Abstract. Motivic Homotopy Theory is oft thought of as the homotopy theory of schemes, or in particular as a means of enriching the study of algebraic varieties in unstable and (especially) stable homotopy theory. Introduced by Morel and Voevodsky in the late 1990’s in pursuit of geometric applications such as proofs of the Milnor and Bloch-Kato conjectures, this framework has indeed proven to be hugely successful in answering scheme-theoretic questions. The goal of this paper, however, is to ask about more topological phenomena. To do this, we first introduce the fundamentals of motivic homotopy theory, constructing and examining the stable motivic homotopy category which is the general object of study. We then interrogate the analogy between motivic spaces and topological spaces by examining the class of cellular motivic spaces, the appropriate motivic analog of CW-complexes. We ask, in line with the classical scenario, whether all motivic spaces are “weak equivalent” to a cell complex. A negative answer to this question is given, and justified with a group of examples including a folklore result which is proven here explicitly.

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1. Motivic (Stable) Homotopy Theory

1.1. Definitions and Preliminaries. In this section, we construct the fundamental categories studied in motivic homotopy theory: in particular, the category $\text{Spc}(k)$ of motivic spaces over a field $k$ and the motivic stable homotopy category.
SH(k) over k. Spc(k) will be the category of sheaves of simplicial sets on a convenient category of schemes with a particularly fortuitous Grothenieck topology. This means that motivic spaces will be generalized “spaces” imbued with a homotopically advantageous notion of locality and enriched with a notion of homotopy type. Then, SH(k) will be defined as a stabilization of Spc(k), constructed in explicit analogy to the classical stable homotopy category for topological spaces. As a guiding principle, the homotopy theory we describe should (and will) respect the intuitive topology and geometry and schemes: for example, the intuition that \( \mathbb{P}^1 \) is a sphere and that \( \mathbb{A}^1 \) is a line. We will thus try to be quite explicit in describing why we take each step, so that one can use said preservation of the “geometry” of schemes as a guidepost: the following is a rich confluence of stable homotopy theory and Grothendieckian algebraic geometry, and we wish to make what comes accessible and intuitive to those with only a passing familiarity with both fields.

1.2. Pursuing Motivic Spaces. In all that follows, fix a field perfect field \( k \); the choice will not matter save when we give certain examples. Then, just as one starts with a convenient category of topological spaces in classical homotopy theory, we wish to work with a category of schemes over \( k \) that are nice enough to actually do homotopy theory with (whatever that will mean). Towards that end, consider the category \( \text{Sm}/k \) of smooth, separated, finite type schemes over \( k \).\(^1\) This is an appropriate starting point on an intuitive geometric level (the objects of \( \text{Sm}/k \) are analogous to manifolds), because of the fact the \( \text{Sm}/k \) is essentially small (so that the category \( \text{Shv}(\text{Sm}/k) \), with the sheaves taken in whatever topology, is actually a topos), and especially because of theorems like Quillen’s proof of the \( \mathbb{A}^1 \)-homotopy invariance of Algebraic K-Theory for regular schemes: that is, that \( K_i(R) = K_i(R[t]) \) for any regular ring \( R \), see Qui.

The last fact is perhaps the most compelling argument to use \( \text{Sm}/k \), for it agrees with the guiding principle we describe in Section 1.1: that is, that for us, the affine line \( \mathbb{A}^1 \) should be also a homotopy theoretic line and thus contractible. The above then says that Algebraic K-Theory for regular schemes sees \( \mathbb{A}^1 \) in exactly this way!

Unfortunately, \( \text{Sm}/k \) is not (co)complete, making naive attempts at homotopy theory intractable. Not only this, but classical Grothendieck topologies on \( \text{Sm}/k \) have a finicky-at-best relationship with the topologies of the “spaces” which these schemes correspond to. For example, the following are true.

- On one hand, for any \( X \in \text{Sm}/\mathbb{C} \), the Zariski topology renders trivial cohomology groups for every constant sheaf on \( X \), while the dimensions

\(^1\)One may also begin with the category \( \text{QcQs}/k \) of quasi-compact quasi-separated schemes over \( k \). Indeed, if objects of \( \text{Sm}/k \) are to be thought of in analogy with manifolds, objects of \( \text{QcQs}/k \) are a close analog of compactly generated weak Hausdorff spaces, a readymade starting place for classical homotopy theory. Furthermore, \( \text{QcQs}/k \) enjoys nice topos theoretic properties: for example, \( T = \text{Shv}_{\text{Zar}}(\text{QcQs}/k) \) is a coherent topos (see Lur, Section A.4), meaning both that cohomology in \( T \) commutes with filtered colimits and that \( T \) has enough points. Unfortunately, as explained in the body, the Zariski topology is not sufficient for “homotopy theory”. Indeed, the appropriate homotopical Grothendieck topology on \( \text{QcQs}/k \) is even finer than the \( \text{\acute{e}tale} \) topology with which we also take issue above, and Algebraic K-Theory and other important “cohomology theories” will only have representing motivic spectra up to homotopy in the resultant stable category (see Denis Nardin’s comment at Nar).
of the cohomology groups of the constant sheaf with values in \( \mathbb{R} \) on the underlying real manifold of \( X(\mathbb{C}) \) are precisely its Betti numbers.

- On the other, not only does the étale topology allow for situations where \( \dim \text{Top}(X) < \dim \text{Cohom}(X) \), but one also has étale non-contractible affine lines in positive characteristic (see Aff).

These are real quandries if we are supposedly doing a “homotopy theory” that cares about the topology of our spaces! Thus if one wants to resolve \( \text{Sm}/k \)'s (co)completeness issues while still retaining the “topological” properties of the objects in \( \text{Sm}/k \), we must consider a third class of morphisms with which to define a site structure. The right one, which ends up being somewhere between open immersions and étale morphisms, is the following.

**Definition 1.1.** A morphism \( f: X \to Y \) of schemes is called Nisnevich if the following conditions hold:

1) \( f \) is étale, and;
2) \( \forall y \in Y \) with \( f^{-1}(y) \neq \emptyset \), \( \exists x \in f^{-1} \) such that the induced map \( \overline{f}: k(x) \to k(y) \) of residue fields is an isomorphism.

A Nisnevich morphism in \( \text{Sm}/k \) is a morphism in \( \text{Sm}/k \) whose underlying morphism of schemes is Nisnevich.

As hinted at above, all Nisnevich morphisms are étale, while one can see that open immersions are Nisnevich on their image. This chain of inclusions is strict. Intuitively, Nisnevich morphisms do not go so far as to require that any given \( f \) is some kind of module isomorphism, which in tandem with Nakayama’s lemma would give an isomorphism of Zariski-local rings at \( x \) and \( y \). Condition 2), however, is equivalent to saying that for every \( y \in Y \) in the image of \( f \), \( \exists x \in X \) such that the henselizations of the local rings \( \mathcal{O}_{X,x} \) and \( \mathcal{O}_{Y,y} \) are isomorphic, while étalement alone only implies that the strict henselizations of \( \mathcal{O}_{X,x} \) and \( \mathcal{O}_{Y,y} \) are isomorphic.

This definition perhaps appears abstruse, but the geometric and topological riches it affords us are substantial when we make it into a topology.

**Definition 1.2.** A family of morphisms \( \{ \phi_a : X_a \to Y \} \) (indexed by a set \( A \)) in \( \text{Sm}/k \) is a Nisnevich cover if:

1) each \( \phi_a \) is étale morphism of schemes, and;
2) \( \forall y \in Y \), \( \exists a \in A \) and \( x \in \phi_a^{-1}(y) \) such that the induced map \( \overline{\phi_a}: k(x) \to k(y) \) of residue fields is an isomorphism.

Nisnevich covers define a Grothendieck pretopology on \( \text{Sm}/k \), which generates the **Nisnevich topology** on \( \text{Sm}/k \). We claim that proceeding by using Nisnevich sheaves on \( \text{Sm}/k \) ends up being the right homotopical thing to do. Indeed, it turns out that the interjacency of Nisnevich morphisms between étale and Zariski results in the Nisnevich topology enjoying the best properties of both worlds. For example, the following is true of \( \text{Shv}_{Nis}(\text{Sm}/k) \).

- As a sheaf topos, it is locally presentable and so complete and cocomplete.
- Since the Nisnevich Topology is coarser than the already subcanonical étale topology, every object \( X \in \text{Sm}/k \) represents a sheaf, which by abuse of notation we will often refer to as \( X \) itself.
- The spectrum of a field has no nontrivial Nisnevich cohomology (similar to the Zariski topology).
• The cohomological dimension of every scheme $X$ (that is, the sheaf $X$ represents) is its Krull dimension (similar again to the Zariski topology).
• Nisnevich cohomology can be computed using Čech cohomology (like in the étale topology), and;
• Algebraic K-Theory of any degree is a Nisnevich sheaf of groups.

See NiS and MoVo pg. 94-95 for a further discussion of this list.

Further, checking the Nisnevich sheaf condition is not so hard a task. Indeed, we may check it on a particular class of Nisnevich coverings.

Definition 1.3. An elementary distinguished square in $\text{Sm}/k$ is a cartesian square of the form:

\[
\begin{array}{ccc}
U \times_X V & \longrightarrow & V \\
\downarrow & & \downarrow p \\
U & \longrightarrow & X \\
\downarrow j & & \\
V & \longrightarrow & U \\
\end{array}
\]

such that $p$ is étale, $j$ is an open embedding, and $p^{-1}((X - U)_{\text{red}}) \to (X - U)_{\text{red}}$ is an isomorphism of schemes, where the subscript “red” means we endow these sets with the reduced induced scheme structure.

Elementary distinguished squares should be thought of as a kind of Nisnevich excisive triads, analogous to those in classical algebraic topology. Indeed, étale maps are open, so $(X; U, \text{im}(p))$ is literally topologically excisive, while the conditions on $p$ mean that every point in $X$ has a preimage on which this cover locally looks like the actual inclusion of subschemes. That is, $(X; U, \text{im}(p))$ is even in some sense algebro-geometrically excisive. The Nisnevichness of this excisive triad comes from explicitly asking that we encode both strictly-Zariski and more general étale information. Then, as nothing but excisive triads, one would perhaps not be surprised that any $\mathcal{F} \in \text{Sh}_{\text{Nis}}(\text{Sm}/k)$ behaves nicely with respect to them. Indeed, the image of any elementary distinguished square will also be cartesian under any such $\mathcal{F}$. For presheaves on $\text{Sm}/k$, it is even true that a converse holds.

