ON THE GENERATORS OF THE REAL COBORDISM RING

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ABSTRACT. In this paper, we compute the Stiefel-Whitney classes of the Dold manifolds and give a proof that some of them are generators of the real cobordism ring, following Thom's results in [1].

CONTENTS

1. INTRODUCTION

The study of the cobordism ring $\mathfrak{N}$ is basically about when a manifold can be realized as the boundary of another manifold. In the celebrated paper [1] by Thom, it is proved that $\mathfrak{N}$ is isomorphic to some polynomial algebra over $\mathbb{F}_2$. He also finds an equivalent description of the generators in terms of the Stiefel-Whitney classes but fails to give all the representatives explicitly. In [3], Dold introduces what are later called the Dold manifolds and proves that some of them are exactly the generators of $\mathfrak{N}$, completing the characterization of the cobordism ring. However, the paper is in German. Hence the author writes down this calculation of the Stiefel-Whitney classes of Dold manifolds independently.

In chapter 2, we briefly overview some backgroud of the cobordism ring and characteristic classes, mostly based on [1]. In chapter 3, we introduce the Dold manifolds and their topological properties. In chapter 4, we compute the characteristic classes of the tangent bundles of Dold manifolds. In chapter 5, we prove that some of the Dold manifolds and the even dimensional real projective spaces together give a complete generating set of $\mathfrak{N}$.

2. THE COBORDISM RING AND STIEFEL-WHITNEY CLASSES

In this paper, by a manifold M^n we mean an n-dimensional C^∞-manifold (possibly with boundary).

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A compact manifold M^n is called *null-cobordant* if there exists a compact manifold W^{n+1} with boundary M^n. Two compact manifolds M_1^n and M_2^n are *cobordant* if the disjoint union $M_1 \sqcup M_2$ is null-cobordant.

Being cobordant is an equivalence relation, and the equivalence classes are called *cobordism classes*. We use $[M]$ to denote the class containing M. We can define an additive structure by setting

$$[M_1] + [M_2] = [M_1 \sqcup M_2].$$

This operation is well-defined because if M_1 and M_1' are cobordant through W^{n+1}, then the boundary of $W^{n+1} \sqcup (M_2 \times I)$ is exactly $(M_1 \sqcup M_2) \sqcup (M_1' \sqcup M_2)$. The class of a single point is the identity and the inverse of $[M]$ is $[M]$ itself since $\partial(M \times I) = M \sqcup M$. The addition is commutative and associative by the definition, making the set of n-dimensional classes into an abelian group. We denote this abelian group by $\mathfrak{N}^n$.

Moreover, we can define a multiplication among the groups $\mathfrak{N}^k$ by setting $[M] \times [N] = [M \times N]$. The multiplication maps from $\mathfrak{N}^m \times \mathfrak{N}^n$ to $\mathfrak{N}^{m+n}$. This is well-defined because if $\partial W = M \sqcup M'$, we have $\partial(W \times N) = (M \times N) \sqcup (M' \times N)$. It is also distributive with respect to the summation. Thus, the direct sum $\mathfrak{N}$ of the abelian groups $\mathfrak{N}^n$ forms a graded ring. We call $\mathfrak{N}$ the (real) cobordism ring. Since every element in $\mathfrak{N}$ has order 2, it is actually a $\mathbb{F}_2$-algebra.

Remark 2.1. Sometimes $\mathfrak{N}$ is called the modulo 2 cobordism ring, in contrast to the oriented version Ω where oriented manifolds are considered.

In [1], Thom computes the multiplicative structure of the cobordism ring $\mathfrak{N}$. One of the main tools he uses is the Stifel-Whitney classes for real vector bundles.

Definition 2.2. The Stiefel-Whitney classes of any real vector bundle are determined by the following axioms:

Axiom 1. For any rank-k real vector bundle over a topological space B, there is a collection of cohomology classes $w_i(E) \in H^i(B;\mathbb{Z}_2)$ for $i \geq 0$ depending on E (where we write $\mathbb{Z}_2$=$\mathbb{Z}/2\mathbb{Z}$). The class $w_0(E)$ equals the unit element in $H^0(B;\mathbb{Z}_2)$ and $w_i(E) = 0$ for $i > k$. The class $w_i(E)$ is called the *i-th Stiefel-Whitney class* of E. The *total Stiefel-Whitney class* $w(E)$ is defined to be $w_0(E) + \cdots + w_k(E)$.

