*) of $V$ .

(2) $V$ is said to have a *Spin-structure* if there exists a principal $\mathrm{Spin}(k)$-bundle $P$ such that $V \cong P \times_{\mathrm{Spin}(k)} \mathbb{R}^k$.

**Theorem 5.2.** *$V$ is KO-orientable if and only if $V$ has a Spin-structure.*

*Proof.* "$\Leftarrow$": First we assume that the dimension $k = 8r$ ($r \in \mathbb{Z}_{\geq 0}$). By the eight-fold periodicity of real $K$-theory, we can shift the degree of the map

$$i_x^* : \widetilde{KO}^k(T(V)) \to \widetilde{KO}^k(T(V_x))$$

to 0, i.e. $i_x^* : \widetilde{KO}(T(V)) \to \widetilde{KO}(T(V_x))$. Hence, to show that $V$ is $KO$-orientable, it suffices to find a class $\mu_V \in \widetilde{KO}(T(V))$ which restricts to the generator at every point $x \in X$. Recall that $\lambda \in A_8 \cong \mathbb{Z}$ is a generator. We claim that $\mu_V := \alpha_P(\lambda^r)$ gives a $KO$-orientation of $V$. Let $y$ be a generator of $KO^{-8}(\mathrm{pt}) \cong \mathbb{Z}$. Since $V$ has a Spin-structure, we can apply the ABS map to $V$. In particular, since the ABS isomorphism is a ring isomorphism, we know that $\alpha(\lambda) = y$, so

$$\begin{array}{ccccc} \alpha : A_k & \xrightarrow{\alpha_P} & \widetilde{KO}(T(V)) & \xrightarrow{i_x^*} & \widetilde{KO}(T(V_x)) = KO^{-k}(\mathrm{pt}) \\ \lambda^r & \mapsto & \mu_V & \mapsto & y^r, \end{array}$$

i.e. $i_x^*(\mu_V) = y^r$ is the generator of $\widetilde{KO}(T(V_x)) \cong \mathbb{Z}$ for any point $x \in X$. This shows that $\mu_V$ gives a $KO$-orientation of $V$, so $V$ is $KO$-orientable.

For the general dimension $k$, let $s$ be the positive integer such that $k + s$ is a multiple of 8. From the fiber sequence

$$B\mathrm{Spin}(k) \to B\mathrm{SO}(k) \xrightarrow{w_2} B^2\mathbb{Z}/2,$$

we see that $V$ having a Spin-structure implies that its second Stiefel-Whitney class vanishes, i.e. $w_2(V) = 0$. Hence, $w_2(V \oplus s) = w_2(V) = 0$ implies that $V \oplus s$ also has a Spin-structure, where $s$ denotes the trivial $s$-bundle over $X$. Since $V \oplus s$ has dimension a multiple of 8, by the previous case we know that $V \oplus s$ is $KO$-orientable. It suffices to show that $V$ is $KO$-orientable if and only if $V \oplus 1$ is $KO$-orientable. This follows from the commutative diagram

$$\begin{tikzcd} \widetilde{KO}^k(T(V)) \arrow[r, "i_x^*"] \arrow[d, "\Sigma"', "\wr"] & \widetilde{KO}^k(T(V_x)) \arrow[d, "\wr"', "\Sigma"] \\ \widetilde{KO}^{k+1}(T(V \oplus 1)) \arrow[r, "i_x^*"'] & \widetilde{KO}^{k+1}(T(V \oplus 1)_x) \end{tikzcd}$$

where the left vertical arrow is an isomorphism because $\widetilde{KO}^{k+1}(T(V \oplus 1)) = \widetilde{KO}^{k+1}(\Sigma T(V)) \cong \widetilde{KO}^k(T(V))$.

"$\Rightarrow$": We follow [19] and use the Atiyah-Hirzebruch spectral sequence (AHSS) to show this implication.

We first use AHSS to show that $KO$-orientation implies $H\mathbb{Z}$-orientation. Let $ko$ be the connective cover of the real $K$-theory spectrum $KO$. Then the AHSS of $ko$ is single-quadrant and hence converges. The second page of the AHSS of $T(V)$ looks like

$$E_2^{p,q} = \tilde{H}^p(T(V); ko^q(\mathrm{pt})) = H^p(D(V), S(V); ko^q(\mathrm{pt})).$$

Since $(D(V), S(V))$ is $(k-1)$-connected, $E_2^{p,q} = 0$ for $p < k$. Hence, the edge homomorphism

$$\tilde{H}^k(T(V); ko^0(\mathrm{pt})) \xrightarrow{\sim} ko^q(T(V))$$

is an isomorphism. Since $ko^0(\mathrm{pt}) = \mathbb{Z}$, this isomorphism maps the $ko$-Thom class maps to a $H\mathbb{Z}$-Thom class, implying the $H\mathbb{Z}$-orientability.

Now, consider the following diagram

$$\begin{tikzcd}
H^p(X;\pi_{-q}(ko)) \arrow[r] \arrow[d, "\Phi:-\cup\mu_V^{\pi_{-q}(ko)}"', "\wr"] & ko^{p+q}(X) \arrow[d, dashed, "\wr"', "-\cdot\mu_V"] \\
\tilde{H}^{p+k}(T(V);\pi_{-q}(ko)) \arrow[r] & \widetilde{ko}^{p+q+k}(T(V))
\end{tikzcd}$$

where the vertical arrows are the Thom isomorphisms, $\mu_V^{\pi_{-q}(ko)}$ and $\mu_V$ are the $H\pi_{-q}(ko)$ and $ko$-Thom classes respectively, and the horizontal arrows refer to the convergence of the AHSS. Since $V$ is both $ko$-orientable and $H\mathbb{Z}$-orientable, the $H\mathbb{Z}$-Thom isomorphism on the $E_2$ pages survives to the $ko$-Thom isomorphism on the $E_\infty$ pages; hence, the diagram above commutes. Then by Leibniz rule, we see that $d_2^{k,q}(\mu_V^{\pi_{-q}(ko)}) = 0$.

The differentials $d_2^{k,q}$ are clearly understood [22, Theorem 3.4]. In particular, when $q = -1$, $\pi_1(ko) = \mathbb{Z}/2$ and $d_2^{k,-1}$ is the $k$-invariant of $ko$ in $H^{k+2}(K(\mathbb{Z}/2,k);\mathbb{Z}/2)$, which is exactly the Steenrod square $Sq^2$. Hence, the class $Sq^2(\mu_V^{\mathbb{Z}/2}) = 0$ implies that $w_2(V) = \Phi^{-1}Sq^2(\mu_V^{\mathbb{Z}/2}) = 0$ where $\Phi$ is the $H\mathbb{Z}/2$-Thom isomorphism; the later one is equivalent to the existence of a Spin-structure. □

Then, by the Thom isomorphism theorem in real $K$-theory, we know that $\widetilde{KO}^*(T(V))$ is a free $\widetilde{KO}^*(X)$-module generated by $\mu_V$, where $\widetilde{KO}^*(X) = \bigoplus_{i=0}^7 \widetilde{KO}^i(X)$. The complex case is totally parallel if we replace Spin-groups by Spin$^c$-groups and eight-fold periodicity by two-fold periodicity.

5.2. **Vector fields on spheres problem.** We start our adventure to vector fields on spheres problem from the well-known result in topology that there does not exist a nowhere-zero vector field on $S^{n-1}$ for $n$ odd and in particular, for $S^2$ this property is called the "hairy ball theorem". The reason follows by calculating the degree of a map between spheres. Assume that there exists a nowhere-zero vector field

$$v : S^{n-1} \to TS^{n-1}.$$

We may normalize the vector field such that every vector attached to the point on $S^{n-1}$ has length 1. Hence the vector field $v$ is reduced to $v : S^{n-1} \to S^{n-1}$. Consider the following homotopy

$$H(x,t) = x\cos\pi t + v(x)\sin\pi t$$

between the identity map and the antipodal map $\alpha(x) = -x$. This implies that $\deg\alpha = \deg\mathrm{id} = 1$. However, $\alpha$ is a composite of $n$ (which is odd) reflections in $\mathbb{R}^n$ and hence $\deg\alpha = (-1)^n = -1$, a contradiction.