Theorem 1.4. A presheaf of sets $\mathcal{F}$ on $\text{Sm}/k$ is a sheaf in the Nisnevich topology if and only if for any elementary distinguished square, the corresponding square of sets:

\[
\begin{array}{ccc}
\mathcal{F}(X) & \longrightarrow & \mathcal{F}(U) \\
\downarrow & & \downarrow p \\
\mathcal{F}(V) & \longrightarrow & \mathcal{F}(U \times_X V) \\
\end{array}
\]

is cartesian.

Proof. See MoVo, Prop 1.4. □

A presheaf which satisfies the above, and thus which is actually an object of $\text{Sh}_{\text{Nis}}(\text{Sm}/k)$, is said to satisfy Nisnevich descent. A corollary of this result says that elementary distinguished squares render a sheaf-theoretic Mayer-Vietoris sequence of spaces. Indeed, for elementary distinguished squares, the following sequence of Chow groups, the algebraic geometer’s analog of singular cohomology, is exact:
(1.5) \[ CH^*(X) \to CH^*(U) \oplus CH^*(V) \to CH^*(U \times_X V) \]
giving a literal Mayer-Vietoris sequence for “Nisnevich excisive triads” (see Aso, Section 1.2). This theorem does not hold for general \( \acute{e} \text{tale} \) covers, explicitly necessitating the use of the Nisnevich topology to witness 
\textit{bona fide} topological facts!

We’re \textit{almost} there. For despite all the upshot of the prequel, we still have little means of working with elements \( \text{Shv}_{Nis}(\text{Sm}/k) \) as if they actually have homotopy types. For example, the above is a \textit{sheaf cohomological}, not \textit{a priori} homotopical, Mayer-Vietoris! Another perk of working with sheaves, however, is that we may take values in any category we like; so why not take them \textit{in homotopy types}? Since \textit{simplicial sets} are both a convenient model for homotopy types and presheaves in their own right, this idea is naturally turned into a definition.

**Definition 1.6.** A \textbf{motivic space} is an object of \( \text{Spc}(k) := \Delta_{op} \text{Shv}_{Nis}(\text{Sm}/k) \).

That is, a motivic space is a simplicial object in \( \text{Shv}_{Nis}(\text{Sm}/k) \); or, and this is the perspective we will take, it is a Nisnevich sheaf on \( \text{Sm}/k \) with values in simplicial sets. From here on out, when we say \textit{space}, we mean a motivic space.

As described above, a representable presheaf is a Nisnevich sheaf, so each \( X \in \text{Sm}/k \) gives a space via Yoneda composed with the embedding of sheaves of sets into sheaves of simplicial sets. On the other extreme, \( \text{Spc}(k) \) contains constant sheaves with value in a \textit{given} simplicial set: that is, fixing a simplicial set \( S \), the presheaf associating \( S \) to every \( U \in \text{Sm}/k \) is a space. In the sequel, we refer to a scheme \( X \) or a simplicial set \( S \), we mean their associated spaces. We will stress this when we introduce new concepts involving the spaces represented by schemes.

1.3. **Fundamentals of Homotopy Theory in \( \text{Spc}(k) \).** With a reasonable notion of “homotopical” scheme theoretic spaces in mind, let’s start trying to do some homotopy theory! To start, recall that one often needs to point their topological spaces to do homotopical constructions. The same fundamental necessity is true motivically, so we say the following.

**Definition 1.7.** A \textbf{pointed space} is a space \( X \) equipped with a map:
\[ x : \text{Spec}(k) \to X \]
from the space representing \( \text{Spec}(k) \) to \( X \).

Note that \( \text{Spec}(k) \) is not only the terminal object in \( \text{Sm}/k \), but also the terminal Nisnevich sheaf (or even just terminal presheaf on \( \text{Sm}/k \)). More explicitly, we have \( \text{Spec}(k)(U) = \text{Hom}(U, \text{Spec}(k)) = \{ \ast \} \) for every \( U \in \text{Sm}/k \), so that a pointing on a space \( X \) is the same as a compatible choice of vertex in every simplicial set \( X(U) \).

That is, a \textit{pointed space} is the same as a Nisnevich sheaf of pointed simplicial sets.

The category of pointed spaces, defined as the undercategory \( \text{Spc}(k)_{\text{Spec}(k)/} \), will be denoted by \( \text{Spc}_*(k) \). If \( X \) is a space, then \( X_+ \) will denote the pointed space \( X \coprod \text{Spec}(k) \), with point the canonical map into the second component.

The category \( \text{Spc}_*(k) \) is equipped with a zero object in the form of \( (\text{Spec}(k), \text{id}) \), and a forgetful functor from \( \text{Spc}_*(k) \) to \( \text{Spc}(k) \) whose left adjoint is the functor
sending $X$ to $X_+$. It also has coproducts: namely, given two spaces $(X, x_0)$ and $(Y, y_0)$, one defines their wedge product $(X, x_0) \lor (Y, y_0)$ to be the pushout in $\text{Spc}_*(k)$ of the following diagram.

\[
\begin{array}{c}
(Spec(k), \text{id}) \\
\downarrow \\
(Y, y_0)
\end{array}
\rightarrow
\begin{array}{c}
(X, x_0)
\end{array}
\]

Equivalently, we may take $(X, x_0) \lor (Y, y_0)$ to be the sheaf of pointed simplicial sets associated to the presheaf $U \rightarrow ((X, x_0))(U) \lor ((Y, y_0))(U)$. One can check that this is really the coproduct in the category $\text{Spc}_*(k)$.

This construction allows us to imbue $\text{Spc}_*(k)$ with a symmetric monoidal structure, in the form of something deserving of the name smash product.

**Definition 1.9.** Let $(X, x_0), (Y, y_0) \in \text{Spc}_*(k)$. Then, their smash product $(X, x_0) \wedge (Y, y_0)$ is the sheaf of pointed simplicial sets associated to the presheaf $U \rightarrow ((X, x_0))(U) \wedge ((Y, y_0))(U) := ((X, x_0) \times (Y, y_0))(U)/((X, x_0) \lor (Y, y_0))(U)$.

**Sanity Check 1.10.** What is the unit for the smash product?

Further, because one already has a notion of homotopy internal to simplicial sets, we can even define homotopy sheaves associated to pointed spaces without yet defining a motivic “circle” (though we will do so soon!). To do this, recall first the definition of homotopy groups of pointed Kan complexes.

**Definition 1.11.** Let $K$ be a Kan complex, $k \in K_0$ a vertex of $K$, and $n \geq 0$. Then $\pi_n(K, k)$ is the set of simplicial homotopy classes of basepoint preserving maps $(\Delta_n/\partial \Delta_n, x_0) \rightarrow (K, k)$, where $\Delta_n/\partial \Delta_n$ is the simplicial $n$-sphere and $x_0$ its unique vertex. If $n = 1$ this is a group, and it is abelian if $n \geq 2$.

To define $\pi_n(X, x)$ for an arbitrary pointed simplicial set $(X, x)$ one applies Kan’s $\text{Ex}^\infty$ functor to $X$, receiving a Kan complex $\text{Ex}^\infty(X)$ and a weak equivalence $f : X \rightarrow \text{Ex}^\infty(X)$. We then set $\pi_n(X, x) := \pi_n(\text{Ex}^\infty(X), f(x))$. This allows us to make the following definition.

**Definition 1.12.** Let $(X, x)$ be a pointed space. Then the $n$-th homotopy sheaf $\pi_n(X, x)$ is the Nisnevich sheaf associated to the presheaf $U \rightarrow \pi_n(X(U), x|U)$.

Here $x|U$ denotes the vertex of $X(U)$ in the image of the map $x : \text{Spec}(k)(U) \rightarrow X(U)$. This is a Nisnevich sheaf of groups for $n \geq 1$, and a Nisnevich sheaf of abelian groups for $n \geq 2$. The association is functorial.

Thus the fundamental operations of algebraic topology carry over swimmingly into the motivic world. Further, in light of (1.12), we claim that we can even define a model structure on $\text{Spc}(k)$ faithful to the classical model structure on the category of simplicial sets. Indeed, since we now have homotopy groups, we can define weak equivalences as one does classically: as the maps which induce isomorphisms of them.

**Definition 1.13.** A morphism of spaces $f : X \rightarrow Y$ in $\text{Spc}(k)$ is a simplicial weak equivalence if for all $n \geq 0$ and any choice of basepoints $x_0 : \text{Spec}(k) \rightarrow X$, $y_0 : \text{Spec}(k) \rightarrow Y$ such that $f \circ x_0 = y_0$, the induced map of sheaves:
\[
\pi_n((X, x_0)) \to \pi_n((Y, y_0))
\]
is an isomorphism (of sets/groups/abelian groups when \(n = 0/1/\geq 2\)).

Then we may copy the definitions used to define the classical model structure on \(\text{sSet}\) almost verbatim.

**Definition 1.15.** The simplicial model structure on \(\text{Spc}(k)\) is given by the following distinguished classes.

- A morphism \(f\) is a **weak equivalence** if it a simplicial weak equivalence.
- A morphism \(f\) is a **cofibration** if it is a monomorphism (i.e., if it is a termwise monomorphism of simplicial sets).
- A morphism \(f\) is a **fibration** if it satisfies the right lifting property with respect to acyclic cofibrations.

This model structure is proper, and we will denote its associated homotopy category by \(H_{\text{sSpc}}(k)\).

*Now* we’re doing something that looks like homotopy theory! But geometrically, we’ve done little to attend to our guiding principle: that is, that the geometry of a scheme should somehow correspond to the homotopy type of the space it represent, or in particular that the affine line \(\mathbb{A}^1\) should be a topological line. The reader might object here: after all, don’t we already have the simplicial interval \(\Delta[1]\) in \(\text{Spc}(k)\)? This is a crucial observation, and one which we will keep in mind. Our philosophy dictates, however, that the pertinence of contracting \(\mathbb{A}^1\) outweighs the need to be true to classical homotopy theory. To contract \(\mathbb{A}^1\), then, we employ the following definitions.