Axiom 2. (Naturality) Suppose $f : B' \to B$ is a continuous map and $E \to B$ is a real vector bundle over B. Then

$$f^*(w_i(E)) = w_i(f^*E)$$

for all $i \geq 0$. Here f^*E denotes the pull-back bundle of E.

Axiom 3. (The Whitney product theorem) If E and E' are two real vector bundles over the same base space B, then

$$w(E \oplus E') = w(E) \smile w(E').$$

Axiom 4. For the tautological line bundle η over $\mathbb{R}P^1$, the first Stiefel-Whitney class $w_1(\eta) \neq 0$.

The Stiefel-Whitney classes are uniquely determined for any real vector bundle by the axioms above. See [2].

For a closed manifold M^n, we have $H^n(M;\mathbb{Z}_2) \cong H_n(M;\mathbb{Z}_2) \cong \mathbb{Z}_2$ and a non-degenerate pairing

$$\langle \cdot, \cdot \rangle : H^n(M;\mathbb{Z}_2) \times H_n(M;\mathbb{Z}_2) \to \mathbb{Z}_2$$

by Poincaré duality. Given a partition $\pi = (i_1, \dots, i_j)$ of integer n with $i_1, \dots, i_j > 0$, define the corresponding Stiefel-Whitney number as

$$w_\pi(M) = \langle w_{i_1}(TM) \smile \cdots \smile w_{i_j}(TM), [M] \rangle \in \mathbb{Z}_2.$$

Here $[M]$ is the $\mathbb{Z}_2$-fundamental class of M.

If M is null-cobordant, say $M^n = \partial W^{n+1}$, let ι be the inclusion map $M \to W$. Then the pull-back bundle $\iota^*(TW)$ is the direct sum of TM and the normal bundle $\nu(M)$ on M. The bundle $\nu(M)$ is a trivial line bundle because the inner normal vector field gives a nowhere zero section. By naturality and the Whitney product theorem, we have

$$\iota^* w_i(TW) = w_i(TM \oplus \nu(M)) = w_i(TM)$$

for all i. Thus

$$\begin{aligned} w_\pi(M) &= \langle w_{i_1}(TM) \smile \cdots \smile w_{i_j}(TM), [M] \rangle \\ &= \langle \iota^* w_{i_1}(TW) \smile \cdots \smile \iota^* w_{i_j}(TW), [M] \rangle \\ &= \langle w_{i_1}(TW) \smile \cdots \smile w_{i_j}(TW), \iota_*[M] \rangle. \end{aligned} \tag{2.3}$$

Consider the long exact sequence for the $\mathbb{Z}_2$-homology of the pair $(W, \partial W)$:

$$0 = H_{n+1}(W) \longrightarrow H_{n+1}(W, \partial W) \xrightarrow{\partial_*} H_n(M) \xrightarrow{\iota_*} H_n(W).$$

By Poincaré duality, $H_{n+1}(W) \cong H_{n+1}(W, \partial W) \cong \mathbb{Z}_2$. The boundary map ∂_* is an injection by exactness, hence an isomorphism, mapping the fundamental class $[W, \partial W]$ to $[M]$. Again by exactness, we have

$$\iota_*[M] = \iota_*(\partial_*[W, \partial W]) = 0.$$

By (2.3) we get $w_\pi(M) = 0$ for every partition π.

Thus one necessary condition for a n-manifold to be null-cobordant is that all of its Stiefel-Whitney numbers vanish. In fact this is also a sufficient condition, and it could be rephrased as a condition for two manifolds to be cobordant:

Theorem 2.4 (Thom [1])**.** *Two manifolds M and M' are cobordant if and only if they have equal Stiefel-Whitney numbers.*

Therefore, we can use Stiefel-Whitney numbers to characterize cobordism classes as well as the structure of $\mathfrak{N}$.

