THE CLASSIFICATION OF VECTOR BUNDLES OVER $\mathbb{P}^1$

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Abstract. This expository paper will begin with a rigorous introduction to the theory of sheaves and schemes. It will then introduce the theory of vector bundles over schemes using locally free modules and quasicoherent sheaves, and it will use the theory developed along with an elementary result from linear algebra to provide a complete classification of vector bundles over $\mathbb{P}^1$.

Contents

1. Introduction

Roughly speaking, a vector bundle in differential topology and algebraic geometry is a way to assign to each point p in a space X a vector space over that point. For example the tangent bundle TM over a manifold M is the vector bundle which assigns to each point p the tangent space T_pM of M at p. In the context of algebraic geometry, a vector bundle over a scheme can be more precisely defined as a locally free sheaf over that scheme.

Despite the necessary abstraction needed to understand vector bundles in an algebro-geometric context, algebraic vector bundles have been ubiquitous in much of algebraic geometry and related fields. For example, in representation theory, we can use the data of a vector bundle of an object to understand representations of its fundamental group, or more precisely, the Riemann Hilbert correspondence tells us that the data of flat connections of vector bundles over smooth algebraic varieties X is sufficient to understand the complex representations of $\pi_1(X)$ (see [1]). The theory of vector bundles is also a component in providing the definition and in understanding the Higgs Bundle which has been useful in much of the development of geometric Langlands (see [8]).

In this paper, we will develop the necessary theory in order to define and motivate the definition of vector bundles over a scheme as locally free sheaves over that scheme, and we will see that the only vector bundles over $\mathbb{A}^1$ are the trivial bundles.

The classification of vector bundles over $\mathbb{P}^1$ then serves as the simplest nontrivial example of a classification of vector bundles, and using the techniques developed throughout this paper alongside an elementary result from linear algebra, we will provide a complete classification of all vector bundles over $\mathbb{P}^1$.

2. Sheaves

Given a topological space X, intuitively, a sheaf on X is the data of functions on open subsets of X along with the data of restriction maps $\text{res}_{U,V}$ sending functions on U to functions on V for any open sets U and V with $V \subset U$. Our motivating examples of a sheaf will come from a differentiable manifold M. For any open subset $U \subset M$, define $\mathcal{O}(U)$ to be the ring of differentiable functions $U \to \mathbb{R}$ and for any open subset $V \subset U$, we have a ring homomorphism $\text{res}_{U,V} : \mathcal{O}(U) \to \mathcal{O}(V)$ sending f to $f\big|_V$. Once we define sheaves, it will follow that $\mathcal{O}$ along with these restriction maps forms a sheaf of rings on the manifold M.

Definition 2.1. A presheaf $\mathcal{F}$ of abelian groups on a topological space X assigns for each open set U an abelian group $\mathcal{F}(U)$ and assigns for each open set $V \subset U$, a group homomorphism $\text{res}_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ such that $\text{res}_{U,U}$ is the identity, and whenever we have open sets $W \subset V \subset U$, we have $\text{res}_{V,W} \circ \text{res}_{U,V} = \text{res}_{U,W}$. We can similarly define a presheaf of rings by assigning for each open set U, a ring $\mathcal{F}(U)$ such that $\text{res}_{U,V}$ is a ring homomorphism instead of a group homomorphism.

A presheaf $\mathcal{F}$ is a sheaf if for any open set U and any open cover $\{U_i\}$ of U, $\mathcal{F}$ satisfies:

(1) Identity. If $f_1, f_2 \in \mathcal{F}(U)$ such that $\text{res}_{U,U_i}(f_1) = \text{res}_{U,U_i}(f_2)$ for every U_i in the open cover, then $f_1 = f_2$.
(2) Gluability. If we have elements $f_i \in \mathcal{F}(U_i)$ for each U_i such that any two elements f_i and f_j agree on their overlap meaning $\text{res}_{U_i,U_i \cap U_j}(f_i) = \text{res}_{U_j,U_i \cap U_j}(f_j)$, then there exists $f \in \mathcal{F}(U)$ such that $\text{res}_{U,U_i}(f) = f_i$ for all i (and it follows from the axiom of identity that this element f is unique).

Notation 2.2. If $f \in \mathcal{F}(U)$ and $V \subset U$, then we will often use the shorthand $f\big|_V$ in place of $\text{res}_{U,V}(f)$, and we will refer to the maps $\text{res}_{U,V}$ as restriction maps.

Remark 2.3. The primary objects we will be studying in this paper are sheaves of rings and of abelian groups and we saw in Definition 2.1 that we can create a similar definition for presheaves and sheaves of rings. We will assume all presheaves and sheaves are of abelian groups, and all of our proofs in this section will generalize with little difficulty in an obvious way to presheaves and sheaves of rings. However, for those who have seen category theory before, we are able to expand the definition of a presheaf and then a sheaf for any category. We can form a category on a topological space X whose objects are open sets of X and whose arrows are inclusion maps $V \to U$ for any open sets $V \subset U$. Then a presheaf $\mathcal{F}$ on any category $\mathcal{C}$ is a contravariant functor from the category of open subsets of X into $\mathcal{C}$, where $\text{res}_{U,V}$ is the image of a morphism $V \to U$ under $\mathcal{F}$. $\mathcal{F}$ is a sheaf if $\text{res}_{U,V}$ satisfy the axioms of identity and gluability.

Example 2.4. It is easy to check that the sheaf of differentiable functions on a manifold as described at the start of this section is a sheaf of rings.

Example 2.5. If $\mathcal{F}$ is a presheaf on X and U is an open subset of X, then we can define the presheaf $\mathcal{F}\big|_U$ on U where for any open subset V of U we define

$\mathcal{F}\big|_U(V) = \mathcal{F}(V)$ and for any pair of open subsets V and W of U with $W \subset V$, we take the restriction map $\mathcal{F}\big|_U(V) \to \mathcal{F}\big|_U(W)$ to just be the restriction map $\mathrm{res}_{V,W}$. It is clear that $\mathcal{F}\big|_U$ is a presheaf and that it is a sheaf whenever $\mathcal{F}$ is a sheaf.

Example 2.6. If $\pi : X \to Y$ is a continuous map with $\mathcal{F}$ a presheaf on X, define $\pi_*\mathcal{F}(V) = \mathcal{F}(\pi^{-1}(V))$ for any set V which is open in Y. Then it is slightly tedious but not difficult to verify that $\pi_*\mathcal{F}$ is a presheaf on Y, and $\pi_*\mathcal{F}$ is a sheaf on Y whenever $\mathcal{F}$ is a sheaf on X.

Definition 2.7. Given two presheaves $\mathcal{F}$ and $\mathcal{G}$ on X, a morphism $f : \mathcal{F} \to \mathcal{G}$ is a homomorphism $f(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for each open set U such that for any open sets $V \subset U$, the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d] & \mathcal{G}(U) \arrow[d] \\ \mathcal{F}(V) \arrow[r] & \mathcal{G}(V) \end{tikzcd}$$

commutes, where the horizontal arrows are the homomorphisms $f(U)$ and $f(V)$, and the vertical arrows are the restriction maps.

Given morphism $f : \mathcal{F} \to \mathcal{G}$ and $g : \mathcal{G} \to \mathcal{H}$ we can define both the composition $g \circ f : \mathcal{F} \to \mathcal{H}$ and the identity morphism $Id : \mathcal{F} \to \mathcal{F}$ in the obvious way.

Definition 2.8. A morphism $f : \mathcal{F} \to \mathcal{G}$ is an isomorphism if there exists a morphism $g : \mathcal{G} \to \mathcal{F}$ such that $f \circ g$ and $g \circ f$ are the identity morphisms. We say that $\mathcal{F}$ and $\mathcal{G}$ are isomorphic if there exists an isomorphism $f : \mathcal{F} \to \mathcal{G}$ and we write this as $\mathcal{F} \cong \mathcal{G}$.

3. Germs

Definition 3.1. If $\mathcal{F}$ is a presheaf on X, a germ of $\mathcal{F}$ at p is an equivalence class of pairs (f, U) with U an open set containing p and $f \in \mathcal{F}(U)$ such that $(f, U) \sim (g, V)$ if there exists an open set $W \subset U \cap V$ such that $f\big|_W = g\big|_W$. The germ given by the equivalence class of the pair (f, U) is called the germ of f at p. The stalk of $\mathcal{F}$ at p which we denote by $\mathcal{F}_p$ is the set of all germs of $\mathcal{F}$ at p. We can define a germ of $\mathcal{F}$ completely identically if $\mathcal{F}$ is a sheaf.

Remark 3.2. If $\mathcal{F}$ is a presheaf of abelian groups, we can define the sum of any two elements in $\mathcal{F}_p$ by $(f, U)+(g, V) = (f\big|_{U\cap V}+g\big|_{U\cap V}, U\cap V)$, and it is straightforward to check that this definition is well-defined up to a choice of representatives (f, U) and (g, V) and that this sum turns $\mathcal{F}_p$ into an abelian group. Similarly, we get a natural ring structure on $\mathcal{F}_p$ whenever $\mathcal{F}$ is a presheaf of rings.

Remark 3.3. Given a morphism $f : \mathcal{F} \to \mathcal{G}$ of two presheaves $\mathcal{F}$ and $\mathcal{G}$ on X, we get an induced homomorphism $f_p : \mathcal{F}_p \to \mathcal{G}_p$ where for any $s_p \in \mathcal{F}_p$ we can choose a representative (s, U) of s_p and define the image $f(s_p) \in \mathcal{G}_p$ to be the germ of $f(U)(s) \in \mathcal{G}(U)$. It is not difficult to verify that this map is independent of our choice of representative (s, U) for the germ s_p and therefore, that this map is a well-defined homomorphism.

Using Remark 3.3, we get a useful criterion to determine if a morphism of sheaves is an isomorphism.

Proposition 3.4. *If $\mathcal{F}$ and $\mathcal{G}$ are sheaves on X, and $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then ϕ is an isomorphism if and only if the induced map ϕ_p is an isomorphism for every $p \in X$.*

Proof. If ϕ is an isomorphism with inverse ψ then ϕ_p is an isomorphism for all p follows since $\phi_p \circ \psi_p$ and $\psi_p \circ \phi_p$ are both the identity. Conversely if ϕ_p is an isomorphism for all p, then it suffices to show that $\phi(U)$ is an isomorphism for all U, because then the morphism $\psi : \mathcal{G} \to \mathcal{F}$ defined by $\psi(U) = \phi^{-1}(U)$ is the desired inverse for ϕ. To prove $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective we note that for $f \in \mathcal{F}(U)$ if we let f_p be the germ of f at p, then whenever $\phi(U)(f) = 0$ it follows that for all $p \in U$, $\phi_p(f_p) = 0$ so $f_p = 0$. By the identity axiom of sheaves, since the germ f_p of f is 0 for all $p \in U$, we can conclude that $f = 0$.