**Definition 1.16.** Call a space \(Z\) **\(\mathbb{A}^1\)-local** if for any \(Y \in \text{Sm}/k\), the induced map:

\[
\text{Hom}_{H_{\text{sSpc}}(k)}(Y \times \mathbb{A}^1, Z) \to \text{Hom}_{H_{\text{sSpc}}(k)}(Y, Z)
\]

is a bijection. A morphism of spaces \(f : X \to Y\) is an **\(\mathbb{A}^1\)-weak equivalence** if for any \(\mathbb{A}^1\)-local \(Z\), the induced map:

\[
\text{Hom}_{H_{\text{Spc}}(k)}(Y, Z) \to \text{Hom}_{H_{\text{Spc}}(k)}(X, Z)
\]

is a bijection.

Intuitively, \(\mathbb{A}^1\)-local spaces are the ones for which \(\mathbb{A}^1\) interacts with them *as if it was contractible* (that is, simplicially weak equivalent to \(\text{Spec}(k)\)), while \(\mathbb{A}^1\)-weak equivalences are the morphisms that \(\mathbb{A}^1\)-local spaces see as weak equivalences. Thus the \(\mathbb{A}^1\)-local spaces and \(\mathbb{A}^1\)-weak equivalences are exactly the objects and maps that see the affine line as homotopically trivial.

Since the simplicial model structure is proper, its left Bousfield localization at the \(\mathbb{A}^1\)-weak equivalences exists, which we will call the **\(\mathbb{A}^1\)-model structure** on \(\text{Spc}(k)\). Explicitly, we say the following.

**Definition 1.19.** The **\(\mathbb{A}^1\)-model structure** on \(\text{Spc}(k)\) is given by the following distinguished classes.

- A morphism \(f\) is a **weak equivalence** if it is an \(\mathbb{A}^1\)-weak equivalences.
- A morphism \(f\) is a **cofibration** if it is a monomorphism, as defined above.
• A morphism \( f \) is a \textbf{fibration} if it satisfies the right lifting property with respect to acyclic cofibrations.

The homotopy category of \( \text{Spc}(k) \) with its \( \mathbb{A}^1 \)-model structure is called the \textbf{\( \mathbb{A}^1 \)-homotopy category}, and will be denoted by \( \text{H}(k) \).

Isomorphisms in \( \text{H}_\ast \text{Spc}(k) \) are certainly \( \mathbb{A}^1 \)-weak equivalences, so the weak equivalences in \( (1.19) \) are a strictly larger class of morphisms than the simplicial weak equivalences. Since the fibrant objects in a Bousfield localized model category are precisely the local and fibrant objects in the original model structure, the fibrant objects in the \( \mathbb{A}^1 \)-model structure are exactly the fibrant, \( \mathbb{A}^1 \)-local spaces. Thus \( \mathbb{A}^1 \) is “contractible”, i.e. weak equivalent to a point, write large in \( \text{H}(k) \), since the objects of \( \text{H}(k) \) can be taken to be the \( \mathbb{A}^1 \)-fibrant-cofibrant objects of \( \text{Spc}(k) \).

More formally, this is to say that the following theorem holds.

**Theorem 1.20.** Assume \( X \in \text{Spc}(k) \). Then the canonical map:

\[
(1.21) \quad X \times \mathbb{A}^1 \rightarrow X
\]

is an \( \mathbb{A}^1 \)-weak equivalence.

We denote the relation of \( \mathbb{A}^1 \)-weak homotopy equivalence by \( \simeq \), so that the above says that the projection of \( X \times \mathbb{A}^1 \) onto \( X \) witnesses that \( X \times \mathbb{A}^1 \simeq X \). This tells us also that \( \mathbb{A}^n \) will be contractible for all \( n \), since the product in \( \text{Sm}/k \) is really the fiber product over \( k \), i.e. \( \mathbb{A}^1 \times \ldots \times \mathbb{A}^1 \cong \mathbb{A}^n \).

One can then define a \textbf{pointed} \( \mathbb{A}^1 \)-model structure on \( \text{Spc}_\ast(k) \): namely, by taking as weak equivalences, cofibrations, and fibrations those morphisms whose underlying morphisms in \( \text{Spc}(k) \) are of said class. Its homotopy category will be called the \textbf{pointed} \( \mathbb{A}^1 \)-\textbf{homotopy category}, and will be denoted by \( \text{H}_\ast(k) \).

This brings us to a reason to contract \( \mathbb{A}^1 \) that goes beyond just intuition.

**Theorem 1.22.** Any functor \( F : \text{Spc}_\ast(k)^{\text{op}} \rightarrow \text{Set} \) which is invariant under \( \mathbb{A}^1 \)-weak equivalence, preserves images of elementary distinguished squares, and sends coproducts to products is representable by an object in \( \text{H}(k)_\ast \).

**Proof.** This is Brown representability on the homotopy category of a pointed model category, namely \( \text{H}_\ast(k) \), combined with \((1.20)\). \hfill \Box

Thus, if we have a (sufficiently nice) \( \mathbb{A}^1 \)-homotopy invariant functor on \( \text{Sm}/k \), one need only define some Nisnevich sheaf-theoretic enrichment of it to ensure that it is actually representable by a space. The power of this statement comes both from the fact that many important invariants, for example Algebraic K-Theory and Chow groups, will satisfy these conditions, and from the fact that it can be used in extremely general scenarios. In fact, the rest of this section of the text is dedicated to finding a natural perspective from which to extend and use this fact.

1.4. **Motivic Spheres.** In classical stable homotopy theory, Brown representability allows us to say that Eilenberg-Steenrod cohomology theories are represented by spectra in the stable homotopy category. We would like to use \((1.22)\) to say an analogous thing motivically, i.e. with some notion of motivic cohomology theories and motivic spectra and in the setting of some stable \( \mathbb{A}^1 \)-homotopy category. This
goal seems far off: after all, we don’t yet have a notion of suspensions of spaces with which to define spectra in first place, much less of circles to suspend with! Luckily for us, the latter, at least, is ready at hand.

Let’s think: what should a motivic circle be? We claim that there’s at least a fairly natural preliminary answer. After all, the simplicial circle $S^1$, i.e. the constant presheaf with values $S^1 = \Delta[1]/\partial\Delta[1]$, is a space. Since smashing a simplicial set with $S^1$ is the classical suspension of simplicial sets, smashing a space with $S^1_s$ (i.e., doing the classical suspension of simplicial sets componentwise and then sheafifying) is a more-than-appropriate Nisnevich analog, and we’re quite happy.

As in the case of motivic intervals, however, where we have both $\Delta[1]$ and $\mathbb{A}^1$, we need to do more than just emulate classical homotopy theory here. For algebraic geometry has its own classical analog for a circle: namely, $\mathbb{G}_m(k)$. Indeed, not only does $\mathbb{G}_m(k)^n$ behave algebraically like the $n$-dimensional torus $S^1 \times \cdots \times S^1 = T^n$ for any field $k$, but the $\mathbb{C}$-valued points of $(\mathbb{G}_m(\mathbb{C})^n)$ even deformation retract onto $T^n$. Since $T^1 \cong S^1$, this means that $\mathbb{G}_m(k)$ must be some kind of circle as well! Thus our principle that the intuitive homotopy types of schemes should dictate the actual homotopy types of the associated spaces demands that we think of $\mathbb{G}_m(k)$ as honest-to-goodness circle as well, the Tate circle. We will denote it by $S^1_t$.

Therefore we can suspend spaces in not just one but two variables, both of which are also naturally pointed spaces: $S^1_t$ by the map which for every $U \in \text{Sm}/k$ sends the vertex of $\text{Spec}(k)(U)$ to the only vertex in $S^1_t(U)$ for every $U$, and $S^1_s$ by the morphism of spaces induced by the morphism of schemes mapping the point $\text{Spec}(k)$ to the closed point $1$. When we write $S^1_t$ or $S^1_s$ in the pointed context, we mean $(S^1_s, *)$ and $(S^1_t, 1)$. In any event, this implies that we have a set of bigraded mixed spheres in the motivic world. Namely, we will define:

**Definition 1.23.** $S^{p, q} = (S^1_t \wedge \cdots \wedge S^1_t) \wedge (S^1_s \wedge \cdots \wedge S^1_s)$

where this implies that $S^1_t = S^{1,0}$, and $S^1_s = S^{1,1}$, and we set $S^{0,0} := \text{Spec}(k) \sqcup \text{Spec}(k)$. The perhaps strange seeming grading is taken in analogy with the situation in equivariant homotopy theory, and because the grading agrees with that of realization functors into a category of “mixed motives” which we have no interest in discussing here. When $p, q \neq 0$ we will write $(S^{p, q}, *)$, or even just $S^{p, q}$, for $S^{p, q}$ with pointing induced by the smash product in (1.23), while one can identify $(S^{0, 0}, *)$ with $\text{Spec}(k)_+$.

These bigraded spheres are not just some abstract collection of spaces to care about purely categorically. Here is a telling example.

**Example 1.24.** Define the model theoretic suspension $\Sigma_M X$ of a pointed space $X \in \text{Spc}_\ast(k)$ to be the $\mathbb{A}^1$-homotopy pushout of the diagram $\ast \leftarrow X \rightarrow \ast$. Then one can see that $\Sigma_M X \simeq S^1_t \wedge X$: indeed, for any $U \in \text{Sm}/k$, $\Sigma_M X(U)$ will be exactly the homotopy pushout of the diagram of simplicial sets $\ast \leftarrow X(U) \rightarrow \ast$. Setting $X = \mathbb{G}_m = S^1_t$ and using that $\ast \simeq \mathbb{A}^1$ in the $\mathbb{A}^1$ model structure, this shows that the following square is a homotopy pushout in $\text{Spc}_\ast(k)$:
where every scheme is pointed at 1, and the top arrow is the map of spaces induced by the scheme theoretic inclusion of \( \mathbb{G}_m \) into \( \mathbb{A}^1 \) sending 1 to 1. Thus we have that \((S^{2,1},\ast) \simeq (\mathbb{A}^1,1)/(\mathbb{G}_m,1)\). But, taking our guiding principle seriously, alarm bells should go off when we think about the quotient of \( \mathbb{A}^1 \) by \( \mathbb{G}_m \) complex analytically. For example, if \( \mathbb{A}^1 \) is thought of as the affine line \( \mathbb{C} \), then \( \mathbb{A}^1/\mathbb{G}_m \) is the quotient of a plane by a circle: that is, it is a sphere. We can make this rigorous, and with little work! For if instead we replace both points in the diagram defining \( \Sigma_M X \) above with copies of \( \mathbb{A}^1 \), we see that the following diagram of schemes:

\[
\begin{array}{ccc}
\mathbb{G}_m & \xrightarrow{1 \to 1} & \mathbb{A}^1 \\
\downarrow & & \downarrow 1 \to \ast \\
\{\ast\} & \xrightarrow{\mathbb{A}^1 \to \mathbb{S}^{2,1}} & S^{2,1}
\end{array}
\]

is a pullback in \( \text{Sm}/k \), where the top and left morphisms are the same as the top morphism in the previous diagram and the right and bottom arrows are the inclusions of the affine charts corresponding to \( \mathbb{P}^1 - \{0\} \) and \( \mathbb{P}^1 - \{\infty\} \), respectively. This renders a pushout in \( \text{Spc}_*(k) \) where every scheme is pointed at 1, and where each space is a cofibrant replacement of the corresponding space in the previous diagram. Thus \((\mathbb{P},1)\) is a homotopy pushout of a cofibrant replacement for the diagram above, and putting everything together gives the following.