A useful viewpoint we will adopt towards the Stiefel-Whitney classes is to consider w_i as the elementary symmetric polynomial of degree i in viariables $t_1, t_2, \dots, t_r$ of degree one. One motivation is the *splitting principle*, which says that for any real bundle $E \to B$, there is a space B' and a map $f : B' \to B$ such that f^*E is a direct sum of line bundles and f induces an injection on cohomology. Write $f^*E = L_1 \oplus \cdots \oplus L_n$. Then by the Whitney product theorem we have

$$f^*w(E) = w(L_1) \dots w(L_n) = (1 + w_1(L_1)) \dots (1 + w_1(L_n))$$

or

$$\begin{aligned}
f^*w_0(E) &= 1,\\
f^*w_1(E) &= w_1(L_1) + \cdots + w_1(L_n),\\
&\cdots\\
f^*w_n(E) &= w_1(L_1)w_1(L_2)\dots w_1(L_n).
\end{aligned}$$

Since the elementary symmetric polynomials generate the ring of symmetric polynomials, the Stiefel-Whitney numbers are in 1-1 correspondence with symmetric polynomials of degree n in t_i.

The following theorem gives a complete characterization of $\mathfrak{N}$.

Theorem 2.5 (Thom [1])**.** *The ring $\mathfrak{N}$ of cobordisms is isomorphic to the polynomial algebra over $\mathbb{Z}_2$ with exactly one generator x_k of degree k for each positive integer $k \neq 2^m - 1$. For any such k, the generator x_k could be chosen to be the cobordism class containing any k-manifold M_k whose Stiefel-Whitney number corresponding to the symmetric polynomial $\sum_i t_i^k$ is non-zero.*

In the following chapters, we will identify suitable representations M_k for all generators.

3. Dold manifolds

In this section, we define the Dold manifolds and give a characterization of their cohomology.

Definition 3.1. Fix $m, n > 1$. Let τ be the involution of $S^m \times \mathbb{C}P^n$ defined by

$$\tau(x, [y]) = (-x, [\bar{y}]), x \in S^m, [y] \in \mathbb{C}P^n.$$

The involution τ gives a free $\mathbb{Z}_2$ action on $S^m \times \mathbb{C}P^n$. The *Dold manifold* $P(m, n)$ is defined to be the $(m + 2n)$-dimensional quotient manifold of $S^m \times \mathbb{C}P^n$ by the $\mathbb{Z}_2$-action. We use $[x, [y]]$ to denote the point in $P(m, n)$ corresponding to the equivalence class containing $(x, [y])$.

Since the $\mathbb{Z}_2$-action on $S^m \times \mathbb{C}P^n$ given by τ is free as the antipodal map on the S^m component, $P(m, n)$ can be viewed as a $\mathbb{C}P^n$-bundle over $\mathbb{R}P^m$. Denote the bundle map by p.

Embed S^m into $\mathbb{R}^{m+1}$ as the unit sphere. We define

$$\begin{aligned}
C_i^+ &= \{(x_0, \dots, x_m) \in S^m : x_i > 0, x_{i+1} = \cdots = x_m = 0\},\\
C_i^- &= \{(x_0, \dots, x_m) \in S^m : x_i < 0, x_{i+1} = \cdots = x_m = 0\}.
\end{aligned}$$

Then for $i = 0, \dots, m$, $C_i^\pm$ is an i-cell in S^m and all $C_i^\pm$ together give a cell decomposition of S^m.

For $\mathbb{C}P^n$, we define

$$D_j = \{[z_0, \dots, z_n] \in \mathbb{C}P^n : z_j \neq 0, z_{j+1} = \cdots = z_n = 0\}.$$

Here D_j is a $2j$-cell in $\mathbb{C}P^n$ and all D_j together give a cell decomposition of $\mathbb{C}P^n$.

