WIGNER'S SEMICIRCLE LAW FOR GAUSSIAN RANDOM MATRICES

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Abstract. In this expository paper, we study Wigner’s semicircle law for two types of Gaussian matrix ensembles, the Gaussian orthogonal ensemble (GOE) and the Gaussian unitary ensemble (GUE). We present two different proofs, an analytic one for GOE and a combinatorial one for GUE.

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1. Introduction

Random matrices are matrices where the entries are random variables. Gaussian random matrices are matrices of size $N$ by $N$ where the entries are independent and identically distributed Gaussian random variables. Figure 1 is the histograms giving the empirical eigenvalues distribution of real Gaussian random matrices of different sizes. We observe a semicircle pattern, which is clearer for larger $N$.

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Figure 1. Empirical Eigenvalue Distribution of GOEs of Different Sizes
The phenomenon above is a special case of Wigner’s semicircle law. Let $A_N$ be a symmetric matrix of size $N$ by $N$ where the entries are independent and identically distributed random variables with bounded moments. Then, Wigner’s semicircle law states that the eigenvalue distribution of $A_N$ converges to a distribution in the shape of a semicircle as $N$ goes to infinity. In this paper, we prove Wigner’s semicircle law for two types of Gaussian matrix ensembles, the Gaussian orthogonal ensemble (GOE) and the Gaussian unitary ensemble (GUE).

This paper is structured as follows. In section 2, we provide the definition and some properties of the Catalan numbers and the semicircle distribution. We also discuss few types of convergence of probability measures. In section 3, we introduce Stieltjes transform, which allows us to prove the convergence of measure $\mu$ by showing the convergence of the function $S_\mu$. Then, using the recurrence relation of Catalan numbers, we show that the Stieltjes transform of the averaged eigenvalue distribution converges to that of the semicircle distribution and then the semicircle law for GOE. In section 4, we use a combinatorial approach involving the Wick formula and genus expansion to show the moments of the eigenvalue distribution converges to the moments of the semicircle distribution and then the semicircle law for GUE.

2. Background

2.1. Catalan Numbers. The Catalan numbers, and the recurrence relation they satisfy, are essential to both the analytic proof of semicircle law for GOE and the combinatorial proof of semicircle law for GUE. In both proofs we will use the fact that the moments of the semicircle distribution are given by the Catalan numbers to show that the moments of the averaged eigenvalue distribution are equal to the moments of the semicircle distribution.

**Definition 2.1. Catalan Numbers.** The Catalan numbers $C_k$ are given by the recurrence relation

$$C_k = \sum_{l=0}^{k-1} C_l C_{k-l-1}, \quad k \geq 1$$

where $C_0 = 1$.

**Theorem 2.2.** The Catalan numbers satisfy the following equation

$$C_k = \frac{1}{k+1} \binom{2k}{k}, \quad k \geq 0.$$

Proof. Consider the generating function of the Catalan numbers

$$G(x) = \sum_{k=0}^{\infty} C_k x^k.$$

Cauchy product of infinite series implies that

$$\left( \sum_{k=0}^{\infty} C_k x^k \right)^2 = \sum_{k=0}^{\infty} \sum_{l=0}^{k} C_l C_{k-l} x^k = \sum_{k=0}^{\infty} C_{k+1} x^k = \sum_{k=1}^{\infty} C_k x^{k-1}.$$

Therefore, $G(x)$ satisfies the following quadratic equation.

$$G(x) = 1 + xG(x)^2.$$
The solutions of this quadratic equation are

\[ G(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x}. \]

Using Newton’s generalized binomial theorem, we have

\[ \sqrt{1 - 4x} = \sum_{k=0}^{\infty} \frac{-1}{2k - 1} \binom{2k}{k} x^k. \]

Substituting this back to the solution with the minus sign, we get

\[ \sum_{k=0}^{\infty} C_k x^k = \frac{1 \pm \sqrt{1 - 4x}}{2x} = \sum_{k=1}^{\infty} \frac{1}{2 (2k - 1)} \binom{2k}{k} x^{k-1} = \sum_{k=0}^{\infty} \frac{1}{k + 1} \binom{2k}{k} x^k. \]

\[ C_k = \frac{1}{k + 1} \binom{2k}{k}. \]

\[ \square \]

2.2. Semicircle Distribution.

**Definition 2.3. Semicircle Distribution.** The (standard) semicircle distribution is the probability measure on \([-2, 2]\) with density

\[ d\mu_{SC}(x) = \frac{1}{2\pi} \sqrt{4 - x^2} dx. \]

**Theorem 2.4.** The moments of the semicircle distribution are given by

\[ \frac{1}{2\pi} \int_{-2}^{2} x^n \sqrt{4 - x^2} dx = \begin{cases} 0, & n \text{ odd} \\ C_k, & n = 2k \text{ even} \end{cases}. \]

**Proof.** Notice that the function inside the integral is odd when \(n\) is odd. Then the integral evaluates to 0. We are now left with the case of \(n\) being even.

Let \(x = 2\sin \theta\) then the \(2k\)-th moment is

\[ \frac{1}{2\pi} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} 2^{2k+2} \sin^{2k} \theta \cos^2 \theta d\theta = \frac{2^{2k+1}}{\pi} \left( \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^{2k} \theta d\theta - \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^{2k+2} \theta d\theta \right). \]

Using the reduction formula, we have

\[ \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^{2k+2} \theta d\theta = \frac{2k - 1}{2k} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^{2k} \theta d\theta \]

\[ = \pi \prod_{l=1}^{k} \frac{2l - 1}{2l}. \]
Then, the $2k$-th moment becomes

$$2^{2k+1} \left( \prod_{l=1}^{k} \frac{2l - 1}{2l} - \prod_{l=1}^{k+1} \frac{2l - 1}{2l} \right) = 2^{2k+1} \frac{1}{2k + 2} \prod_{l=1}^{k} \frac{2l - 1}{2l}$$

$$= 2^{2k+1} \frac{1}{2k + 2} \prod_{l=1}^{k} \frac{2l - 1}{2l} \prod_{l=1}^{k} \frac{2l}{2l}$$

$$= \frac{1}{k+1} \binom{2k}{k}$$

$$= C_k.$$  

\[ \Box \]

### 2.3. Types of Convergence.

We want to show that the eigenvalue distribution “converges” to the semicircle distribution as $N$ goes to infinity. The type of convergence we will show is weak convergence, or convergence in distribution. This type of convergence means that the averaged eigenvalue distribution converges to the semicircle distribution. In fact, based on this paper, one can show a stronger form of convergence called almost sure convergence, or convergence in probability, by showing the concentration phenomenon of the variance. In this subsection, we define weak and vague convergence of measures and show their equivalence on the real line.

**Definition 2.5. Convergence of Measures.** Let $(\mu_N)_{N \in \mathbb{N}}$ be a sequence of finite measures and $\mu$ be a finite measure, then

1. $\mu_N$ converges weakly to $\mu$, denoted by $\mu_N \stackrel{w}{\rightarrow} \mu$ if

$$\int_{\mathbb{R}} f(x) \, d\mu_N(x) \rightarrow \int_{\mathbb{R}} f(x) \, d\mu(x)$$

for all bounded continuous functions on $\mathbb{R}$.

2. $\mu_N$ converges vaguely to $\mu$, denoted by $\mu_N \stackrel{v}{\rightarrow} \mu$ if

$$\int_{\mathbb{R}} f(x) \, d\mu_N(x) \rightarrow \int_{\mathbb{R}} f(x) \, d\mu(x)$$

for all continuous functions on $\mathbb{R}$ that vanish at infinity.

**Lemma 2.6.** If a sequence of probability measures weakly converges to a limit, then the limit is also a probability measure.