**Theorem 1.25.** \((\mathbb{P},1) \simeq (\mathbb{A}^1,1)/(\mathbb{G}_m,1) \simeq (S^{2,1},\ast)\)

That is, the topological truism that the plane is weak equivalent to a closed disc, and that a disc modulo its boundary circle is a sphere, becomes true for schemes.

### 1.5. Stable Motivic Homotopy Theory

Now we can define motivic spectra, and realize the goal outlined at the beginning of Section 1.4. We’ll end this section by doing so, by describing also the stable motivic homotopy category, and by talking briefly about some important motivic cohomology theories. At this point, the first task boils down to emulating the classical definition of spectra, save respecting suspensions in both variables.

**Definition 1.26.** Let \( m,n \geq 0 \) be integers. An \((s,t)\)-bispectrum \( E \) consists of pointed spaces \( E_{n,m} \), along with structure maps:

\[
\sigma_s : S^1_s \wedge E_{n,m} \to E_{n+1,m}
\]

\[
\sigma_t : S^1_t \wedge E_{n,m} \to E_{n,m+1}
\]

such that the following diagram commutes for all \( m,n \):
where \( \tau \) is the isomorphism \( S^1_s \wedge S^1_t \xrightarrow{\sim} S^1_t \wedge S^1_s \) given by the symmetric monoidal structure on \( \text{Spec}(k)_* \). A morphism \( f : E \to E' \) of bispectra is a sequence of maps of pointed spaces \( E_{n,m} \to E'_{n,m} \) that commute with the structure maps. We will denote the category of \((s,t)\)-bispectra by \( \text{Spt}_{s,t}(k) \).

Note that this is, indeed, precisely the homotopical definition of spectra, save that having two motivic circles requires that one demand explicitly that the structure maps are compatible with swapping the order in which one suspends.

The coproduct of two bispectra \( X, Y \in \text{Spt}_{s,t}(k) \) is given by taking \( X_{p,q} \vee Y_{p,q} \) componentwise, so that it is called the wedge product of bispectra and denoted \( X \vee Y \). There is a functor \( \Sigma^{p,q} \) which sends a bispectrum with components \( E_{n,m} \) to the bispectrum with components \( S^{p,q} \wedge E_{n,m} \) and which smashes morphisms componentwise with \( S^{p,q} \) as well, which we will call suspension in \( S^{p,q} \).

Every pointed space \( X \) has a naturally associated suspension bispectrum \( \Sigma^\infty X \): that is, \( \Sigma^\infty X \) is the bispectrum whose \((n,m)\)-th component is \((S^1_s \wedge ... \wedge S^1_s) \wedge (S^1_t \wedge ... \wedge S^1_t) \wedge X \), and whose structure maps are the identity. When we talk about the suspension bispectrum of an \emph{a priori} unpointed space \( X \), we mean the suspension bispectrum of \( X_* \); when we talk about the suspension bispectrum of a scheme, we mean the suspension bispectrum of its associated space. \( \text{Spt}_{s,t}(k) \) is pointed by the suspension bispectrum of \( (\text{Spec}(k), \text{id}) \): that is, \( \Sigma^\infty(\text{Spec}(k), \text{id}) \) is the zero object of \( \text{Spt}_{s,t}(k) \), and one can check that every component pointed space of \( \Sigma^\infty(\text{Spec}(k), \text{id}) \) is \( (\text{Spec}(k), \text{id}) \).

\( \text{Spt}_{s,t}(k) \) also has a smash product, denoted \( \wedge \), whose definition is as difficult as that of the smash product for classical spectra. We urge the reader to take such a thing on faith; for an actual discussion of difficulties in defining a smash product of motivic spectra, see Nord, Remark 2.14 (and surrounding). We claim, however, that there is an appropriate \( \mathbb{A}^1 \)-stable homotopy category of \( \text{Spt}_{s,t}(k) \) on which suspension in \( S^{p,q} \) becomes an autoequivalence, the smash product becomes symmetric monoidal, and spectra represent cohomology theories, all derived in one go. Our task now is to define the right model structures to realize such a dream.

One often wants to consider only singly graded spectra: that is, sequences of spaces for which one chooses a sphere, and iterates the definition above modulo the \( \tau \) compatibility. Indeed, we actually must do exactly such a thing here. The case we will care about is the following.
Definition 1.29. An $S^1$-spectrum (or simply s-spectrum) $E$ consists of pointed spaces $E_n$, along with structure maps of the following form.

\[
\sigma_s : S^1_s \wedge E_n \to E_{n+1}
\]

A morphism $f : E \to E'$ of s-spectra is a sequence of maps of pointed spaces $E_n \to E'_n$ that commute with the structure maps. The category of s-spectra will be denoted $\mathbf{Spt}_s(k)$. Similarly to the case for bispectra, every pointed space $X \in \text{Sm}/k$ has an associated s-suspension spectrum $\Sigma_\infty^s X$, whose $n$-th space is $S^1_s \wedge X$ and whose structure maps are the identity. Just as in the case of bispectra, there is a functor $\Sigma_n$ which sends an s-spectrum with components $E_n$ to one with components $S^{n,0}_n \wedge E_n$ which we will call simplicial suspension in $S^{n,0}_n$.

One can take the definition above, save replacing $S^1_s$ with $S^{2,1}_s \simeq \mathbb{P}^1$, to define the category of $\mathbb{P}^1$-spectra, which gives an equivalent theory to the one described here. We won’t use the latter category for technical reasons, however, while $\mathbf{Spt}_s(k)$ is especially nice because objects in it have a notion of stable homotopy groups taken directly from the usual definition for spectra. Namely, we may say the following.

Definition 1.31. If $E$ is an s-spectrum and $n \in \mathbb{Z}$, denote by $\pi_n(E)$ the $n$-th sheaf of stable homotopy groups associated to the presheaf of abelian groups on $\text{Sm}/k$, defined as follows.

\[
\pi_n(E) := \text{colim}_{m>0} \pi_{n+m}(E_m)
\]

We study s-spectra because the relative simplicity and naturality of the definition of s-stable homotopy groups makes defining $A^1$-stable homotopy groups of $(s,t)$-bispectra tractable. Indeed, it will turn out that $A^1$-stable homotopy sheaves are naturally a colimit of s-stable homotopy groups.

Just as for classical spectra, we say a map of s-spectra is an s-stable weak equivalence if it induces an isomorphism on all stable homotopy groups. With this in mind, one can define an s-simplicial model structure on $\mathbf{Spt}_s(k)$: namely, with weak equivalences the s-stable weak equivalences and cofibrations the morphisms of s-spectra such that each component morphism is an $A^1$-homotopical cofibration of pointed spaces. The associated homotopy category is proper, and will be denoted $\text{SH}_s(k)$.

We claimed above that s-spectra are a means of putting $A^1$-homotopy groups of $(s,t)$-bispectra on firm footing. But of course, to do this, we must define an $A^1$-homotopy category of s-spectra. One can do this by starting with $\text{SH}_s(k)$ and localizing along spaces and maps which are $A^1$-“local” and “equivalences”, just as we did before in $\text{Spc}(k)$. That is, we define an $A^1$-model structure on $\mathbf{Spt}_s(k)$ by emulating exactly (1.16) and (1.19), save that here an s-spectrum is $E$ is called $A^1$-local if for all $U \in \text{Sm}/k$ and $n \in \mathbb{Z}$, there is a bijection:

\[
\text{Hom}_{\text{SH}_s(k)}(\Sigma_s^n(U \times A^1)_+, \Sigma_s^n E) \to \text{Hom}_{\text{SH}_s(k)}(\Sigma_s^n U_+, \Sigma_s^n E)
\]

induced by the canonical map $U \times A^1 \to U$. We invite the reader to assure themselves that the rest of (1.16) works verbatim, defining $A^1$-weak equivalences as needed.
Then, we may give \( \text{Spt}_s(k) \) its \( s \)-stably \( A^1 \) model structure by left Bousfield localizing along the \( A^1 \)-weak equivalences. This is to say that the \( s \)-stably \( A^1 \) model structure on \( \text{Spt}_s(k) \) will have weak equivalences the morphisms whose images in \( \text{SH}_s(k) \) are \( A^1 \)-weak equivalences and cofibrations the pointwise cofibrations of spaces. Its associated homotopy category will be denoted by \( \text{SH}_{A^1}^s \).

Now we may finally define stable homotopy groups of bispectra. In the following, if \( E \) is an \((s,t)\)-bispectrum, let \( E_m \) denote the \( s \)-spectrum given by restricting to terms of the form \( E_{\ast,m} \), with structure maps their bispectrum structure maps.

**Definition 1.34.** If \( E \) is an \((s,t)\)-bispectrum and \( p, q \in \mathbb{Z} \), denote by \( \pi_{p,q} \) the sheaf of bigraded stable homotopy groups associated to the presheaf of abelian groups on \( \text{Sm/k} \).

\[
(1.35) \quad U \to \text{colim}_m \text{Hom}_{\text{SH}_A^s(k)}(\Sigma^\infty (S^p - q \land S^q + m \land U_+), E_m)
\]

This seems convoluted, but one can note that the wedge of spheres in this definition for fixed \((p, q)\) is exactly \( S^{p+m,q+m} \). Thus, what we’re doing here is the usual definition of stable homotopy groups of spectra, save that now instead of ratcheting up through suspensions of individual spaces until the Freudenthal Suspension Theorem assures us that the homotopy groups under consideration stabilize, we’re doing the same for entire \( s \)-spectra. This is useful enough that from here on out, when we say “spectrum” without qualification, we mean “\((s,t)\)-bispectrum”.