To show surjectivity, let $g \in \mathcal{G}(U)$. At each point p, the germ g_p of g at p is the image $\phi_p(f_p)$ where f_p is the germ at p of $\widetilde{f_p} \in \mathcal{F}(U_p)$ for some U_p containing p. Then $\phi(U_p)(\widetilde{f_p})$ has the same germ as $\phi(U)(f)$ at p, so after taking a smaller open set U_p containing p if necessary, we may assume $\phi(U_p)(f_p) = \phi(U)(f)\big|_{U_p}$. Then for any $p, q \in U$, we have $\phi(U_p \cap U_q)(f_p\big|_{U_p \cap U_q}) = \phi(U_p \cap U_q)(f_q\big|_{U_p \cap U_q})$, so since we proved that $\phi(U_p \cap U_q)$ is injective earlier in this proof, we conclude that $f_p\big|_{U_p \cap U_q} = f_q\big|_{U_p \cap U_q}$. Therefore, by gluability, there exists an element $f \in \mathcal{F}(U)$ such that $f\big|_{U_p} = f_p$. It follows that the germ of $\phi(U)(f)$ at p is g_p, and therefore by identity, that $\phi(U)(f) = g$ so $\phi(U)$ is surjective. $\square$

Definition 3.5. Let $\mathcal{F}$ be a presheaf. An element $(s_p)_{p \in U} \in \prod_{p \in U} \mathcal{F}_p$ is said to consist of compatible germs if for all $p \in U$ there exists an open set $U_p \subset U$ and an element $\widetilde{s_p} \in \mathcal{F}_p(U_p)$ such that the germ of $\widetilde{s_p}$ at q is s_q for each q in U_p.

The remaining results of this section all use similar techniques using compatible germs in order to construct sheaves from a simpler collection of data.

Theorem 3.6. *For any presheaf $\mathcal{F}$ on a space X, there exists a unique sheaf $\mathcal{F}^{sh}$ on X along with a morphism $sh : \mathcal{F} \to \mathcal{F}^{sh}$ such that for any sheaf $\mathcal{G}$ and any morphism $g : \mathcal{F} \to \mathcal{G}$ there exists a unique morphism $f : \mathcal{F}^{sh} \to \mathcal{G}$ such that the diagram*

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[dr] & \mathcal{F}^{sh} \arrow[d] \\ & \mathcal{G} \end{tikzcd}$$

commutes. Furthermore, the sheaf $\mathcal{F}^{sh}$ is unique up to unique isomorphism.

In order to motivate this proof, we should think of elements of $\mathcal{F}(U)$ as functions on U, and then an element $(s_p)_{p \in U}$ consisting of a compatible germ means that at each point p, we can find a neighborhood U_p such that s_q is the germ of some function $f \in U_p$ for every $q \in U_p$. Intuitively but somewhat imprecisely, this means that at each point p, s_p "looks like" a function in a neighborhood U_p around

s_p, and therefore, since s_p locally, looks like a function, if we can glue together functions (which by gluability we can do in a sheaf but not necessarily a presheaf) we should be able to glue each s_p together to form a function on all of U. Since each compatible germ "looks like" a function, we should be able to construct a sheaf consisting of all compatible germs of a presheaf $\mathcal{F}$. We will see in the proof of Theorem 3.6, that this sheaf of compatible germs is precisely the sheaf $\mathcal{F}^{sh}$ which we are trying to construct.

Proof. The claim that $\mathcal{F}^{sh}$ is unique up to unique isomorphism is an immediate consequence of the universal property that $\mathcal{F}^{sh}$ must satisfy.

For an open set $U \subset X$ define

$$\mathcal{F}^{sh}(U) = \{(f_p)_{p\in U} | f_p \in \mathcal{F}_p \text{ and } (f_p)_{p\in U} \text{ consists of compatible germs}\}.$$

For an open set $V \subset U$ define $\mathrm{res}_{U,V}((f_p)_{p\in U}) = (f_p)_{p\in V}$. Define the map $\mathrm{sh} : \mathcal{F} \to \mathcal{F}^{sh}$ such that for each U, $\mathrm{sh}(U) : \mathcal{F}(U) \to \mathcal{F}^{sh}(U)$ sends an element f to $(f_p)_{p\in U}$ where f_p is the germ of f at p. It is not hard to check that $\mathcal{F}^{sh}$ is a sheaf and that the map $\mathcal{F} \to \mathcal{F}^{sh}$ is a morphism.

To show uniqueness of the morphism f, we note that by the commutative diagram any element $(s_p)_{p\in U} \in \mathcal{F}^{sh}(U)$ must be sent to an element $t \in \mathcal{G}(U)$ satisfying the germ t_p of t at p is $g_p(s_p)$. It follows from the identity axiom that if t and t' have the same germ at every point p then $t = t'$, so this element is necessarily unique. Therefore, to show the existence of the morphism f, we define $f(U)((s_p)_{p\in U})$ to be the unique element $t \in \mathcal{G}(U)$ such that the germ of t at p is $g_p(s_p)$. We will show the existence of such an element t, and therefore the map $f(U) : \mathcal{F}^{sh}(U) \to \mathcal{G}(U)$ is well-defined, and it is straightforward to verify that the maps $f(U)$ are each homomorphisms, and that they turn f into a morphism $\mathcal{F}^{sh} \to \mathcal{G}$. Since $(s_p)_{p\in U}$ consists of compatible germs, for all $p \in U$, there exists an open set $U_p \subset U$ containing p and an element $\widetilde{s_p} \in \mathcal{F}(U_p)$ such that the germ of $\widetilde{s_p}$ at r is s_r for all $r \in U_p$. Therefore, if we define $\widetilde{t_p} = g(U_p)(\widetilde{s_p})$, then for all $r \in U_p$, the germ of $\widetilde{t_p}$ is t_r. Therefore, $\widetilde{t_q}\big|_{U_p\cap U_q}$ and $\widetilde{t_p}\big|_{U_p\cap U_q}$ have the same germ at every point $r \in U_p \cap U_q$ so by the gluability axiom of $\mathcal{G}$ there exists an element $t \in \mathcal{G}(U)$ such that the germ of t at p is t_p for all $p \in U$. Therefore, the map $f(U) : \mathcal{F}^{sh}(U) \to \mathcal{G}(U)$ is well-defined. $\square$

Definition 3.7. We call the unique sheaf $\mathcal{F}^{sh}$ constructed from a presheaf $\mathcal{F}$ the sheafification of $\mathcal{F}$.

Remark 3.8. It follows from the construction of $\mathcal{F}^{sh}$ that $(\mathcal{F}\big|_U)^{sh} = \mathcal{F}^{sh}\big|_U$. This is because for any $V \subset U$, $(\mathcal{F}\big|_U)^{sh}(V)$ and $\mathcal{F}^{sh}\big|_U(V)$ are both the group containing the elements $(s_p)_{p\in V} \in \prod_{p\in V} \mathcal{F}_p$ such that $(s_P)_{p\in V}$ consists of compatible germs. It also follows from the definition of $\mathcal{F}^{sh}$ that for any point p, $(\mathcal{F}^{sh})_p = \mathcal{F}_p$ and for any $g : \mathcal{F} \to \mathcal{G}$ and unique map $f : \mathcal{F}^{sh} \to \mathcal{G}$ such that $f \circ sh = g$, we have that $f_p = g_p$ for all p.

The following definition allows us to generalize our definition of a sheaf to a sheaf on a basis. The definition is slightly long, but most of it should look almost identical to our definition of a sheaf.

Definition 3.9. Given a basis $\{B_\alpha\}$ for a topological space X. A sheaf F of abelian groups on the basis $\{B_\alpha\}$ assigns for each $B_i \in \{B_\alpha\}$ an abelian group $F(B_i)$, and for any pair $B_j \subset B_i$ a group homomorphism $\mathrm{res}_{B_i,B_j} : F(B_i) \to F(B_j)$ such that res_{B_i,B_i} is the identity, and if $B_k \subset B_j \subset B_i$, then $\mathrm{res}_{B_j,B_k} \circ \mathrm{res}_{B_i,B_j} = \mathrm{res}_{B_i,B_k}$. Additionally, F must satisfy that for any open cover $\{B_{ij}\}$ of B_i, if for $f, g \in F(B_i)$, $\mathrm{res}_{B_i,B_{ij}}(f) = \mathrm{res}_{B_i,B_{ij}}(g)$ for all j then $f = g$, and if $f_j \in F(B_{ij})$ and $f_k \in F(B_{ik})$ such that $\mathrm{res}_{B_{ij},B_l} f_j = \mathrm{res}_{B_{ik},B_l} f_k$ for any basic set $B_l \subset B_{ij} \cap B_{ik}$ then there exists an element $f \in B_i$ such that for all j, $\mathrm{res}_{B_i,B_{ij}}(f) = f_j$. It should be clear how we can similarly define a sheaf of rings on the basis $\{B_\alpha\}$.

Theorem 3.10. *Given a sheaf F on a basis $\{B_\alpha\}$ for a space X, there exists a unique sheaf $\mathcal{F}$ on X such that $\mathcal{F}(B_i) = F(B_i)$, and the restriction map $\mathcal{F}(B_i) \to \mathcal{F}(B_j)$ is the same as the map $res_{B_i,B_j} F(B_i) \to F(B_j)$ for any basis sets $B_j \subset B_i$.*

Proof. The techniques of this proof are similar to Theorem 3.6. We can define the stalk F_p of F at p to be the equivalence class of all elements (f, B_i) with $p \in B_i$ and $f \in F(B_i)$ such that $(f, B_i) \sim (g, B_j)$ if there exists $B_k \subset B_i \cap B_j$ with $f\big|_{B_k} = g\big|_{B_k}$. We can similarly define the germ f_p of $f \in F(B)$ at p to be the class (f, B) in F_p. Define

$$\begin{aligned}\mathcal{F}(U) = \{(s_p)_{p \in U} | s_p \in F_p \text{ and for all } p \in U \text{ there exists } B \text{ such that } p \in B \subset U,\\ \text{and there exists } \widetilde{s} \in F(B) \text{ such that } \widetilde{s_q} = s_q \text{ for all } q \in B\}\end{aligned}$$

We have a natural map $F(B) \to \mathcal{F}(B)$ sending s to $(s_p)_{p \in B}$. The main part of our proof that $\mathcal{F}$ extends F is showing that this natural map is an isomorphism and that the restriction maps $\mathrm{res}_{B_i,B_j} : F(B_i) \to F(B_j)$ will be the same as the restriction maps $\mathrm{res}_{B_i,B_j} : \mathcal{F}(B_i) \to \mathcal{F}(B_j)$, and the result follows easily from there. To prove that this map is injective we use the identity axiom for sheaves on a basis to show that if $s, t \in F(B)$ such that $s_p = t_p$ for all $p \in B$ then $s = t$. To show the map is surjective, note that for any element $(s_p)_{p \in B} \in \mathcal{F}(B)$, we can cover B with basic open sets B_p with $p \in B_p$ such that there exists $\widetilde{s_p} \in F(B_p)$ where $\widetilde{s_q} = s_q$ for all $q \in B_p$. We can then use the gluability axiom of sheaves on a basis to find an element $\widetilde{s} \in F(B)$ such that the germ of $\widetilde{s}$ at p is s_p for every $p \in B$. To show that the restriction maps $\mathrm{res}_{B_i,B_j} : F(B_i) \to F(B_j)$ will be the same as the restriction maps $\mathrm{res}_{B_i,B_j} : \mathcal{F}(B_i) \to \mathcal{F}(B_j)$ follows without too much difficulty ☐

Definition 3.11. If $\{B_i\}$ is a basis for X, then a morphism $F \to G$ where F and G are sheaves on a basis is a map $F(B_i) \to G(B_i)$ for each set B_i in the basis such that if $B_j \subset B_i$ are both in the basis, then the diagram

$$\begin{tikzcd} F(B_i) \arrow[r] \arrow[d] & G(B_i) \arrow[d] \\ F(B_j) \arrow[r] & G(B_j) \end{tikzcd}$$

commutes.