On the other hand, when $n$ is even, we construct a nowhere-zero vector field on $S^{n-1}$ by regarding $S^{n-1}$ as the sphere in $\mathbb{C}^{n/2}$ and consider $v(x) = ix$. We can further ask the following question:

**Question 5.3.** What is the maximum number of linearly independent vector fields on $S^{n-1}$?

This question has been completely answered by Hurwitz-Radon-Eckmann and Adams in the following theorem.

**Theorem 5.4** (Hurwitz-Radon-Eckmann, Adams)**.** *For any positive integer $n$, write $n = (2a+1)\cdot 2^b$ and $b = 4d + c$, where $a, b, c, d$ are nonnegative integers*

*and* $0 \leq c \leq 3$*, and set* $\rho(n) = 8d + 2^c$*. Then the maximal number of linearly independent vector fields on* $S^{n-1}$ *is* $\rho(n) - 1$*.*

**Definition 5.5.** The number $\rho(n) = 8d + 2^c$ is called the *Radon-Hurwitz number.*

Since $\rho(n)$ only depends on the power of two in $n$, it grows very slowly. The complete proof of vector fields on spheres problem splits into two parts: the proof of the upper bound and the construction realizing the upper bound. The construction is given by Hurwitz, Radon and Eckmann using Clifford algebras, and the proof of the upper bound is given by Adams [1] using real $K$-theory.

5.2.1. *Adams' Proof of the upper bound.*

**Definition 5.6.** $\mathbb{R}P_k^{k+n} = \mathbb{R}P^{k+n}/\mathbb{R}P^{k-1}$ is called a *real stunted projective space*

The name comes from the CW structure of $\mathbb{R}P_k^{k+n}$ that it consists of one cell in each dimension 0 and $k, k+1, \cdots, k+n$. The proof of the upper bound, which is given by Adams [1], uses real $K$-theory and some stable homotopy theory, and then is reduced to the following theorem concerning real stunted projective spaces.

**Theorem 5.7.** *There is no map*

$$f : \mathbb{R}P_n^{n+\rho(n)} \to S^n$$

*such that the composition with the inclusion map*

$$S^n = \mathbb{R}P^n/\mathbb{R}P^{n-1} \xrightarrow{i} \mathbb{R}P^{n+\rho(n)}/\mathbb{R}P^{n-1} \xrightarrow{f} S^n \tag{5.8}$$

*has degree 1.*

The proof uses Adams operations, which are a series of a ring homomorphisms

$$\Psi^k : KO(X) \to KO(X)$$

for each positive integer $k$ mapping the line bundles to a power of them, and computation of the real $K$-theory of real stunted projective spaces, to obtain a contradiction that the composition of the induced maps of (5.8) on real $K$-theories

$$\widetilde{KO}(S^n) \xleftarrow{i^*} \widetilde{KO}(\mathbb{R}P_n^{n+\rho(n)}) \xleftarrow{f^*} \widetilde{KO}(S^n)$$

cannot be an identity. The real $K$-theory of real stunted projective spaces is related to Clifford modules by the following result.

**Proposition 5.9** ([4, Proposition 15.7])**.** *When* $X = \mathrm{pt}$*, there is an exact sequence*

$$M(C_k) \xrightarrow{i^*} M(C_r) \to \widetilde{KO}(\mathbb{R}P_r^{k-1}) \to 0 \tag{5.10}$$

*where* $r \leq k$*. A similar result also holds in the complex case.*

In particular, taking $r = 1$, we get $\widetilde{KO}(\mathbb{R}P^{k-1}) \cong \mathbb{Z}/a_k$. It also worth remarking that the isomorphism

$$\mathrm{coker}(M(C_k) \xrightarrow{i^*} M(C_r)) \xrightarrow{\sim} \widetilde{KO}(\mathbb{R}P^{k-1}/\mathbb{R}P^{r-1})$$

actually comes from the Euler characteristic defined in Section 4.3.

5.2.2. *Constructing vector fields.* We use the following proposition to relate the vector fields on spheres with Clifford algebras.

**Proposition 5.11.** *If $\mathbb{R}^n$ admits the structure of $C_k$-module, then $S^{n-1}$ admits $k$ linearly independent vector fields.*

*Proof.* First of all, we may assume without loss of generality that $C_k$ acts on $\mathbb{R}^n$ orthogonally, i.e. there exists an inner product $\langle - \mid - \rangle$ on $\mathbb{R}^n$ such that $\langle e_i x \mid e_i y\rangle = \langle x \mid y\rangle$ for any $e_i$ the generators of $C_k$ by an averaging argument.

Then for any $x \in \mathbb{R}^n$, $x$ is orthogonal to $e_i x$ because $\langle x \mid e_i x\rangle = \langle e_i x \mid e_i^2 x\rangle = \langle e_i x \mid -x\rangle = -\langle x \mid e_i x\rangle$. For $i \neq j$, $e_i x$ is also orthogonal to $e_j x$ because $\langle e_i x \mid e_j x\rangle = \langle e_i e_j e_i x \mid e_i e_j e_j x\rangle = \langle -e_j e_i e_i x \mid -e_i x\rangle = \langle e_j x \mid -e_i x\rangle = -\langle e_i x \mid e_j x\rangle$. Therefore, the $k$ vector fields $v_i : x \mapsto e_i x$, $1 \leq i \leq k$ are orthogonal to each other and hence linearly independent. □

**Definition 5.12.** Let $n_k$ be the $\mathbb{R}$-dimension of the ungraded irreducible $C_k$-module.

If $M$ is a graded irreducible $C_k$-module, $M^0$ will be an ungraded irreducible $C_k^0$-module and hence an ungraded irreducible $C_{k-1}$-module according to the isomorphism $C_k^0 \cong C_{k-1}$ (Proposition 2.14 (2)). Thus

$$n_k = a_{k+1},$$

where the computations of $a_k$ are listed in Table 2. For convenience, we rewrite the values of $n_k$ in the following table.

| $k$ | $C_k$ | $n_k$ |
|---|---|---|
| 1 | $\mathbb{C}$ | 2 |
| 2 | $\mathbb{H}$ | 4 |
| 3 | $\mathbb{H} \oplus \mathbb{H}$ | 4 |
| 4 | $\mathbb{H}(2)$ | 8 |
| 5 | $\mathbb{C}(4)$ | 8 |
| 6 | $\mathbb{R}(8)$ | 8 |
| 7 | $\mathbb{R}(8) \oplus \mathbb{R}(8)$ | 8 |
| 8 | $\mathbb{R}(16)$ | 16 |

Table 5.

It is worth noticing that the $n_k$'s are actually all powers of 2. This follows from the data listed in Table 5 and $n_{k+8} = 16n_k$ (2.18). Our next step towards realizing the upper bound is to reduce the dimension of the spheres according to the following lemma.