This definition imbues \( \text{Spt}_{s,t} \) with its own notion of weak equivalence. Namely, we say that a morphism of \((s,t)\)-bispectra \( f : E \to F \) is a stable weak equivalence if it induces an isomorphism of all sheaves of bigraded stable homotopy groups. With this definition in mind, we may finally say the following.

**Definition 1.36.** The \( A^1 \)-model structure on \( \text{Spt}_{s,t}(k) \) is given by the following distinguished classes:

- A morphism \( f \) is a weak equivalence if it a stable weak equivalence;
- A morphism \( f \) is a cofibration if it is a componentwise cofibration of pointed spaces;
- A morphism \( f \) is a fibration if it satisfies the right lifting property with respect to acyclic cofibrations.

The homotopy category of \( \text{Spt}_{s,t}(k) \) with its \( A^1 \)-model structure is called the \( A^1 \)-stable homotopy category, and will be denoted by \( \text{SH}(k) \).

The fact that we’ve arrived at the right notion of a stable \( A^1 \)-homotopy category is first cemented by the following theorem, the stable analog of \((1.20)\). As in the unstable case, we indicate the relation of stable weak equivalence by \( \simeq \); it will be clear from context which one we mean.

**Theorem 1.37.** Let \( X \in \text{Sm/k} \). Then the canonical map:

\[
(1.38) \quad X \times A^1 \to X
\]

induces a stable weak equivalence of suspension spectra \( \Sigma^\infty (X \times A^1) \simeq \Sigma^\infty X \).

Two spaces are called stably equivalent if their suspension (bi)spectra are stably weak equivalent. One can see that (unstable) weak \( A^1 \)-weak equivalent spaces give
stably weak equivalent suspension bispectra. When it is clear from context that we are talking about spectra, however, we will often abbreviate “stably weak equivalent” to simply “weak equivalent” or even just “equivalent”.

The notions of $\mathbb{A}^1$-weak equivalence and of stable equivalence are the desired realization of our principle that schemes should represent spaces with the homotopy type matching their intuitive geometry. Indeed, in a vast multitude of situations where one has a scheme $X$ which in some senses looks locally like $\mathbb{A}^1 \times Y$ for another scheme $Y$, we’ll have $X \simeq Y$. For example, the following is true.

**Theorem 1.39.** Line bundles are $\mathbb{A}^1$-weak equivalences.

Intuitively, this is true because of the following. If one has a line bundle $E \to X$, then one may take a Zariski cover $\{U_\alpha\}$ of $X$ which trivializes $E$. Then $X$ is a colimit of the diagram $D$ induced by all inclusions $\cap_n U_n \hookrightarrow \cap_{n \neq i} U_n$, and $E$ is a colimit of a diagram $\tilde{D}$ consisting of the same objects colimit but with different gluing morphisms. But then, twists in these gluings are parametrized exactly by invertible regular maps, so that up to $\mathbb{A}^1$-homotopy, these twists are trivial. Thus the colimits of each diagram are weak equivalent.

This same intuition shows the following, which is strictly more general.

**Theorem 1.40.** Vector bundles are $\mathbb{A}^1$-weak equivalences.

See MoVo, Example 3.2.2. Thus our homotopy theory of schemes is one where even local products with $\mathbb{A}^n$, the space we went out of the way to contract, are trivial; one can simply homotope $\mathbb{A}^n$ away!

Another example where $\mathbb{A}^1$-weak equivalence makes intuitive geometry into an interesting homotopical reality comes from the class of toric varieties, which are normal varieties containing $(\mathbb{C}^*)^n$ as a dense open subscheme and with the property that the multiplication action of $(\mathbb{C}^*)^n$ on itself extends to an action on the whole variety. Since we only work with schemes in $\text{Sm}/k$ throughout, for us all toric varieties are smooth (this is not true in general). Then for starters, we have the following lemma, whose proof is Proposition 5.3 in Duls1.

**Lemma 1.41.** Any affine toric variety $X$ is $\mathbb{A}^1$-weak equivalent to $(\mathbb{A}^1 - 0)^m$ for some $m \leq \dim X$.

In fact, any toric variety can be written as a disjoint union of tori, and the $m$ above will be precisely the dimension of the smallest torus in this disjoint union. Thus, the above then says that for a variety $X$ containing a dense torus and controlled in some rigid arithmetic way by said torus’s geometry, one can simply $\mathbb{A}^1$-homotope it to a torus anyway!

In fact, one can characterize all toric varieties up to stable weak equivalence, though the proof assumes familiarity with the theory of toric varieties.

**Theorem 1.42.** Let $X \in \text{Sm}/k$ be a toric variety. Then, $X$ is stably equivalent to a homotopy colimit of tori $(\mathbb{A}^1 - 0)^\flat$ with disjoint basepoints.

**Proof.** Given a fan $\Phi$ defining $X$, the Zariski cover $\{U_\alpha\}$ of $X$ by the affine toric varieties corresponding to the maximal cones of $\Phi$ and their (also affine toric) intersections renders a diagram $D$ induced by all inclusions $\cap_n U_n \hookrightarrow \cap_{n \neq i} U_n$ whose
colimit is $X$. This induces a diagram of cofibrant spaces, and then of suspension bispectra, whose homotopy colimit is $\Sigma^\infty X$. By (1.41), any affine toric variety is weak equivalent, and thus stably weak equivalent, to $(\mathbb{A}^1 - 0)^k$ for some $k$. Thus, replacing each affine in our cover of $X$ with an $\mathbb{A}^1$-weak equivalent torus does not alter the induced homotopy colimit, so the theorem follows.

For those familiar with the combinatorial theory of toric varieties, the above pair of statements say that, motivically, there is no such theory of toric varieties beyond determining the codimension of a given variety’s representing fan in the corresponding $\mathbb{R}$-vector space!

On the other hand, $\text{SH}(k)$ is a category properly deserving of the name “stable homotopy”. For example, as we hoped above, the smash product on $S\text{pt}_{s,t}(k)$ induces a symmetric monoidal structure on $\text{SH}(k)$ with unit the sphere spectrum $\Sigma^\infty (\text{Spec}(k) +)$. Further, stable suspending even interacts nicely with unstable suspension. Indeed, the following theorem holds.

**Theorem 1.43.** For any $X,Y \in \text{Sm}/k$, $\Sigma^\infty X_+ \land \Sigma^\infty Y_+ \cong \Sigma^\infty (X \times Y)_+$.

Suspension in $S^1_s$ and $S^1_t$ are both autoequivalences of $\text{SH}(k)$, with inverse deserving of the name simplicial and algebraic loop functors and denoted $\Omega_s$ and $\Omega_t$, respectively. $\text{SH}(k)$ is a triangulated category, with distinguished triangles given by the homotopy (co)fiber sequences (where fiber and cofiber sequences agree). Thus $\text{SH}(k)$ behaves algebraically exactly like the classical stable category!

Finally, recall (1.22), which says that Brown representability still holds for homotopy invariant functors on pointed motivic spaces. Then, using suspensions in both motivic spheres, one can construct motivic Eilenberg-Steenrod cohomology theories $\mathbb{A}^1$-homotopically: namely, by taking a sequence of functors $\{F_{n,m}\}$ such that 1) $F_{n,m}$ satisfies the hypotheses of (1.22) and 2) $F_{n+1,m}(S^1_s \land X) \simeq F_{n,m}(X) \simeq F_{n,m+1}(S^1_t \land X)$ for all $n,m \in \mathbb{Z}$, $X \in \text{Spc}(k)$. Since 1) is the same as saying that every $F_{n,m}$ obeys all of the Eilenberg-Steenrod axioms besides suspension, the inclusion of 2) means that such a sequence is rightfully called a motivic cohomology theory. But then, using (1.22), each $F_{n,m}$ is represented as a functor on $H(k)_*$ by a space $X_{n,m}$. Further, 2) tells us that the sequence of spaces $\{X_{n,m}\}$ satisfy $S^1_s \land X_{n,m} \simeq X_{n+1,m}$ and $S^1_t \land X_{n,m} \simeq X_{n,m+1}$ for all $n,m$. But such a sequence of spaces and maps is an $(s,t)$-bispectrum by definition! Thus the discussion above shows that, just as in the classical stable world, motivic cohomology theories are represented by motivic spectra.

As a downside, the classical fact that there are in general more morphisms between the sequences of functors constituting a pair of cohomology theories and the objects of $\text{SH}(k)$ representing them still holds. Thus why we are careful in saying that spectra represent cohomology theories: the two categories are not the same!

On the other hand, the above means that a great deal of “cohomological” functors will have representing spectra. For example, there is an Eilenberg-Maclane Spectrum $H\mathbb{Z}$ in $\text{SH}(k)$ such that for any scheme $X$, one has a sequence of motivic singular cohomology groups $H\mathbb{Z}^{p,q}(X)$ associated to $X$ defined by:
(1.44) $H(Z)^{p,q}(X) := \text{Hom}_{SH(k)}(\Sigma^\infty(X), S^p_s \wedge S^q_t \wedge HZ) = CH^q(X, 2q - p)$

where the groups $CH^q(X, 2q - p)$ are Bloch’s higher Chow groups, about which we will say nothing more. Instead, we will simply give the following non-exhaustive list of outputs for $H(Z)^{p,q}$, which shows that the ostensibly boastful name “motivic singular cohomology” is more than appropriate.

- $H(Z)^{0,0}(X) = CH^0(X, 0) = \mathbb{Z}^{\pi_0(X)}$, the free abelian group generated by path components of $X$.
- For an $n$-dimensional scheme $X$ in $\text{Sm}/k$, $H(Z)^{2n-2,n-1}(X) = \text{Pic}(X)$, the Picard group of $X$.
- In general, $H(Z)^{2(n-k),n-k}(X)$ is $X$’s $k$th Chow group, which is the classical analog of singular cohomology used in algebraic geometry.
- For any field $\ell$ over $k$, $H(Z)^{n,n}(\ell)$ is the $n$-th Milnor $K$-Theory of $\ell$.

