BESICOVITCH SETS, KAKEYA SETS, AND THEIR PROPERTIES

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ABSTRACT. In this paper, we construct a Besicovitch set, prove that Besicovitch sets must have Hausdorff dimension equal to 2, and show that there does not exist an analogous set in $\mathbb{R}^3$. We then discuss the Kakeya problem and construct a Kakeya set of arbitrarily small area.

CONTENTS

1. INTRODUCTION

In 1917, the Russian mathematician Abram Besicovitch was working on a problem in Riemann integration: if $f : \mathbb{R}^2 \to \mathbb{R}$ is Riemann integrable, then are there orthogonal coordinate axes such that the Riemann integral $\int f(x, y)dx$ exists for all y and such that $\int f(x, y)dx$ is Riemann integrable as a function of y? The answer is no, and a counterexample is based on the fact that there exists a set of measure zero in the plane which contains a unit line segment in every direction. In 1919, Besicovitch constructed such a set, and this set would come to be known as a Besicovitch set.

Also in 1917, the Japanese mathematician Sōichi Kakeya asked the following question: what is the smallest area of a convex planar set in which a needle can be rotated by 180 degrees? Sets in which a unit line segment can be rotated by 180 degrees are known as Kakeya sets. In 1921, Pál showed that a convex Kakeya set of smallest area must be an equilateral triangle. However, Kakeya's problem without the convex condition was still an open question. When Besicovitch later found out about this question, he realized that he could construct a Kakeya set of arbitrarily small area by adapting his construction of a Besicovitch set.

Besicovitch and Kakeya sets are not just interesting from a geometric perspective. They are also related to harmonic analysis, solutions to the wave equation, and additive combinatorics (the interested reader can see [4] and [7]). In this paper, we will not focus on such applications, but instead, we will provide an introduction to Besicovitch and Kakeya sets. We will begin by constructing a Besicovitch set, and then we will prove that Besicovitch sets must have Hausdorff dimension equal to 2. We will discuss how there does not exist an analogue of a Besicovitch set in $\mathbb{R}^3$. Finally, we will construct a Kakeya set of arbitrarily small area.

Throughout the paper, we will assume that the reader has an understanding of basic real analysis and measure theory. The following is a list of notation we will use:

Notation

$\mathcal{L}^n$	n-dimensional Lebesgue measure
H^s	s-dimensional Hausdorff measure or outer measure
$\dim(E)$	Hausdorff dimension of E
$\overline{E}$	closure of E
$\\|\cdot\\|$	Euclidean norm
$\cdot$	dot product
$B_r(x)$	closed ball of radius r centered at x
$\mathrm{Gr}(n,k)$	Grassmanian of k-dimensional linear subspaces of $\mathbb{R}^n$

2. Constructing a Besicovitch set

A *Besicovitch set* is a subset of $\mathbb{R}^2$ with Lebesgue measure zero which contains a unit line segment in every direction. To construct a Besicovitch set, we will follow the construction given in Falconer's book [1, p. 96-99]. The basic idea behind the construction is the following geometric argument.

Lemma 2.1. *Consider a triangle T whose base lies along a line L. Suppose T has base length $2b$ and height h. Split the triangle T into subtriangles T_1 and T_2 with base length b by connecting a median from the apex of T to the base of T (see Figure 1). Fixing some $\alpha \in (1/2, 1)$, we slide T_2 along L by a distance of $2b(1-\alpha)$ in order to overlap T_1 such that the resulting figure F consisting of the overlapped triangles has area $\mathcal{L}^2(F) = (\alpha^2 + 2(1-\alpha)^2)\mathcal{L}^2(T)$. The reduction in area is then $\mathcal{L}^2(T) - \mathcal{L}^2(F) = (1-\alpha)(3\alpha-1)\mathcal{L}^2(T)$.*

Proof. Note that the figure F contains a triangle T' that is similar to T (see Figure 1). T' has a base length of

$$2b(1-\alpha) + 2(b - 2b(1-\alpha)) = 2b - 2b\alpha + 2b - 4b + 4b\alpha = 2b\alpha.$$

Thus, we find that the similarity ratio is α. It follows that the area of T' is $\mathcal{L}^2(T') = \alpha^2\mathcal{L}^2(T)$. Now consider the triangles A, B, C, and D, as indicated in Figure 1. We see that A and D are similar to triangle T_2, and B and C are similar to triangle T_1. Because each of the triangles A, B, C and D has base length $b(1-\alpha)$, we find that the similarity ratio is $1-\alpha$ for A and D as well as B and C. Therefore, we have $\mathcal{L}^2(A) = \mathcal{L}^2(D)$ and $\mathcal{L}^2(B) = \mathcal{L}^2(C)$, and we can compute these areas:

$$\mathcal{L}^2(A) = \mathcal{L}^2(D) = (1-\alpha)^2\mathcal{L}^2(T_1) = \frac{1}{2}(1-\alpha)^2\mathcal{L}^2(T),$$

$$\mathcal{L}^2(B) = \mathcal{L}^2(C) = (1-\alpha)^2\mathcal{L}^2(T_2) = \frac{1}{2}(1-\alpha)^2\mathcal{L}^2(T).$$