**Lemma 5.13.** *If $S^{n-1}$ admits $k$ linearly independent vector fields, then $S^{mn-1}$ also admits $k$ linearly independent vector fields for $m$ any positive integer.*

*Proof.* Let $v_1, \cdots, v_k$ be the $k$ linearly independent vector fields. By Gram-Schmidt argument we may assume that they are normalized and orthogonal to each other. Consider

$$S^{mn-1} = \underbrace{S^{n-1} * \cdots * S^{n-1}}_{m \text{ times}}$$

where $*$ denotes the join operation. For any $x \in S^{mn-1}$, write $x = (\alpha_1 x_1, \cdots, \alpha_m x_m)$ where $x_i \in S^{n-1}$, $\alpha_i \geq 0$, and $\sum_{i=m}^{q} \alpha_i^2 = 1$. Define

$$\begin{aligned} v_i^* : S^{mn-1} &\to \mathbb{R}^{mn-1} \\ x &\mapsto (\alpha_1 v_i(x_1), \cdots, \alpha_m v_i(x_m)). \end{aligned}$$

Then $\langle x \mid v_i^*(x) \rangle = \sum_j \alpha_j^2 \langle x_j \mid v_i(x_j) \rangle = 0$ and

$$\langle v_i^*(x) \mid v_j^*(x) \rangle = \sum_k \alpha_k^2 \langle v_i^*(x_k) \mid v_j^*(x_k) \rangle = \begin{cases} \sum_k \alpha_k^2 = 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

These tell us that $v_1^*, \cdots, v_k^*$ are orthonormal vector fields over on $S^{mn-1}$ and hence linearly independent. □

Now we are able to prove the following theorem which realizes the upper bound of the main theorem of vector fields on spheres problem (Theorem 5.4) using Clifford algebras.

**Theorem 5.14.** *$S^{n-1}$ admits $\rho(n) - 1$ linearly independent vector fields.*

*Proof.* First, write $n = (2a+1) \cdot 2^b$, where $b = 4d + c$, $a, b, c, d$ are all nonnegative integers, and $0 \leq c \leq 3$. By Lemma 5.13 it suffices to show that $S^{2^b - 1}$ admits $\rho(n) - 1 = \rho(2^b) - 1$ linearly independent vector fields.

From our previous observation there exists an integer $k$ such that $2^b = n_k$. Choose $k$ to be the maximal such integer. Then $\mathbb{R}^{n_k}$ has the structure of $C_k$-module and Proposition 5.11 ensures that $S^{n_k - 1}$ (equivalently, $S^{2^b - 1}$) admits $k$ linearly independent vector fields.

The next step is to show that $\rho(2^k) - 1 = k$. Write $k = 8q + r$, where $q, r$ are nonnegative integers and $0 \leq r \leq 7$. Then

$$2^{4d} 2^c = 2^b = n_k = n_{8q+r} = 16^q n_r = 2^{4q} n_r.$$

Since $0 \leq r \leq 7$, $n_r \leq 8$ according to the computations in Table 5, the only possibility is

$$\begin{cases} d = q \\ 2^c = n_r. \end{cases}$$

Thus $k = 8d + r$. Moreover, since $k$ is the maximal choice that $n_k = 2^b$, $r$ is also the maximal choice that $n_r = 2^c$, where $0 \leq c \leq 3$. Compute all the possibilities by comparing with the data in Table 5 and we get the following result:

| $c$ | $2^c$ | $r$ |
|---|---|---|
| 0 | 1 | 0 |
| 1 | 2 | 1 |
| 2 | 4 | 3 |
| 3 | 8 | 7 |

TABLE 6.

From Table 6 we conclude that $r = 2^c - 1$. Hence, $k = 8d + r = 8d + 2^c - 1 = \rho(2^r) - 1$. □

5.2.3. *Parallelizable spheres.* A special case of the vector fields on spheres problem is that $S^{n-1}$ is parallelizable if and only if $n = 1, 2, 4, 8$.

**Definition 5.15.** The sphere $S^{n-1}$ is *parallelizable* if its tangent bundle $TS^{n-1}$ is trivial; this is equivalent to the existence of $(n-1)$ linearly independent vector fields over $S^{n-1}$.

**Theorem 5.16.** *$S^{n-1}$ is parallelizable if and only if $n = 1, 2, 4, 8$.*

*Proof.* According to the main theorem of vector fields on spheres problem, Theorem 5.4, $S^{n-1}$ is parallelizable if and only if we have the equation $\rho(n) - 1 = n - 1$, or equivalently, $\rho(n) = n$.

Solving this equation, we get $n = 1, 2, 4, 8$. Recall that we write $n = (2a+1)2^b = (2a+1)2^{4d+c} = (2a+1)16^d 2^c$, for $a, b, c, d$ nonnegative integers, $0 \leq c \leq 3$, and $\rho(n) = 8d + 2^c$. Hence, $\rho(n) = n$ is reduced to $(2a+1)16^d 2^c = 8d + 2^c$; that is,

$$2^c((2a+1)16^d - 1) = 8d. \tag{5.17}$$

Then, as an odd integer, $(2a+1)16^d - 1 | d$, implying that $(2a+1)16^d - 1 \leq d$. This only happens when $d = 0$. Hence, (5.17) is reduced to $2^{c+1}a = 0$, which implies that $a = 0$. Therefore, $n = 2^c$, where $0 \leq c \leq 3$. □

5.3. **Normed division algebras.** Recall that a *normed division algebra* over $\mathbb{R}$ is a not necessarily associative division algebra over $\mathbb{R}$ equipped with a norm $\|\cdot\|$ such that $\|xy\| = \|x\|\|y\|$ for any two elements $x, y$. Using Clifford algebras and our previous results on vector fields on spheres problem, we can prove that the possible dimensions of finite dimensional normed division $\mathbb{R}$-algebras are 1, 2, 4 or 8 [26].

The normed division algebras over $\mathbb{R}$ are related to Clifford algebras by the following lemma.

**Lemma 5.18.** *If there exists an $n$-dimensional normed division algebra over $\mathbb{R}$, then $\mathbb{R}^n$ admits a structure of a $C_{n-1}$-module.*

*Proof.* Let $\mathbb{K}$ be an $n$-dimensional normed division algebra over $\mathbb{R}$ and let $\|\cdot\|$ be the norm on $\mathbb{K}$. We first show that $\|\cdot\|$ is induced by an inner product $\langle -|- \rangle$ on $\mathbb{K}$. As $\mathbb{R}$-vector spaces, $\mathbb{K} \cong \mathbb{R}^n$. Let $(-|-)$ be the standard inner product on $\mathbb{R}^n$; it naturally becomes an inner product on $\mathbb{K}$ and induces a new norm $\|\cdot\|'$ on $\mathbb{K}$. Notice that $\mathbb{K}^* = \{k \in \mathbb{K} | \|k\| = 1\}$ is a compact subspace. By averaging $(-|-)$ over $\mathbb{K}^*$ we get an inner product $\langle -|- \rangle$ which is invariant under left multiplication maps $L_x$ where $x \in \mathbb{K}^*$. Then, for any $a \in \mathbb{K}, a/\|a\| \in \mathbb{K}^*$; hence

$$\langle a \mid a \rangle = \|a\|^2 \left\langle \frac{a}{\|a\|} \middle| \frac{a}{\|a\|} \right\rangle = \|a\|^2.$$

Now, let $\operatorname{Im}\mathbb{K} = \{k \in \mathbb{K} \mid \langle k \mid 1 \rangle = 0\}$. Then we have an $\mathbb{R}$-vector space isomorphism $\operatorname{Im}\mathbb{K} \cong \mathbb{R}^{n-1}$. For any $k \in \operatorname{Im}\mathbb{K}$, we associate it with the left multiplication map $L_k$. Then we can regard $L_k$ as an element in $M_n(\mathbb{R})$ under the identification $\mathbb{K} \cong \mathbb{R}^n$. This induces a linear map

$$L : \operatorname{Im}\mathbb{K} \to M_n(\mathbb{R}), \quad k \mapsto L_k.$$

Next, we show that $L_k^2 = -1$ for any $k \in \operatorname{Im}(\mathbb{K})$ with $\|k\| = 1$. Let $k$ be such an element and define $l = (k+1)/\sqrt{2}$. Then $\|l\| = 1$ and $L_k, L_l \in \mathrm{O}(n)$. Since

$$1 = L_l L_l^t = \frac{1}{2}(L_k + 1)(L_k^t + 1) = 1 + \frac{1}{2}(L_k + L_k^t),$$

we have $L_k = -L_k^t$. Thus, $L_k^2 = L_k(-L_k^t) = -L_k L_k^t = -1$, as desired.