Thus, $H(Z)$ represents a cohomology theory which interpolates between the topology of $X$, algebraic singular cohomology, and suitably refined $K$-theories. See MVW for proofs and more generalities.

$H(Z)$ will turn out to be a motivic ring spectrum, in the usual sense of being a monoid under the smash product. This means in particular that $H(Z)$ induces a motivic singular cohomology ring for any space $X$, or more explicitly that the groups $H(Z)^{m,n}(X)$ can be arranged into a graded ring $H(Z)^{*,*}(X)$ analogous to the singular cohomology ring $H^*(X'; \mathbb{Z})$ associated to any topological space $X'$. These properties along with the important generic outputs of $H(Z)(X)$ lead many sources to call $H(Z)$ the motivic cohomology ring. One can define an Eilenberg-Maclane ring spectra $HA$ for any abelian group $A$ which will render a motivic cohomology ring with $A$ coefficients, however, just as one can take singular cohomology with coefficients in any abelian group in classical homotopy theory. As such, we avoid the blanket name “motivic cohomology”.

There are a host of exotic motivic cohomology theories. $K$-theories are representable in $\text{SH}(k)$, for example: indeed, Algebraic $K$-Theory itself is representable, with associated spectrum $KGL$. Algebraic Cobordism is representable in $\text{SH}(k)$. There is even a motivic analog to the spectrum of Topological Modular Forms! The moral is, as argued above, $\text{SH}(k)$’s relationship to “cohomology theories” for schemes is quite similar to the classical stable homotopy category’s relation to cohomology theories of topological spaces.

2. Cellular Schemes

2.1. From Cohomology Theories to “Point-Set” Topology. Thus far, our main task has been to import stable homotopy theory into scheme theory. But this means that we more or less ran straight towards classical stable phenomena like spectra/cohomology theories, leaving many questions about the purely topological properties of $\text{Spc}(k)$ and $\text{SH}(k)$ open. The more abstract structures are all there, from Eilenberg-Maclane spaces to cobordism spectra, but we’ve jumped past a lot of down to earth concerns. So let’s inquire about something which non-trivially depends on point-set matters, and which is so important as to take up the early
chapters of many widely used topology textbooks. That is, let’s ask: what about motivic CW-complexes?

2.2. Abstract (Non)-Cellularity. Cell-complexes are a cornerstone of algebraic topology because of their tractability, their usefulness, and the because of cellular/Whitehead approximation, which says that every (compactly generated weak Hausdorff) topological space is weak equivalent to one. To have a bona fide theory of motivic cell complexes, then, would be a huge boon. For the sake of keeping the object of study in mind, recall that a topological space $X$ is a cell-complex if it can be defined inductively as follows.

1. Start with a discrete set of points $X_0$.
2. Define inductively the $n$-skeleton $X_n$ attaching a collection $\{e^n_\alpha\}$ of $n$-cells to $X_{n-1}$ via maps $\phi_\alpha : S^{n-1} \rightarrow X_{n-1}$. Equivalently, we can define $X_n$ as the pushout of the following diagram.

\[
\begin{array}{ccc}
\biguplus \alpha S^{n-1} & \longrightarrow & X_{n-1} \\
\downarrow & & \downarrow \\
\biguplus \alpha e^n & \longrightarrow & 
\end{array}
\]

3. Stop at a finite step $k$ and give $X = X_k$ the quotient topology, or proceed transfinitely and give $X$ the weak topology.

Unfortunately, such a definition does not adapt well to the motivic scenario. Indeed, if one tries to define a motivic cell-complex by emulating the above, then item (1) goes through well enough by setting $X_0 = \biguplus \alpha \text{Spec}(k)$, but we’re already in a bind at step (2): we can try to take some kinds of pushouts of disjoint unions of motivic spheres $S^{p,q}$ along a bigraded collection of skeleta $X_{p,q}$, but we’re missing the classical topology fact that $\pi_k(S^{n-1}) = 0$ for $q < n - 1$ and, more cuttingly, we’re missing the relations $S^{n-1} = \partial e^n = e^{n-1}/\partial e^{n-1}$ in the first place. Neither of the above, unfortunately, have convenient motivic analogs.

On the other hand, say instead that we retain only the intuition that cell-complexes are colimits of disks and spheres, or more broadly (in the spirit of homotopy theory) that a space is cellular if it is weak equivalent to such a gluing. This motivates the following definition.

Definition 2.1. Let $\mathcal{M}$ be a pointed model category, and $\mathcal{A}$ a set of objects in $\mathcal{M}$. The class of $\mathcal{A}$-cellular objects is the smallest class of objects in $\mathcal{M}$ such that the following hold.

- Every object of $\mathcal{A}$ is $\mathcal{A}$-cellular.
- If $X$ is weak equivalent to an object in $\mathcal{A}$, then $X$ is $\mathcal{A}$-cellular.
- If $D : I \rightarrow \mathcal{M}$ is a diagram where each object $D_i \in D$ is cellular, then so is hocolim $D$.

That is, cellular objects are those which can be built up to homotopy out of gluings of objects in $\mathcal{A}$. The motivating example is, naturally, the following.

Example 2.2. Let $\mathcal{M} = \text{Top}_*$ with the Quillen model structure, and let $\mathcal{A} = \{(S_0, -1) \cup (S_0, 1)\}$, where think of $S_0$ as the set of points of distance 1 from the origin in $\mathbb{R}^1$ (equivalently, we could let $\mathcal{A}$ be the set of all spheres with all possible
pointings). The weak equivalence axiom ensures first that $\forall n \in \mathbb{N}$ and with every possible pointing, $D^n$ is $A$-cellular (we will often say is in $\text{Cell}(A)$). But then, all inclusions of spheres into discs are maps between objects in $\text{Cell}(A)$, so the colimit axiom says all $\text{CW}$-complexes are in $\text{Cell}(A)$. By Whitehead approximation and the weak equivalence axiom, this gives that $\text{Top}_* = \text{Cell}(A)$.

Definition (2.1) is thus an uninteresting one in the classical case, since we need only a select choice of colimits and the weak equivalence axiom in order to recover the class of spaces weak equivalent to $\text{CW}$ complexes. For more general model categories, however, it turns out that even these relaxed conditions applied to reasonable sets $A$ can fail to render an analogous cellular approximation theorem. That is, there are a plethora of model categories $M$ and nice candidate sets $A \subset \text{Ob}(M)$ for which $\text{Cell}(A) \neq \text{Ob}(M)$. This begs the question: can one cellu-
larly approximate motivic spaces starting from spheres? To investigate this question, we must start with the following definition.

Example 2.3. Let $M = \text{Spc}_*(k)$ with its $A^1$-model structure, and let $A = \{S^{p,q} | p, q \in \mathbb{N}\}$. An object in $\text{Spc}_*(k)$ is called unstably cellular if it is in $\text{Cell}(A)$.

$\mathbb{G}_m$, $A^1$, $\mathbb{P}^1$, and $A^n - 0$ have all been shown to be unstably cellular with 1 as their basepoint, which is the basepoint we’ll use when discussing these schemes if not otherwise indicated. It will turn out they are unstably cellular no matter the choice of basepoint. When we say an a priori unpointed space $X \in \text{Spc}(k)$ is unstably cellular, we mean that $X_+$ is unstably cellular. When we say a scheme is unstably cellular, we mean that the space it represents is unstably cellular.

Our preoccupation will often be with asking, more restrictively, whether even certain schemes are cellular. After all, schemes are in many senses a more restrictive class of objects than topological spaces (morphisms between them are strictly more rigid, since they’re given by continuous maps with the addition of sheaf theoretic data), so perhaps it seems a bit strong to ask that all schemes are unstably cellular. Indeed, as an answer to our preliminary question, it turns out that we do not even know if tori, schemes of the form $(A^1 - 0)^n$, are unstably cellular - we actually strongly expect them not to be (see Duls1, Example 3.5). More to the point, as an answer to our initial question, there is no unstable Whitehead approximation for schemes, much less for spaces (we promise counterexamples shortly!). On the other hand, the prequel leaves some hope: after all, moving from $H(k)$ to $\text{SH}(k)$ gave us plenty of constructions for free, so perhaps it will also rectify whatever is preventing spaces from being cellular. Thus we imitate (2.3) stably.

Definition 2.4. Let $M = \text{Spt}_{s,t}(k)$ with the $A^1$-model structure, and let $A = \{\Sigma^\infty S^{p,q} | p, q \in \mathbb{N}\}$. An object in $\text{Spt}_{s,t}(k)$ is called stably cellular if it is in $\text{Cell}(A)$.

When we say a pointed space $X$ is stably cellular we mean that $\Sigma^\infty X$ is stably cellular. When we say an a priori unpointed space $X \in \text{Spc}(k)$ is stably cellular, we mean that $X_+$ is stably cellular, and saying a scheme is stably cellular means the same for the space it represents.

A few fundamental properties are ready at hand with these definitions. Their proofs involve a fair amount of model category theory, so we only prove the last one and refer the reader to Duls1, Section 3 for the rest.
Property 2.5. (Stability properties of cellular spaces)

- All unstably cellular spaces are stably cellular.
- If $X$ and $Y$ are stably (unstably) cellular, then $X \wedge Y$ is stably (unstably, resp.) cellular.
- If $X$ and $Y$ are stably cellular, so is $X \times Y$.
- A disjoint union of stably cellular spaces is stably cellular.

Proof. For last the claim, say one has a space $X = \bigsqcup_i X_i$. Since the functor $X \to X_+$ is a left adjoint, we have that $X_+ = \vee_i X_{i+}$. Further, since the smash product of simplicial sets commutes with the wedge product pointwise, we have that $S^{p,q} \wedge X_+ \cong \vee_i S^{p,q} \wedge X_{i+}$ for any $p, q$. This implies that $\Sigma^\infty X_+ \simeq \vee_i \Sigma^\infty X_{i+}$, and then the claim follows from the homotopy colimit axiom. \hfill \square

Even these minimal tools immediately give us powerful theorems.

Theorem 2.6. For all $n \in \mathbb{N}$, the $n$-dimensional torus $(\mathbb{A}^1 - 0)^n$ is stably cellular.

Proof. Exercise. \hfill \square

Corollary 2.7. All toric varieties are stably cellular.