**Proof.** For the identity function we have

$$1 = \mu_N(\mathbb{R}) = \int_{\mathbb{R}} 1(x) \, d\mu_N(x) \rightarrow \int_{\mathbb{R}} 1(x) \, d\mu(x) = \mu(\mathbb{R}).$$

\[ \Box \]

**Theorem 2.7.** (Helly’s Selection Theorem.) Every sequence $\mu_N$ of probability measures has a subsequence that converges vaguely to a measure $\mu$ such that $\mu(\mathbb{R}) \leq 1$.

**Proof.** This theorem is an exercise on [3, page 167].

**Theorem 2.8.** Let $(\mu_N)_{N \in \mathbb{N}}$ be a sequence of probability measures and $\mu$ be a probability measure, then $\mu_N \stackrel{w}{\rightarrow} \mu$ if and only if $\mu_N \stackrel{v}{\rightarrow} \mu$. 

Proof.
Proof. The forward direction is clear since continuous functions that vanish at infinity are bounded.

Now we are left with the backward direction. Notice that for every $\epsilon_1 > 0$, there exists $R > 0$ such that $\mu([-R, R]) > 1 - \epsilon_1$. Define a continuous real function as

$$1_{\epsilon_2[-R, R]}(x) = \begin{cases} 1, & [-R, R] \\ \text{some value between 1 and 0}, & (-R - \epsilon_2, -R) \cup (R, R + \epsilon_2) \\ 0, & (-\infty, -R - \epsilon_2] \cup [R + \epsilon_2, \infty) \end{cases}.$$

Since $\mu_N$ converges to $\mu$ vaguely, we have for every $\epsilon_3 > 0$, there exists $N_0 \in \mathbb{N}$ such that $N \geq N_0$ implies that

$$\left| \int_{\mathbb{R}} 1_{\epsilon_2[-R, R]}(x) \, d\mu_N(x) - \int_{\mathbb{R}} 1_{\epsilon_2[-R, R]}(x) \, d\mu(x) \right| < \epsilon_3.$$

Then, for $N \geq N_0$ we have $\mu_N([-R, R]) > 1 - \epsilon_1 - 2\epsilon_2 - \epsilon_3$. We are left with finitely many measures so we can take $R$ large enough such that $\mu_N([-R, R]) > 1 - \epsilon_1 - 2\epsilon_2 - \epsilon_3$ for all $N \in \mathbb{N}$. Since the choice of the epsilons is arbitrary, the sequence of measures $(\mu_N)_{N \in \mathbb{N}}$ is tight.

By Prokhorov’s theorem, the sequence of measures $(\mu_N)_{N \in \mathbb{N}}$ contains a weakly convergent subsequence. Because weak convergence implies vague convergence, all weakly convergent subsequences must converge weakly to $\mu$. Therefore, $\mu_N$ converges to $\mu$ weakly. \hfill $\Box$

3. Semicircle Law for GOE

3.1. Stieltjes Transform. We want to show that the probability measure of the averaged eigenvalue distribution converges weakly to the probability measure of the semicircle distribution. However, it is hard to show the convergence of probability measures directly. In this section, we will introduce Stieltjes transform. Stieltjes transform enables one to prove convergence of probability measures by showing the convergence of a specific function $S(z)$.

**Definition 3.1. Stieltjes Transform.** Let $\mu$ be a finite Borel measure on $\mathbb{R}$, then we define its Stieltjes transform $S_\mu$ as

$$S_\mu(z) = \int_{\mathbb{R}} \frac{1}{x - z} \, d\mu(x), \text{ where } z \in \mathbb{C}^+.$$

**Proposition 3.2.** Let $\mu$ be a probability measure on $\mathbb{R}$ and $S_\mu$ be its Stieltjes transform. Then, we have the following:

1. $S_\mu(z) \in \mathbb{C}^+$
2. $S_\mu$ is analytic on $\mathbb{C}^+$

**Proof.** Recall that the reciprocal of a complex number $z$ is given by $\frac{1}{\overline{z}}$. Since the imaginary part of $z$ is positive, the imaginary part of $\frac{1}{x-z}$ is also positive. Therefore, we have (1).

Since $\frac{1}{x-z}$ is analytic on $\mathbb{C}^+$, its integral over the real line is also analytic on $\mathbb{C}^+$. Then (2) is true. \hfill $\Box$
**Proposition 3.3.** (Stieltjes Inversion Formula) We can recover \( \mu \) from \( S_\mu \) using the Stieltjes inversion formula:

\[
\lim_{\epsilon \to 0} \frac{1}{\pi} \int_a^b \text{Im} \, S_\mu(t + i\epsilon) \, dt = \mu((a, b)) + \frac{1}{2} (\mu(a) + \mu(b))
\]

where \( a < b \).

**Proof.** Notice that

\[
\text{Im} \, S_\mu(t + i\epsilon) = \int_{\mathbb{R}} \text{Im} \left( \frac{1}{x - t - i\epsilon} \right) \, d\mu(x) = \int_{\mathbb{R}} \frac{\epsilon}{(x - t)^2 + \epsilon^2} \, d\mu(x).
\]

Then, for the integral we have

\[
\frac{1}{\pi} \int_a^b \text{Im} \, S_\mu(t + i\epsilon) \, dt = \int_{\mathbb{R}} \frac{1}{\pi} \int_a^b \frac{\epsilon}{(x - t)^2 + \epsilon^2} \, dt \, d\mu(x).
\]

Let \( u = \frac{t-x}{\epsilon} \), then

\[
\frac{1}{\pi} \int_a^b \frac{\epsilon}{(x - t)^2 + \epsilon^2} \, dt = \frac{1}{\pi} \int_{\frac{b-x}{\epsilon}}^{\frac{a-x}{\epsilon}} \frac{1}{u^2 + 1} \, du = \frac{1}{\pi} \left( \tan^{-1} \left( \frac{b-x}{\epsilon} \right) - \tan^{-1} \left( \frac{a-x}{\epsilon} \right) \right).
\]

As \( \epsilon \to 0 \), the expression above goes to 1 when \( x \in (a, b) \), \( \frac{1}{2} \) when \( x = a \) or \( x = b \), and 0 otherwise. Then, the proposition follows. \( \square \)

**Theorem 3.4.** Let \( \mu \) and \( \eta \) be finite measures. If \( S_\mu(z) = S_\eta(z) \) for all \( z \in \mathbb{C}^+ \), then \( \mu = \eta \).

**Proof.** For all \( a, b \) such that \( a < b \) and \( a, b \) are not atoms of \( \mu \) or \( \eta \), it follows from the Stieltjes inversion formula that \( \mu((a, b)) = \eta((a, b)) \). Because uncountably many positive numbers sum to infinity, there can only be countably many atoms. Then, for any interval \( (a, b) \), we can find a positive sequence \( \epsilon_n \to 0 \) such that all \( a + \epsilon_n \) and \( b - \epsilon_n \) are not atoms of \( \mu \) or \( \eta \) and

\[
(a, b) = \bigcup_{n=1}^{\infty} (a + \epsilon_n, b - \epsilon_n).
\]

By the monotone convergence theorem, we have

\[
\mu((a, b)) = \lim_{n \to \infty} \mu((a + \epsilon_n, b - \epsilon_n)) = \lim_{n \to \infty} \eta((a + \epsilon_n, b - \epsilon_n)) = \eta((a, b)).
\]

\( \square \)

**Theorem 3.5.** Let \( (\mu_N)_{N \in \mathbb{N}} \) be a sequence of probability measures and \( \mu \) be a probability measure, then \( \mu_N \xrightarrow{w} \mu \) if and only if \( \lim_{N \to \infty} S_{\mu_N}(z) = S_\mu(z) \) for all \( z \in \mathbb{C}^+ \).