For any $k \in \operatorname{Im}\mathbb{K}$,

$$L_k^2 = L^2_{\|k\| \cdot \frac{k}{\|k\|}} = \|k\|^2 L^2_{\frac{k}{\|k\|}} = -\|k\|^2$$

where the last equality follows from the fact that $k/\|k\|$ has norm one. Then, from the universal property of Clifford algebras (2.4),

$$\begin{tikzcd} \operatorname{Im}\mathbb{K} \arrow[r] \arrow[dr, "L"'] & C_{n-1} \arrow[d, dashed, "\bar{L}"] \\ & M_n(\mathbb{R}) \end{tikzcd}$$

we have a map $\bar{L} : C_{n-1} \to M_n(\mathbb{R})$ and this gives $\mathbb{R}^n$ a $C_{n-1}$-module structure. □

Using Lemma 5.18 and the maximum number $\rho(n)$ of vector fields on spheres, we prove the following theorem by Hurwitz [8],[21].

**Theorem 5.19** (Hurwitz)**.** *The finite dimensional normed division algebras over* $\mathbb{R}$ *have dimensions* $1, 2, 4$ *or* $8$.

*Proof.* Again, let $\mathbb{K}$ be a finite dimensional normed division algebra over $\mathbb{R}$ and let $n = \dim \mathbb{K}$. By Lemma 5.18, $\mathbb{R}^n$ admits a $C_{n-1}$-module structure. This implies that $S^{n-1}$ admits $n-1$ linearly independent vector fields by Proposition 5.11,. Then, Theorem 5.4 reduces this theorem to solving the equation $\rho(n) - 1 = n - 1$. As we have solved this equation in Theorem 5.16, we know that the only solutions are $n = 1, 2, 4, 8$. On the other hand, we know that $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ are the normed division algebras of dimensions $1, 2, 4, 8$ respectively. □

One construction of $n-1$ linearly independent vector fields over $S^{n-1}$ when $n = 1, 2, 4, 8$ can be given by taking $S^{n-1}$ as the unit sphere in $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ respectively, and use the division algebra structure.

One can further show that $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ are the only four possible normed division $\mathbb{R}$-algebras of dimension 1, 2, 4, 8. It is still true that the finite dimensional division $\mathbb{R}$-algebras which are not necessarily equipped with a norm can only have dimension 1, 2, 4 or 8, but they are not restricted to $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ anymore.

## Appendix A. Obstruction theory and the relative lifting problem

One problem that we will encounter in Lemma B.1 can be abstracted as follows.

**Question A.1** (The relative lifting problem)**.** Given a CW pair $(X, Y)$, a fibration $E \to B$ and a map $X \to B$, does there exist a lift $X \to E$ extending the given lift $Y \to E$?

$$\begin{tikzcd} Y \arrow[r] \arrow[d, hook] & E \arrow[d] \\ X \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

This problem reduces to the lifting problem when $Y = \varnothing$ and reduces to the extension problem when $B = \mathrm{pt}$.

The relative lifting problem is solved by *obstruction theory* which refers to a procedure for defining a sequence of cohomology classes that are the obstructions to finding a solution to the relative lifting problem.

Before we state the main theorem, let's recall an important notion in homotopy theory. Notice that since we are talking within the category of based finite CW complexes, we denote $\pi_n(X) = \pi_n(X, *)$ for $(X, *)$.

**Definitions A.2.** For an integer $n \geq 1$, a connected space $X$ is said to be *$n$-simple* if $\pi_1(X)$ is abelian and acts trivially on the homotopy groups $\pi_q(X)$ for $q \leq n$.

$X$ is said to be *simple* if it is $n$-simple for all $n$.

Let $(X, Y)$ be a CW pair, $p : E \to B$ be a fibration where $B$ is a path-connected CW complex with fiber $F$ $n$-simple. Let $f : X \to B$ be a map and $g : X^n \to E$ a lift of $f$ extending the given lift $Y \to E$, where $X^n$ refers to the relative $n$-skeleton of $(X, Y)$.

Recall that if $F$ is $n$-simple, then the fibration $F \hookrightarrow E \to B$ defines a local coefficient system over $B$ with fiber $\pi_n(F)$, or equivalently speaking, giving a group homomorphism $\rho : \pi_1(B) \to \mathrm{Aut}(\pi_n(F))$ [10, Proposition 6.62]. Pullback this local coefficient system over $X$ via $f : X \to B$ and we get a local coefficient system over $X$, which we continue to denote by $\rho$; that is,

$$\rho : \pi_1(X) \xrightarrow{f_*} \pi_1(B) \xrightarrow{\rho} \mathrm{Aut}(\pi_n(F)).$$

Now we can state the main theorem for this section.

**Theorem A.3.** *There exists a cohomology class*

$$[\theta^{n+1}(g)] \in H^{n+1}(X, Y; \pi_n(F)_\rho),$$

*which is called the obstruction class, such that if $[\theta^{n+1}(g)]$ vanishes, then $g$ can be redefined over $X^n$ $\mathrm{rel}\, X^{n-1}$ and then extended to $X^{n+1}$ such that the extended map $g : X^{n+1} \to E$ fits into the commutative diagram*

$$\begin{tikzcd} Y \arrow[r] \arrow[d, hook] & E \arrow[d] \\ X^{n+1} \arrow[r, "f"] \arrow[ur, dashed, "g"] & B \end{tikzcd}$$

*In particular, the lift $X \to E$ of $X \to B$ which extends $Y \to E$ exists if all the obstruction classes vanish.*

*Moreover, given two lifts $g, g' : X^n \to E$ of $f$ both extending the given lift $Y \to E$, and a homotopy $\mathrm{rel}\, Y$ of their restrictions to $X^{n-1}$, there exists another obstruction class*

$$d^n(g, g') \in H^n(X, Y; \pi_n(F)_\rho)$$

*such that if it vanishes, then the restriction of the given homotopy to $X^{n-2}$ extends to a homotopy $g \simeq g'$ $\mathrm{rel}\, Y$. In particular, if all these new obstruction classes vanish, then the lift $X \to E$ is unique up to homotopy equivalence $\mathrm{rel}\, Y$.* [10, Chapter 7], [14, Proposition 3.19].

## Appendix B. Proofs in Section 3

B.1. **Proof of Proposition 3.4.** The proof of Proposition 3.4 is developed from the following lemma.