Proof. (1.42) states that non-singular toric varieties are stably equivalent to homotopy colimits of tori, namely via a diagram parametrized by and termwise stably weak equivalent to the diagram of schemes given by its affine cover. (2.6) tells us that tori are stably cellular, and so the suspension spectra of toric varieties are homotopy colimits of stably cellular schemes. Thus, by the homotopy colimit axiom for cellular objects, the theorem follows. \hfill \square

Toric varieties are a fairly broadly class of schemes, and one such that the truth or falsity of a conjecture on it often accurately predicts the truth or falsity of the proposition writ large. Thus, one may thus be convinced that some kind of stable Whitehead approximation holds for schemes, and perhaps even for spaces writ large. Unfortunately, such a thought fails even in simple cases.

Theorem 2.8. No nontrivial field extension of the base field $k$ is stably cellular.

This shocked the author upon first encounter. Indeed, how bad can the failure of a scheme to be (un)stably cellular be, if the most trivial of algebraic test cases fails in all possible cases? How big is the gap between the class of cellular schemes and the class of all schemes? The rest of this paper is dedicated to trying to answer this question. First, however, we will talk about the power of the theory of cellularity. We will focus in general on stable cellularity, both because knowing that a space isn’t stably cellular suffices to show that it also isn’t unstably cellular and because it is a much simpler condition to work with, as evidenced by our agnosticism with regards to even the unstable cellularity of tori.

A great deal of what follows can be found with proof in DuIs1; however, some proofs take us too far afield, and in general our presentations trades their technical comments for a more expository handling of the same material.

2.3. Tools and First Results in (Stable) Cellularity. In a sea of a priori arbitrarily hard to work with motivic spaces, cellular spaces are a fairly vast subclass that behave like a convenient category of topological spaces. They are also quite
a rigid subclass. For example, they are closed under taking homotopy cofibers in $\mathbf{Spc}_\bullet(k)$, by the colimit axioms, and are thus closed under model theoretic suspension, defined in (1.24). But we showed in said example that $\Sigma M X = S^1_s \wedge X$ for a pointed space $X$, so that the following theorem holds.

**Theorem 2.9.** If $X \in \mathbf{Spc}(K)_\bullet$ is unstably cellular, then so is $S^1_s \wedge X$.

Further, one will have noticed that we haven’t been so cautious in pointing our schemes. Everything we’ve said so far is rigorous with the conventions we’ve taken, but perhaps it feels (or at least, perhaps our bold tacitness has implied) that two different pointings of a scheme which satisfy some homotopic condition should give weak equivalent pointed objects. Indeed, the following is true.

**Theorem 2.10.** Suppose $X$ is an object of $\mathbf{Spc}(k)$, and $a,b: X \to X$ are two choices of basepoint. If $a$ and $b$ are $\mathbb{A}^1$-weakly homotopic maps in $\mathbf{Spc}(k)$, then $(X,a)$ is $\mathbb{A}^1$-weak equivalent to $(X,b)$ in $\mathbf{Spc}_\bullet(k)$.

_Proof._ This is DuIs1, Proposition 2.16. $\square$

Hence in such a situation, $(X,a)$ is unstably cellular if and only if $(X,b)$ is. All of the schemes with which we will work will satisfy the above for any two choices of basepoint. Further, when we work stably, we need not be worried at all.

**Theorem 2.11.** If $(X, x_0) \in \mathbf{Spc}_\bullet(k)$, then $(X, x_0)$ is stably cellular if and only if $X_+$ is.

_Proof._ Consider the cofiber sequence $\Sigma^\infty_k(\text{Spec}(k)_+) \to \Sigma^\infty_k(X_+) \to \Sigma^\infty_k(X, x_0)$ induced by the scheme theoretic identity map from $X_+$ to $(X, x_0)$. The colimit axiom implies one direction, and the other follows from a fair amount of model category theory; see DuIs1 Lemma 2.5. $\square$

**Corollary 2.12.** If $X \in \mathbf{Spc}(k)$ and $x_0, x_1 \in X$, then $(X, x_0)$ is stably cellular if and only if $(X, x_1)$ is.

Using the theory of algebraic vector bundles, one can even show the following.

**Theorem 2.13.** A scheme $X$ is stably cellular if and only if for any closed point $x$ of $X$, $X - \{x\}$ is stably cellular.

Thus, the class of cellular schemes and spaces enjoys profound homotopical closure properties: one can work stably as if one is not dealing with pointed spaces, or even as if one has thrown out a given base point entirely, and be certain they are not leaving the cellular realm.

The class of cellular schemes in particular is closed under a plethora of geometric constructions, too. As a first gesture, recall that (1.42) and (2.6) together show that toric varieties are stably cellular, and that (1.40) says that vector bundles bundles over (un)stably cellular schemes are (un)stably cellular by the weak equivalence axiom. In particular, recall the proofs of (1.42) and (1.40), which the reader may have noted went pretty similarly: we found a Zariski cover $\{U_\alpha\}$ of the schemes in question, showed all finite intersections $\cap_\alpha U_\alpha$ of elements of the cover to be cellular, and then took a homotopy colimit. That is, we exploited Zariski covers of the following type.
Definition 2.14. A Zariski cover \( \{ U_\alpha \} \) of a scheme \( X \) is completely stably cellular if each finite intersection \( \cap_n U_\alpha \) is stably cellular.

In the toric case, for example, the cover of a toric variety \( X \) by the affine toric varieties \( U_\alpha \) corresponding to the maximal cones of its defining fan \( \Phi \) is stably cellular by (1.42), (2.6), and the fact that the intersection of any two \( U_\alpha \) is another toric affine. Then, as the two cases where we have surreptitiously applied it may indicate, the following is true.

Theorem 2.15. If a variety \( X \) has a Zariski cover which is completely stable cellular, then \( X \) is also stably cellular.

Proof. This is DuIs1 Lemma 3.8, but the way we’ve used it above essentially already proves it: if a completely stably cellular cover \( \{ U_\alpha \} \) of \( X \) exists, then the diagram \( D \) associated to it induced by all inclusions \( \cap_n U_n \hookrightarrow \cap_n U_n \) can be weak-equivalent-replaced by a diagram \( \tilde{D} \) of stably cellular objects. Thus \( D \) and its \( \tilde{D} \) induce equivalent homotopy colimits of suspension spectra. But the homotopy colimit of \( D \) is weak equivalent to \( \Sigma^\infty X \), so the theorem follows. \( \Box \)

As argued above, (2.15) has (1.42) and (1.40) as a special case. It also generalizes the two of them almost immediately. For example, recall that an algebraic fiber bundle with fiber \( F \) is a map of schemes \( \phi: E \rightarrow B \) such that \( B \) admits a cover \( \{ U_\alpha \} \) that trivializes \( \phi \), i.e. an open cover on which \( \forall i, \phi|_{\phi^{-1}(U_\alpha)} \) is isomorphic to a projection \( U_\alpha \times F \rightarrow U_\alpha \). Then the following holds.

Theorem 2.16. Let \( p: E \rightarrow B \) be an algebraic fiber bundle with fiber \( F \) such that \( F \) is stably cellular, and such that \( B \) has a completely stably cellular cover that trivializes \( p \). Then, \( E \) is also stably cellular.

Proof. Let \( \{ U_\alpha \} \) be such a completely stably cellular and trivializing cover of \( B \). Let \( V_\alpha = p^{-1}(U_\alpha) \), and consider the cover \( \{ V_\alpha \} \) of \( E \). By hypothesis, we have \( V_\alpha \cong U_\alpha \times F \), so each element of \( \{ V_\alpha \} \) is stably cellular by the closure of stably cellular spaces under products. Further, we have that \( V_{\alpha_1} \cap \ldots \cap V_{\alpha_n} \cong U_{\alpha_1} \cap \ldots \cap U_{\alpha_n} \times F \) by the algebraic fiber bundle condition. But \( U_\alpha \) is completely stably cellular, so every \( \cap_n U_n \) is stably cellular, and so every \( \cap_m V_m \) is a product of stably cellular objects. Thus \( \{ V_\alpha \} \) is completely stably cellular, and we’re done by 2.15. \( \Box \)

With a bit more work, (2.15) is general enough that, using it and its corollaries, one can show that all of the following are stably cellular:

- Grassmannians;
- Stiefel Varieties;
- Flag varieties;
- Complements of affine and projective quadrics in \( \mathbb{A}^n \) and \( \mathbb{P}^n \), respectively;
- And even the Algebraic K-Theory and Algebraic Cobordism spectra.

The proofs of these claims are precisely the contents of DuIs1, Sections 3 - 6.

2.4. Applications of Motivic Cellularity to the Topology of Schemes. One is perhaps convinced at this point that cellularity is at least a powerful organizational principle. Cellular spaces, however, also enjoy extremely rich cohomological structures. For example, they enjoy in general a host of motivic cohomological
K"unneth spectral sequences, when most motivic spaces only enjoy some kind of derived equivalent. A preliminary example is given by Proposition 7.7 in DuIs1, which holds for motivic spectra only under some cellularity assumption.

**Theorem 2.17.** Let $E$ be a motivic ring spectrum, $M$ be a right $E$-module, and $N$ be a left $E$-module. Assume that $E$ and $M$ are cellular. Then there is a strongly convergent tri-graded spectral sequence with $E^2$ page as follows.

\[(2.18) \quad E^2_{a,(b,c)} = \text{Tor}^{\pi_\ast E}_{a,(b,c)}(M \wedge E N) \Rightarrow \pi_{a+b,c}(X \wedge E Y)\]

$\wedge_E$ denotes the smash product of spectra over the ring spectrum $E$, which we avoid defining precisely and insist that it acts analogously to the same construction in the classical stable scenario. One also has a conditionally convergent dual spectral sequence in terms of $\text{Ext}$ groups, see again DuIs1, Proposition 7.7.

One can say even stronger things for certain nice subclasses of cellular spaces. Namely, define the class $\text{FinCell}$ of finite cell complexes in $\text{Spt}_{s,t}(k)$ to be the smallest class with the following properties.

- $\text{FinCell}$ contains the suspension spectra $\Sigma^\infty S^{p,q}$ for every $p, q$;
- $\text{FinCell}$ is closed under weak equivalence, and;
- If $X \to Y \to Z$ is a homotopy cofiber sequence and any two of the objects are in $\text{FinCell}$, then so is the third.