Corollary 3.12. *Any morphism $F \to G$ of sheaves on a basis of X can be uniquely extended to a morphism $\mathcal{F} \to \mathcal{G}$ where $\mathcal{F}$ and $\mathcal{G}$ are the unique sheaves extending F and G respectively to all of X.*

Proof. Similar to Remark 3.3, given a morphism $f : F \to G$, we obtain a natural map $f_p : F_p \to G_p$. As in Theorem 3.10, viewing $\mathcal{F}(U)$ and $\mathcal{G}(U)$ as subsets of

$\prod_{p\in U} F_p$ and $\prod_{p\in U} G_p$ respectively, we can define the image of $(s_p)_{p\in U} \in \mathcal{F}(U)$ to be $(f_p(s_P))_{p\in U}$, and similar to Theorem 3.6 and Theorem 3.10, we can prove that this is the unique morphism extending the morphism $F \to G$. □

Theorem 3.13. *Let $\{U_i\}$ an open cover of X, and let $\mathcal{F}_i$ a sheaf on U_i. If there exists isomorphisms $\phi_{ij} : \mathcal{F}_i\big|_{U_i\cap U_j} \to \mathcal{F}_j\big|_{U_i\cap U_j}$ such that for any triple $U_i \cap U_j \cap U_k$ we have $\phi_{jk}\circ\phi_{ij} = \phi_{ik}$, then there exists a unique sheaf $\mathcal{F}$ on X such that $\mathcal{F}\big|_{U_i} \cong \mathcal{F}_i$*

Proof. This proof is similar to the proofs of Theorems 3.6 and 3.10 so we will omit most of the details, but the main idea of the proof is to take a basis for each set U_i, and the union of all these bases form a basis on X. In order to construct the sheaf $\mathcal{F}$, we know how $\mathcal{F}$ must behave when restricted to its basis (as each element in the basis is contained in some U_i) so using the techniques with compatible germs as in Theorems 3.6 and 3.10, we can extend this to a sheaf on all open sets of $\mathcal{F}$. □

Definition 3.14. The functions ϕ_{ij} from the previous theorem are called transition functions, and the condition that $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ is called the cocycle condition.

4. Schemes

Definition 4.1. Given a ring A, let $\mathrm{spec}(A) = \{\mathfrak{p}|\mathfrak{p} \text{ is a prime ideal of } A\}$. For $S \subset A$, define $V(S) = \{\mathfrak{p} \in \mathrm{spec}(A)|S \subset \mathfrak{p}\}$. For an element $f \in A$ define $D(f) = \mathrm{spec}(A) \setminus V(f)$. We call subsets of $\mathrm{spec}(A)$ of the form $V(S)$ for some set $S \subset A$ closed subsets of $\mathrm{spec}(A)$, and we call sets of the form $D(f)$ for some $f \in A$ distinguished open subsets of A.

We can motivate the above definitions by thinking about elements $f \in A$ in an imprecise way as functions on $\mathrm{spec}(A)$ where $f(\mathfrak{p}) = f \mod \mathfrak{p}$. This definition can take some getting used to since when thinking of f as a function, the range of f depends on $\mathfrak{p}$ as for a given prime $\mathfrak{p}$ the range of f is $A/\mathfrak{p}$. We say f vanishes at $\mathfrak{p}$ if $f(\mathfrak{p}) = 0$ or equivalently, if $f \in \mathfrak{p}$. We say that a set $S \subset A$ vanishes at $\mathfrak{p}$ if every $f \in S$ vanishes at $\mathfrak{p}$. We refer to the set of $\mathfrak{p}$ such that S vanishes at $\mathfrak{p}$ by the vanishing set of S, and we can observe that the vanishing set of S is the set $V(S)$ which we defined above.

For an explicit example of what the spectrum of a ring looks like, we can consider $\mathrm{spec}(\mathbb{C}[x])$, or more generally, $\mathrm{spec}(\mathbb{C}[x_1, ..., x_n])$. Since $\mathbb{C}$ is an algebraically closed field, every prime ideal in $\mathbb{C}[x]$ is the 0-ideal or of the form $(x-a)$ for some a in $\mathbb{C}$. So, if we define $\mathrm{maxspec}(\mathbb{C}[x]) \subset \mathrm{spec}(\mathbb{C}[x])$ to be the set of maximal ideals of $\mathbb{C}[x]$, by the identification of each element $(x - a)$ in $\mathrm{maxspec}(\mathbb{C}[x])$ with the element $a \in \mathbb{C}$, we can view $\mathrm{maxspec}(\mathbb{C}[x])$ as the set $\mathbb{C}$, and we can view $\mathrm{spec}(\mathbb{C}[x])$ as $\mathbb{C}$ along with one additional point for the 0-ideal. Similarly, we can use that the maximal ideals in $\mathbb{C}[x_1, ..., x_n]$ are precisely the ideals of the form $(x_1-a_1, ..., x_n-a_n)$ for $a_1, ..., a_n \in \mathbb{C}$ (see chapter 15 of [2]) to identify $\mathrm{maxspec}(\mathbb{C}[x_1, ..., x_n])$ with $\mathbb{C}^n$. If $\mathfrak{p} = (x_1 - a_1, ..., x_n - a_n)$, then we treat f as a function on $\mathrm{maxspec}(\mathbb{C}[x_1, ..., x_n])$ with $f(\mathfrak{p}) = f \mod \mathfrak{p}$. Note that $\mathbb{C}[x_1, ...x_n]/\mathfrak{p}$ is isomorphic to $\mathbb{C}$ under the isomorphism sending a polynomial $f(x_1, ..., x_n)$ to $f(a_1, ..., a_n)$. In particular, if $f \in \mathbb{C}[x_1, ..., x_n]$, then thinking of f as a function on $\mathrm{spec}(\mathbb{C}[x_1, ..., x_n])$ we have $f(\mathfrak{p}) = f \mod \mathfrak{p} = f(a_1, ..., a_n) \in \mathbb{C}$. Thus, after associating $\mathrm{maxspec}(\mathbb{C}[x_1, ..., x_n])$ with $\mathbb{C}^n$ our way of defining elements of $\mathbb{C}[x_1, ..., x_n]$ as functions on $\mathrm{maxspec}(\mathbb{C}[x_1, ..., x_n])$ by defining

$f(\mathfrak{p}) = f \mod \mathfrak{p}$ matches the more normal definition of elements $\mathbb{C}[x_1, ..., x_n]$ as functions defining $f(a_1, ..., a_n)$ in the usual way as polynomial functions.

Proposition 4.2.

(1) *$V(S) = V((S))$ where (S) is the ideal generated by S.*
(2) *We can obtain a topology on spec(A) by taking as closed sets the sets of the form $V(S)$. We call this topology the Zariski Topology.*
(3) *The distinguished open sets $D(f)$ form a basis for the Zariski Topology on spec(A).*
(4) *For any $f, g \in A$, $D(f) \cap D(g) = D(fg)$, and in particular, $D(f^n) = D(f)$.*

Proof. (1) is an easy verification. From (1), we can from now on assume that every closed set is of the form $V(I)$ for some ideal I. To prove (2), note that $\emptyset = V(1)$ and $\mathrm{spec}(A) = V(0)$. To show that the finite union and arbitrary intersection of closed sets is closed note that for any ideals I, J we have $V(IJ) = V(I) \cup V(J)$ and $V(\sum_j I_j) = \bigcap_j V(I_j)$. For (3) note that for any open set $\mathrm{spec}(A) \setminus V(S)$ we have $\mathrm{spec}(A) \setminus V(S) = \bigcup_{f \in S} D(f)$. Finally, (4) is because for any prime ideal $\mathfrak{p}$, $\mathfrak{p} \in D(f) \cap D(g)$ if and only if $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$ if and only if $fg \notin \mathfrak{p}$ if and only if $\mathfrak{p} \in D(fg)$. □

We would like to define a sheaf $\mathcal{O}_{\mathrm{spec}}(A)$ on spec(A). By Theorem 3.10, it suffices to define a sheaf on the distinguished open sets $D(f)$. If we think of elements of a ring A as functions on spec(A), since $D(f)$ is the complement of $V(f)$, the function f does not vanish on $D(f)$ so it is reasonable to suggest that f should have an inverse in $\mathcal{O}_{\mathrm{spec}(A)}(D(f))$. In particular, this suggests a candidate for the definition of $\mathcal{O}_{\mathrm{spec}(A)}(D(f))$ to be A_f. The next few results show that this construction does give a sheaf of rings on spec(A).

Definition 4.3. We call a set $S \subset \mathrm{spec}(A)$ quasicompact if any open cover of S contains a finite subcover.

Remark 4.4. Depending on one's background in topology, this definition of quasicompactness might look identical to the standard definition of compactness. The convention in much of algebraic geometry is to only call a set compact if it is quasicompact and Hausdorff. The next result shows that $\mathrm{spec}(A) = D(1)$ is quasicompact for any ring A, so if we remove the Hausdorff condition from our definition of compactness, many spaces in algebraic geometry will end up being compact.

Lemma 4.5. *Given an index set I, let $D(f_i) \subset D(f)$ for all $i \in I$. Then:*

(1) *$\bigcup_{i \in I} D(f_i) = D(f)$ if and only if f^N can be written as a finite sum $f^N = \sum r_i f_i$ for some $N \in \mathbb{N}$.*
(2) *D(f) is quasicompact.*

Proof. We use that the radical $\sqrt{I}$ of an ideal I is equal to the intersection of all prime ideals containing I (see chapter 1 of [5]). In particular, for all i we have

$$\{\mathfrak{p} | f_i \notin \mathfrak{p}\} = D(f_i) \subset D(f) = \{\mathfrak{p} | f \notin \mathfrak{p}\}$$

so the set of prime ideals containing f is a subset of the set of prime ideals containing f_i, and thus $\sqrt{(f)} \subset \sqrt{(f_i)}$. Therefore, $\sqrt{(f)} \subset \sqrt{I}$ where I is the ideal generated by the set of f_i, so $f^N \in I$, for some N, and therefore, we can write $f^N = \sum r_i f_i$.