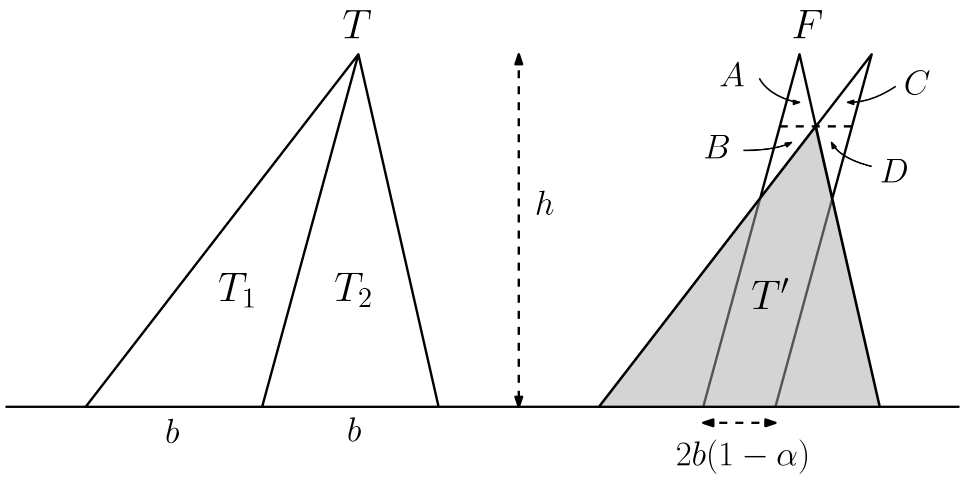

FIGURE 1

Putting this all together, we find that the area of figure F is given by

$$\mathcal{L}^2(F) = \mathcal{L}^2(T') + \mathcal{L}^2(A) + \mathcal{L}^2(B) + \mathcal{L}^2(C) + \mathcal{L}^2(D) = (\alpha^2 + 2(1-\alpha)^2)\mathcal{L}^2(T).$$

Then the reduction in area is

$$\begin{aligned}\mathcal{L}^2(T) - \mathcal{L}^2(F) = (1-\alpha^2 - 2(1-\alpha)^2)\mathcal{L}^2(T) &= (1-\alpha^2-2+4\alpha-2\alpha^2)\mathcal{L}^2(T)\\ &= (-3\alpha^2+4\alpha-1)\mathcal{L}^2(T)\\ &= (1-\alpha)(3\alpha-1)\mathcal{L}^2(T).\end{aligned}$$

□

Using this lemma, we can obtain the following theorem.

Theorem 2.2. *Consider a triangle T whose base lies along a line L. For $k \in \mathbb{N}$, we can divide the base of T into 2^k equal pieces that form the bases of 2^k subtriangles of T sharing an apex with T (see Figure 2(a)). For large enough k, we can slide these 2^k subtriangles along L such that the resulting figure F has arbitrarily small area. Moreover, if we have $T \subseteq U$ for some open set U, we can also ensure that $F \subseteq U$.*

Proof. Fix $\alpha \in (1/2, 1)$ and $k \in \mathbb{N}$, which we will determine later. Divide the base of T into 2^k equal pieces to form the bases of 2^k subtriangles $T_1, \ldots, T_{2^k}$ which share an apex with T. For $1 \leq i \leq 2^{k-1}$, we slide T_{2i} towards T_{2i-1} along L as in Lemma 2.1 to produce a figure $F_i^{(1)}$. We can do this such that $F_i^{(1)}$ contains a triangle $T_i^{(1)}$ which is similar to $T_{2i-1} \cup T_{2i}$ with similarity ratio α, by the argument in Lemma 2.1. By Lemma 2.1, we also have the following reduction in area:

$$\mathcal{L}^2(T_{2i-1} \cup T_{2i}) - \mathcal{L}^2(F_i^{(1)}) = (1-\alpha)(3\alpha-1)\mathcal{L}^2(T_{2i-1} \cup T_{2i}).$$

We then repeat this process by sliding $F_{2i-1}^{(1)}$ and $F_{2i}^{(1)}$ towards each other to produce a figure $F_i^{(2)}$ for $1 \leq i \leq 2^{k-2}$. Note that the triangles $T_{2i-1}^{(1)}$ and $T_{2i}^{(1)}$ contained in $F_{2i-1}^{(1)}$ and $F_{2i}^{(1)}$ have sides that are parallel to each other and have equal length. Therefore, applying Lemma 2.1, we can ensure that $F_i^{(2)}$ contains a triangle $T_i^{(2)}$

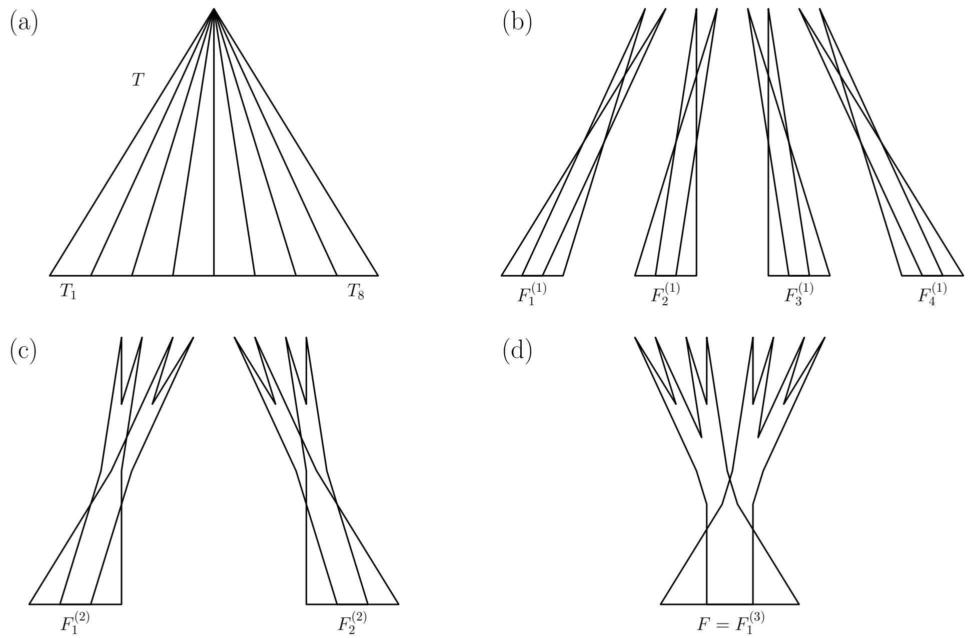

FIGURE 2

which is similar to $T^{(1)}_{2i-1} \cup T^{(1)}_{2i}$ with similarity ratio α. This allows us to place a lower bound on the reduction in area due to forming $F^{(2)}_i$ from $F^{(1)}_{2i-1}$ and $F^{(1)}_{2i}$:

$$\begin{aligned}\mathcal{L}^2(F^{(1)}_{2i-1} \cup F^{(1)}_{2i}) - \mathcal{L}^2(F^{(2)}_i) &\geq (1-\alpha)(3\alpha-1)\mathcal{L}^2(T^{(1)}_{2i-1} \cup T^{(1)}_{2i}) \\ &= (1-\alpha)(3\alpha-1)\alpha^2\mathcal{L}^2(T_{4i-3} \cup T_{4i-2} \cup T_{4i-1} \cup T_{4i}).\end{aligned}$$

We proceed in this way for another $k-2$ steps. The final figure is $F^{(k)}_1$, which we denote by F. Figure 2 shows this process when $k = 3$. In the jth step, we create the figures $F^{(j)}_i$ by sliding $F^{(j-1)}_{2i}$ and $F^{(j-1)}_{2i-1}$ towards each other for $1 \leq i \leq 2^{k-j}$. Note that when $j = 1$, $F^{(0)}_{2i-1}$ and $F^{(0)}_{2i}$ are T_{2i-1} and T_{2i}, respectively. We have the following lower bound on the reduction in area due to forming $F^{(j)}_i$ from $F^{(j-1)}_{2i}$ and $F^{(j-1)}_{2i-1}$:

$$\begin{aligned}\mathcal{L}^2(F^{(j-1)}_{2i-1} \cup F^{(j-1)}_{2i}) - \mathcal{L}^2(F^{(j)}_i) &\geq (1-\alpha)(3\alpha-1)\mathcal{L}^2(T^{(j-1)}_{2i-1} \cup T^{(j-1)}_{2i}) \\ &= (1-\alpha)(3\alpha-1)\alpha^{2(j-1)}\mathcal{L}^2(T_{i2^j-2^j+1} \cup \cdots \cup T_{i2^j}).\end{aligned}$$

From this, we obtain a lower bound on the total reduction in area in the jth step:

$$\begin{aligned}\sum_{i=1}^{2^{k-j}} (\mathcal{L}^2(F^{(j-1)}_{2i-1} \cup F^{(j-1)}_{2i}) - \mathcal{L}^2(F^{(j)}_i)) &\geq (1-\alpha)(3\alpha-1)\alpha^{2(j-1)}\mathcal{L}^2(T_1 \cup \cdots \cup T_{2^k}) \\ &= (1-\alpha)(3\alpha-1)\alpha^{2(j-1)}\mathcal{L}^2(T).\end{aligned}$$

We then add up the following j inequalities:

$$\sum_{i=1}^{2^{k-1}} (\mathcal{L}^2(T_{2i-1} \cup T_{2i}) - \mathcal{L}^2(F_i^{(1)})) \geq (1-\alpha)(3\alpha-1)\mathcal{L}^2(T),$$

$$\sum_{i=1}^{2^{k-2}} (\mathcal{L}^2(F_{2i-1}^{(1)} \cup F_{2i}^{(1)}) - \mathcal{L}^2(F_i^{(2)})) \geq (1-\alpha)(3\alpha-1)\alpha^2\mathcal{L}^2(T),$$

$$\cdots$$

$$\mathcal{L}^2(F_1^{(k-1)} \cup F_2^{(k-1)}) - \mathcal{L}^2(F_1^{(k)}) \geq (1-\alpha)(3\alpha-1)\alpha^{2(k-1)}\mathcal{L}^2(T).$$

We obtain

$$\sum_{i=1}^{2^{k-1}} \mathcal{L}^2(T_{2i-1} \cup T_{2i}) - \mathcal{L}^2(F_1^{(k)}) \geq (1-\alpha)(3\alpha-1)(1+\alpha^2+\cdots+\alpha^{2(k-1)})\mathcal{L}^2(T).$$

Equivalently, we have

$$\begin{aligned} \mathcal{L}^2(F) &\leq \mathcal{L}^2(T) - (1-\alpha)(3\alpha-1)(1+\alpha^2+\cdots+\alpha^{2(k-1)})\mathcal{L}^2(T) \\ &= \mathcal{L}^2(T) - \frac{(1-\alpha)(3\alpha-1)(1-\alpha^{2k})}{1-\alpha^2}\mathcal{L}^2(T) \\ &= \left(1 - \frac{(3\alpha-1)(1-\alpha^{2k})}{1+\alpha}\right)\mathcal{L}^2(T). \end{aligned}$$

We will now find a choice of k and α that makes $\mathcal{L}^2(F)$ arbitrarily small. Let $\varepsilon > 0$ be given. Let $\varepsilon' = \frac{\varepsilon}{\mathcal{L}^2(T)}$. Then there exists $\delta > 0$ (we can assume $\delta < 1/2$) such that for all $1-\delta < \alpha < 1$, we have

$$1 - \frac{3\alpha-1}{1+\alpha} < \frac{\varepsilon'}{2}.$$

For any α such that $1-\delta < \alpha < 1$, by picking $k \in \mathbb{N}$ large enough, we also have

$$\frac{3\alpha-1}{1+\alpha} - \frac{(3\alpha-1)(1-\alpha^{2k})}{1+\alpha} < \frac{\varepsilon'}{2}.$$

Thus, for any $\alpha \in (1-\delta, 1)$ and large enough k, we have

$$\begin{aligned} \mathcal{L}^2(F) &\leq \left(1 - \frac{(3\alpha-1)(1-\alpha^{2k})}{1+\alpha}\right)\mathcal{L}^2(T) \\ &= \left(1 - \frac{3\alpha-1}{1+\alpha}\right)\mathcal{L}^2(T) + \left(\frac{3\alpha-1}{1+\alpha} - \frac{(3\alpha-1)(1-\alpha^{2k})}{1+\alpha}\right)\mathcal{L}^2(T) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This shows that by picking α and k appropriately, we can make $\mathcal{L}^2(F)$ arbitrarily small.