**Lemma B.1.** *Let $E, F$ be vector bundles over $X$, and $f : E \to F$ be a monomorphism of vector bundles restricted over $Y$. If*

$$\dim F > \dim E + \dim X$$

*where $\dim F$ and $\dim E$ are the dimensions of vector bundles, and $\dim X$ is the dimension of a finite CW complex, then $f$ can be extended as a monomorphism of vector bundles over $X$. Moreover, any two such extensions are homotopic $\mathrm{rel}\, Y$.*

*Proof.* We will only prove the case for real vector bundles; the complex version carries out similarly. Consider the fiber bundle $\mathrm{Mon}(E, F) \to X$ whose fiber at $x \in X$ is the space of all monomorphisms $E_x \to F_x$ between vector spaces (equipped with compact-open topology). Consider the surjective map

$$\begin{aligned} \mathrm{GL}(n, \mathbb{R}) &\to \mathrm{Mon}(E_x, F_x) \\ A &\mapsto (A^1, \cdots, A^m) \end{aligned}$$

where $\dim F = n, \dim E = m, m \leq n, A^i$ refers to the $i$-th column of the matrix corresponding to $A$ once we have fixed a set of orthonormal bases for $E_x$ and $F_x$. This passes to a homeomorphism

$$\mathrm{GL}_n(\mathbb{R})/\mathrm{GL}_{n-m}(\mathbb{R}) \cong \mathrm{Mon}(E_x, F_x)$$

and hence gives us a fibration

$$\mathrm{GL}_{n-m}(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R}) \to \mathrm{Mon}(\mathbb{R}^m, \mathbb{R}^n).$$

where we used the homeomorphism $\mathrm{Mon}(E_x, F_x) \cong \mathrm{Mon}(\mathbb{R}^m, \mathbb{R}^n)$. In particular, we have the following fibration

$$\mathrm{GL}_{n-m}(\mathbb{R}) \to \mathrm{GL}_{n-m+1}(\mathbb{R}) \to \mathrm{Mon}(\mathbb{R}^1, \mathbb{R}^{n-m+1}).$$

Since $\mathrm{Mon}(\mathbb{R}^1, \mathbb{R}^{n-m+1}) \cong S^{n-m}$, the long exact sequence of a fibration indicates that $\mathrm{GL}_{n-m}(\mathbb{R}) \to \mathrm{GL}_{n-m+1}(\mathbb{R})$ is an $(n-m-1)$-equivalence. Hence $\mathrm{GL}_{n-m}(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$, which factors through $\mathrm{GL}_{n-m}(\mathbb{R}) \to \mathrm{GL}_{n-m+1}(\mathbb{R})$, is also an $(n-m-1)$-equivalence, indicating that $\mathrm{Mon}(E_x, F_x) \cong \mathrm{Mon}(\mathbb{R}^m, \mathbb{R}^n)$ is $(n-m-1)$-connected.

Notice that a monomorphism $E \to F$ of vector bundles over $X$ is the same thing as a section $X \to \mathrm{Mon}(E, F)$, so it suffices to show that given a section $f : Y \to \mathrm{Mon}(E, F)$, we can extend it over $X$. By considering the fibration

$$\mathrm{Mon}(E_x, F_x) \to \mathrm{Mon}(E, F) \to X,$$

the lemma is further reduced to the relative lifting problem

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, hook] & \mathrm{Mon}(E, F) \arrow[d] \\ X \arrow[r, "\mathrm{id}"] \arrow[ur, dashed] & X \end{tikzcd}$$

(Question A.1). We will apply the obstruction theory (Theorem A.3) to show the existence and uniqueness (up to homotopy $\operatorname{rel} Y$) of the section $X \to \mathrm{Mon}(E, F)$ which extends $f$.

The computation of $H^{k+1}(X, Y; \pi_k(\mathrm{Mon}(E_x, F_x))_\rho)$, where

$$\rho : \pi_1(X) \to \mathrm{Aut}(\pi_k(\mathrm{Mon}(E_x, F_x)))$$

gives the local coefficient system over $X$, can be divided into two cases:

(1) for $k \leq n-m-1$, $H^{k+1}(X, Y; \pi_k(\mathrm{Mon}(E_x, F_x))_\rho) = 0$ because $\mathrm{Mon}(E_x, F_x)$ is $(n-m-1)$-connected;
(2) for $k \geq n-m$, $H^{k+1}(X, Y; \pi_k(\mathrm{Mon}(E_x, F_x))_\rho) = 0$ because

$$\dim X \leq \dim F - \dim E - 1 = n - m - 1 \leq k - 1 < k + 1.$$

Therefore, $H^{k+1}(X, Y; \pi_k(\mathrm{Mon}(E_x, F_x))_\rho)$ is always zero and hence all the obstruction classes vanish, implying the existence of the lift $X \to \mathrm{Mon}(E, F)$ extending $f$.

Such a lift $X \to \mathrm{Mon}(E, F)$ is unique up to homotopy equivalence $\operatorname{rel} Y$ as one can argue similarly that $H^k(X, Y; \pi_k(\mathrm{Mon}(E_x, F_x))_\rho)$ is zero for any $k$ (here we use $\dim F > \dim E + \dim X$). Hence all the obstruction classes for homotopy vanish as well. □

*Proof of Proposition 3.4.* To show that $j_n : L_1(X, Y) \xrightarrow{\sim} L_n(X, Y)$ for each $n \geq 1$, it suffices to prove the isomorphism $L_n(X, Y) \xrightarrow{\sim} L_{n+1}(X, Y)$ for each $n$. To show this, we will construct a map $L_{n+1}(X, Y) \to L_n(X, Y)$ and show that it is the inverse.

Define $\bar{\mathscr{C}}_{n+1}(X,Y) = \{E \in \mathscr{C}_{n+1}(X,Y) \mid \dim E_n > \dim E_{n+1} + \dim X\}$. $\bar{\mathscr{C}}_{n+1}(X,Y)$ is nonempty because for every $E \in \mathscr{C}_{n+1}(X,Y)$, we can add an elementary sequence to $E$ so that it lies in $\bar{\mathscr{C}}_{n+1}(X,Y)$. Hence the natural map $\bar{\mathscr{C}}_{n+1}(X,Y) \to L_{n+1}(X,Y)$ is surjective.

Given $E \in \bar{\mathscr{C}}_{n+1}$, by Lemma B.1 the monomorphism $\sigma_{n+1} : E_{n+1} \to E_n$ of vector bundles over $Y$ can be extended as a monomorphism of vector bundles over $X$ $\sigma'_{n+1} : E_{n+1} \to E_n$. Hence, let $E'_n = \operatorname{coker} \sigma'_{n+1}$, then $E'_n$ is a well-defined vector bundle over $X$. Define

$$E' = (0 \to E'_n \xrightarrow{\rho'_n} E_{n-1} \xrightarrow{\sigma_{n-1}} \cdots \xrightarrow{\sigma_1} E_0 \to 0)$$

where $\rho'_n$ is defined by the following commutative diagram over $Y$

$$\begin{tikzcd} E_n \arrow[rr, two heads] \arrow[dr, "\sigma_n"'] & & E'_n \arrow[dl, "\rho'_n"] \\ & E_{n-1} & \end{tikzcd}$$

If $\sigma''_{n+1}$ is another extension of $\sigma_{n+1}$ leading to a sequence $E''$, then Lemma B.1 ensures that $E'_n \cong E''_n$ and the diagram

$$\begin{tikzcd} E'_n \arrow[r, "\rho'_n"] \arrow[d] & E_{n-1} \arrow[d, "\mathrm{id}"] \\ E''_n \arrow[r, "\rho''_n"] & E_{n-1} \end{tikzcd}$$

is commutative over $Y$. Hence $E' \cong E''$ in $\mathscr{C}_n$. This gives us a well-defined map

$$\bar{\mathscr{C}}_{n+1} \to \mathscr{C}_n, \quad E \mapsto E'. \tag{B.2}$$

Moreover, if

$$Q = (0 \to Q_{n+1} \to Q_n \to 0), \qquad R = (0 \to R_i \to R_{i-1} \to 0) \quad (i \le n)$$

are elementary sequences, then $(E \oplus Q)' \cong E'$ and $(E \oplus R)' \cong E' \oplus R$. Hence the equivalence class of $E'$ in $L_n$ only depends on the equivalence class of $E$ in $L_{n+1}$. Since $\bar{\mathscr{C}}_{n+1} \to L_{n+1}$ is surjective, (B.2) induces a well-defined map

$$L_{n+1} \to L_n, \quad E \mapsto E'. \tag{B.3}$$

The next step is to show that (B.3) is the inverse of

$$L_n \to L_{n+1}, \quad E' \mapsto E'. \tag{B.4}$$

Consider the elementary sequence $P$ with $P_{n+1} = P_n = E_{n+1}$. Then the splitting exact sequence of vector bundles over $X$

$$0 \to E_{n+1} \xrightarrow{\sigma'_{n+1}} E_n \to E'_n \to 0$$

gives us an isomorphism in $\mathscr{C}_{n+1}$ $P \oplus E' \cong E$, so we see that the the composition of (B.3) and (B.4) in either direction is the identity, which completes the proof. □

B.2. **Proof of Theorem 3.5.** We have already constructed the Euler characteristic $\chi : L_1 \to K$ in Section 3.2. It suffices to show that this construction is unique and is a natural isomorphism to finish the proof of Theorem 3.5.