Conversely, if $f^N = \sum r_i f_i$, and $\mathfrak{p} \notin D(f_i)$ for all f_i appearing in the sum, then we have $f_i \in \mathfrak{p}$ for all i in this sum so $f^N \in \mathfrak{p}$ implying $\mathfrak{p} \notin D(f)$. Therefore, the union of the $D(f_i)$ appearing in our sum is $D(f)$ which completes the proof of (1). Since the collection of f_i such that the union of $D(f_i)$ covers $D(f)$ is a finite collection, we see that any open cover of $D(f)$ by basic open sets contains a finite subcover from which it is a basic exercise from point-set topology to show that $D(f)$ is quasicompact. $\square$

Lemma 4.6. *If A is a ring with $f, g \in A$ such that $D(g) \subset D(f)$ then $\frac{1}{f} \in A_g$.*

Proof. $D(g) \subset D(f)$ implies $V(f) \subset V(g)$ so if $f \in \mathfrak{p}$, then $g \in \mathfrak{p}$. In particular g is contained in the intersection of all prime ideals containing $\mathfrak{p}$ so using that the intersection of all prime ideals containing f is $\sqrt{f}$, we conclude that $g \in \sqrt{f}$. Thus $g^n = af \in (f)$ for some $a \in A$. Then $\frac{a}{g^n} = \frac{1}{f} \in A_g$. $\square$

Lemma 4.6 gives a natural map $A_f \to A_g$ whenever $D(g) \subset D(f)$ given by sending an element $\frac{a}{f} \in A_f$ to the element $\frac{a}{f} \in A_g$

Theorem 4.7. *Let $X = spec(A)$. For $D(f) \subset X$, define $\mathcal{O}_{spec(A)}(D(f)) = A_f$, and for $D(g) \subset D(f)$ define the restriction map $A_f \to A_g$ in the obvious way given by the result of Lemma 4.6. Then this forms a sheaf on the basis sets $D(f)$ on X. In particular by Theorem 3.10, we can uniquely extend $\mathcal{O}_{spec(A)}$ to a sheaf of rings on $spec(A)$.*

Proof. Checking that this is a presheaf is easy, so we will only show the proofs that the identity and gluability axioms hold. To show identity, let $\frac{a}{f^n} \in D(f)$ be such that $\frac{a}{f^n} = 0$ in $D(f_i) = A_{f_i}$ for all i. Therefore, for all i, there exists $N_i \in \mathbb{N}$ such that $af_i^{N_i} = 0$. By quasicompactness of $D(f)$, there exists a finite subcover $D(f_1) \cup ... \cup D(f_k) = D(f)$. By proposition 4.2 (4), $D(f) = D(f_1^{N_1}) \cup ... \cup D(f_k^{N_k})$ so by Lemma 4.5, $f^N = \sum_{i=1}^{k} r_i f_i^{N_i}$ for some $N \in \mathbb{N}$. Therefore, since $af_i^{N_i} = 0$,

$$\frac{a}{f^n} = f^N \frac{a}{f^{n+N}} = \sum r_i \frac{af_i^{N_i}}{f^{n+N}} = 0.$$

To show gluability, let $D(f) = \bigcup_{i \in I} D(f_i)$ and let $\frac{a_i}{f_i^{N_i}} \in D(f_i)$ for all i be such that $\frac{a_i}{f_i^{N_i}} = \frac{a_j}{f_j^{N_j}}$ in $D(f_i) \cap D(f_j) = D(f_i f_j)$. We will first prove glaubility in the case where I is a finite set. Let $g_i = f_i^{N_i}$ so $\frac{a_i}{g_i} = \frac{a_j}{g_j}$ in $D(f_i f_j)$ which implies for some N, $(a_i g_j - a_j g_i)(f_i f_j)^N = 0$, and therefore, $(a_i g_j - a_j g_i)(g_i g_j)^N = 0$. Since I is finite, we can choose our value N large enough such that $(a_i g_j - a_j g_i)(g_i g_j)^N = 0$ holds for all $i, j \in I$. If $b_i = a_i g_i^N$ and $h_i = g_i^{N+1}$, then, $(g_i g_j)^N (g_j a_i - g_i a_j) = 0$ implies $h_j b_i = h_i b_j$. By Proposition 4.2 (4),

$$D(f) = \bigcup_{i \in I} D(f_i) = \bigcup_{i \in I} D(g_i) = \bigcup_{i \in I} D(h_i),$$

so, by Lemma 4.5 (1), there exists M such that $f^M = \sum_{i \in I} r_i h_i$ with $r_i \in A$. Let $r = \sum r_i b_i$. Then

$$rh_j = \sum r_i b_i h_j = \sum r_i h_i b_j = f^M b_j$$

so

$$\frac{r}{f^M}\Big|_{D(f_j)} = \frac{b_j}{h_j} = \frac{a_j g_j^N}{g_j^{N+1}} = \frac{a_j}{f_j^{N_j}}$$

completes the proof of the gluability axiom when I is finite.

If I is infinite, by Lemma 4.5 (2), choose $f_1, ..., f_k \in I$ such that $D(f) = D(f_1) \cup ... \cup D(f_k)$. Exactly as we did in the finite case, we can obtain an element $\frac{r}{f^M} \in D(f)$ such that $\frac{r}{f^M}\big|_{D(f_i)} = \frac{a_i}{f_i^{N_i}}$ for $i \in \{1, ..., k\}$. We want to show that for any $y \in I$ that $\frac{r}{f^M}\big|_{D(f_y)} = \frac{a_y}{f_y^{N_y}}$. Since $D(f_1) \cup ... \cup D(f_k) \cup D(f_y)$ is an open cover of $D(f)$, exactly as we did in the finite case, we can find an element $\frac{r'}{f^{M'}}$ such that $\frac{r'}{f^{M'}}\big|_{D(f_i)} = \frac{a_i}{f_i^{N_i}}$ for $i \in \{1, ..., k, y\}$. Since $\frac{r'}{f^{M'}}\big|_{D(f_i)} = \frac{r}{f^M}\big|_{D(f_i)}$ for $i \in \{1, ..., k\}$, by the identity axiom, we conclude that $\frac{r'}{f^{M'}} = \frac{r}{f^M}$. Therefore, $\frac{r}{f^M}\big|_{D(f_y)} = \frac{a_y}{f_y^{N_y}}$. $\square$

Definition 4.8. A pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X is called a ringed space.

Definition 4.9. Given ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$, an isomorphism from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a homeomorphism $\pi : X \to Y$ and an isomorphism of sheaves $\mathcal{O}_Y \to \pi_* \mathcal{O}_X$, where $\pi_* \mathcal{O}_X$ is the morphism defined in Example 2.6.

Definition 4.10. A ringed space $(X, \mathcal{O}_X)$ is an affine scheme if $(X, \mathcal{O}_X)$ is isomorphic to $(\mathrm{spec}(A), \mathcal{O}_{\mathrm{spec}(A)})$ for some ring A. A ringed space $(X, \mathcal{O}_X)$ is a scheme if for every $x \in X$, there exists an open set U containing x such that $(U, \mathcal{O}_X\big|_U)$ is an affine scheme, or equivalently, there is a cover of X by open sets U where for each U, the ringed space $(U, \mathcal{O}_X\big|_U)$ is an affine scheme. Since $(U, \mathcal{O}_X\big|_U) \cong (\mathrm{spec}(A), \mathcal{O}_{spec(A)})$ for some ring A, by abuse of notation, we will often write $\mathrm{spec}(A) \subset X$ in place of $U \subset X$.

Proposition 4.11. *If A is a ring and $f \in A$, the space $(D(f), \mathcal{O}_X\big|_{D(f)})$ is isomorphic as a ringed space to $(spec(A_f), \mathcal{O}_{spec(A_f)})$.*

Proof. There is a natural bijection between the prime ideals in A_f and the prime ideals in A which do not contain f, given by sending a prime ideal $\mathfrak{p} \subset A_f$ to the ideal given by the preimage of $\mathfrak{p}$ under the natural map $A \to A_f$. (see chapter 3 of [5]). Let $X = A_f$ and $Y = D(f) \subset \mathrm{spec}(A)$. Choose a basis of sets of the form $D(\frac{g}{f^n})$ for X where $\frac{g}{f^n} \in A_f$, and choose a basis $D(g) \cap D(f)$ for Y where $g \in A$. Note that in A_f, $\frac{g}{f^n} \notin \mathfrak{p}$ if and only if $g \notin \mathfrak{p}$ so every basis element in X is of the form $D(g)$ for some $g \in A$. Then our map $\pi : X \to Y$ satisfies that for any $\mathfrak{p} \in X$ and $g \in A$, $g \notin \mathfrak{p}$ if and only if $g \notin \pi(\mathfrak{p})$ so π sends basis elements $D(g)$ of X to $D(g) \cap D(f)$ and the preimage of any basis element $D(g) \cap D(f)$ in Y is $D(g) \subset X$. Therefore, the image of any basis element under π is open and the preimage of any basis element under π is open, from which it is an easy consequence of basic point-set topology that π is a homeomorphism.

Additionally,

$$\pi_*\mathcal{O}_X(D(g)\cap D(f)) = \mathcal{O}_X(\pi^{-1}(D(g)\cap D(f))) = \mathcal{O}_X(D(g)) = (A_f)_g,$$

and similarly since $D(f)\cap D(g) = D(fg)$,

$$\mathcal{O}_Y(D(f)\cap D(g)) = \mathcal{O}_{\mathrm{spec}(A)}(D(f)\cap D(g)) = \mathcal{O}_{\mathrm{spec}(\mathcal{A})}(D(fg)) = A_{fg}.$$

Since $(A_f)_g = A_{fg}$ we get an isomorphism $\pi_*\mathcal{O}_X \to \mathcal{O}_Y$ on basis elements. Thus, by Corollary, 3.12, $\pi_*\mathcal{O}_X$ and $\mathcal{O}_Y$ are isomorphic as sheaves, and therefore, $(X,\mathcal{O}_X)$ and $(Y,\mathcal{O}_Y)$ are isomorphic as ringed spaces. $\square$

Corollary 4.12. *If $(X,\mathcal{O}_X)$ is a scheme and $U \subset X$ is open, then $(U,\mathcal{O}_X\big|_U)$ is a scheme.*

Proof. Since X is a scheme, we can cover it with affine open sets $\mathrm{spec}(A_i) \subset X$. For each affine open set, $\mathrm{spec}(A_i)$ choose a basis of sets of the form $D(f_i)$ where $f_i \in A_i$, and taking the collection of all $D(f_i)$ for all i gives a basis for X. Each $D(f_i)$ is an affine scheme by Proposition 4.11, so since $D(f_i)$ form a basis for X, we can cover any open subset $U \subset X$ by sets of the form $D(f_i)$ so U is a scheme. $\square$

Definition 4.13. For a field k, we write $\mathbb{A}_k^n$ in place of $\mathrm{spec}(k[x_1, ..., x_n])$. We call $\mathbb{A}_k^n$ affine n space, and we call $\mathbb{A}_k^1$ the affine line. When the field k is implied, we will often only write $\mathbb{A}^n$ in place of $\mathbb{A}_k^n$.