The product cell structure of $S^m \times \mathbb{C}P^n$ consists of the cells $C_i^\pm \times D_j$ of dimension $i+2j$. The involution τ is compatible with this cell structure of $S^m \times \mathbb{C}P^n$, mapping $C_i^\pm \times D_j$ to $(-1)^{i+j+1} C_i^\mp \times D_j$. Thus the cell structure of $S^m \times \mathbb{C}P^n$ induces a cell structure of the quotient manifold $P(m,n)$. Denote the image of $C_i^{\pm} \times D_j$ in $P(m,n)$ by $[C^i, D^j]$. The boundary map of the cell complex is given by

$$\begin{cases} \partial([C_i, D_j]) = (1 + (-1)^{i+j+1})[C_{i-1}, D_j] & \text{for } i = 1, 2, \dots, m, j = 0, 1 \dots, n, \\ \partial([C_0, D_j]) = 0 & \text{for } j = 0, 1 \dots, n. \end{cases}$$

If we take $\mathbb{Z}_2$ as coefficient, the cellular boundary map is zero for every dimension and all $[C_i, D_j]$ form a basis of $H_*(P(m,n); \mathbb{Z}_2)$. Define a 1-cochain c and a 2-cochain d by

$$\langle c, [C_i, D_j] \rangle = \delta_1^i \delta_0^j,$$
$$\langle d, [C_i, D_j] \rangle = \delta_0^i \delta_1^j,$$

where $\delta_i^j = 0$ if $i \neq j$ and $\delta_i^j = 1$ if $i = j$.

The coboundary map is zero for every dimension by duality, so $c, d \in H^*(P(m,n); \mathbb{Z}_2)$. It is proven in [3] that the $\mathbb{Z}_2$-coefficient cohomology ring of $P(m,n)$ is isomorphic to the polynomial ring

$$\mathbb{Z}_2[c,d]/(c^{m+1}, d^{n+1}).$$

4. Stiefel-Whitney classes of Dold manifolds

Let $TP(m,m)$ be the tangent bundle of $P(m,n)$.

Theorem 4.1. *The total Stiefel-Whitney class of $TP(m,n)$ is given by*

$$(1 + c + d)^{n+1}(1 + c)^m.$$

Proof. We first describe two models for the tangent spaces of S^m and $\mathbb{C}P^n$.

Embed S^m into $\mathbb{R}^{m+1}$ canonically. For any unit vector x, identify $T_x S^m$ with vectors in $\mathbb{R}^{m+1}$ that are perpendicular to x.

For any non-zero vector y in $\mathbb{C}^{n+1}$, identify $T_{[y]}\mathbb{C}P^n$ with the complex linear space $\mathrm{Hom}_{\mathbb{C}}(\langle y \rangle, \langle y \rangle^{\perp})$. Here $\langle y \rangle$ means the 1-dimensional complex subspace of $\mathbb{C}^{n+1}$ spanned by y.

By definition, the Dold manifold $P(m,n)$ is the quotient of $S^m \times \mathbb{C}P^n$ by the order 2 involution τ. The tangent space of $P(m,n)$ at the point $[x, [y]]$ can be identified with the real vector space consisting of the pairs

$$\{(x, [y], v, \alpha), (-x, [\bar{y}], -v, \bar{\alpha})\}$$

with $v \in T_x S^m$ and $\alpha \in T_{[y]}\mathbb{C}P^n$. We are using the above mentioned interpretation of TS^m and $T\mathbb{C}P^n$ so the notation $-v$ and $\bar{\alpha}$ will cause no confusion.

Let E_1 be the subbundle of $TP(m,n)$ consisting of the pairs

$$\{(x, [y], 0, \alpha), (-x, [\bar{y}], 0, \bar{\alpha})\}$$

and E_2 be the subbundle consisting of the pairs

$$\{(x, [y], v, 0), (-x, [\bar{y}], -v, 0)\}.$$

Both are given the natural coordinate-wise linear structures and bundle maps. Then

$$TP(m,n) \cong E_1 \oplus E_2.$$

By the Whitney product theorem, the total Stiefel-Whitney classes of the bundles have the relation

$$w(TP(m,n)) = w(E_1)w(E_2). \tag{4.2}$$

We will calculate $w(E_1)$ and $w(E_2)$ separately.