**Proof.** \((\Rightarrow)\) Suppose that \( \mu_N \xrightarrow{w} \mu \). Consider the function \( f : \mathbb{R} \to \mathbb{C} \) with

\[
f(x) = \frac{1}{x - z}, \text{ where } z \in \mathbb{C}^+.
\]
Notice that $f$ is continuous and vanishes at infinity, then $f$ is also bounded. By the definition of weak convergence we have

$$
\int_{\mathbb{R}} f(x) \, d\mu_N(x) \to \int_{\mathbb{R}} f(x) \, d\mu(x).
$$

which means that $S_{\mu_N}(z) \to S_{\mu}(z)$ for all $z \in \mathbb{C}^+$.

($\Leftarrow$) Theorem 2.3 shows that $(\mu_N)_{N \in \mathbb{N}}$ has a subsequence that converges vaguely to a measure $\eta$ such that $\eta(\mathbb{R}) \leq 1$. By similar arguments as the forward direction, we have $S_{\mu}(z) = S_{\eta}(z)$ for all $z \in \mathbb{C}^+$. By Theorem 3.4, $\mu = \eta$. Then, Theorem 2.8 implies that the subsequence converges to $\mu$ also weakly. Since this is true for all convergent subsequences, the whole sequence converges weakly to $\mu$. \ hfill \Box

**Proposition 3.6.** Let $\mu$ be a compactly supported probability measure such that $\mu([-R,R]) = 1$ for some $R > 0$. Then $S_{\mu}$ has the following power series expansion for $|z| > R$

$$
S_{\mu}(z) = -\sum_{n=0}^{\infty} \frac{m_n}{z^{n+1}}.
$$

where $m_n$ is the $n$-th the moment of $\mu$.

**Proof.** For $|z| > R$ we have

$$
S_{\mu}(z) = \int_{\mathbb{R}} \frac{1}{x - z} \, d\mu(x) = \int_{\mathbb{R}} -\frac{1}{z} \frac{1}{1 - \frac{x}{z}} \, d\mu(x) = \int_{\mathbb{R}} -\frac{1}{z} \sum_{n=0}^{\infty} \left( \frac{x}{z} \right)^n \, d\mu(x).
$$

Since the convergence is uniform on $[-R,R]$, we can put the summation out of the integral:

$$
S_{\mu}(z) = \int_{-R}^{R} -\frac{1}{z} \sum_{n=0}^{\infty} \left( \frac{x}{z} \right)^n \, d\mu(x) = -\sum_{n=0}^{\infty} \int_{-R}^{R} \frac{x^n}{z^{n+1}} \, d\mu(x) = -\sum_{n=0}^{\infty} \frac{m_n}{z^{n+1}}.
$$

\ hfill \Box

**Proposition 3.7.** The Stieltjes transform of the semicircle distribution is given by

$$
S_{\mu_{\text{SC}}}(z) = \frac{-z + \sqrt{z^2 - 4}}{2}, \text{ where } z \in \mathbb{C}^+.
$$

Note: This is the solution of the quadratic equation $S_{\mu_{\text{SC}}}(z)^2 + zS_{\mu_{\text{SC}}}(z) + 1 = 0$ that is in $\mathbb{C}^+$.

**Proof.** By Theorem 2.4 and Proposition 3.6, for large $|z|$ we have

$$
S_{\mu_{\text{SC}}}(z) = -\sum_{k=0}^{\infty} \frac{C_k}{z^{2k+1}}
$$

The recurrence relation of Catalan numbers implies that

$$
S_{\mu_{\text{SC}}}(z)^2 = \left( \sum_{k=0}^{\infty} \frac{C_k}{z^{2k+1}} \right)^2 = \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{C_l C_{k-l}}{z^{2k+1}} = \sum_{k=1}^{\infty} \frac{C_k}{z^{2k+1}} = -S_{\mu_{\text{SC}}}(z) - 1.
$$

Therefore, $S_{\mu_{\text{SC}}}(z)$ must satisfy the following equation

$$
S_{\mu_{\text{SC}}}(z)^2 + zS_{\mu_{\text{SC}}}(z) + 1 = 0.
$$
The two solutions of the equation are
\[ S_{\mu_{\text{SC}}}(z) = \frac{-z \pm \sqrt{z^2 - 4}}{2}. \]

Since \( z \in \mathbb{C}^+ \), by Proposition 3.2, we have \( S_{\mu_{\text{SC}}}(z) \in \mathbb{C}^+ \). Then \( S_{\mu_{\text{SC}}}(z) \) must be the solution with the plus sign. \( \square \)

3.2. Analytic Proof. Now we have showed that the Stieltjes transform of the semicircle distribution is a solution of a quadratic equation. In this subsection, we show that the Stieltjes transform of the averaged eigenvalue distribution also converges to the same solution of the same equation.

First, we formally define Gaussian orthogonal ensemble. Using the properties of resolvents, we show that the Stieltjes transform of the averaged eigenvalue distribution satisfies by (3.14). Then, in the rest of the subsection, we show that as a few terms go to 0, a solution of (3.14) goes to the Stieltjes transform of the semicircle distribution.

**Definition 3.8. GOE.** We define the space \( \Omega_N \) of symmetric \( N \times N \) matrices as
\[ \Omega_N = \{ A_N = (a_{ij})_{i,j=1}^N \mid \forall i,j, a_{ij} \in \mathbb{R} \text{ and } a_{ij} = a_{ji} \}. \]

The Gaussian Orthogonal Ensemble (GOE) is described by the probability measure
\[ dP_N(A_N) := Ce^{-\frac{N}{2} \text{Tr}(A_N^2)} \prod_{i \leq j} da_{ij}, \]
where \( C \) is a normalizing constant that makes the measure a probability measure.

**Remark 3.9.** Notice that
\[ \text{Tr}(A_N^2) = \sum_{j=1}^N \sum_{i=1}^N a_{ij}a_{ji} = 2 \sum_{i<j \leq N} a_{ij}^2 + \sum_{i=1}^N a_{ii}^2. \]

Therefore, the variables on and off the diagonal have different variances
\[ \text{Var}(a_{ij}) = \begin{cases} \frac{1}{N}, & i \neq j \\ \frac{2}{N}, & i = j \end{cases}. \]

**Definition 3.10.** Suppose \( A_N \in \Omega_N \) has eigenvalues \( \lambda_1, \ldots, \lambda_N \) counted with multiplicity. Then we define the eigenvalue distribution of \( A_N \) as a measure \( \mu_N \) on \( R \)
\[ \mu_{A_N} := \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i}, \]
where \( \delta_{\lambda_i} \) is the delta measure that is 1 at \( \lambda_i \) and 0 elsewhere.

Then, we define the averaged eigenvalue distribution under the probability measure \( P_N \) as
\[ \mu_N := E[\mu_{A_N}] = \int_{\Omega_N} \mu_{A_N} dP_N. \]
Lemma 3.11. The Stieltjes transform of $\mu_N$ can be written in the following compact form

$$S_{\mu_N}(z) = E\left[\frac{1}{N} \text{Tr}\left((A_N - zI_N)^{-1}\right)\right].$$

Proof. Notice that

$$S_{\mu_N}(z) = \int_{\mathbb{R}} \frac{1}{x - z} \, d\mu_N(x) = E\left[\int_{\mathbb{R}} \frac{1}{x - z} \, d\mu_{A_N}(x)\right].$$

Denote the eigenvalues of $A_N$ by $\lambda_1, \ldots, \lambda_N$, then

$$\int_{\mathbb{R}} \frac{1}{x - z} \, d\mu_{A_N}(x) = \frac{1}{N} \sum_{i=1}^{N} (\lambda_i - z)^{-1}$$

$$= \frac{1}{N} (\text{Tr}(A_N) - \text{Tr}(zI_N))^{-1}$$

$$= \frac{1}{N} (\text{Tr}(A_N - zI_N))^{-1}.$$

Therefore,

$$S_{\mu_N}(z) = E\left[\frac{1}{N} \text{Tr}\left((A_N - zI_N)^{-1}\right)\right].$$