Remark 4.14. We can note that given a homomorphism of rings $\phi : A \to B$, we get a continuous map $\pi : B \to A$ sending a prime ideal $\mathfrak{p} \in \mathrm{spec}(B)$ to $\phi^{-1}(\mathfrak{p})$. We can show that this map is continuous by showing that for any closed set $V(I) \subset \mathrm{spec}(A)$, $\pi^{-1}(V(I)) = V(I^e)$ where I^e is the ideal generated by $\phi(I)$ in B, and thus, the preimage of every closed set is closed under π. For example, the continuous function $\pi : A_f \to D(f) \subset A$ from Proposition 4.11 is induced by the obvious homomorphism $\pi : A \to A_f$. Therefore, given an isomorphism of rings $\phi : A \to B$ it is not hard to show that π is a homeomorphism and that π induces an isomorphism of ringed spaces $\mathcal{O}_{\mathrm{spec}(B)} \to \mathcal{O}_{\mathrm{spec}(A)}$.

Definition 4.15. Let k be a field and let $\mathbb{A}_x^1 = \mathrm{spec}(k[x])$ and $\mathbb{A}_y^1 = \mathrm{spec}(k[y])$. Let $U_1 = D(x) \subset \mathbb{A}_x^1$ and $U_2 = D(y) \subset \mathbb{A}_y^1$. Proposition 4.11 gives us a natural identification of U_1 with $\mathrm{spec}(k[x,x^{-1}])$ and U_2 with $\mathrm{spec}(k[y,y^{-1}])$. If we take the isomorphism $k[x,x^{-1}] \to k[y,y^{-1}]$ sending x to y^{-1} we get an induced isomorphism of ringed spaces from $k[y,y^{-1}] \to k[x,x^{-1}]$ We can glue $U_1 \subset \mathbb{A}_x^1$ to $U_2 \subset \mathbb{A}_y^1$ by gluing U_1 to U_2 with the quotient topology. Then as in Theorem 3.13, we have a space $\mathbb{A}_x^1 \cup \mathbb{A}_y^1$ and an isomorphism of sheaves along their intersection, so we can form the sheaf $\mathbb{P}_k^1$ on all of $\mathbb{A}_x^1 \cup \mathbb{A}_y^1$ via this isomorphism. To show that $\mathbb{P}_k^1$ is a scheme, we note that each point p is either an element of $\mathbb{A}_x^1 = \mathrm{spec}(k[x])$ or $\mathbb{A}_y^1 = \mathrm{spec}(k[y])$ so $\{\mathbb{A}_x^1, \mathbb{A}_y^1\}$ give an open cover of $\mathbb{P}_k^1$ by affine schemes. We call $\mathbb{P}_k^1$ the projective line, and when k is implicit, we will write it as $\mathbb{P}^1$.

Remark 4.16. We would like to define projective n-space written $\mathbb{P}^n$ for $n \neq 1$. The definition given of $\mathbb{P}^1$ does not generalize in an obvious way to $n \neq 1$. Details on an alternative definition of $\mathbb{P}^1$ which will allow us to more naturally generalize to projective n-space, can be found in chapter 4.5 of [9].

5. Locally Free Sheaves

Definition 5.1. A sheaf of modules on a ringed space $(X, \mathcal{O}_X)$, is a sheaf $\mathcal{F}$ of abelian groups on X such that for every open set U, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module and such that for any open sets $V \subset U$, the diagram

$$\begin{tikzcd} \mathcal{O}_X(U) \times \mathcal{F}(U) \arrow[r] \arrow[d] & \mathcal{F}(U) \arrow[d] \\ \mathcal{O}_X(V) \times \mathcal{F}(V) \arrow[r] & \mathcal{F}(V) \end{tikzcd}$$

commutes, where the horizontal arrows are given by the action of the rings $\mathcal{O}_X(U)$ and $\mathcal{O}_X(V)$ on the $\mathcal{O}_X(U)$-module $\mathcal{F}(U)$ and the $\mathcal{O}_X(V)$-module $\mathcal{F}(V)$ respectively, and the vertical arrows are the restriction maps.

Definition 5.2. A morphism of $\mathcal{O}_X$-modules $f : \mathcal{F} \to \mathcal{G}$, is a morphism of sheaves of abelian groups $\mathcal{F} \to \mathcal{G}$ such that for each U, $f(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is an $\mathcal{O}_X(U)$-module homomorphism. An isomorphism is a morphism $f : \mathcal{F} \to \mathcal{G}$ such that there exists a morphism $g : \mathcal{G} \to \mathcal{F}$ with $f \circ g$ and $g \circ f$ the identities on $\mathcal{G}$ and $\mathcal{F}$ respectively. If there exists an isomorphism $f : \mathcal{F} \to \mathcal{G}$ we say $\mathcal{F}$ and $\mathcal{G}$ are isomorphic and denote this by $\mathcal{F} \cong \mathcal{G}$.

Example 5.3. For a ringed space $(X, \mathcal{O}_X)$ and $\mathcal{O}_X$ modules $\mathcal{F}_i$ for each $i \in I$, we can construct the $\mathcal{O}_X$-module $\oplus_{i\in I}\mathcal{F}_i$ which assigns for each open set U, the module $\oplus_{i\in I}\mathcal{F}_i(U)$ and for any open sets $V \subset U$ the restriction map sending an element $(a_i)_{i\in I} \in \oplus_{i\in I}\mathcal{F}_i(U)$ to $(b_i)_{i\in I}$ where b_i is the restriction of a_i given by $\mathcal{F}_i(U) \to \mathcal{F}_i(V)$. We write $\mathcal{O}_X^{\oplus I}$ in place of $\oplus_{i\in I}\mathcal{O}_X$. A free $\mathcal{O}_X$-module is any $\mathcal{O}_X$-module isomorphic to $\mathcal{O}_X^{\oplus I}$. The free $\mathcal{O}_X$-module of rank n is the $\mathcal{O}_X$-module isomorphic to $\mathcal{O}_X^n$.

Example 5.4. If $\mathcal{F}$ and $\mathcal{G}$ are both $\mathcal{O}_X$-modules, then for any open set $U \subset X$, define $\mathscr{H}\mathrm{om}(\mathcal{F}, \mathcal{G})(U) = \mathrm{Mor}(\mathcal{F}\big|_U, \mathcal{G}\big|_U)$ where $\mathrm{Mor}(\mathcal{F}\big|_U, \mathcal{G}\big|_U)$ is the set of all morphisms $\mathcal{F}\big|_U \to \mathcal{G}\big|_U$. If $f, g \in \mathscr{H}\mathrm{om}(\mathcal{F}, \mathcal{G})(U)$ and V is an open subset of U, then we can define $(f+g)(V)$ by $(f+g)(V)(x) = f(V)(x)+g(V)(x)$ for any $x \in \mathcal{F}$, and similarly, if $a \in \mathcal{O}_X\big|_U(V)$, we can define $(af)(V)$ in the obvious way. It is slightly tedious, but not difficult to check that this definition turns $\mathscr{H}\mathrm{om}(\mathcal{F}, \mathcal{G})(U)$ into an $\mathcal{O}_X\big|_U$-module, and turns $\mathscr{H}\mathrm{om}(\mathcal{F}, \mathcal{G})$ into an $\mathcal{O}_X$-module.

Definition 5.5. If $\mathcal{F}$ is an $\mathcal{O}_X$-module, define $\mathcal{F}^\vee = \mathscr{H}\mathrm{om}(\mathcal{F}, \mathcal{O}_X)$, and call $\mathcal{F}^\vee$ the dual of $\mathcal{F}$.

Example 5.6. If $\mathcal{F}$ and $\mathcal{G}$ are both $\mathcal{O}_X$-modules, then define $(\mathcal{F} \otimes_{\mathrm{pre}} \mathcal{G})(U) = \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$. This turns $\mathcal{F} \otimes_{\mathrm{pre}} \mathcal{G}$ into a presheaf. However, this presheaf is not in general a sheaf. We define the sheaf $\mathcal{F} \otimes \mathcal{G}$ to be the sheafification $(\mathcal{F} \otimes_{\mathrm{pre}} \mathcal{G})^{sh}$. (An example that shows that $\mathcal{F} \otimes_{\mathrm{pre}} \mathcal{G}$ need not be a sheaf appears in section 7, when we construct $\mathcal{O}_{\mathbb{P}^1}$-modules $\mathcal{O}(n)$ as we can check that $\mathcal{O}(1) \otimes \mathcal{O}(-1) \neq \mathcal{O}(1) \otimes_{\mathrm{pre}} \mathcal{O}(-1)$).

Definition 5.7. An $\mathcal{O}_X$-module $\mathcal{F}$ is a locally free sheaf of rank n if for every $x \in X$ there is an open set U containing x such that $\mathcal{F}\big|_U$ is a free sheaf of rank n over $\mathcal{O}_X\big|_U$. When X is a scheme, we will call locally free sheaves of rank n vector bundles, and we will call the free sheaf of rank n the trivial vector bundle. We call a locally free sheaf of rank 1 a line bundle.

It is not immediately obvious the reason for calling locally free sheaves vector bundles. For those who have seen some differential geometry, the upcoming definition and remark provide some motivation for this terminology.

Definition 5.8. If M is a smooth manifold, then a vector bundle of rank n over M is a smooth manifold V along with a map $\pi : V \to M$ such that for every $x \in M$, $\pi^{-1}(x)$ is given the structure of a real vector space. Additionally, the vector bundle is locally trivial meaning for every $x \in M$ there exists an open set U containing x and a diffeomorphism $h : \pi^{-1}(U) \to U \times \mathbb{R}^n$ such that the diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr] \arrow[dr] & & U \times \mathbb{R}^n \arrow[dl] \\ & M & \end{tikzcd}$$

commutes where the map $U \times \mathbb{R}^n \to U \subset M$ is the projection map. The diffeomorphism must additionally satisfy that for every $y \in U$, $h : \pi^{-1}(y) \to y \times \mathbb{R}^n$ is a vector space isomorphism where $y \times \mathbb{R}^n$ is given the structure of a vector space in the obvious way. For a set U where such a commutative diagram exists, we say that the vector bundle is trivial over U.