We first consider E_1. Viewing the real projective space $\mathbb{R}P^m$ as the quotient of S^m by the antipodal map, the tangent space of $\mathbb{R}P^m$ at the point $\{x, -x\}$ can be written in the form of pairs $\{(x, v), (-x, -v)\}$ where $x \in S^m$ and $v \in T_x S^m$. Recall that $P(m, n)$ is a $\mathbb{C}P^n$-bundle over $\mathbb{R}P^m$ and the following map p is the bundle map:

$$p : P(m, n) \to \mathbb{R}P^m$$
$$p([x, [y]]) = \{x, -x\}.$$

It follows from the definition that $p^* T\mathbb{R}P^m \cong E_1$.

There exists a section for the bundle $P(m, n) \to \mathbb{R}P^m$ defined as

$$s : \mathbb{R}P^m \to P(m, n)$$
$$\{x, -x\} \mapsto [x, [1, 0, \ldots, 0]].$$

The composition $p \circ s$ is the identity on $\mathbb{R}P^m$ and the induced map p^* from $H^*(\mathbb{R}P^m)$ to $H^*(P(m, n))$ is injective. In particular, if t is the non-trivial element in $H^1(\mathbb{R}P^m; \mathbb{Z}_2)$, then $p^*(t) = c$ and

$$w(E_1) = w(p^* T\mathbb{R}P^m) = p^*(w(T\mathbb{R}P^m)) = p^*((1 + t)^{m+1}) = (1 + c)^{m+1}. \tag{4.3}$$

Now we turn to the computation of $w(E_2)$. Let η be the tautological bundle over $\mathbb{R}P^m$. Consider the bundle

$$E_2' = E_2 \oplus \underline{\mathbb{R}} \oplus p^*\eta,$$

where the underlined notation $\underline{\mathbb{R}}$ stands for the trivial real line bundle. Each vector of E_2' based at the point $[x, [y]]$ has three coordinates: an ordered pair $(\alpha, \bar{\alpha})$ for some $\alpha \in \mathrm{Hom}_{\mathbb{C}}(\langle y\rangle, \langle y\rangle^\perp)$, a real number μ and a real vector λx lying in the real subspace of $\mathbb{R}^{m+1}$ spanned by x. Note that a different choice of the representitive of the point $[x, [y]]$ will result in a change of the sign of λ, so each vector of E_2' can also be represented by a pair

$$\{(x, [y], \alpha, \mu + \lambda i), (-x, [\bar{y}], \alpha, \mu - \lambda i)\}$$

where $a, \lambda \in \mathbb{R}$, or equivalently, a pair of the form

$$\{(x, [y], \alpha, z), (-x, [\bar{y}], \alpha, \bar{z}), z \in \mathbb{C}\}.$$

Define ξ to be a real vector bundle over $P(m, n)$, whose fiber at the point $[x, [y]]$ is the real vector space of the pairs $\{(x, [y], u), (-x, [\bar{y}], \bar{u})\}$ with $u \in \langle y\rangle \subset \mathbb{C}^{n+1}$. The real vector bundle ξ has rank 2. Note that ξ is not a complex bundle, at least not in the obvious way. Actually the computation will show that $w_1(\xi) \neq 0$.

It is known that $T\mathbb{C}P^n \oplus \underline{\mathbb{C}} \cong \bar{\eta}_{\mathbb{C}}^{\oplus(n+1)}$ where the $\eta_{\mathbb{C}}$ is the tautological bundle of $\mathbb{C}P^n$ (see [2] for example). Let

$$\phi : T\mathbb{C}P^n \oplus \underline{\mathbb{C}} = \mathrm{Hom}_{\mathbb{C}}(\eta_{\mathbb{C}}, \eta_{\mathbb{C}}^\perp) \oplus \underline{\mathbb{C}} \to \bar{\eta}_{\mathbb{C}}^{\oplus(n+1)}$$

be the isomorphism between the bundles and let $\phi_{[y]}$ be the restriction of ϕ on the fiber of $T\mathbb{C}P^n \oplus \underline{\mathbb{C}}$ over $[y]$, mapping from $\mathrm{Hom}_{\mathbb{C}}(\langle y\rangle, \langle y\rangle^\perp) \oplus \mathbb{C}$ to $\langle \bar{y}\rangle^{\oplus(n+1)}$.