□

Definition 3.12. Resolvent. The resolvent of a matrix $A$ is

$$R_A(z) := (A - zI)^{-1}.$$  

Lemma 3.13. The Stieltjes transform of $\mu_N$ is given by

(3.14) $$S_{\mu_N}(z) = -\frac{1}{z} + \frac{1}{z} E[\text{Tr}(A_N R_{A_N}(z))].$$

Proof. By the definition of resolvents, we have

$$(A_N - zI_N) R_{A_N}(z) = I_N$$

$$A_N R_{A_N}(z) - z R_{A_N}(z) = I_N$$

$$R_{A_N}(z) = -\frac{1}{z} I_N + A_N R_{A_N}(z).$$

Taking trace on both sides, we have

$$\frac{1}{N} \text{Tr}(R_{A_N}(z)) = -\frac{1}{z} + \frac{1}{z} E[\text{Tr}(A_N R_{A_N}(z))].$$

Taking expectation on both sides, we have

$$S_{\mu_N}(z) = E\left[\frac{1}{N} \text{Tr}(R_{A_N}(z))\right] = -\frac{1}{z} + \frac{1}{z} E[\text{Tr}(A_N R_{A_N}(z))].$$

□

Theorem 3.15. (Stein’s Identity) Let $X_1, \ldots, X_k$ be independent Gaussian random variables with $E[X_i] = 0$ and $\text{Var}(X_i) = \sigma_i^2$. Let $g : \mathbb{R}^k \to \mathbb{R}$ be a continuously
differentiable function. If \( g \) and the partial derivatives of \( g \) are of polynomial growth, then for all \( i \) we have

\[
\mathbb{E}[g(X_1, \ldots, X_k)X_i] = \sigma_i^2 \mathbb{E} \left[ \frac{\partial g}{\partial X_i}(X_1, \ldots, X_k) \right].
\]

**Proof.** When \( k = 1 \), the left hand side is

\[
\int_{-\infty}^{\infty} xg(x)e^{-\frac{x^2}{2\sigma_i^2}} \, dx = \int_{-\infty}^{\infty} g(x) \left[ -\sigma_i^2 e^{-\frac{x^2}{2\sigma_i^2}} \right]' \, dx.
\]

Integrating by parts, we have

\[
\int_{-\infty}^{\infty} g(x) \left[ -\sigma_i^2 e^{-\frac{x^2}{2\sigma_i^2}} \right]' \, dx = \int_{-\infty}^{\infty} g'(x)\sigma_i^2 e^{-\frac{x^2}{2\sigma_i^2}} \, dx - \left[ h(x)\sigma_i^2 e^{-\frac{x^2}{2\sigma_i^2}} \right]_{-\infty}^{\infty} = \sigma_i^2 \mathbb{E} \left[ \frac{\partial g}{\partial X_i}(X) \right].
\]

When \( k \geq 1 \), we do the same steps for the \( i \)-th random variable. \( \square \)

**Proposition 3.16.** The last term of (3.14) can be further decomposed as

\[
\frac{1}{N} \mathbb{E} [\text{Tr}(A_N R_{A_N}(z))] = -\frac{1}{N^2} \mathbb{E} \left[ \text{Tr} \left( R_{A_N}(z)^2 \right) \right] - \frac{1}{N^2} \mathbb{E} \left[ \text{Tr} \left( R_{A_N}(z) \right)^2 \right]
\]

**Proof.** Notice that

\[
\text{Tr}(A_N R_{A_N}(z)) = \sum_{i,j=1}^{N} a_{ij}[R_{A_N}(z)]_{ji}.
\]

Then, by Stein’s identity we have

\[
\frac{1}{N} \mathbb{E} [\text{Tr}(A_N R_{A_N}(z))] = \frac{1}{N} \mathbb{E} \left[ \sum_{i,j=1}^{N} a_{ij}[R_{A_N}(z)]_{ji} \right]
\]

\[
= \frac{1}{N} \sum_{i,j=1}^{N} \text{Var}(a_{ij}) \mathbb{E} \left[ \frac{\partial}{\partial a_{ij}}[R_{A_N}(z)]_{ji} \right].
\]

Recall that \((A_N - z\mathbb{I}_N) R_{A_N}(z) = \mathbb{I}_N\). Taking partial derivatives with respect to \( a_{ij} \) on both sides yields

\[
\frac{\partial R_{A_N}(z)}{\partial a_{ij}} (A_N - z\mathbb{I}_N) + R_{A_N}(z) \frac{\partial A}{\partial a_{ij}} = 0
\]

\[
\frac{\partial R_{A_N}(z)}{\partial a_{ij}} = -R_{A_N}(z) \frac{\partial A}{\partial a_{ij}} R_{A_N}(z).
\]

Let \( M_{ij} \) be a matrix that has 1 at the position \((i, j)\) and 0 elsewhere, then the partials of \( R_{A_N} \) are given by

\[
\frac{\partial A_N(z)}{\partial a_{ij}} = \begin{cases} M_{ii}, & i = j \\ M_{ij} + M_{ji}, & i \neq j \end{cases}.
\]
Then, when $i = j$
\[
\left[ \frac{\partial}{\partial a_{ii}} R_{A_N}(z) \right]_{ii} = -[R_{A_N}(z)M_{ii} R_{A_N}(z)]_{ii} = -[R_{A_N}(z)]_{ii}^2.
\]
and when $i \neq j$
\[
\left[ \frac{\partial}{\partial a_{ij}} R_{A_N}(z) \right]_{ji} = -[R_{A_N}(z)M_{ij} R_{A_N}(z)]_{ji} - [R_{A_N}(z)M_{ji} R_{A_N}(z)]_{ij} = -[R_{A_N}(z)]_{ij}^2.
\]

Since $(A_N - zI_N)$ is symmetric, $R_{A_N}(z)$ is also symmetric, which means that $[R_{A_N}(z)]_{ij} = [R_{A_N}(z)]_{ji}$. Recall that by Remark 3.9, the elements on the diagonal have variance $\frac{2}{N}$ and the elements off the diagonal have variance $\frac{1}{N}$. Then, the term $\frac{1}{N} \mathbb{E}[\text{Tr}(A_N R_{A_N}(z))]$ can be expanded as
\[
\frac{1}{N} \sum_{i,j=1}^{N} \text{Var}(a_{ij}) \mathbb{E} \left[ \frac{\partial}{\partial a_{ij}} [R_{A_N}(z)]_{ij} \right]
\]
\[
= -\frac{1}{N} \sum_{i=1}^{N} \frac{2}{N} \mathbb{E} \left[ (R_{A_N}(z))_{ii}^2 \right] - \frac{1}{N} \sum_{1 \leq i \neq j \leq N} \frac{2}{N} \mathbb{E} \left[ (R_{A_N}(z))_{ij}^2 + (R_{A_N}(z))_{ji} (R_{A_N}(z))_{ii} \right]
\]
\[
= -\frac{1}{N^2} \sum_{i,j=1}^{N} \mathbb{E} [R_{A_N}(z)]_{ij}^2 - \frac{1}{N^2} \sum_{i,j=1}^{N} \mathbb{E} \left[ (R_{A_N}(z))_{jj} (R_{A_N}(z))_{ii} \right]
\]
\[
= -\frac{1}{N^2} \mathbb{E} \left[ \text{Tr}(R_{A_N}(z)^2) \right] - \frac{1}{N^2} \mathbb{E} \left[ \text{Tr}(R_{A_N}(z)^2) \right].
\]

Now, we want to show two things. First, the first term $\frac{1}{N^2} \mathbb{E}[\text{Tr}(R_{A_N}(z)^2)]$ goes to 0. Second, the second term $\frac{1}{N^2} \mathbb{E}[\text{Tr}(R_{A_N}(z)^2)]$ goes to $S_{\mu_N}(z)^2$. These two results imply that if $S_{\mu_N}(z)$ converges, the limit satisfies the same equation as $S_{\mu_{SC}}(z)$ as $N \to \infty$. Then, by Theorem 3.4, we can finally show that $\mu_N \stackrel{w}{\to} \mu_{SC}$.