Now suppose $T \subseteq U$ for some open set U. Because T and U^C are disjoint, T is compact, and U^C is closed, we have

$$\operatorname{dist}(T, U^C) = \inf\{\|x-y\| : x \in T, y \in U^C\} > 0.$$

Choose $\eta < \operatorname{dist}(T, U^C)$. Note that in the procedure above, none of the subtriangles of the original triangle move more than the base length of the original triangle after we fix the position of the first subtriangle and move all other subtriangles relative to the first one. Now divide T into subtriangles with base length $b < \eta$. We may then perform the procedure above on each subtriangle of T such that the resulting figure F has arbitrarily small area. Furthermore, we have $F \subseteq U$ because all of the subtriangles of each subtriangle of T have moved less than η. □

Theorem 2.3. *There exists a Besicovitch set.*

Proof. Let F_1 be an equilateral triangle whose base lies along the x-axis and whose height is equal to 1. Let U_1 be a bounded open set such that $F_1 \subseteq U_1$ and such that $\mathcal{L}^2(\overline{U}_1) \leq 2\mathcal{L}^2(F_1)$. By Theorem 2.2, we can divide F_1 into subtriangles and move them to form a new figure $F_2 \subseteq U_1$ that satisfies $\mathcal{L}^2(F_2) \leq 2^{-2}$. We can then find a new open set U_2 satisfying $F_2 \subseteq U_2 \subseteq U_1$ and $\mathcal{L}^2(\overline{U}_2) \leq 2\mathcal{L}^2(F_2)$. Again, applying Theorem 2.2, we divide each of the triangles moved to create F_2 into further subtriangles and move these subtriangles to create a new figure $F_3 \subseteq U_2$ that satisfies $\mathcal{L}^2(F_3) \leq 2^{-3}$. We then find an open set U_3 satisfying $F_3 \subseteq U_3 \subseteq U_2$. Continuing this process, we obtain figures $\{F_i\}_{i=1}^{\infty}$ with $\mathcal{L}^2(F_i) \leq 2^{-i}$ for all $i \in \mathbb{N}$ as well as open sets $\{U_i\}_{i=1}^{\infty}$ with

$$F_i \subseteq U_i \subseteq U_{i-1}$$

and

$$\mathcal{L}^2(\overline{U}_i) \leq 2\mathcal{L}^2(F_i) \leq 2^{-i+1}$$

for all $i \in \mathbb{N}$.

Let $E = \bigcap_{i=1}^{\infty} \overline{U}_i$. We must have $\mathcal{L}^2(E) = 0$ because for all $i \in \mathbb{N}$, we have

$$\mathcal{L}^2(E) \leq \mathcal{L}^2(\overline{U}_i) \leq 2^{-i+1}.$$

Now note that for all $i \in \mathbb{N}$, we have that F_i, and therefore also $\overline{U}_i$, contains a unit line segment in any direction ρ such that the angle between the positive x-axis and the line segment is between $\pi/3$ and $2\pi/3$. This is true because it is true for the original equilateral triangle F_1. We will show that E also satisfies this property. Let ρ be a given direction that makes an angle between $\pi/3$ and $2\pi/3$ with the positive x-axis. For each $i \in \mathbb{N}$, there exists a unit line segment $\gamma_i \subseteq \overline{U}_i$ in the direction ρ. For all $i \in \mathbb{N}$, we have that γ_i is a subset of $\overline{U}_1$, which is compact. Let x_i be the lowermost endpoint of γ_i for each $i \in \mathbb{N}$ (lowermost meaning closest to the x-axis). Then because $\overline{U}_1$ is compact, by passing to a subsequence if necessary, we can assume that $\{x_i\}_{i=1}^{\infty}$ converges to some x. Consider the unit line segment γ in the direction ρ with lowermost endpoint x. Since all the γ_i are unit line segments in the same direction, we must have that the γ_i converge to γ pointwise. Given $j \in \mathbb{N}$, since $\{\overline{U}_i\}_{i=1}^{\infty}$ is decreasing, we have $\gamma_i \subseteq \overline{U}_j$ if $i \geq j$, and because $\overline{U}_j$ is closed, we must have $\gamma \subseteq \overline{U}_j$. We thus have $\gamma \subseteq \overline{U}_j$ for all $j \in \mathbb{N}$, and so $\gamma \subseteq E$. This means that E contains a unit line segment in the direction ρ. Let $B = E \cup E' \cup E''$, where E' and E'' are counterclockwise rotated copies of E by $\pi/3$ and $2\pi/3$, respectively. Then B is a set of measure zero which contains a unit line segment in every direction. □

It is possible to construct an even better Besicovitch set that contains a *line* in every direction. See [1, p. 103] for such a construction.