Thus the difference between finite cell complexes and general cell complexes is that the former are only homotopy cofibers of spheres, while the latter are general homotopy colimits. One can think of restricting to homotopy cofibers as rendering a more faithful, albeit usually model theoretically lacking, allusion to CW-complexes than is given by the general class of stably cellular spaces. Indeed, a topological point is homotopic to an $n$-cell, so that a homotopy cofiber is like taking a pushout of disks, and thus $\text{FinCell}$ is in some sense the class of spectra which are pushouts of disks (things weak equivalent to $(\text{Spec}(k), \text{id})$) along spheres $(\Sigma^\infty S^{p,q})$.

Less handwavingly, a natural reason to consider $\text{FinCell}$ is because, recalling that $\text{SH}(k)$ is a triangulated category with distinguished triangles the co/fiber sequences, the image of $\text{FinCell}$ in $\text{SH}(k)$ is the smallest subclass of $\text{Obj}(\text{SH}(k))$ containing the spheres and closed under taking distinguished triangles. We will say that $\text{FinCell}$ is the triangulated closure of the set $\{\Sigma^\infty S^{p,q}\}$ of all spheres. Further, it turns out that (2.15) has the following refinement in the finite context.

**Theorem 2.19.** If a variety $X$ has a Zariski cover $\{U_n\}$ which is completely stable cellular and such that every $\cap_n U_n$ is in $\text{FinCell}$, then $X$ is also stably cellular and in $\text{FinCell}$.

Thus the restriction to $\text{FinCell}$ has both intuitive topological and algebraic upshots, without even losing our favorite tool for investigating cellularity. Further, finite cell complexes enjoy even richer K"unneth style theorems.

**Theorem 2.20.** Let $X, Y$ be two motivic spectra, and let $X$ be a finite cell complex. Let $E$ be a ring spectrum. Then, there exists strongly-convergent tri-graded K"unneth spectral sequence of the following form.

\[(2.21) \quad \text{Tor}_{a,(b,c)}^{E^\ast,\ast}(E^\ast,\ast X, E^\ast,\ast Y) \Rightarrow E^{b-a,c}(X \wedge_E Y)\]
Proof. This is DuIs1 Theorem 8.6.

Unfortunately, the author knows of no general way to show that a scheme is a finite cell complex, nor of any way of model theoretically working with them in terms nearly as convenient as one can use for cellular schemes writ large.

One may ponder why we only have the theorems of Section 2.4 for schemes with some variety of cellularity assumption. But in fact, one shouldn’t expect Künneth style theorems to hold for general motivic spaces! After all, motivic cohomology theories often encode sheaf theoretic information about objects in $\text{Sm}/k$ ($\mathbb{H}^i_{\mathbb{Z}}(X) = \text{Pic}(X)$ for any scheme $X$, for example), and it is a fact in classical algebraic geometry that in general, one needs to work with projective schemes and coherent sheaves in order to actually have a Künneth formula in sheaf cohomology. Thus one must assume some kind of topological compactness and algebraic finiteness conditions algebro-geometrically to have Künneth formulas for the cohomology of schemes, and so we should expect the same motivically.

Finite cellularity is exactly such a condition. Indeed, as alluded to above, the use of homotopy cofibers works double duty in that is means that 1) topologically, $\text{FinCell}$ consists only of finite gluings of spheres, and 2) algebraically, $\text{FinCell}$ is the triangulated closure of the spheres in $\text{SH}(k)$. Finite cellularity is therefore a smallness phenomenon internal to $\text{SH}(k)$ that ensures motivic cohomology behaves as it scheme-theoretically should. This intuition also explains why would expect (2.17) to hold even in the situation of more general cellular spaces: for though we lose some algebraic control when we substitute all homotopy colimits for just homotopy cofibers in expanding to the class of cellular spaces writ large, we retain enough topological control to ensure that we can easily describe sheaves of homotopy groups.

This line of argument suggests that other “topological” motivic cohomology theories should behave nicely with respect to cellular schemes. Indeed, the following fact which we will use below to prove (2.8) cements the above.

**Theorem 2.22.** Let $X \in \text{Sm}/k$ be stably cellular, and let $\mathbb{H}$ be the motivic singular cohomology spectrum. Then, there is a Künneth spectral sequence:

$$\text{Tor}_{\mathbb{H}^*(\text{Spec}(k))}(\mathbb{H}^*\cdot^*(X), \mathbb{H}^*\cdot^*(Y)) \Rightarrow \mathbb{H}^*\cdot^*(X \times Y)$$

for all $Y \in \text{Sm}/k$.

**Proof.** This is DuIs1 Theorem 8.12 applied to $\mathbb{H}$, where the discussion directly above said theorem there shows that $\mathbb{H}$ satisfies the necessary hypotheses.

On the other hand, we would be remiss not to briefly mention the applications of cellularity to classical algebraic topology. Indeed, it turns out that spectral sequences like the above reflect information about actual topological spaces. For example, there is a motivic analog of the Adams spectral sequence which, stronger than just computing motivic stable homotopy groups of spheres, is in special cases isomorphic to a class of Novikov spectral sequences which is useful in calculating the stable homotopy groups of spheres. A yoga of ideas surrounding this isomorphism
have been powerful enough to push forward said calculations immensely, to the
point that the following is true.

**Theorem 2.24** (Isaksen, Wang, Xu). *Up to dimension 90, only 4 Adams differentials have values which are not completely determined.*

For a much deeper discussion and proof of the above, see IWX.

### 2.5. Non-Cellular Schemes

Unfortunately, the class of stably cellular schemes is far from all encompassing. Indeed, (2.8) says that there are absolutely basic examples of schemes which are not cellular, and as a corollary that the class of stably cellular schemes is not closed under algebro-geometric operations like base change or closed immersions.

As a sort of pre-mortem, let’s ask: why would one expect (2.8) to be true? One line of argument could be that a field extension simply contains a great amount of arithmetic information which motivic cell complexes, as a class of spaces designed in line with a topological principal, cannot keep control of. Indeed, all of our arguments and intuition so far have been (more or less) geometrically and topologically based, while for any field extension \( \ell/k \) it is only purely algebraic information that separates the spaces represented by the topological points \( \text{Spec}(k) \) and \( \text{Spec}(\ell) \). In what follows, this arithmetic argument turns out to be a powerful heuristic.

**Proof of Theorem 2.8.** We spell the proof out fairly explicitly because the general result is folklore which the author could not find written up in this form in the literature. The special case of \( \mathbb{C} \) over \( \mathbb{R} \), however, can be found in Wil, Section 2.6.

Fix \( \ell/k \) any field extension, and assume \( \ell \) is stably cellular. We claim that finding a scheme \( X \) that is connected over \( k \) but disconnected upon changing bases to \( \ell \) suffices to render a contradiction. Such a scheme would violate the convergence of 2.23 at the \((0,0)\) grade: since \( H^{0,0}_X \) of any scheme \( X \) is \( \mathbb{Z}^{\pi_0(X)} \) and we’re given a first quadrant spectral sequence whose initial \((0,0)\) graded term is \( \text{Tor}_2(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z} \), the \((0,0)\) graded term of any page is a subobject of \( \mathbb{Z} \), while the spectral sequence should converge to \( \mathbb{Z}^{|\text{Aut}(\ell/k)|} \).

To see that such a scheme exists, it suffices to consider \( \ell \) itself. Indeed, recall from commutative algebra that for \( \ell \) a field extension of \( k \), if one has a \( k \)-algebra \( M \) which admits an \( \ell \)-algebra structure, then one has an isomorphism of \( k \)-algebras \( \ell \otimes_k M \cong \prod_{|\text{Aut}(\ell/k)|} M \) (if this is unfamiliar, it is worth thinking through what the \( k \)-algebra isomorphism is). Then the following diagram in Scheme is a pullback, induced by the pushout in CRing:

\[
\begin{array}{ccc}
\prod_{|\text{Aut}(\ell/k)|} \text{Spec}(\ell) & \longrightarrow & \text{Spec}(\ell) \\
\downarrow & & \downarrow \\
\text{Spec}(\ell) & \longrightarrow & \text{Spec}(k)
\end{array}
\]

Where \( \iota \) is the map induced by the inclusion of \( k \) into \( \ell \). This shows that \( \prod_{|\text{Aut}(\ell/k)|} \text{Spec}(\ell) \cong \text{Spec}(\ell) \times \text{Spec}(\ell) \) in \( \text{Sm}/k \), and so we’re done by the above.
More generally, this shows the following, whose proofs proceed exactly like that of (2.8) and which we split into two cases because of their slightly different characters.

**Corollary 2.25.** Let \( X \in \text{Sm}/k \) be a smooth \( k \)-scheme that is connected but not geometrically connected over \( k \). Then \( X \) is not stably cellular.

**Corollary 2.26.** Let \( X, Y \in \text{Sm}/k \) be smooth connected \( k \)-schemes such that \( X \times Y \) is disconnected. Then \( X \) is not stably cellular.

As surmised above, this line of argument went through precisely because of the overabundance of arithmetic information encoded in \( \text{Spec}(\ell) \). Indeed, the failure of (2.23) in the setup of (2.8) is parameterized exactly by the non-triviality of Galois theory for \( \ell \) over \( k \). Let’s follow this heuristic, and guess that perhaps schemes which are too algebraically or arithmetically rich will fail to be cellular in general. Scouring number theory for such schemes, a class to vet against cellularity immediately comes to mind: elliptic curves. Indeed, the following holds.

**Theorem 2.27.** Let \( E \in \text{Sm}/k \) be an elliptic curve. Then \( E \) is not stably cellular.

*Proof.* The machinery necessary for a rigorous proof would require 10 more pages of paper to motivate no matter the direction we took, so we send the reader to DuIs1 Section 1.2 and Tot1 Section 7 for discussions on how to proceed. \( \square \)

More broadly, recall that an abelian variety over a field \( k \) is a connected and projective variety over \( k \) which is also group scheme. An elliptic curve is an abelian variety of dimension one, and indeed elliptic curves are a motivating example for this definition. Then since our entire motto at the moment is to proceed by interrogating arithmetically or algebraically rich schemes (i.e. group varieties) and we’ve seen that elliptic curves fail to be stably cellular, one might question the cellularity of general abelian varieties. Indeed, the following fact holds.

**Theorem 2.28** (Private Correspondence from Aravind Asok). No (non-trivial) abelian variety in \( \text{Sm}/\mathbb{C} \) is stably cellular.

Thus cellularity can fail spectacularly for schemes encoding an abundance of arithmetic information.

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