**Proposition 3.17.** The term
\[
\frac{1}{N^2} \mathbb{E} \left[ \text{Tr} \left( R_{A_N}(z)^2 \right) \right]
\]
goes to 0 as $N \to \infty$.

**Proof.** Denote the eigenvalues of $A_N$ by $\lambda_1, ..., \lambda_N$, then the eigenvalues of $R_{A_N}(z)$ are $\frac{1}{(\lambda_i - z)^2}, ..., \frac{1}{(\lambda_N - z)^2}$. Note that for $z \in \mathbb{C}^+$ and any $\lambda_i \in \mathbb{R}$ we have
\[
(\lambda_i - z)^2 = \text{Im}(z)^2 + (\text{Re}(z) - \lambda_i)^2 \geq \text{Im}(z)^2.
\]
Therefore,
\[
\left| \frac{1}{N^2} \mathbb{E} \left[ \text{Tr} \left( R_{A_N}(z)^2 \right) \right] \right| \leq \frac{1}{N^2} \mathbb{E} \left[ \left| \text{Tr} \left( R_{A_N}(z)^2 \right) \right| \right]
\]
\[
\leq \frac{1}{N^2} \mathbb{E} \left( \sum_{i=1}^{N} \frac{1}{(\lambda_i - z)^2} \right)
\]
\[
\leq \frac{1}{N} \frac{1}{\text{Im}(z)^2} \to 0 \text{ as } N \to \infty.
\]
Theorem 3.18. (Gaussian Poincaré Inequality.) Let \(X_1, \ldots, X_k\) be independent Gaussian random variables with \(E[X_i] = 0\) and \(\text{Var}(X_i) = \sigma_i^2\). Let \(f : \mathbb{R}^k \to \mathbb{R}\) be a continuously differentiable function, then

\[
\text{Var}(f(X_1, \ldots, X_k)) \leq \max_i (\sigma_i^2) \mathbb{E}[|\nabla f(X_1, \ldots, X_k)|^2].
\]

Proof. This theorem follows from the introduction of [2]. \(\square\)

Proposition 3.19. The term

\[
\frac{1}{N^2} \mathbb{E} \left[ \text{Tr} \left( R_{A_N} (z) \right)^2 \right]
\]

goes to \(S_{\mu_N}(z)^2\) as \(N \to \infty\).

Proof. Note that for any random variable \(X\) we have

\[
\mathbb{E}[X^2] = \mathbb{E}[X]^2 + \text{Var}(X).
\]

Let \(X = \frac{1}{N} \text{Tr}(R_{A_N}(z))\), then

\[
\mathbb{E} \left[ \frac{1}{N} \text{Tr} \left( R_{A_N} (z) \right)^2 \right] = \mathbb{E} \left[ \frac{1}{N} \text{Tr} (R_{A_N} (z)) \right]^2 + \text{Var} \left( \frac{1}{N} \text{Tr}(R_{A_N}(z)) \right)
\]

\[
= S_{\mu_N}(z)^2 + \text{Var} \left( \frac{1}{N} \text{Tr}(R_{A_N}(z)) \right).
\]

We want to show the variance goes to 0 as \(N \to \infty\) using the Gaussian Poincaré inequality. Since \(\frac{1}{N} \text{Tr}(R_{A_N}(z))\) is complex, we can not apply the inequality directly. However, for any complex random variable \(Z\) we have

\[
\text{Var}(Z) = \text{Var}(\text{Re}(Z)) + \text{Var}(\text{Im}(Z)).
\]

Then, it suffice to show the variance of the real and imaginary parts both go to 0. From the proof of Lemma 3.16 we can see that when \(i \neq j\)

\[
\frac{\partial}{\partial a_{ij}} \frac{1}{N} \text{Tr}(R_{A_N}(z)) = \frac{1}{N} \sum_{k=1}^{N} \left[ \frac{\partial R_{A_N}(z)}{\partial a_{ij}} \right]_{kk}
\]

\[
= -\frac{1}{N} \sum_{k=1}^{N} [(R_{A_N}(z)M_{ij}R_{A_N}(z)]_{kk} + [R_{A_N}(z)M_{ji}R_{A_N}(z)]_{kk}
\]

\[
= -\frac{2}{N} \sum_{k=1}^{N} [R_{A_N}(z)]_{ik} [R_{A_N}(z)]_{kj}
\]

\[
= -\frac{2}{N} [R_{A_N}(z)^2]_{ij}.
\]

Similarly, for \(i = j\), we have

\[
\frac{\partial}{\partial a_{ij}} \frac{1}{N} \text{Tr}(R_{A_N}(z)) = \frac{1}{N} [R_{A_N}(z)^2]_{ii}.
\]
Then, by the proof of Lemma 3.17, we have
\[
\frac{\partial}{\partial a_{ij}} \text{Re} \left( \frac{1}{N} \text{Tr}(R_{AN}(z)) \right) = \text{Re} \left( \frac{\partial}{\partial a_{ij}} \frac{1}{N} \text{Tr}(R_{AN}(z)) \right) \\
\leq \frac{2}{N} \left| [R_{AN}(z)^2]_{ij} \right| \\
\leq \frac{2}{N} \left| [R_{AN}(z)^2] \right| \\
= \frac{2}{N} \left| \text{Tr} (R_{AN}(z)^2) \right| \\
\leq \frac{2}{N \text{Im}(z)^2}.
\]
Applying Gaussian Poincaré inequality to the real part gives
\[
\text{Var} \left( \frac{1}{N} \text{Tr}(R_{AN}(z)) \right) \leq \frac{2}{N} \sum_{i,j=1}^{N} \left( \frac{\partial}{\partial a_{ij}} \text{Re} \left( \frac{1}{N} \text{Tr}(R_{AN}(z)) \right) \right)^2 \\
\leq \frac{8}{N \text{Im}(z)^4} \to 0 \text{ as } N \to \infty.
\]
The same result holds for the imaginary part so \(\text{Var}(\frac{1}{N} \text{Tr}(R_{AN}(z)))\) goes to 0. Therefore, the lemma holds.

**Theorem 3.20.** *(Wigner’s Semicircle Law for GOE)* The averaged eigenvalue distribution \(\mu_N\) converges weakly to the semicircle distribution \(\mu_{SC}\).

**Proof.** Note that
\[
|S_{\mu_N}| = \left| \int_{\mathbb{R}} \frac{1}{x-z} d\mu_N \right| \leq \int_{\mathbb{R}} \left| \frac{1}{x-z} \right| d\mu_N \leq \frac{1}{\text{Im}(z)}.
\]
This means that for every \(z\), \((S_{\mu_N}(z))_N\) is a bounded sequence of complex numbers. Then, it contains a convergent subsequence that converges to some complex number \(w\). Then, combining the results of Proposition 3.16, Proposition 3.17, and Proposition 3.19, we can see that \(w\) satisfies the following equation
\[
w = -\frac{1}{z} - \frac{1}{z} w^2.
\]
Then,
\[
w = -\frac{z \pm \sqrt{z^2 - 4}}{2}.
\]
Since all \(S_{\mu_N}(z)\) are in \(\mathbb{C}^+\), \(w\) must be in \(\overline{\mathbb{C}}^+\). Because \(z\) can not be real, \(w \in \mathbb{C}^+\). By Proposition 3.7, \(w = S_{\mu_{SC}}(z)\). Since this is true for any convergent subsequence of \((S_{\mu_N}(z))_N\), the whole sequence also converges to \(S_{\mu_{SC}}(z)\). Therefore, by Theorem 3.4, we have \(\mu_N \xrightarrow{w} \mu_{SC}\).