3. Hausdorff Measure and Properties of Besicovitch sets

Although we know that Besicovitch sets have measure zero, we can ask how "large" they are in comparison to other sets of measure zero in the plane. To investigate this, we can examine the dimension of Besicovitch sets. Intuitively, the dimension of a set is a reflection of its scaling behavior. For a one-dimensional set, like a line segment, scaling the set by $n \in \mathbb{N}$ results in a set which is composed of n copies of the original set. Scaling a two-dimensional set, like a square, by n results in a set which is composed of n^2 copies of the original set. Extending this idea, we could imagine that an s-dimensional set scaled by n would result in a set which is composed of n^s copies of the original set. We might guess that the dimension of Besicovitch sets is between 1 and 2. Since they contain unit line segments in every direction, they contain one-dimensional subsets, and so their dimension cannot be less than 1. On the other hand, they are planar sets, so their dimension cannot exceed 2. However, it is not immediately clear what value between 1 and 2 their dimension must be. The way to make this discussion formal is to introduce the notion of Hausdorff measure and dimension.

3.1. **Hausdorff Measure and Dimension.**

Definition 3.1. Let a subset $A \subseteq \mathbb{R}^n$ and a non-negative real number s be given. For $\delta > 0$, let $H^s_\delta(A)$ be the following:

$$H^s_\delta(A) = \inf\left\{\sum_{i=1}^{\infty} (\operatorname{diam}(E_i))^s : A \subseteq \bigcup_{i=1}^{\infty} E_i,\ \operatorname{diam}(E_i) \leq \delta \text{ for all } i \in \mathbb{N}\right\}.$$

Then the *s-dimensional Hausdorff outer measure* of A is given by

$$H^s(A) = \lim_{\delta \to 0} H^s_\delta(A).$$

Notice that as δ decreases to 0, $H^s_\delta(A)$ increases, so $H^s(A) = \lim_{\delta\to 0} H^s_\delta(A)$ exists (although it is possibly infinite). It can be shown that H^s is a metric outer measure [2, p. 326-327]. From this, it follows that H^s is a measure when restricted to the sigma algebra of Borel sets [2, p. 267-269]. We call this measure the *s-dimensional Hausdorff measure.* H^s has the following regularity property.

Proposition 3.2. *Let $A \subseteq \mathbb{R}^n$ be a subset. Then there is a G_δ-set G such that $A \subseteq G$ and such that $H^s(A) = H^s(G)$. Furthermore, if A is H^s-measurable and $H^s(A) < \infty$, then there is an F_σ-set F such that $F \subseteq A$ and such that $H^s(A) = H^s(F)$.*

For a proof of this, see [1, p. 8-9]. Another important property of the Hausdorff measure is that for any Borel set A, $H^s(A)$ is infinite for small values of s, and it is 0 for large values of s. There exists some cut-off value of s called the *Hausdorff dimension* of A at which $H^s(A)$ switches from being infinity to being 0.

Proposition 3.3. *Let $A \subseteq \mathbb{R}^n$ be a Borel set. Then there exists a unique value α such that for all $s < \alpha$, we have $H^s(A) = \infty$, and for all $s > \alpha$, we have $H^s(A) = 0$. We call α the Hausdorff dimension of A, which we denote by* $\dim(A)$.

Proof. We first show that if $H^\beta(A) < \infty$ for some β, then we have $H^s(A) = 0$ for all $s > \beta$. Let $s > \beta$ be given. Consider a covering of A by sets $\{E_i\}$, where $\operatorname{diam}(E_i) \leq \delta$ for all $i \in \mathbb{N}$. For $i \in \mathbb{N}$, we have

$$(\operatorname{diam}(E_i))^s = (\operatorname{diam}(E_i))^\beta (\operatorname{diam}(E_i))^{s-\beta} \leq (\operatorname{diam}(E_i))^\beta \delta^{s-\beta}.$$

Therefore,

$$H^s_\delta(A) \leq \delta^{s-\beta} H^\beta_\delta(A).$$

Note that $H^\beta_\delta(A)$ is finite, and note that $s-\beta>0$, so that $\delta^{s-\beta}H^\beta_\delta(A)$ tends to 0 as δ tends to 0. Taking the limit as δ tends to 0 of both sides of the above inequality, we obtain $H^s(A)=0$. Furthermore, we have that if $H^\beta(A)>0$ for some β, then we have $H^s(A)=\infty$ for all $s<\beta$. This follows immediately from the above because if we had $H^t(A)<\infty$ for some $t<\beta$, then we would have $H^s(A)=0$ for all $s>t$, which in particular means $H^\beta(A)=0$, in contradiction. Set $\alpha=\sup\{s : H^s(A)=\infty\}$. From the above results, it follows that for all $s<\alpha$, we have $H^s(A)=\infty$, and for all $s>\alpha$, we have $H^s(A)=0$. □

Using the regularity property of H^s, we can prove the following useful proposition.

Proposition 3.4. *Let $A\subseteq\mathbb{R}^n$ be any subset. Then there is a G_δ-set G containing A such that* $\dim(G)=\dim(A)$.

Proof. Let s be the Hausdorff dimension of A. For each $i\in\mathbb{N}$, by Proposition 3.2, there exists a G_δ-set G_i containing A such that

$$H^{s+1/i}(G_i)=H^{s+1/i}(A)=0.$$

Let $G=\bigcap_{i=1}^\infty G_i$. Note that G is G_δ and contains A. Consider any $t>s$, and pick $k\in\mathbb{N}$ such that $s+1/k<t$. Then we have

$$H^t(G)\le H^t(G_k)=0.$$

On the other hand, for any $t<s$, we have

$$H^t(G)\ge H^t(A)=\infty.$$

It follows that $\dim(G)=s$. □

3.2. Properties of Besicovitch Sets.

We will now prove that Besicovitch sets must have Hausdorff dimension equal to 2. Informally, this means that among sets of measure zero in $\mathbb{R}^2$, Besicovitch sets are the "largest" of them in some sense. To prove this, we follow Mattila [3, p. 144]. We first need the result that projection does not increase Hausdorff measure.