B.2.1. *Uniqueness.* We want to show that if $\chi, \chi' : L_1 \to K$ are two Euler characteristics, then $\chi = \chi'$. Since $\chi$ is a natural isomorphism, it has a well-defined inverse $\chi^{-1}$. Let $T = \chi'\chi^{-1} : K(X,Y) \to K(X,Y)$.

For $Y = \varnothing$, by definition of Euler characteristics we have $\chi(E) = \chi'(E) = \sum_i (-1)^i E_i$, so $T = \chi'\chi^{-1} = \mathrm{id}$. Hence $T = \mathrm{id} : K(X) \to K(X)$.

Now, consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K(X/Y, Y/Y) \arrow[r] \arrow[d, "T"] & K(X/Y) \arrow[r] \arrow[d, "T"] & K(Y/Y) \arrow[r] \arrow[d, "T"] & 0 \\
0 \arrow[r] & K(X/Y, Y/Y) \arrow[r] & K(X/Y) \arrow[r] & K(Y/Y) \arrow[r] & 0
\end{tikzcd}$$

where the two horizontal lines are given by the long exact sequence of $K$-theory of $(X/Y, Y/Y)$, and the last two vertical arrows are the identity by the discussion above. Then we know that $T = \mathrm{id} : K(X/Y, Y/Y) \to K(X/Y, Y/Y)$. Since $K(X/Y, Y/Y) \cong K(X,Y)$, we conclude that $T = \mathrm{id} : K(X,Y) \to K(X,Y)$; that is, $\chi = \chi'$. □

B.2.2. *Natural isomorphism.* The Euler characteristic $\chi : L_1(X,Y) \to K(X,Y)$ connects a semi-group $L_1(X,Y)$ with a group $K(X,Y)$. Indeed, $L_1(X,Y)$ inherits a natural group structure from $\chi$ in the following sense.

**Lemma B.5.** *Let $A$ be a semi-group with identity element $1$, $B$ a group, $\phi : A \to B$ an epimorphism of semi-groups with $\phi^{-1}(1) = \{1\}$. Then $A$ has the structure of a group and $\phi$ is a group isomorphism.*

*Proof.* Once we have shown that $A$ is a group, $\phi$ is naturally an isomorphism by our assumption. So now it suffices to find an inverse for every element in $A$.

Pick an arbitrary $a \in A$. Since $\phi$ is an epimorphism, there exists an element $a' \in A$ such that $\phi(a') = \phi(a)^{-1}$. Hence $\phi(aa') = \phi(a)\phi(a') = \phi(a)\phi(a)^{-1} = 1$. This implies that $aa' \in \phi^{-1}(1) = \{1\}$; that is, $aa' = 1$. We denote $a'$ as the inverse of $a$, so $A$ becomes a group. □

**Proposition B.6.** *$L_1(X)$ is a group and $\chi : L_1(X) \to K(X)$ is a group isomorphism.*

*Proof.* Since $\chi : L_1(X) \to K(X)$ maps $E$ to $E_0 - E_1$ and every element in $K(X)$ can be written as a difference of two vector bundles over $X$, we know that $\chi$ is an epimorphism of abelian semi-groups.

Next we show that $\chi^{-1}(0) = 0$. Suppose $E \in L_1(X)$ satisfies $\chi(E) = 0$. Then, $E_0 - E_1 = 0$ in $K(X)$. This implies an $F \in K(X)$ such that $E_0 \oplus F \cong E_1 \oplus F$. Let $P = (0 \to F \to F \to 0)$ be an elementary sequence. Then $E \oplus F \cong (0 \to E_1 \oplus F \to E_1 \oplus F \to 0)$ and the right hand side is an elementary sequence, implying that $E = 0$ in $L_1(X)$. Then we can apply Lemma B.5 directly. □

**Proposition B.7.** *$L_1(X, \mathrm{pt})$ is a group and $\chi : L_1(X, \mathrm{pt}) \to K(X, \mathrm{pt})$ is a group isomorphism.*

*Proof.* By Lemma B.5, it suffices to show that $\chi : L_1(X, \mathrm{pt}) \to K(X, \mathrm{pt})$ is an epimorphism of abelian semi-groups and $\chi^{-1}(0) = 0$. Consider the follwoing diagram

$$\begin{tikzcd}
0 \arrow[r] & L_1(X, \mathrm{pt}) \arrow[r, "\alpha"] \arrow[d, "\chi"] & L_1(X) \arrow[r, "\beta"] \arrow[d, "\chi"] & L_1(\mathrm{pt}) \arrow[r] \arrow[d, "\chi"] & 0 \\
0 \arrow[r] & K(X, \mathrm{pt}) \arrow[r] & K(X) \arrow[r] & K(\mathrm{pt}) \arrow[r] & 0
\end{tikzcd}$$

Since the middle vertical arrow is an isomorphism by Proposition B.6 and the bottom horizontal sequence is exact, it suffices to show that the top horizontal sequence is also exact, so by diagram chasing the left vertical arrow is also an isomorphism.

The part $\operatorname{im}\alpha \subseteq \ker\beta$ is trivial. For any $E = (0 \to E_1 \to E_0 \to 0) \in L_1(X)$ which lies in $\ker\beta$, since in Proposition B.6 we have already proved that $\chi : L_1(\mathrm{pt}) \xrightarrow{\sim} K(\mathrm{pt})$, $\beta(E) = 0$ is equivalent to $\chi \circ \beta(E) = 0$, and the latter one is equivalent to $\dim E_1|_{\mathrm{pt}} = \dim E_0|_{\mathrm{pt}}$. This tells us that there is an isomorphism $\sigma : E_1|_{\mathrm{pt}} \to E_0|_{\mathrm{pt}}$, so $0 \to E_1 \xrightarrow{\sigma} E_0 \to 0$ is an element in $L_1(X, \mathrm{pt})$ which maps to $E$ through $\alpha$. This shows that $E \in \operatorname{im}\alpha$. Hence, $\ker\beta \subseteq \operatorname{im}\alpha$. We have proved the exactness at $L_1(X)$.

Now we show that $\alpha$ is injective. Consider an element $E = (0 \to E_1 \xrightarrow{\sigma} E_0 \to 0)$in $L_1(X, \mathrm{pt})$ such that $\alpha(E) = 0$ in $L_1(X)$. Then $\chi\circ\alpha(E) = E_0 - E_1 = 0$ in $K(X)$. Without loss of generality, we may assume that $E_1 \cong E_0$ as vector bundles over $X$; otherwise we can replace them by adding a same vector bundle. Let $\tau : E_1 \to E_0$ be the isomorphism over $X$. Though $\tau|_{\mathrm{pt}}$ might not be equal to $\sigma$, we can consider $\sigma \circ (\tau^{-1}|_{\mathrm{pt}}) \in \mathrm{Aut}(E_0|_{\mathrm{pt}})$.