4. **Semicircle Law for GUE**

4.1. **Wick Formula.** The Wick formula, some times called Isserlis’ theorem, expands the expectation of the product of multiple random variables. Based on it, we develop the genus expansion of the moments of the GUE.
Definition 4.1. A standard (real) Gaussian random variable is a random variable $X$ with distribution

$$ \mathbb{P}[t \leq X \leq b] = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{x^2}{2}} dx. $$

Proposition 4.2. The moments of a standard Gaussian random variable $X$ are given by

$$ \mathbb{E}[X^n] = \begin{cases} 0, & n \text{ odd} \\ (n-1)!!, & n \text{ even} \end{cases} $$

where the double factorial is defined for odd number $m$ as

$$ m!! = m(m-2)\ldots3\cdot1. $$

Proof. The $n$-th moment of a standard Gaussian random variable $X$ is

$$ \mathbb{E}[X^n] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n e^{-\frac{x^2}{2}} dx. $$

When $n$ is odd, the function inside the integral is odd so the integral evaluates to 0. When $n$ is even, integrating by parts yields

$$ \mathbb{E}[X^n] = \frac{1}{\sqrt{2\pi}} \left[ x^{n-1} e^{-\frac{x^2}{2}} \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} (n-1)x^{n-2} e^{-\frac{x^2}{2}} dx $$

$$ = \frac{n-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{n-2} e^{-\frac{x^2}{2}} dx $$

Then, for $n$ even we have

$$ \mathbb{E}[X^n] = (n-1)(n-3)\ldots5\cdot3\cdot\mathbb{E}[X^0] = (n-1)(n-3)\ldots5\cdot3\cdot1 = (n-1)!!. $$

Note: As a direct consequence, for a standard Gaussian variable $X$ we have $\mathbb{E}[X] = 0$ and $\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 1.$

Definition 4.3. For a natural number $N$, we denote $\{1, \ldots, n\}$, the set of natural numbers up to $n$ by $[n]$. A pairing $\pi$ of $[n]$ is a collection $\{P_1, \ldots, P_k\}$ of subsets of $\pi$ such that the following hold:

1. Each $P_i$ has two elements
2. $P_i \cap P_j = \phi$
3. $\bigcup_{i=1}^k P_i = [n]$

Denote the set of all pairings of $[n]$ by $\mathcal{P}(n)$.

Example 4.4.

1. Consider the set $[4] = \{1, 2, 3, 4\}$. The pairings of $[4]$ are $\{(1,2), (3,4)\}$, $\{(1,3), (2,4)\}$, and $\{(1,4), (2,3)\}$. Then, the number of elements in $\mathcal{P}(4)$ is 3.
2. Consider the set $[5] = \{1, 2, 3, 4, 5\}$. No matter how we pair the elements, there is one left. Then, $\mathcal{P}(5) = \phi$.
3. Consider the set $[6] = \{1, 2, 3, 4, 5, 6\}$. If we pair 1 with 2, there are 4 elements left so there are 3 ways to pair them. There are always 3 ways to pair the
4 elements left no matter what we pair 1 with. Then, the number of elements in $\mathcal{P}(6)$ is $3 \cdot 5 = 15$.

**Proposition 4.5.** The number of elements in $\mathcal{P}(n)$ is given by

$$|\mathcal{P}(n)| = \begin{cases} 0, & n \text{ odd} \\ (n - 1)!!, & n \text{ even} \end{cases}$$

which is equal to $E[X^n]$.

**Proof.** When $n$ is odd, there cannot be pairings of $[n]$ so $|\mathcal{P}(n)| = 0$. Now, we are left with the case of $n$ being even. For any pairing, there must be a subset that contains 1. There are $(n - 1)$ ways to choose the subset that contains 1. After choosing this subset, there are $(n - 2)$ elements left. Therefore, $|\mathcal{P}(n)|$ satisfies the recurrence relation

$$|\mathcal{P}(n)| = (n - 1) |\mathcal{P}(n - 2)|.$$

Then, we have

$$|\mathcal{P}(n)| = (n - 1)(n - 3) \ldots 5 \cdot 3 \cdot |\mathcal{P}(2)| = (n - 1)(n - 3) \ldots 5 \cdot 3 \cdot 1 = (n - 1)!!.$$

□

**Example 4.6.** Let $A$, $B$, and $C$ be standard Gaussian random variables. Then we have the following:

1. $E[AA] = 1$
2. $E[AAA] = 0$
4. $E[AAAAAB] = E[A^4] \cdot E[B] = 3 \cdot 0 = 0$

Note that $E[A^2] \cdot E[B^4] \cdot E[C^2]$ is the number of ways to pair $A$ to $A$, $B$ to $B$, and $C$ to $C$. Therefore, $E[AABBBBCC]$ is the number of pairings that pair $A$ to $A$, $B$ to $B$, and $C$ to $C$. Similarly, $E[AAAAAB] = 0$ because there are no pairings that pair $B$ to $B$.

Now, we want to generalize the pattern we just found to any number of standard Gaussian random variables.

**Theorem 4.7.** (Wick Formula) Let $Y_1, \ldots, Y_m$ be independent standard Gaussian random variables. Let $x_1, \ldots, x_n$ be elements of the set $\{Y_1, \ldots, Y_m\}$. Then, we have

$$E \left[ \prod_{i=1}^n x_i \right] = \sum_{\pi \in \mathcal{P}(n)} \prod_{(i,j) \in \pi} E[x_i x_j]$$

**Proof.** For $i \in [m]$, let $k_i$ be the number of times that the random variable $Y_i$ occurs in $x_1, \ldots, x_n$. Then, by independence, the left hand side can be written as

$$E \left[ \prod_{i=1}^n x_i \right] = E \left[ \prod_{i=1}^m Y_i^{k_i} \right] = \prod_{i=1}^m E \left[ Y_i^{k_i} \right] = \prod_{i=1}^m |\mathcal{P}(k_i)|$$

The number of ways to pair $x_i$ with $x_j$ for each $i$ is the number of pairings in which $x_i$ is paired with $x_i$ for all $i$. Formally,

$$\prod_{i=1}^m |\mathcal{P}(k_i)| = |\{\pi \in \mathcal{P}(n) | x_i = x_j \text{ for all } (x_i, x_j) \in \pi\}|.$$
Note that
\[ \prod_{(i,j) \in \pi} E[x_i x_j] = 1 \]
if and only if \( x_i = x_j \) for all \((i,j) \in \pi \). Otherwise, it is equal to 0. Therefore,
\[ |\{\pi \in \mathcal{P}(n) | x_i = x_j \text{ for all } (x_i, x_j) \in \pi\}| = \sum_{\pi \in \mathcal{P}(n)} \prod_{(i,j) \in \pi} E[x_i x_j] \]

\[ \square \]

4.2. Combinatorial Proof.

Definition 4.8. A standard complex Gaussian random variable is in the form
\[ Z = \frac{X + iY}{\sqrt{2}} \]
where \( X \) and \( Y \) are standard (real) Gaussian random variables.

Remark 4.9. Let \( Z = \frac{X + iY}{\sqrt{2}} \) be a standard complex random variable. Then we have the following:

1. \( E[Z] = \frac{1}{\sqrt{2}} E[X] + \frac{i}{\sqrt{2}} E[Y] = 0 \)
2. \( E[Z^2] = \frac{1}{2} E[X^2] - \frac{1}{2} E[Y^2] + i E[XY] = 0 \)
3. \( E[ZZ^\ast] = \frac{1}{2} E[X^2] + \frac{1}{2} E[Y^2] + \frac{i}{2} E[XY - YX] = 1 \)

Definition 4.10. GUE. A complex Gaussian random matrix, or a Gaussian unitary ensemble, is a unitary matrix of the form
\[ A_N = \frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} a_{ij} \]
where \( a_{ij} \) are independent standard complex Gaussian random variables for \( i \leq j \).