Lemma 3.5. *Let $\mathrm{Gr}(k,n)$ denote the Grassmanian of k-dimensional linear subspaces of $\mathbb{R}^n$. Suppose $W\in\mathrm{Gr}(n,k)$ is a subspace. Let $\pi_W:\mathbb{R}^n\to\mathbb{R}^k$ be the orthogonal projection map onto W, and let $A\subseteq\mathbb{R}^n$ be a subset. Then we have*

$$H^s(\pi_W(A))\le H^s(A).$$

Proof. First note that for $a,b\in A$, we have

$$\|\pi_W(a)-\pi_W(b)\|\le\|a-b\|.$$

Let $\{E_i\}$ be a cover of A with $\mathrm{diam}(E_i)\le\delta$ for all $i\in\mathbb{N}$. Then $\{\pi_W(E_i)\}$ is a cover of $\pi_W(A)$ with $\mathrm{diam}(\pi_W(E_i))\le\mathrm{diam}(E_i)\le\delta$ for all $i\in\mathbb{N}$. We have

$$\sum_{i=1}^\infty(\mathrm{diam}(\pi_W(E_i)))^s\le\sum_{i=1}^\infty(\mathrm{diam}(E_i))^s,$$

and so it follows that

$$H^s_\delta(\pi_W(A))\le H^s_\delta(A).$$

Taking the limit as δ tends to 0, we obtain

$$H^s(\pi_W(A))\le H^s(A).$$

□

We also require the following proposition which states that if a Borel set has dimension less than or equal to one, dimension is preserved under projection onto almost all lines through the origin, and if a Borel set has dimension greater than one, the one-dimensional Hausdorff measure of projections onto almost all lines through the origin is positive.

Proposition 3.6. *Let $A \subseteq \mathbb{R}^n$ be a Borel set. For $e \in S^{n-1}$ (the $(n-1)$-sphere), let $\text{proj}_e : \mathbb{R}^n \to \mathbb{R}$ be the orthogonal projection map onto the line $\{te : t \in \mathbb{R}\}$. If $\dim(A) \leq 1$, then*

$$\dim(\text{proj}_e(A)) = \dim(A) \text{ for almost all } e \in S^{n-1}.$$

If $\dim(A) > 1$, then

$$H^1(\text{proj}_e(A)) > 0 \text{ for almost all } e \in S^{n-1}.$$

Note that in the second case, we have $\dim(\text{proj}_e(A)) = 1$ for almost all $e \in S^{n-1}$. A proof of this proposition requires some Fourier analysis which is not directly related to the main topics in this paper, so we will not give one. See [3, p. 56] for a proof. Finally, we need a result about the Hausdorff measure of intersections of a set with translated linear subspaces.

Proposition 3.7. *Let $A \subseteq \mathbb{R}^n$ be a set with $H^s(A) < \infty$. Then for $k \in \mathbb{N}$ with $k \leq s$ and for any $W \in \text{Gr}(k, n)$, we have*

$$H^{s-k}(A \cap (W^\perp + x)) < \infty \text{ for } H^k \text{ almost all } x \in W.$$

The proof of this result is an application of Fatou's lemma, but because the proof does not involve techniques directly relevant to this paper, we will omit it for the sake of space. For a proof, see [3, p. 94]. We can now prove our desired result about the Hausdorff dimension of Besicovitch sets. The main idea of the proof is the concept of duality, which means that for a certain line segment in $\mathbb{R}^2$, we can associate with it a unique point in $\mathbb{R}^2$. For example, a line segment of the form $\{(t, at + b) : 0 \leq t \leq 1/2\}$ for $a, b \in \mathbb{R}$ can be associated with the point (a, b). Given a set A in $\mathbb{R}^2$, it is often useful to consider the dual set of line segments that are associated with A. There is somewhat of a continuity property in that varying points in A by a small amount results in a variation of line segments by a small amount in the dual set. Furthermore, we can also use duality in the opposite direction by considering a set of line segments and looking at properties of the set of points with which those line segments are associated. This is what we do in the proof of the following theorem.

Theorem 3.8. *Besicovitch sets have Hausdorff dimension equal to 2.*

Proof. Let $B \subseteq \mathbb{R}^2$ be a Besicovitch set. By Proposition 3.4, we can assume without loss of generality that B is a G_δ-set. For $a \in (0, 1)$, $b \in \mathbb{R}$, and $q \in \mathbb{Q}$, let $\gamma(a, b, q)$ denote the line segment $\{(q + t, at + b) : 0 \leq t \leq 1/2\}$, that is, the line segment beginning at (q, b) and ending at $(q + 1/2, a/2 + b)$. For $q \in \mathbb{Q}$, define the set E_q as

$$E_q = \{(a, b) : a \in (0, 1), b \in \mathbb{R}, \gamma(a, b, q) \subseteq B\}.$$

Note that for any open set U, the set

$$\{(a, b) : a \in (0, 1), b \in \mathbb{R}, \gamma(a, b, q) \subseteq U\}$$

is open. Since B is G_δ, it follows that each E_q is G_δ. Because each $\gamma(a, b, q)$ is of length less than 1, given any $a \in (0, 1)$, there exists $b \in \mathbb{R}$ and $q \in \mathbb{Q}$ such that

$\gamma(a, b, q) \subseteq B$, by definition of a Besicovitch set. Let $\pi_x : \mathbb{R}^2 \to \mathbb{R}$ be the orthogonal projection onto the x-axis. Then we see that we must have $\pi_x(\bigcup_{q\in\mathbb{Q}} E_q) = (0, 1)$. I now claim that there must exist $q_0 \in \mathbb{Q}$ such that $H^1(E_{q_0}) > 0$. To see why this is the case, suppose for the sake of contradiction that $H^1(E_q) = 0$ for all $q \in \mathbb{Q}$. Then $H^1(\bigcup_{q\in\mathbb{Q}} E_q) = 0$. By Lemma 3.5, this implies that $H^1(0, 1) = 0$, in contradiction. Thus, there exists $q_0 \in \mathbb{Q}$ with $H^1(E_{q_0}) > 0$, and so $\dim(E_{q_0}) \geq 1$.