In the category of complex vector bundles, $\mathrm{Aut}(E_0|_{\mathrm{pt}}) \cong \mathrm{GL}_n(\mathbb{C})$is path-connected, where $n = \dim_{\mathbb{C}} E_0$. Hence $\sigma \circ (\tau^{-1}|_{\mathrm{pt}})$ is homotopic to id in $\mathrm{Aut}(E_0|_{\mathrm{pt}})$, and we can use this homotopy to extend $\sigma \circ (\tau^{-1}|_{\mathrm{pt}})$ to an element $\rho \in \mathrm{Aut}(E_0)$. In the category of real vector bundles, $\mathrm{Aut}(E_0|_{\mathrm{pt}}) \cong \mathrm{GL}_n(\mathbb{R})$ has two path components: if $\sigma\circ(\tau^{-1}|_{\mathrm{pt}})$ has positive determinant, it lies in the component of id so the argument above still applies; otherwise, we may replace $E_i$ by $E_i \oplus 1$ where 1 is the trivial bundle, and consider $\sigma \oplus 1$ and $\tau \oplus (-1)$ instead to get an element with positive determinant, and repeat the argument above. In summary, we still get the extension $\rho \in \mathrm{Aut}(E_0)$.

Now, the map $\rho \circ \tau : E_1 \to E_0$ is an isomorphism of vector bundles over $X$ extending $\sigma$, so $E = 0$ in $L_1(X, \mathrm{pt})$. This shows that $\ker\alpha = 0$. □

**Proposition B.8.** *$L_1(X, Y)$ is a group and $\chi : L_1(X, Y) \to K(X, Y)$ is a group isomorphism.*

*Proof.* Observe that $\chi$ is a natural transformation, so it suffices to fix a $(X, Y)$ and prove that $\chi : L_1(X, Y) \to K(X, Y)$ is a group isomorphism. Consider the following commutative diagram

$$\begin{tikzcd} L_1(X/Y, Y/Y) \arrow[r, "\chi"] \arrow[d, "\phi"'] & K(X/Y, Y/Y) \arrow[d, "\psi"] \\ L_1(X, Y) \arrow[r, "\chi"] & K(X, Y) \end{tikzcd}$$

where the maps $\phi, \psi$ are induced by the natural quotient map $(X, Y) \to (X/Y, Y/Y)$. By definition of relative $K$-groups, $\psi$ is an isomorphism. By Proposition B.7, the top horizontal map $\chi$ is an isomorphism as well. Hence $\chi \circ \phi = \psi \circ \chi$ is an isomorphism, implying that $\phi$ is a monomorphism and $\chi$ is an epimorphism.

It suffices to show that $\phi$ is an epimorphism. If that is true, then $\chi^{-1}(0) = 0$ and we can apply Lemma B.5 to get the result. For any element $E = (0 \to E_1 \xrightarrow{\sigma} E_0 \to 0)$ in $L_1(X, Y)$, we may assume $E_0$ to be trivial; otherwise we take the direct sum of $E$ with the elementary sequence $(0 \to E_0^{\perp} \to E_0^{\perp} \to 0)$. By definition of the sequence $E$, $\sigma : E_1|_Y \xrightarrow{\sim} E_0|_Y$ is an isomorphism and hence $E_1|_Y$ is a trivial bundle as well. Hence we are allowed to transform $E_1$ to a vector bundle $E_1'$ over $X/Y$ by pinching $E_1|_Y$ to a vector bundle over a point. Let $E_0'$ be the trivial bundle obtained from $E_0$ in the same way. Define $E' = (0 \to E_1' \to E_0' \to 0) \in L_1(X/Y, Y/Y)$. Then $\phi(E') = E$ and so $\phi$ is an epimorphism. □

B.3. **Proof of Proposition 3.11.** The surjectivity follows directly from the lemma below.

**Lemma B.9.** *Let $E_0, \cdots, E_n$ be vector bundles over $X$. If*

$$E = (0 \to E_n \xrightarrow{\sigma_n} \cdots \xrightarrow{\sigma_1} E_0 \to 0)$$

*where the $\sigma_i$'s are defined over $Y$ that $\sigma_{i-1}\sigma_i = 0$, then the $\sigma_i$'s can be extended to $X$ and the extensions also satisfy that $\sigma_{i-1}\sigma_i = 0$.*

*Proof.* We will show that this is true by induction on the cells of $X - Y$. Hence, it suffices to consider the case when $X$ is obtained from $Y$ by attaching one cell. Let $X = Y \cup_f e^k$ where $f : S^{k-1} \to Y$ is the attaching map.

Let $\pi : Y \sqcup D^k \to X$ be the quotient map. Then we have a pullback bundle $\pi^* E_i \to Y \sqcup D^k$. Since $D^k$ is contractible, the restriction $\pi^* E_i|_{D^k}$ is a trivial bundle $D^k \times V_i$, where $V_i$ is the fiber of $E_i \to X$. Hence, $\pi^* E_i$ is the disjoint union $\pi^* E_i = E_i|_Y \sqcup (D^k \times V_i)$.

The vector bundle homomorphism $\sigma_i : E_i|_Y \to E_{i-1}|_Y$ is the same thing as a section $Y \to \mathrm{Hom}(V_i, V_{i-1})$. Composing it with the map $f : S^{k-1} \to Y$, we get a section

$$\tau_i : S^{k-1} \xrightarrow{f} Y \to \mathrm{Hom}(V_i, V_{i-1}).$$

We extend $\tau_i$ to $D^k$ and the new section is still called $\tau_i$

$$\tau_i : D^k \to \mathrm{Hom}(V_i, V_{i-1})$$
$$u \mapsto \begin{cases} \|u\|\tau_i(u/\|u\|), & u \neq 0 \\ 0, & u = 0 \end{cases}$$

Then we get a vector bundle homomorphism induced by $\tau_i$ (and again we call it $\tau_i$)

$$\tau_i : D^k \times V_i \to D^k \to V_{i-1}.$$

Combining $\sigma_i$ and $\tau_i$ gives us a vector bundle homomorphism $E_i \to E_{i-1}$ over $X$ which extends $\sigma_i$. Since $\sigma_{i-1}\sigma_i = 0$, we have $\tau_{i-1}\tau_i = 0$ and hence the extended vector bundle homomorphism also has this property. □

*Proof of Proposition 3.11.* Lemma B.9 shows us the surjectivity.

For injectivity, consider the pair $(X \times I, X \times \{0\} \cup X \times \{1\} \cup Y \times I)$. Suppose that $E, E' \in \mathscr{D}_n(X, Y)$ satisfy that $\varphi(E) \simeq \varphi(E')$ in $\mathscr{C}_n(X, Y)$; that is, there exists $F \in \mathscr{C}_n(X \times I, Y \times I)$ such that $\varphi(E) \cong F|_{X\times\{0\}}$ and $\varphi(E') \cong F|_{X\times\{1\}}$. Since both $E$ and $E'$ lie in $\mathscr{D}_n(X, Y)$, they have vector bundle homomorphisms defined on $X$ and hence we extend the vector bundle homomorphisms in $F$ naturally to $X \times \{0\} \cup X \times \{1\} \cup Y \times I$. Then $F \in \mathscr{C}_n(X \times I, X \times \{0\} \cup X \times \{1\} \cup Y \times I)$.