Proposition 4.11. The moments of a Gaussian random matrix \( A_N \) are given by
\[ E\left[ \frac{1}{N} \text{Tr}(A_N^m) \right] = \frac{1}{N^{\frac{m}{2}} + 1} \sum_{i(1), \ldots, i(m)} a_{i(1)i(2)} a_{i(2)i(3)} \cdots a_{i(m)i(1)} \]
where the indices \( i(1), \ldots, i(m) \) are viewed as a function \( i : [m] \to [N] \).

Proof. This equation is \[1, \text{equation 3.4}]\. \hspace{1cm} \square

Definition 4.12. Permutation. For a natural number \( n \) we define permutations of \([n]\) as bijective \([n] \to [n]\) mappings. Let \( i_1, \ldots, i_n \) be elements of \( n \). Then a permutation that maps \( i_1 \) to \( i_2 \), \( i_2 \) to \( i_3 \), \ldots, and \( i_n \) to \( i_1 \) can be written as \((i_1, i_2, i_3, \ldots, i_n)\).

Note: We can consider a pairing \( \pi \) of \([n]\) as a permutation \( \pi \) of \([n]\) such that \( \pi(i) = j \) and \( \pi(j) = i \) for all \((i,j) \in \pi \).

Example 4.13. Consider the set \([4] = \{1, 2, 3, 4\}\). The permutation \( \sigma = (1, 2, 3, 4) \) maps 1 to 2, 2 to 3, 3 to 4, and 4 back to 1. Let \( \pi = (1, 4), (2, 3) \) be a pairing of \([4]\). Then we have \( \pi(1) = 4, \pi(4) = 1, \pi(2) = 3, \) and \( \pi(3) = 2 \).
Now we compose \( \sigma \) with \( \pi \) as \( \sigma \circ \pi \). We know that \( \pi \) maps 1 to 4 and \( \sigma \) maps 4 to 1. Then, \( \sigma \circ \pi \) maps 1 to 1. Doing the same of all indices, we can see that \( \sigma \circ \pi \) maps 1 to 1, 2 to 4, 3 to 3, and 4 to 2. We can then write the permutation \( \sigma \circ \pi \)
as \{(1), (2, 4), (3)\}. Therefore, the number of cycles in \(\sigma \circ \pi\), denoted by \#(\sigma \circ \pi)\), is 3.

**Theorem 4.14.** (Genus Expansion) Let \(A_N\) be a complex Gaussian random matrix of size \(N\). Let \(\sigma = (1, 2, 3, \ldots, m)\) be a permutation of \([m]\). Then the moments of \(A_N\) are given by

\[
\mathbb{E}\left[\frac{1}{N} \text{Tr}(A_N^m)\right] = \sum_{\pi \in \mathcal{P}(m)} N^{\#(\sigma \circ \pi) - \frac{m}{2} - 1}
\]

where \#(\sigma \circ \pi) denotes the number of cycles in the permutation \(\sigma \circ \pi\).

**Proof.** Combining Theorem 4.7 and Proposition 4.11 we get

\[
\mathbb{E}\left[\frac{1}{N} \text{Tr}(A_N^m)\right] = \frac{1}{N^\frac{m}{2} + \frac{1}{2}} \sum_{i(1), \ldots, i(m) \in \mathcal{P}(m)} \sum_{j, k \in \mathbb{Z}} \mathbb{E}\left]\left[ a_{i(j)i(\sigma(j))} a_{i(k)i(\sigma(k))} \right]\right.\
\]

Notice that the expression above is 0 when \(m\) is odd. We are left with the case of \(m\) being even.

We can see that the product in the expression above is 1 if and only if \(i(j) = i(\sigma(k))\) and \(i(\sigma(j)) = i(k)\) for all \((j, k) \in \pi\). Then, the product of the expectations is 1 if and only if \(i(j) = i(\sigma \circ \pi(j))\) for all \(k\) in \([m]\). Therefore, we have

\[
\mathbb{E}\left[\frac{1}{N} \text{Tr}(A_N^m)\right] = \frac{1}{N^\frac{m}{2} + \frac{1}{2}} \sum_{\pi \in \mathcal{P}(m)} \sum_{i(1), \ldots, i(m) \in \mathbb{Z}} \prod_{j, k \in \mathbb{Z}} [i(j) = i(\sigma \circ \pi(j))]
\]

Let us fix \(\pi\). The previous equation implies that \(i(j) = i(\sigma \circ \pi(j)) = i((\sigma \circ \pi)^2(j)) = \cdots = i(j)\) for all \(k\). To satisfy the condition, \(i\) must be constant on the cycles of the permutation \(\sigma \circ \pi\). Then, the condition is satisfied \(N^{\#(\sigma \circ \pi)}\) times. \(\square\)

**Definition 4.15.** A pairing \(\pi \in \mathcal{P}(m)\) is called non-crossing if \(i < k < j < l\) implies that \((i, j)\) and \((k, l)\) can both be pairs in \(\pi\). We denote the set of non-crossing pairings in \(\mathcal{P}(m)\) by \(\mathcal{NC}(m)\).

**Example 4.16.** Let us again consider the set \([4]\) = \(\{1, 2, 3, 4\}\). The pairings \(\{(1, 2), (3, 4)\}\) and \(\{(1, 4), (2, 3)\}\) are non-crossing. In pairings \(\{(1, 3), (2, 4)\}\), the pair \((1, 3)\) crosses with the pair \((2, 4)\). Then, \(|\mathcal{NC}(4)| = 2\).

\[
\begin{align*}
1 & \ 2 & \ 3 & \ 4 \\
\{(1, 2), (3, 4)\} & \ & \ & \\
1 & \ 2 & \ 3 & \ 4 \\
\{(1, 4), (2, 3)\} & \ & \\
1 & \ 2 & \ 3 & \ 4 \\
\{(1, 3), (2, 4)\} & \\
\end{align*}
\]

**Example 4.17.** Let us consider the three pairings of \([4]\). We can see that \(\sigma \circ \pi\) for each of the pairings is given by:

- When \(\pi = \{(1, 2), (3, 4)\}\), \(\sigma \circ \pi = \{(1, 3), (2, 4)\}\), \(#(\sigma \circ \pi) = 3 = \frac{3}{2} + 1\).
- When \(\pi = \{(1, 4), (2, 3)\}\), \(\sigma \circ \pi = \{(1, 2), (4, 3)\}\), \(#(\sigma \circ \pi) = 3 = \frac{3}{2} + 1\).
- When \(\pi = \{(1, 3), (2, 4)\}\), \(\sigma \circ \pi = \{(1, 4, 3, 2)\}\), \(#(\sigma \circ \pi) = 1 \leq \frac{3}{2} + 1\).
Therefore, as \( N \) goes to infinity, the 4-th moment of a complex Gaussian random matrix \( A_N \) is
\[
\lim_{N \to \infty} \mathbb{E}\left[ \frac{1}{N} \text{Tr}(A_N^4) \right] = \lim_{N \to \infty} \left( N^0 + N^0 + N^{-2} \right) = 2 = |\mathcal{NC}(4)|.
\]

In fact, the equation above holds for all the moments of the \( A_N \). We will see in the next Proposition why this is true.

**Proposition 4.18.** Let \( \sigma \) and \( #(\sigma \circ \pi) \) be the same as in Theorem 4.14. Let \( m \) be even and \( \pi \) be a pairing of \([m]\). Then we have
\[
#(\sigma \circ \pi) \leq \frac{m}{2} + 1
\]
Furthermore, equality holds if and only if \( \pi \in \mathcal{NC}(m) \).

**Proof.** Notice that for any \( \pi \in \mathcal{NC}(m) \) we have \( \pi(1) = 2k \) for some \( k \). This is because numbers between 1 and \( 2k \) can not be paired with numbers greater than \( 2k \). Then, we can consider the pairs between 1 and \( 2k \) as a pairing in \( \pi \in \mathcal{NC}(2k-2) \) and apply the argument above again. Repeating the process, we can always find a number \( i \) such that \((i, \sigma(i)) \in \mathcal{NC}(m)\).