Using ϕ we can define a bundle map

$$\tilde{\phi} : E_2' \to \bar{\xi}^{\oplus(n+1)}$$
$$\{(x, [y], \alpha, z), (-x, [\bar{y}], \bar{\alpha}, \bar{z})\} \mapsto \{(x, [y], \phi_{[y]}(\alpha, z)), (-x, [\bar{y}], \overline{\phi_{[y]}(\alpha, z)})\}.$$

The map $\tilde{\phi}$ fixes the base points and is linear on each fiber. Since ϕ is a bundle isomorphism, the map $\tilde{\phi}$ is injective on every fiber hence a bundle isomorphism too.

Since the Stiefel-Whitney classes of conjugate complex bundles are the same, we get

$$\begin{aligned}(w(\xi))^{n+1} &= w(\xi^{\oplus n+1}) = w(\bar{\xi}^{\oplus(n+1)}) \\ &= w(E_2') = w(E_2 \oplus \mathbb{R} \oplus p^*\eta_{\mathbb{C}}) \\ &= w(E_2)w(p^*\eta_{\mathbb{C}}) \\ &= (1+c)w(E_2)\end{aligned}$$

or

$$w(E_2) = \frac{(w(\xi))^{n+1}}{1+c}. \tag{4.4}$$

Now we only need to compute $w(\xi)$.

Let q be the covering map $S^m \times \mathbb{C}P^n \to P(m,n)$, π_1 be the projection $S^m \times \mathbb{C}P^n \to S^m$ and π_2 be the projection $S^m \times \mathbb{C}P^n \to \mathbb{C}P^n$. Consider the diagram below:

$$\begin{tikzcd} \pi_2^*\eta_{\mathbb{C}} \arrow[r, "f"] \arrow[d] & \xi \arrow[d] \\ S^m \times \mathbb{C}P^n \arrow[r, "q"] & P(m,n) \end{tikzcd}$$

where

$$f(x,[y],u) = \{(x,[y],u),(-x,[\bar{y}],\bar{u})\}.$$

The diagram commutes and f is a linear fiber-wise injection. Thus it is a pull-back diagram and

$$\pi_2^*(\eta_{\mathbb{C}}) \cong q^*(\xi). \tag{4.5}$$

By the Künneth Formula, we have

$$H^*(S^m \times \mathbb{C}P^n; \mathbb{Z}_2) \cong H^*(S^m;\mathbb{Z}_2) \otimes H^*(\mathbb{C}P^n;\mathbb{Z}_2).$$

The induced map π_2^* sends generator of degree 2 of $H^*(\mathbb{C}P^n)$ to the generator $\tilde{d}$ of $H^0(S^m) \otimes H^2(\mathbb{C}P^n) \cong \mathbb{Z}_2$. Combined with (4.5) we get

$$q^*(w(\xi)) = \pi_2^*(w(\eta_{\mathbb{C}})) = 1 + \tilde{d}. \tag{4.6}$$

When $m > 1$, $S^m \times \mathbb{C}P^n$ is simply connected so that $q^*(c)$ vanishes. When $m = 1$, the induced homomorphism q_* between fundamental groups is the map from $\mathbb{Z}$ to $\mathbb{Z}$ multiplying by 2. Thus the restriction of q^* on $H^1(P(m,n))$ is the zero map if we use $\mathbb{Z}_2$-coefficient.

To conclude, $q^*(c) = 0$ holds for all m. Using (4.6) and the fact that ξ is a rank-2 vector bundle, there are only four possibilities for $w(\xi)$:

$$1+d, 1+c+d, 1+d+c^2, 1+c+d+c^2.$$

In order to determine the coefficient of terms c and c^2, consider again the section $s : \mathbb{R}P^m \to P(m,n)$ which sends $\{x,-x\}$ to $[x,[1,0,\ldots,0]]$. The induced map s^* of cohomology is a surjection mapping c to a, where a is the generator of degree 1 of the polynomial ring $H^*(\mathbb{R}P^m;\mathbb{Z}_2) \cong \mathbb{Z}_2[a]/(a^{m+1})$. Consider a cell structure of $\mathbb{R}P^m$, where the unique i-cell is the image of $C_i^{\pm} \in S^m$ under the 2-sheet covering $S^m \to \mathbb{R}P^m$. The map s is compatible with the cell structures of $\mathbb{R}P^m$ and $P(m,n)$, inducing a chain map between the cellular chain complexes. The cell $[C_0, D_1]$ does not lie in the image of s, so by duality $s^*(d) = 0$.