By Proposition 3.6, we have $\dim(p_t(E_{q_0})) = 1$ for almost all $t \in \mathbb{R}$, where $p_t : \mathbb{R}^2 \to \mathbb{R}$ is the map given by $p_t(x, y) = tx + y$. Note that for $0 \leq t \leq 1/2$, we have

$$\{q_0 + t\} \times p_t(E_{q_0}) = \{(q_0 + t, at + b) : (a, b) \in E_{q_0}\} \subseteq B \cap \{(x, y) \in \mathbb{R}^2 : x = q_0 + t\}.$$

For $t \in \mathbb{R}$, consider the vertical section $B_t = \{(x, y) \in B : x = q_0 + t\}$. Because $\dim(p_t(E_{q_0})) = 1$ for almost all $t \in \mathbb{R}$, and $\{q_0 + t\} \times p_t(E_{q_0}) \subseteq B_t$ for $0 \leq t \leq 1/2$, we see that the set

$$A = \{t \in [0, 1/2] : \dim(B_t) = 1\}$$

has positive measure. This means that for all $\alpha \in [1, 2)$, we must have $H^{\alpha-1}(B_t) = \infty$ on a set of t of positive measure. By the contrapositive of Proposition 3.7, this implies that $H^\alpha(B) = \infty$ for all $\alpha \in [1, 2)$. Hence, $\dim(B) = 2$. □

Another question we can ask about Besicovitch sets is how essential is the fact that we are working in two dimensions? It turns out that this is important, as we will see in the following theorem. We follow Falconer [1, p. 105-106]. Again, we use the concept of duality, and we associate planes in $\mathbb{R}^3$ with points.

Theorem 3.9. *Let $A \subseteq \mathbb{R}^3$ be a subset of Lebesgue measure zero. Then A cannot contain a translate of every plane.*

Proof. Before beginning the proof, we will set up some notation. For $a, b, c \in \mathbb{R}$, let $P(a, b, c)$ be the plane given by

$$P(a, b, c) = \{(x, y, z) \in \mathbb{R}^3 : z = a + bx + cy\}.$$

For a subset $E \subseteq \mathbb{R}^3$, we define $P(E)$ to be the union of planes given by

$$P(E) = \bigcup_{(a,b,c)\in E} P(a, b, c).$$

Now suppose that a subset $A \subseteq \mathbb{R}^3$ contains a translate of every plane. By outer regularity of the Lebesgue measure, we may assume without loss of generality that A is G_δ so that the set

$$\begin{aligned}\Omega &= \{(a, b, c) \in \mathbb{R}^3 : P(a, b, c) \subseteq A\}\\ &= \bigcap_{r=1}^{\infty} \{(a, b, c) \in \mathbb{R}^3 : P(a, b, c) \cap B_r(0) \subseteq A \cap B_r(0)\}\end{aligned}$$

is G_δ and hence Borel. Let V be the yz-plane and let π_V be the orthogonal projection map onto V. Because A contains a translate of every plane, for every $(b, c) \in \mathbb{R}^2$, there exists $a \in \mathbb{R}$ such that the plane $P(a, b, c)$ is contained in A. Thus, $\pi_V(\Omega) = V$. Note that $H^2(V) = \infty$. Therefore, by Lemma 3.5, we have that $H^2(\Omega) = \infty$, which implies that $\dim(\Omega) \geq 2$.

We will now show that the $\mathcal{L}^1$ measure of $P(\Omega)$ intersected with a line perpendicular to the xy-plane is positive for almost all such lines. Given $d, e \in \mathbb{R}$, we define $\gamma_{d,e}$ to be the line given by

$$\gamma_{d,e} = \{(x, y, z) \in \mathbb{R}^3 : x = d, y = e\}.$$

Note that for $a, b, c, d, e \in \mathbb{R}$, we have

$$\gamma_{d,e} \cap P(a, b, c) = \{(d, e, a + bd + ce)\} = \{(d, e, (a, b, c) \cdot (1, d, e))\}.$$

For $E \subseteq \mathbb{R}^3$ and $d, e \in \mathbb{R}$, we obtain

$$\gamma_{d,e} \cap P(E) = \{(d, e, (a, b, c) \cdot (1, d, e)) : (a, b, c) \in E\}.$$

Let $\ell_{d,e}$ be the line given by $\{\alpha(1, d, e) : \alpha \in \mathbb{R}\}$, and let $\text{proj}_{d,e} : \mathbb{R}^3 \to \mathbb{R}$ be the orthogonal projection map onto the line $\ell_{d,e}$. We find that $\mathcal{L}^1(\text{proj}_{d,e}(E)) = 0$ if and only if $\mathcal{L}^1(\gamma_{d,e} \cap P(E)) = 0$. For $u \in S^2$, let $\text{proj}_u : \mathbb{R}^3 \to \mathbb{R}$ be the orthogonal projection map onto the line $\{tu : t \in \mathbb{R}\}$. By Proposition 3.6, we have $\mathcal{L}^1(\text{proj}_u(\Omega)) > 0$ for almost all $u \in S^2$, which implies that $\mathcal{L}^1(\gamma_{d,e} \cap P(\Omega)) > 0$ for almost all $(d, e) \in \mathbb{R}^2$.