Suppose the pair \((i, \sigma(i))\) is in \( \pi \in \mathcal{NC}(m) \). We have \( \sigma \circ \pi(i) = \sigma(\sigma(i)) \) and \( \sigma \circ \pi(\sigma(i)) = \sigma(i) \). Then, \((\sigma(i)) \) and \((\ldots, i, \sigma^2(i), \ldots)\) are circles in the permutation \( \sigma \circ \pi \). When we remove the pair \((i, \sigma(i))\) from \( \pi \), the set of the remaining pairs can be considered as a pairing is \( \pi \in \mathcal{NC}(m-2) \). Denote the set of the remaining pairs by \( \pi_1 \), we can see that \( \pi_1 \) has 1 less cycle, which is \((\sigma(i))\), than \( \pi \) does.

Repeating the process above \( \frac{m}{2} - 1 \) times, we have \( \pi_\frac{m}{2} - 1 = (1, 2) \in \mathcal{NC}(2) \). Since \( \pi_{\frac{m}{2} - 1} = (1, 2) \) and \( \sigma = (1, 2) \), the cycles in \( \sigma \circ \pi \) are \((1)\) and \((2)\). Then, the number of cycles in \( \sigma \circ \pi \) is given by \( \frac{m}{2} - 1 + 2 = \frac{m}{2} + 1 \).

When we apply the same process for a pairing \( \pi \in (\mathcal{P}(m) \setminus \mathcal{NC}(m)) \), there will not be a pair in the form of \((i, \sigma(i))\) after some iterations. Otherwise, the pairing is non-crossing. Therefore, the number of cycles in \( \sigma \circ \pi \) is strictly less than \( \frac{m}{2} + 1 \) for \( \pi \in (\mathcal{P}(m) \setminus \mathcal{NC}(m)) \). \( \square \)

**Example 4.19.** Let a non-crossing pairing of \([8]\) be \( \pi = \{(1, 8), (2, 3), (4, 7), (5, 6)\} \). Then, \( \sigma \circ \pi = \{(1), (2, 4, 8), (3), (5, 7), (6)\} \).

We can see the two adjacent pairs are \((2, 3)\) and \((5, 6)\). These two pairs corresponds to the cycles \((3)\) and \((6)\). Removing them, \( \sigma \circ \pi \) becomes \{(1), (4, 8), (7)\}.  

\[
|\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{array}|
\]

We can see the two adjacent pairs are \((2, 3)\) and \((5, 6)\). These two pairs corresponds to the cycles \((3)\) and \((6)\). Removing them, \( \sigma \circ \pi \) becomes \{(1), (4, 8), (7)\}.
Now the adjacent pair is \((4, 7)\), which corresponds to the cycle \((7)\). Removing it, we have \(\{(1), (8)\}\).

\[
\begin{array}{c}
1 \\
\hline
8
\end{array}
\]

Summing up, we can see that the total number of cycles in the original \(\sigma \circ \pi\) is \(8 + 1 = 5\). Consider the crossing pairing \(\pi = \{(1, 2), (3, 5), (4, 6)\}\). The composed permutation is \(\sigma \circ \pi = \{(1, 3, 6, 5, 4), (2)\}\). Removing the adjacent pair \((1, 2)\), \(\sigma \circ \pi\) becomes \(\{(3, 6, 5, 4)\}\). There are no adjacent pairs. Therefore, the number of cycles in the original \(\sigma \circ \pi\) is \(2\leq \frac{6}{2} + 1\).

**Proposition 4.20.** The number of pairings in \(NC(2k)\), denoted by \(|NC(2k)|\), is equal to the Catalan number \(C_k\).

**Proof.** Recall from Proposition 4.18 that \((1, 2l)\) is in \(\pi\) for some natural number \(l\). The pair \((1, 2l)\) splits \(\pi\) into two pairings: \(\pi_1\) contains the pairs between 1 and \(2l\) and \(\pi_2\) contains rest of the pairs. We can see that \(\pi_1 \in NC(2l-2)\) and \(\pi_2 \in NC(2k-2l)\). Since \(l\) can be any number in \([k]\), we have

\[
|NC(2k)| = \sum_{l=1}^{k} |NC(2(l-1))||NC(2(k-l))|
\]

Note that we need to set \(|NC(0)|\) to 1. The recurrence relation above is exactly the one that defines the Catalan numbers. Therefore, \(|NC(2k)| = C_k\). \(\square\)

**Proposition 4.21.** Let \(A_N\) be a complex Gaussian random matrix, then

\[
\lim_{N \to \infty} \mathbb{E}\left[ \frac{1}{N} \text{Tr}(A_N^m) \right] = \frac{1}{2\pi} \int_{-2}^{2} x^m \sqrt{4-x^2} \, dx
\]

**Proof.** From Theorem 4.14 we know that for even \(m\) we have

\[
\lim_{N \to \infty} \mathbb{E}\left[ \frac{1}{N} \text{Tr}(A_N^m) \right] = \sum_{\pi \in P(m)} \lim_{N \to \infty} \left( N^{\#(\pi)} \frac{2}{2(m-1)} \right)
\]

By Theorem 4.18, the term \(N^{\#(\pi)} \frac{2}{2(m-1)}\) does not go to 0 if and only if \(\pi \in NC(m)\). Then, we have

\[
\sum_{\pi \in P(m)} \lim_{N \to \infty} \left( N^{\#(\pi)} \frac{2}{2(m-1)} \right) = |NC(m)| = C_m^2
\]

for \(m\) even. Since both sides of the equation that we want to prove are 0 when \(m\) is odd, the equation holds for all \(m \in \mathbb{N}\). \(\square\)

**Theorem 4.22.** (Wigner’s Semicircle Law for GUE) Let the measures \(\mu_{AN}\) and \(\mu_N\) be defined similarly as in Definition 3.10. Then the averaged eigenvalue distribution \(\mu_N\) converges weakly to the semicircle distribution \(\mu_{SC}\).

**Proof.** Let \(f : \mathbb{R} \to \mathbb{R}\) that is continuous and vanishes at infinity. Fix \(\epsilon > 0\). Then there exist \(r\) such that \(\int_{-\infty}^{\infty} f(x) \mathbb{I}_{[r, \infty]} \, dx < \epsilon\). By the Stone-Weierstrass
Theorem, there exists a polynomial \( P \) such that \( |P(x) - f(x)| < \epsilon \) for all \( x \in [-r, r] \). Then, by triangular inequality
\[
\left| \int_{-\infty}^{\infty} f(x) \, d\mu_N(x) - \int_{-\infty}^{\infty} f(x) \, d\mu_{SC}(x) \right| \\
\leq \left| \int_{-\infty}^{\infty} f(x) \mathbb{1}_{[r, r]}(x) \, d\mu_N(x) \right| + \left| \int_{-\infty}^{\infty} (f(x) - P(x)) \mathbb{1}_{[-r, r]}(x) \, d\mu_N(x) \right| \\
+ \left| \int_{-\infty}^{\infty} P(x) \, d\mu_N(x) - \int_{-\infty}^{\infty} P(x) \, d\mu_{SC}(x) \right| \\
+ \left| \int_{-\infty}^{\infty} (f(x) - P(x)) \mathbb{1}_{[-r, r]}(x) \, d\mu_{SC}(x) \right| + \left| \int_{-\infty}^{\infty} P(x) \mathbb{1}_{[r, r]}(x) \, d\mu_N(x) \right|.
\]

We can see that the first, second, and fourth term are all bounded by \( \epsilon \). Note that a polynomial is a linear combination of the monomials. Then, by Proposition 4.21, the third term is 0. Taking \( r \geq 2 \), the last term is also 0. Since \( \epsilon \) is arbitrary, the sum can be arbitrarily small. Then, \( \mu_N \) converges to \( \mu_{SC} \) vaguely. By Theorem 2.8, \( \mu_N \) converges to \( \mu_{SC} \) weakly. \( \square\)

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References