Remark 5.9. If $\pi : V \to M$ is a rank n vector bundle on the manifold M, then, we can form the sheaf of rings $\mathcal{O}$ of differentiable functions on M as in Example 2.4. For any open subset $U \subset M$, we can define $\mathcal{F}(U)$ to be the set of all sections of U, where a section of U is a smooth map $U \to V$ such that every $x \in U$ is mapped to an element in $\pi^{-1}(x)$. For any $f, g \in \mathcal{F}(U)$ we can define $f + g$ by $(f + g)(x) = f(x) + g(x)$ where we can take the sum $f(x) + g(x)$ since $f(x)$ and $g(x)$ are both in the vector space $\pi^{-1}(x)$. Similarly, for any $s \in \mathcal{O}(U)$ we can define $(sf)(x)$ by $s(x)f(x)$ since $s(x) \in \mathbb{R}$ and we can multiply by scalars in $\pi^{-1}(x)$. Under these operations, $\mathcal{F}(U)$ is a module over $\mathcal{O}(U)$, and more generally, one can show that $\mathcal{F}$ becomes a sheaf of modules over the ringed space $(M, \mathcal{O})$. Furthermore, if U is an open set such that the vector bundle is trivial over U, then $\mathcal{F}\big|_U$ is isomorphic to a free sheaf of rank n over $\mathcal{O}$ and thus $\mathcal{F}$ is a locally free sheaf of rank n over $\mathcal{O}$.

Definition 5.10. Given a locally free $\mathcal{O}_X$-module $\mathcal{F}$, a choice of an open cover $\{U_i\}$ such that $\mathcal{F}\big|_{U_i}$ is free is called a trivialization of $\mathcal{F}$.

If $\mathcal{F}$ is a locally free module of rank n over $\mathcal{O}_X$, then we can choose a trivialization $\{U_i\}$ of $\mathcal{F}$ such that $\mathcal{F}\big|_{U_i}$ is free. In particular, we get an isomorphism of sheaves $\phi_i : \mathcal{F}\big|_{U_i} \to (\mathcal{O}_X\big|_{U_i})^n$ for all i. For any open sets U_i and U_j and isomorphisms ϕ_i and ϕ_j we can restrict our isomorphisms to $\mathcal{F}\big|_{U_i \cap U_j}$, and use them to construct a map $T_{ij} : (\mathcal{O}_X\big|_{U_i \cap U_j})^n \to \mathcal{F}\big|_{U_i \cap U_j} \to (\mathcal{O}_X\big|_{U_i \cap U_j})^n$ where this map is the composition $\phi_j \circ \phi_i^{-1}$.

Remark 5.11. The process of starting with a locally free sheaf $\mathcal{F}$ and constructing the functions T_{ij} is a reversible process. More precisely, given the functions T_{ij}, we can construct the sheaf $\mathcal{F}$. In particular, the maps T_{ij} satisfy the cocycle condition of Definition 3.14. since

$$T_{jk} \circ T_{ij} = \phi_k \circ \phi_j^{-1} \circ \phi_j \circ \phi_i^{-1} = \phi_k \circ \phi_i^{-1} = T_{ik}.$$

Therefore, given sheaves $\mathcal{F}_i \cong (\mathcal{O}_X\big|_U)^n$ and isomorphisms T_{ij} on the intersections $\mathcal{F}_i \cap \mathcal{F}_j$, by Theorem 3.13, we can construct a unique sheaf $\mathcal{F}'$ such that $\mathcal{F}'\big|_{U_i} \cong \mathcal{F}_i$ for all i, and since this sheaf is unique, we must have that $\mathcal{F}' = \mathcal{F}$.

Remark 5.12. The transition functions T_{ij} are clearly not in general unique as they depend on our choices of isomorphisms ϕ_i and ϕ_j. In particular, if one chooses an isomorphism $\psi_i : (\mathcal{O}_X\big|_{U_i})^n \to (\mathcal{O}_X\big|_{U_i})^n$, $\phi_i' = \psi_i \circ \phi_i$ gives another isomorphism $\mathcal{F}\big|_{U_i} \to (\mathcal{O}_X\big|_{U_i})^n$ so we obtain a new function $T'_{ij} = \phi_j \circ \phi_i'^{-1} = \phi_j \circ \phi_i^{-1} \circ \psi_i^{-1}$. Therefore, replacing T_{ij} with T'_{ij} which we obtain from T_{ij} by a right action of an isomorphism $(\mathcal{O}_X\big|_{U_i})^n \to (\mathcal{O}_X\big|_{U_i})^n$ and constructing $\mathcal{F}$ from the transition functions as in Remark 5.9 gives the same sheaf $\mathcal{F}$. Similarly, replacing T_{ij} with new transition maps T'_{ij} obtain be left action on T_{ij} by an isomorphism of $(\mathcal{O}_X\big|_{U_j})^n$ does not alter the sheaf $\mathcal{F}$ obtained from the transition functions.

Proposition 5.13. *Let $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ be locally free $\mathcal{O}_X$-modules of finite rank.*

(1) *If $\mathcal{F}$ is locally free of rank n, then $\mathcal{F}^\vee$ is locally free of rank n.*
(2) *$\mathcal{F} \otimes \mathcal{G}$ is a locally free sheaf.*
(3) *$\mathcal{F} \otimes \mathcal{F}^\vee \cong \mathcal{O}_X$.*
(4) *$\mathcal{F} \otimes (\mathcal{G} \oplus \mathcal{H}) \cong (\mathcal{F} \otimes \mathcal{G}) \oplus (\mathcal{F} \otimes \mathcal{H})$.*
(5) *$\mathcal{F} \otimes (\mathcal{G} \otimes \mathcal{H}) \cong (\mathcal{F} \otimes \mathcal{G}) \otimes \mathcal{H}$.*

Proof. To prove (1), It is easy to prove that if $\mathcal{F}$ is free of rank n then so is $\mathcal{F}^\vee$. When $\mathcal{F}$ is locally free of rank n, we can cover X by open sets U such that $\mathcal{F}\big|_U$ is free of rank n and thus, $\mathcal{F}^\vee\big|_U$ is free of rank n so $\mathcal{F}^\vee$ is locally free of rank n.

To prove (2), we note that if $\mathcal{F} \otimes_{\text{pre}} \mathcal{G}$ is a locally free sheaf so that $(\mathcal{F} \otimes_{\text{pre}} \mathcal{G})\big|_U$ is isomorphic to $\mathcal{O}_X^n$ for some n then, $(\mathcal{F} \otimes_{\text{pre}} \mathcal{G})\big|_U$ is a sheaf so by Remark 3.8, $(\mathcal{F} \otimes \mathcal{G})\big|_U = (\mathcal{F} \otimes_{\text{pre}} \mathcal{G})\big|_U$ and therefore if $\mathcal{F} \otimes_{\text{pre}} \mathcal{G}$ is locally free then so is $\mathcal{F} \otimes \mathcal{G}$. Then it is easy to show when $\mathcal{F}$ and $\mathcal{G}$ are free that $\mathcal{F} \otimes \mathcal{G}_{pre}$ is a free sheaf so $\mathcal{F} \otimes \mathcal{G}$ is a free sheaf. In the case where $\mathcal{F}$ and $\mathcal{G}$ are locally free but not necessarily free, for every point $p \in X$, we can find a set U containing p such that $\mathcal{F}\big|_U$ and $\mathcal{G}\big|_U$ are both free so $(\mathcal{F} \otimes_{\text{pre}} \mathcal{G})\big|_U$ is free, and thus, $\mathcal{F} \otimes \mathcal{G}$ is locally free.

To prove (3), we note that for any ring R and any R-module M there is a natural homomorphism $M \otimes \operatorname{Hom}(M, R) \to R$ sending a pair $m \otimes f$ to $f(m)$, and this homomorphism is an isomorphism whenever M is a finitely generated projective module (and therefore, whenever M is free of finite rank). This gives a morphism of presheaves $g : \mathcal{F} \otimes_{\text{pre}} \mathcal{F}^\vee \to \mathcal{O}_X$ sending $\mathcal{F} \otimes_{\text{pre}} \mathcal{F}^\vee(U)$ to $\mathcal{O}_X(U)$ via the homomorphism $\mathcal{F}(U) \otimes \mathscr{H}\text{om}(\mathcal{F}(U), \mathcal{O}_X(U)) \to \mathcal{O}_X(U)$ described above. For any $p \in X$, if we choose an open set U containing p such that $\mathcal{F}\big|_U$ and $\mathcal{F}^\vee\big|_U$ are both free, then $g(V)$ is an isomorphism for any $V \subset U$ so so it follows that the induced map g_p is an isomorphism. By the universal property of sheaffification, we get a unique induced map $f : \mathcal{F} \otimes \mathcal{F}^\vee \to \mathcal{O}_X$ such that $f \circ sh = g$, and by Remark 3.8, since g_p is an isomorphism for all p, f_p is an isomorphism for all p so therefore, by Proposition 3.4, $\mathcal{F} \otimes \mathcal{F}^\vee \cong \mathcal{O}_X$.

The proof of (4) is similar to the proof of (3). Namely, we can construct a morphism $\mathcal{F} \otimes_{\text{pre}} (\mathcal{G} \oplus \mathcal{H}) \to (\mathcal{F} \otimes_{\text{pre}} \mathcal{G}) \oplus (\mathcal{F} \otimes_{\text{pre}} \oplus \mathcal{H})$ given by for each U the natural isomorphism $\mathcal{F}(U) \otimes (\mathcal{G}(U) \oplus \mathcal{H}(U)) \to (\mathcal{F}(U) \otimes \mathcal{G}(U)) \oplus (\mathcal{F}(U) \otimes \mathcal{H}(U))$. Then we have the obvious sheafification map $(\mathcal{F} \otimes_{\text{pre}} \mathcal{G}) \oplus (\mathcal{F} \otimes_{\text{pre}} \mathcal{H}) \to (\mathcal{F} \otimes \mathcal{G}) \oplus (\mathcal{F} \otimes \mathcal{H})$, and thus, by composing our maps, we get a map $g : \mathcal{F} \otimes_{\text{pre}} (\mathcal{G} \oplus \mathcal{H}) \to (\mathcal{F} \otimes \mathcal{G}) \oplus$

$(\mathcal{F} \otimes \mathcal{H})$. By the universal property of sheafification, this induces a unique map $f : \mathcal{F} \otimes (\mathcal{G} \oplus \mathcal{H}) \to (\mathcal{F} \otimes \mathcal{G}) \oplus (\mathcal{F} \otimes \mathcal{H})$. Finally, similar to (3), if we choose a set U such that $\mathcal{F}\big|_U, \mathcal{G}\big|_U$, and $\mathcal{H}\big|_U$ are all free, then it is clear that $g(V)$ is an isomorphism for all $V \subset U$ so g_p is an isomorphism, and therefore by Remark 3.8 and Proposition 3.4, f_p is an isomorphism for all p so f is an isomorphism.