Note that if a set $E \subseteq \mathbb{R}^3$ is open, then $P(E)$ is also open, so that if $E \subseteq \mathbb{R}^3$ is G_δ, then $P(E)$ is also G_δ. Thus, $P(\Omega)$ is $\mathcal{L}^3$-measurable, so we may apply Fubini's theorem to obtain that $\mathcal{L}^3(P(\Omega)) > 0$. Because $P(\Omega) \subseteq A$, we see that A must have positive Lebesgue measure. □

4. The Kakeya problem

Very much related to a Besicovitch set, a *Kakeya set* is a set such that a unit line segment in the set can be rotated and translated continuously to its original position with its endpoints reversed without leaving the set. The Kakeya problem is the problem of determining whether or not there exists some minimum area that a Kakeya set must have. Note that being able to rotate a line segment inside a Kakeya set forces the set to have positive area. However, it turns out that there exist Kakeya sets of arbitrarily small area. To prove this fact, we follow Falconer [1, p. 100-101]. We require the following lemma.

Lemma 4.1. *Consider parallel lines L_1 and L_2 in $\mathbb{R}^2$. Let $\varepsilon > 0$ be given. Then there exists a set A containing L_1 and L_2 such that a unit line segment $\gamma \subseteq L_1$ can be moved continuously onto L_2 without leaving A and such that $\mathcal{L}^2(A) < \varepsilon$.*

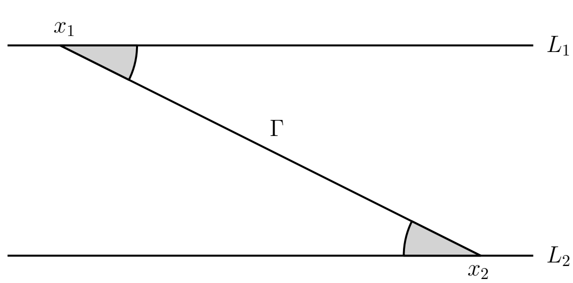

Figure 3

Proof. Consider points $x_1 \in L_1$ and $x_2 \in L_2$ as well as the line segment Γ connecting x_1 and x_2. Let A be the set composed of L_1, L_2, and Γ as well as the sector of radius 1 between L_1 and Γ centered at x_1 and the sector of radius 1 between L_2 and Γ centered at x_2 (see Figure 3). By choosing x_1 and x_2 to be very far apart, we can make the angles of the sectors arbitrarily small such that $\mathcal{L}^2(A) < \varepsilon$. To move γ continuously onto L_2 without leaving A, we move γ along L_1 to the first sector, rotate it to line up with Γ, translate it along Γ, and rotate it in the second sector to line up with L_2. □

We combine this lemma with Theorem 2.2 to construct a Kakeya set of arbitrarily small measure.

Theorem 4.2. *Let $\varepsilon > 0$ be given. Then there exists a Kakeya set K with $\mathcal{L}^2(K) < \varepsilon$.*

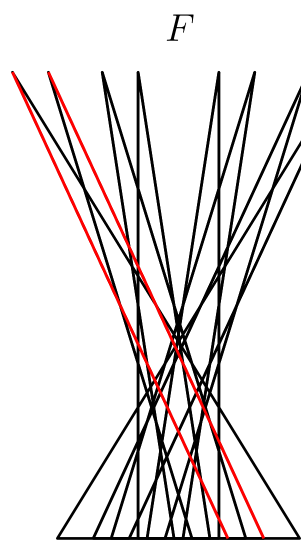

FIGURE 4

Proof. Let T be an equilateral triangle with its base along the line L and with a height equal to 1. By Theorem 2.2, we can divide T into subtriangles $T_1, \ldots, T_{2^k}$ and move these subtriangles along L to new positions $S_1, \ldots, S_{2^k}$ to create a new figure F with $\mathcal{L}^2(F) < \varepsilon/12$. For $1 \leq i \leq 2^k - 1$, triangles S_i and S_{i+1} have sides that are parallel to each other. In Figure 4, one such pair of parallel sides is highlighted in the case when $k = 3$. Using Lemma 4.1, for $1 \leq i \leq 2^k - 1$, we can then add a set A_i of measure less than $\frac{\varepsilon}{12(2^k-1)}$ that allows a unit line segment to be moved continuously from S_i to S_{i+1}. This results in a set $K_1 = F \cup \left(\bigcup_{i=1}^{2^k-1} A_i\right)$ with measure $\mathcal{L}^2(K_1) < \varepsilon/6$ in which a unit segment can be moved continuously to a position at an angle of $\pi/3$ with its original position. We may take the initial segment to be along the leftmost side of S_1 and move the segment in K_1 to reach the rightmost side of S_{2^k}. Let K_2 and K_3 be counterclockwise rotated copies of K_1 by angles of $\pi/3$ and $2\pi/3$, respectively. Using Lemma 4.1, we may add sets A'_1,A'_2, and A'_3 with combined measure less than $\varepsilon/2$ in which we can move the segment from K_1 to K_2, from K_2 to K_3, and from K_3 to K_1, respectively. Note that we can move the segment from the rightmost side of S_{2^k} in K_1 to the leftmost side of the rotated copy of S_1 in K_2. Similar movements take the segment from K_2 to K_3 and from K_3 to K_1. Let $K = K_1 \cup K_2 \cup K_3 \cup A'_1 \cup A'_2 \cup A'_3$. Then we see that we have $\mathcal{L}^2(K) < \varepsilon$. Furthermore, we can move a line segment inside K such that it returns to its original position with reversed endpoints (we rotate the segment by $\pi/3$ in each K_i for $1 \leq i \leq 3$). $\square$

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