By the definition of pull-back bundles, the fiber of $s^*\xi$ at $\{x, -x\}$ consists of the pairs

$$\{(x, [1, 0, \dots, 0], [z, 0, \dots, 0]), (-x, [1, 0, \dots, 0], [\bar{z}, 0, \dots, 0])\},$$

for $z \in \mathbb{C}$, or equivalently, the pairs

$$\{(x, z), (-x, \bar{z})\}$$

for $z \in \mathbb{C}$.

The pull-back bundle $s^*\xi$ can be decomposed into a direct sum of two real line bundles ξ_1 and ξ_2 by separating the real and imaginary parts of z. To be precise, the descriptions of ξ_1 and ξ_2 as subbundles of $s^*\xi$ are given by

$$\begin{aligned}
\xi_1 &: \{(x, s), (-x, s) : s \in \mathbb{R}\} \to \{x, -x\}, \\
\xi_2 &: \{(x, ti), (-x, -ti) : t \in \mathbb{R}\} \to \{x, -x\}.
\end{aligned}$$

The line bundle ξ_1 is trivial while ξ_2 is non-trivial because no nowhere zero section of ξ_2 can be found. Since w_1 classifies real line bundles completely and $H^1(\mathbb{R}P^m; \mathbb{Z}_2)$ has only one non-zero element a, it can be deduced that

$$s^*(w(\xi)) = w(s^*\xi) = w(\xi_1)w(\xi_2) = 1 + a. \tag{4.7}$$

By (4.6) and (4.7) we know that

$$w(\xi) = 1 + c + d. \tag{4.8}$$

By (4.2), (4.3), (4.4) and (4.8) we can deduce that

$$w(E) = w(E_1)w(E_2) = (1 + c)^m(1 + c + d)^{n+1}.$$

□

5. The generators of the unoriented cobordism ring $\mathfrak{N}$

By [1], the ring $\mathfrak{N}$ is isomorphic to the polynomial ring $\mathbb{F}_2[x_2, x_4, x_5, \dots]$ with exactly one generator x_k for each dimension $k \neq 2^n - 1$. We will identify representatives M_k for each dimension k.

Theorem 5.1. *For any $k \neq 2^n - 1$: if k is even, let $M_k = \mathbb{R}P^k$; if k is odd, let $M_k = P(2^r - 1, 2^r s)$, where $r \geq 1$, s is a positive integer and $k + 1 = 2^r(2s + 1)$. Let $x_k \in \mathfrak{N}$ be the cobordism class containing M_k. Then $\mathfrak{N}$ is isomorphic to the polynomial ring $\mathbb{F}_2[x_2, x_4, x_5, \dots]$.*

Proof. Recall that for the manifold M_k, we may view the n-th Stifel-Whitney class of TM_k as the elementary symmetric polynomial of degree n in formal variables $\{t_i\}_{i=1}^N$ in degree 1. In practice, one can consider the polynomial

$$q(t) = \sum_{j=0}^{N} t^j w_{N-j}(TM_k) \tag{5.2}$$

and the t_i's are just roots of $q(t)$. Here we adopt the convention that $w_0 = 1$.

In order to prove that the cobordism ring is freely generated by the classes $[M_k]$, we only need to verify that M_k is not cobordant to the sum of products of lower dimensional manifolds. By Thom's result, this is equivalent to proving that $\sum_{i=1}^N t_i^k \in H^k(M_k; \mathbb{Z}_2)$ is non-zero.