To prove (5), we will first show that for any presheaf $\mathcal{F}$, $\mathcal{F}^{sh} \otimes \mathcal{G} \cong (\mathcal{F} \otimes_{\text{pre}} \mathcal{G})^{sh}$. In particular, there exists a map $\mathcal{F} \otimes_{\text{pre}} \mathcal{G} \to \mathcal{F}^{sh} \otimes_{\text{pre}} \mathcal{G}$ given by for each U, the homomorphism $(\mathcal{F} \otimes_{\text{pre}} \mathcal{G})(U) \to (\mathcal{F}^{sh} \otimes_{\text{pre}} \mathcal{G})(U)$ sending $a \otimes b$ to $sh(U)(a) \otimes b$ where sh is the sheafification morphism $\mathcal{F} \to \mathcal{F}^{sh}$. Then, composing with the obvious sheafification map from $\mathcal{F}^{sh} \otimes_{\text{pre}} \mathcal{G} \to \mathcal{F}^{sh} \otimes \mathcal{G}$ we get a map from $\mathcal{F} \otimes_{\text{pre}} \mathcal{G} \to \mathcal{F}^{sh} \otimes \mathcal{G}$. By showing that $\mathcal{F}^{sh} \otimes \mathcal{G}$ along with this morphism satisfy the universal property of sheafification, we can conclude that $\mathcal{F}^{sh} \otimes \mathcal{G} \cong ((F \otimes_{\text{pre}} \mathcal{G})^{sh}$.

Therefore, for any sheaves $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$,

$$(\mathcal{F} \otimes \mathcal{G}) \otimes \mathcal{H} = (\mathcal{F} \otimes_{\text{pre}} \mathcal{G})^{sh} \otimes \mathcal{H} = ((\mathcal{F} \otimes_{\text{pre}} \mathcal{G}) \otimes_{\text{pre}} \mathcal{H}))^{sh}.$$

We can similarly prove that $\mathcal{F} \otimes (\mathcal{G} \otimes \mathcal{H}) = (\mathcal{F} \otimes_{\text{pre}} (\mathcal{G} \otimes_{\text{pre}} \mathcal{H}))^{sh}$, and it follows from the associativity of tensor products for modules over rings that $(\mathcal{F} \otimes_{\text{pre}} \mathcal{G}) \otimes_{\text{pre}} \mathcal{H} = \mathcal{F} \otimes_{\text{pre}} (\mathcal{G} \otimes_{\text{pre}} \mathcal{H})$ so $(\mathcal{F} \otimes \mathcal{G}) \otimes \mathcal{H} = \mathcal{F} \otimes (\mathcal{G} \otimes \mathcal{H})$. □

Corollary 5.14. *The set of line bundles on X up to isomorphism form an abelian group under $\otimes$.*

Proof. If $\mathcal{F}$ and $\mathcal{G}$ are both line bundles, we want to show that $\mathcal{F} \otimes \mathcal{G}$ is a line bundle. We know that $\mathcal{F} \otimes \mathcal{G}$ is locally free by Proposition 5.13 (1), so we only need to show that $\mathcal{F} \otimes \mathcal{G}$ has rank 1. Similar to Proposition 5.13, this follows easily if $\mathcal{F}$ and $\mathcal{G}$ are free, and otherwise, when we restrict $\mathcal{F}$ and $\mathcal{G}$ to an open cover of sets U where both $\mathcal{F}\big|_U$ and $\mathcal{G}\big|_U$ are free, we can conclude that $\mathcal{F} \otimes \mathcal{G}\big|_U$ is locally free of rank 1. It is clear to see that $\mathcal{O}_X$ is the identity, and thus, that the inverse of an element $\mathcal{F}$ is $\mathcal{F}^\vee$ follows from Proposition 5.13 (1) and (3). Associativity follows from Proposition 5.13 (5). Since tensor product of modules is commutative, we have that $\mathcal{F} \otimes_{\text{pre}} \mathcal{G} = \mathcal{G} \otimes_{\text{pre}} \mathcal{F}$ so $\mathcal{F} \otimes \mathcal{G} = \mathcal{G} \otimes \mathcal{F}$. □

Definition 5.15. The group of line bundles on X up to isomorphism under $\otimes$ is called the Picard group of X and it is denoted Pic$X$.

6. Quasicoherent Sheaves

Definition 6.1. If A is a ring and M is an A-module, then we define an $\mathcal{O}_{\text{spec}(A)}$-module $\widetilde{M}$ by first defining $\widetilde{M}$ on open sets $D(f)$ by $\widetilde{M}(D(f)) = M_f$. If $D(g) \subset D(f)$ and $m \in M$, then similar to Lemma 4.6, $\frac{m}{f} \in M_g$ so we get an obvious restriction map $M_f \to M_g$. The proof that this is a sheaf on the basis elements $D(f)$ is similar to Theorem 4.7, and in particular, by Theorem 3.10, we can uniquely extend this sheaf on the basis to obtain the $\mathcal{O}_{\text{spec}(A)}$-module, $\widetilde{M}$.

Definition 6.2. If $(X, \mathcal{O}_X)$ is a scheme, then an $\mathcal{O}_X$-module $\mathcal{F}$ is quasicoherent if for every affine open subset spec(A) of X, $\mathcal{F}\big|_{\text{spec}(A)} \cong \widetilde{M}$ for some A-module M. Equivalently a quasicoherent sheaf $\mathcal{F}$ is a sheaf in which there exists an open cover by affine open sets spec(A_i) such that $\mathcal{F}\big|_{\text{spec}(A_i)} \cong \widetilde{M}$ for some A_i-module M.

Remark 6.3. It is not obvious that the two definitions of quasicoherent sheaves given in Definition 6.2 are equivalent. To prove the equivalence of these two definitions see chapter 2.5 of [10].

Remark 6.4. Let $X = \text{spec}(A)$ be an affine scheme and let $\widetilde{M}$ and $\widetilde{N}$ be two quasicoherent $\mathcal{O}_X$-modules. Given a module homomorphism $\phi : M \to N$, we can define a module homomorphism $\phi_f : M_f \to N_f$ sending $\frac{m}{f^n}$ to $\frac{\phi(m)}{f^n}$ for any element $\frac{m}{f^n}$ in M_f. In particular, we get a homomorphism $\widetilde{M}(D(f)) \to \widetilde{N}(D(f))$ for each $D(f)$, and after checking that these morphisms satisfy the necessary condition in Definition 3.11 to be a morphism on the basis $D(f)$ of X, by Corollary 3.12, we obtain a unique morphism $\widetilde{M} \to \widetilde{N}$ induced by ϕ. Therefore, to specify a morphism $\widetilde{M} \to \widetilde{N}$, it suffices to define a module homomorphism $M \to N$.

Proposition 6.5. *Every vector bundle $\mathcal{F}$ over a scheme X is quasicoherent.*

Proof. First note that if M is a free module and $X = \text{spec}(A)$ for some ring A, then for any $f \in A$, $\widetilde{M}(D(f)) = (\mathcal{O}_{\text{spec}(A)}(D(f)))^n$. Thus, if $\mathcal{F}$ is a vector bundle of rank n over $(\text{spec}(A), \mathcal{O}_{\text{spec}(A)})$, then $\mathcal{F}$ agree with $\widetilde{M}$ over basis open sets $D(f))$ so there exists an obvious morphism on a basis $\mathcal{F} \to \widetilde{M}$ so by Corollary 3.12, they are isomorphic and therefore, $\mathcal{F}$ is quasicoherent.

If $\mathcal{F}$ is locally free, and X is any scheme, then we can cover X by affine open sets $\text{spec}(A_i)$ and we can choose a basis of sets of the form $D(f_i)$ for each affine open set $\text{spec}(A_i)$. In particular, the collection of distinguished open sets $D(f_i)$ form a basis for $\mathcal{F}$. Therefore, since we can cover X by open sets U such that $\mathcal{F}\big|_U$ is free, we can find an open cover by basic sets $D(f_i)$ such that $\mathcal{F}\big|_{D(f_i)}$ is free. By the isomorphism of ringed spaces between $D(f)$ and $\text{spec}(A_f)$ given in Proposition 4.11, we can conclude that $\mathcal{F}\big|_{D(f_i)} \cong \widetilde{M}$ where $\widetilde{M}$ is the quasicoherent sheaf over $\text{spec}(A_f)$ formed by a free module over A_f. Therefore, since the collection of open sets $D(f_i)$ where $\mathcal{F}\big|_{D(f_i)}$ are free form an affine open cover, $\mathcal{F}$ is quasicoherent. □

Proposition 6.6. *The only rank n vector bundle on $\mathbb{A}^1$ is the trivial vector bundle.*

Proof. Let $\mathcal{F}$ be a vector bundle over $\mathcal{O}_{\mathbb{A}^1}$. Since $\mathbb{A}^1$ is an affine open subset of $\mathbb{A}^1$, by the previous proposition, $\mathcal{F}$ is quasicoherent, so $\mathcal{F} = \widetilde{M}$ for some A-module $\widetilde{M}$, and if $M = A^n$, then as in the previous proposition it is easy to verify that $\widetilde{M} = \mathcal{O}_{\mathbb{A}^1}^n$ by verifying that they agree on the basis of sets of the form $D(f)$. Note that $M = \mathcal{F}(\mathbb{A}^1)$ is a module over $\mathcal{O}_{\mathbb{A}^1}(\mathbb{A}^1) = k[x]$ where k is a field. Since $k[x]$ is a principal ideal domain, by the classification of modules over principal ideal domains (see chapter 12.1 of [2]), in order to show that M is a free A-module, it suffices to show that M is torsion free. Let $am = 0$ with $0 \neq a \in k[x]$ and $m \in M$, and let $\{D(f)\}$ be an open cover of $\mathbb{A}^1$ such that $\mathcal{F}\big|_{D(f)}$ is free. Then $(am)\big|_{D(f)} = 0$ so by the commutative diagram in Definition 5.1, we get that $a\big|_{D(f)} m\big|_{D(f)} = 0$ for all $D(f)$ in the open cover. If $a\big|_{D(f)} = 0$ for some $D(f)$, then $af^n = 0$ in $k[x]$ so $a = 0$ in $k[x]$. Therefore, $a\big|_{D(f)} \neq 0$ for all $D(f)$, so $m\big|_{D(f)} = 0$ since $k[x]_f$ is a principal ideal domain. Thus, since $a\big|_{D(f)} m\big|_{D(f)} \in \mathcal{F}(D(f)) = \mathcal{O}_{\mathbb{A}^1}(D(f))^n$ is torsion free $m\big|_{D(f)} = 0$. Therefore, by the identity axiom $m = 0$, so M is torsion free. □

7. A Classification of Vector Bundles on $\mathbb{P}^1$

Our first step in classifying all vector bundles on $\mathbb{P}^1$ is to classify all line bundles on $\mathbb{P}^1$. $\mathbb{P}^1$ is given by the union of two copies of $\mathbb{A}^1$. We will refer to these two copies of $\mathbb{A}^1$ by $\mathbb{A}^1_x$ and $\mathbb{A}^1_y$ with $\mathbb{A}^1_x = \operatorname{spec}(k[x])$ and $A^1_y = \operatorname{spec} k[y]$ for some field k. Since any line bundle over $\mathbb{A}^1$ is trivial, the restriction of any line bundle on $\mathbb{P}^1$ to each copy of $\mathbb{A}^1$ gives a trivialization of that line bundle. In particular, a line bundle on $\mathbb{P}^1$ is given by the transition function $T_{12} : \mathcal{O}_{\mathbb{P}^1}\big|_{U_1\cap U_2} \to \mathcal{O}_{\mathbb{P}^1}\big|_{U_1\cap U_2}$ where $U_1 = \mathbb{A}^1_x$ and $U_2 = \mathbb{A}^1_y$. $U_1 \cap U_2 = \operatorname{spec}(k[x]_x) = \operatorname{spec}(k[x, x^{-1}])$, so by Remark 6.4, the transition function is given by an isomorphism of modules $k[x, x^{-1}] \to k[x, x^{-1}]$. Let $\mathcal{O}(n)$ be the line bundle obtained by the isomorphism T_{12} sending f to $x^n f$ for each $f \in k[x, x^{-1}]$.