Apply the similar argument in the previous paragraph to $F$, we can extend $F$ to be an element in $\mathscr{D}_n(X \times I, X \times \{0\} \cup X \times \{1\} \cup Y \times I)$. Moreover, we have $F|_{X\times\{0\}} \cong E$ and $F|_{X\times\{1\}} \cong E'$ in $\mathscr{D}_n(X, Y)$. This shows that $\varphi(E)$ being homotopic to $\varphi(F)$ implies that $E$ is homotopic to $F$; that is, the injectivity. □

B.4. **Proof of Proposition 3.16.** By Proposition 3.12, $\chi(E \otimes E') = \chi(E)\chi(E')$. Although we have constructed an inverse of $j_n : L_1(X, Y) \to L_n(X, Y)$ in the proof of Proposition 3.4, we introduce a different construction of $j_n^{-1}$ here which gives us $j_2^{-1}(E \otimes E') = F$. Hence, $\chi(E \otimes E') = \chi(F)$.

Let $E \in \mathscr{C}_n(X, Y)$. By introducing metrics we can define the adjoint sequence of $E$ by $E^* = (0 \to E_0 \xrightarrow{\sigma_1^*} E_1 \xrightarrow{\sigma_2^*} \cdots \xrightarrow{\sigma_n^*} E_n \to 0)$. Consider the sequence

$$G = (0 \to G_1 \xrightarrow{\tau} G_0 \to 0)$$

where $G_0 = \bigoplus_i E_{2i}, G_1 = \bigoplus_i E_{2i+1}$ and

$$\tau(e_1, e_3, e_5, \cdots) = (\sigma_1 e_1, \sigma_2^* e_1 + \sigma_3 e_3, \sigma_4^* e_3 + \sigma_5 e_5, \cdots).$$

Since we have the decomposition $E_{2i} = \sigma_{2i+1}(E_{2i+1}) \oplus \sigma_{2i}^*(E_{2i-1})$ by the exactness on $Y$, $\tau$ is an isomorphism of vector bundles over $Y$, so $G \in \mathscr{C}_1(X, Y)$. This gives us a map $\mathscr{C}_n(X, Y) \to \mathscr{C}_1(X, Y), E \mapsto G$. Observe that this map preserves the equivalence relations $\sim$ in $\mathscr{C}_n(X, Y)$ and $\mathscr{C}_1(X, Y)$, so it passes to a map $L_n(X, Y) \to L_1(X, Y)$, which we denote as $j_n^{-1}$. Notice that the construction of $G$ depends on the choice of metric on $E$, but $j_n^{-1} : L_n(X, Y) \to L_1(X, Y)$ turns out to be independent of the choice of metric on $E$ because two choices of metric on $E$ give homotopic results on $G$ in $\mathscr{C}_1(X, Y)$, and hence gives the same element when we pass to $L_1(X, Y)$ (Proposition 3.10). Finally, it is a check by definition that $j_n^{-1}$ is an inverse of $j_n$, and particularly, $j_2^{-1}(E \otimes E') = F$. □

## Acknowledgments

I am very thankful to my mentors Professor Peter May and Dr. Danny Xiaolin Shi for their support, the fructifying conversations, offering advice of resources, and detailed feedback on revising drafts of this paper. Their help enlightens me through understanding the paper [4] and many correlated materials. I would also like to thank Xiuwu Zhu with whom I collaborated in working out the algebraic details of [4], and Weinan Lin for patiently answering my questions on Bott periodicity. I am particularly grateful towards Professor Peter May for his devotion in the REU and for giving the illuminating lectures on algebraic topology which significantly broaden my horizon in mathematics. Finally, I want to thank Professor Yang Su for teaching me topological $K$-theory, Yuqiao Huang and Xiuwu Zhu for catching errors, Professor Ye Tian for his lectures on linear algebra in my freshman year which sparked my interests in Clifford algebras, and the undergraduate program of the University of Chinese Academy of Sciences for their financial support.

## References

[1] Adams, J. Frank. "Vector fields on spheres." *Annals of Mathematics* (1962): 603-632.
[2] Atiyah, Michael Francis. "Thom complexes." *Proceedings of the London Mathematical Society* 3, no. 1 (1961): 291-310.
[3] Atiyah, Michael Francis, and Donald Werner Anderson. *K-theory.* CRC Press, 2018.
[4] Atiyah, Michael F., Raoul Bott, and Arnold Shapiro. "Clifford modules." *Topology* 3, no. Suppl 1 (1969).
[5] Atiyah, Michael F., and Isadore M. Singer. "Index theory for skew-adjoint Fredholm operators." *Publications Mathématiques de l'IHÉS 37* (1969): 5-26.
[6] Bailie, James. "Vector fields on spheres." Australian National University, 2017.
[7] Bott, Raoul. "The stable homotopy of the classical groups." *Annals of Mathematics* (1959): 313-337.
[8] Chevalley, Claude. *The algebraic theory of spinors and Clifford algebras: collected works. Vol. 2.* Springer Science & Business Media, 1996.
[9] Chua, Dexter. "Clifford Algebras and Bott Periodicity." https://dec41.user.srcf.net/exp/clifford/clifford.pdf
[10] Davis, James Frederic, and Paul Kirk. *Lecture notes in algebraic topology.* Vol. 35. American Mathematical Soc., 2001.
[11] Eckmann, Beno. "Gruppentheoretischer Beweis des Satzes von Hurwitz-Radon über die Komposition quadratischer Formen." *Commentarii Mathematici Helvetici* 15, no. 1 (1942): 358-366.
[12] Friedrich, Thomas. *Dirac operators in Riemannian geometry.* Vol. 25. American Mathematical Soc., 2000.
[13] Hatcher, Allen. *Algebraic topology.* Cambridge University Press, 2002.
[14] Hatcher, Allen. *Vector bundles and K-theory, version 2.2.* https://pi.math.cornell.edu/~hatcher/VBKT/VB.pdf, 2017.
[15] Henriques, Andre. "A proof of Bott periodicity via Clifford algebras." http://andreghenriques.com/PDF/BottPer.pdf, 2009.

[16] N. Jacobson. *Basic Algebra II.* Dover Publications, second edition. 2009.

[17] James, I. M. "Spaces associated with Stiefel manifolds." *Proceedings of the London Mathematical Society* 3, no. 1 (1959): 115-140.

[18] Karoubi, Max. "Algébres de Clifford et $K$-théorie." In *Annales scientifiques de l'École normale supérieure*, vol. 1, no. 2, pp. 161-270. 1968.

[19] Knudsen, Ben. "Notes for NU pre-talbot seminar 4/18/13". http://www.hiroleetanaka.com/pretalbot2013/notes/2013-04-18-Ben-Kundsen-TMF.pdf, 2013.

[20] Lam, Tsit-Yuen. *A first course in noncommutative rings.* Vol. 131. Springer Science & Business Media, second edition. 2013.

[21] Lee, H. C. "Sur le théorème de Hurwitz-Radon pour la composition des formes quadratiques." *Commentarii Mathematici Helvetici* 21, no. 1 (1948): 261-269

[22] Maunder, C. R. F. "The spectral sequence of an extraordinary cohomology theory." In *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 59, no. 3, pp. 567-574. Cambridge University Press, 1963.

[23] May, J. Peter. *A concise course in algebraic topology.* University of Chicago Press, 1999.

[24] May, J. Peter. "Bott Periodicity." https://math.uchicago.edu/~chicagotopology2/BottTalk.pdf, 2016.

[25] Milnor, John Willard, and James D. Stasheff. *Characteristic classes.* No. 76. Princeton university press, 1974.

[26] Siegel, Kyler. "Bott Periodicity and Clifford Algebras." http://www.math.columbia.edu/~kyler/BottPeriodicityAndCliffordAlgebras.pdf, 2012.

[27] Steenrod, N. E. and J. H. Whitehead. "Vector Fields on the $n$-Sphere." *Proceedings of the National Academy of Sciences* - PNAS 37, no. 1 (1951): 58-63.

[28] Wood, Reginald. "Banach algebras and Bott periodicity." *Topology* 4, no. 4 (1966): 371-389.