If k is even, $M_k = \mathbb{R}P^k$ and $w(TM_k) = w(T\mathbb{R}P^k) = (1+c)^{k+1}$, c being the generator of degree 1 of $H^*(\mathbb{R}P^k;\mathbb{Z}_2)$. Thus

$$q(t) = \sum_{j=0}^{N} t^j w_{N-j}(TM_k) = (t+c)^{k+1}.$$

So $t_1 = \cdots = t_{k+1} = c$ and

$$\sum_{i=1}^{k+1} t_i^k = (k+1)c^k = c^k \neq 0.$$

If k is odd, write $k+1$ as the form $2^r(2s+1)$ where $r \geq 1$ and $s \geq 1$. Let $m = 2^r - 1$ and $n = 2^r s$. By the computation in the first section, we have

$$w(TM_k) = w(TP(m,n)) = (1+c+d)^{n+1}(1+c)^m.$$

Therefore one can take the t_i to be the roots of the polynomial

$$q(t) = (t^2+ct+d)^{n+1}(t+c)^m.$$

Let x, y be the two roots of t^2+ct+d. Then x and y satisfy the relation

$$x+y=c,\ xy=d.$$

Let $s_r = x^r + y^r$ for any positive integer r. Then we have

$$\sum_i t_i^k = (n+1)(x^k+y^k) + mc^k = s_k. \tag{5.3}$$

Thus we only need to prove that $s_k \neq 0$.

In low degrees we have $s_1 = c$ and $s_2 = (x+y)^2 - 2xy = d$. Moreover, we have the following inductive relation for $r \geq 3$:

$$s_r = s_1 s_{r-1} + s_2 s_{r-1} = cs_{r-1} + ds_{r-2}. \tag{5.4}$$

Note that we are working with $\mathbb{Z}_2$-coefficient so the sign does not matter.

We can show by induction that

$$s_r = \sum_{i=0}^{[r/2]} \binom{r-i-1}{i} c^{r-2i} d^i \tag{5.5}$$

for $r \geq 1$. Here we assume that $\binom{a}{b} = 0$ if $b > a$ or $a \leq 0$, with the exception that $\binom{0}{0} = 1$.. Indeed, (5.5) holds for $r = 1, 2$. Now assume it holds for all degrees below

r. For degree r, by (5.4) we have

$$\begin{aligned}
s_r &= s_1 s_{r-1} + s_2 s_{r-1} \\
&= c s_{r-1} + d s_{r-2} \\
&= c \sum_{i=0}^{[(r-1)/2]} \binom{r-i-2}{i} c^{r-2i-1} d^i + d \sum_{i=0}^{[(r-2)/2]} \binom{r-i-3}{i} c^{r-2i-2} d^i \\
&= \sum_{i=0}^{[(r-1)/2]} \binom{r-i-2}{i} c^{r-2i} d^i + \sum_{i=0}^{[(r-2)/2]} \binom{r-i-3}{i} c^{r-2i-2} d^{i+1} \\
&= \sum_{i=0}^{[r/2]} \binom{r-i-2}{i} c^{r-2i} d^i + \sum_{i=1}^{[r/2]} \binom{r-i-2}{i-1} c^{r-2i} d^i \\
&= \sum_{i=0}^{[r/2]} \binom{r-i-1}{i} c^{r-2i} d^i
\end{aligned}$$

which indicates that (5.5) holds for degree r.

When $r = k$,

$$s_k = \sum_{i=0}^{\infty} \binom{k-i-1}{i} c^{k-2i} d^i = \binom{m+n-1}{n} c^m d^n = \binom{2^r(s+1)-2}{2^r s} c^m d^n.$$

By (5.3), in order to prove that M_k is indecomposable, we only need to show that $\binom{2^r(s+1)-2}{2^r s}$ is odd. One way to compute $\binom{a}{b}$ mod 2 is to write a and b in binary. The binomial number $\binom{a}{b}$ is odd if and only if every binary digit of a is greater than or equal to that of b. In our case where $a = 2^r(s+1) - 2$ and $b = 2^r s$, the binary representation of a can be obtained from the representation of b by turning all the 0's in the last r digits to 1's. This shows that $\binom{2^r(s+1)-2}{2^r s}$ is indeed an odd integer.

Combining the two cases together, we can deduce that the M_k's are generators of the cobordism ring $\mathfrak{N}^*$. □

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