Theorem 7.1. *Pic* $\mathbb{P}^1 \cong \mathbb{Z}$.

Proof. Our goal is to show the map $\phi : \mathbb{Z} \to \operatorname{Pic} \mathbb{P}^1$ sending n to $\mathcal{O}(n)$ is an isomorphism. To show this map is a homomorphism, we want to show that $\mathcal{O}(n) \otimes \mathcal{O}(m) = \mathcal{O}(n+m)$. We have an isomorphisms $\phi_m, \psi_m : \mathcal{O}(m)\big|_{U_1\cap U_2} \to \mathcal{O}_X\big|_{U_1\cap U_2}$ such that $\psi_m \circ \phi_m^{-1}$ is the map given by multiplication by x^m. We similarly have $\phi_n, \psi_n : \mathcal{O}(n)\big|_{U_1\cap U_2} \to \mathcal{O}_X\big|_{U_1\cap U_2}$. Since $\mathcal{O}(n) \otimes \mathcal{O}(m)$ is free when restricted to $U_1 \cap U_2$, $\mathcal{O}(n) \otimes \mathcal{O}(m) = \mathcal{O}(n) \otimes_{\text{pre}} \mathcal{O}(m)$ on $U_1 \cap U_2$, so we get an isomorphism $\mathcal{O}(n) \otimes \mathcal{O}(m)\big|_{U_1\cap U_2} \to \mathcal{O}_X\big|_{U_1\cap U_2} \otimes \mathcal{O}_X\big|_{U_1\cap U_2}$ given by sending $a \otimes b \mapsto \phi_m(a) \otimes \phi_n(b)$. Then composing with the isomorphism $(O)_X\big|_{U_1\cap U_2} \otimes \mathcal{O}_X\big|_{U_1\cap U_2}$ given by sending $a \otimes b$ to ab we get a map $\phi : (O)_X\big|_{U_1\cap U_2} \otimes \mathcal{O}_X\big|_{U_1\cap U_2} \to \mathcal{O}\big|_{U_1\cap U_2}$. We can get a map $\psi : (O)_X\big|_{U_1\cap U_2} \otimes \mathcal{O}_X\big|_{U_1\cap U_2} \to \mathcal{O}\big|_{U_1\cap U_2}$ in an almost identical way from the maps ψ_m and ψ_n. Then the map $\psi \circ \phi^{-1}$ is the map given by multiplication by x^{n+m} so we conclude that $\mathcal{O}(n) \otimes \mathcal{O}(m) = \mathcal{O}(n+m)$.

We next want to show this map is injective by showing if $n \neq 0$ then $\mathcal{O}(n) \neq \mathcal{O}(0)$, and therefore, $\ker \phi = \{0\}$. An element $f \in \mathcal{O}(n)(\mathbb{P}^1)$ can be obtained by taking $f_1 = f\big|_{U_1}$ and $f_2 = f\big|_{U_2}$. Then $f_1 \in \mathcal{O}_n\big|_{\mathbb{A}^1_x} = k[x]$ and similarly $f_2 \in k[y]$ and they are related by the transition function that tells us that in $U_1 \cap U_2$, $f_2 x^n = f_1$. In particular, $f\big|_{U_1\cap U_2} \in k[x] \cap x^n k[x^{-1}]$. When $n \geq 0$, the set of polynomials in $k[x] \cap x^n k[x^{-1}]$ is an $n+1$ dimension vector space over k and otherwise, when $n < 0$ $k[x] \cap x^n k[x^{-1}] = \{0\}$. In particular, the only time when $k[x] \cap x^n k[x^{-1}]$ is 1-dimensional is when $n = 0$ so if $n \neq 0$, the space $k[x] \cap x^n k[x^{-1}]$ is not isomorphic to the space $k[x] \cap x^0 k[x^{-1}]$ so $\mathcal{O}(n) \neq \mathcal{O}(0)$.

Finally, to show surjectivity, we want to show that every line bundle over $\mathbb{P}^1$ is isomorphic to $\mathcal{O}(n)$ for some n. As we remarked at the start of this section, any line bundle $\mathcal{L}$ has a trivialization given by the sets U_1 and U_2, so by Remark 5.11, the line bundle can be obtained by knowing the data of its transition function $T_{ij} \in \mathrm{GL}_1(k[x, x^{-1}])$. The isomorphism T_{ij} is uniquely determined by where it sends the element $1 \in k[x, x^{-1}]$ and it sends 1 to some unit in $k[x, x^{-1}]$, so $T_{ij}(1) = cx^n$ for some $c \in k^\times$ and $n \in \mathbb{Z}$. By Remark 5.12, right action on T_{ij} by the isomorphism $\mathcal{O}_{\mathbb{P}^1}\big|_{U_1} \to \mathcal{O}_{\mathbb{P}^1}\big|_{U_1}$ has no effect on the structure of the line bundle $\mathcal{L}$. In particular, since $\mathcal{O}_{\mathbb{P}^1}\big|_{U_1} = k[x]$, the action given by the isomorphism $k[x] \to k[x]$ sending $f(x)$ to $c^{-1} f(x)$ turns the isomorphism given by cx^n into the isomorphism given by x^n so $\mathcal{L} = \mathcal{O}(n)$. $\square$

To classify rank n vector bundles, by Remark 5.12, we want to classify all isomorphisms in $\mathrm{GL}_n(k[x,x^{-1}])$ up to right action by elements in $\mathrm{GL}_n(k[x^{-1}])$ and left action by $\mathrm{GL}_n(k[x])$. Therefore, the next lemma is the main step in classifying all vector bundles over $\mathbb{P}^1$.

Lemma 7.2. *Any matrix A in $GL_n(k[x,x^{-1}])$ is of the form $VD(d_1,....,d_n)U$ with $d_1 \geq ... \geq d_n$, where $V \in GL_n(k[x^{-1}])$, $V \in GL_n(k[x])$ and $D(d_1,...,d_n)$ is the diagonal matrix*

$$D(d_1,...,d_n) = \begin{bmatrix} x^{d_1} & & \\ & \ddots & \\ & & x^{d_n} \end{bmatrix}.$$

Furthermore, if each entry of A is in $k[x]$, we can choose our matrix $D(d_1,...,d_n)$ such that $d_1,...,d_n \geq 0$.

Proof. We will prove this by induction. The case when $n = 1$ follows since the elements of $\mathrm{GL}_n(k[x,x^{-1}])$ are the elements in $k[x,x^{-1}]^\times$, so they are of the form cx^n with $c \in k^\times$ and $n \in \mathbb{Z}$. Let $A = [a_{ij}]$ be an $n \times n$ matrix with $n > 1$. We will first assume the case that A is a matrix whose entries are all in $k[x]$. We can change A by any matrix B where B is obtained by multiplying A on the left by a matrix $U \in \mathrm{GL}_n(k[x])$ and by multiplying A on the right by any matrix $V \in \mathrm{GL}_n(k[x^{-1}])$. In particular, this allows us to do the column operations of A consisting of adding to one column a $k[x]$-multiple of any other column, and this allows us to similarly add any $k[x^{-1}]$-multiple of any row to any other row. We can also multiply A by any nonzero element in k and do row and column swaps. We also note that $\det(A) \in k[x,x^{-1}]^\times$ and since A has entries in $k[x]$, we also have $\det(A) \in k[x]$ so $\det(A) = cx^t$ for $c \in k^\times$ and $t \geq 0$, and any matrix in $\mathrm{GL}_n(k[x]) \cup \mathrm{GL}_n k[x^{-1}]$ has determinant in $k^\times$. Therefore, multiplying A by matrices U and V multiplies the determinant of A by some element in $k^\times$ so the t appearing in $\det(A) = cx^t$ will remain fixed throughout this proof.

Since $k[x]$ is a Euclidean domain, our first step is to use the Euclidean algorithm to add $k[x]$-multiples of the columns to obtain a matrix $B = [b_{ij}]$ so that $b_{1i} = 0$ for all $i \neq 0$ and b_{11} is the gcd of all of the elements in the first row in B. If B_{11} is the matrix obtained from B by removing the first row and the first column, then $\det(B) = b_{11}\det(B_{11})$ so B_{11} is invertible since B is invertible, and $\det(B) = c_0x^s$ so $b_{11} = c_1x^{d_1}$ for some $c_1 \in k^\times$ and $d_1 \in \mathbb{Z}$. Since each element of A was in $k[x]$, and the matrix B was obtained by adding $k[x]$-multiples of columns to each other, it follows that every entry of B is in $k[x]$, so we may assume $d_1 \geq 0$. Multiplying B by the constant c_1^{-1}, we may assume $b_{11} = x^{d_1}$, and by induction on n we may assume $B_{11} = V'D(d_2,...,d_n)U'$ with $V' \in \mathrm{GL}_{n-1}(k[x^{-1}])$ and $U' \in \mathrm{GL}_{n-1}(k[x])$, and since B_{11} has entries in $k[x]$, we can further assume that $d_2 \geq ... \geq d_n \geq 0$. Therefore, we can replace the matrix B with the matrix C where

$$C = \begin{bmatrix} 1 & 0 \\ 0 & V' \end{bmatrix} B \begin{bmatrix} 1 & 0 \\ 0 & U' \end{bmatrix} = \begin{bmatrix} x^{d_1} & & & \\ c_{12} & x^{d_2} & & \\ \vdots & & \ddots & \\ c_{1n} & & & x^{d_n} \end{bmatrix}.$$

Consider the set Σ of all matrices $C = [c_{ij}]$ which are equivalent to A up to right action by $\mathrm{GL}_n(k[x^{-1}])$ and left action by $\mathrm{GL}_n(k[x])$ where for all i, $c_{ii} = x^{d_i}$ where $d_i \geq 0$, $c_{1i} \in k[x]$, and every other element of C is 0. By our above computation

$\Sigma \neq \emptyset$. For any element $C \in \Sigma$ we have $\deg(\det(C)) = d_1 + ... + d_n \geq d_1$ and $\det(A) = cx^t$ so $d_1 + ... + d_n = t$. Therefore, since d_1 has an upper bound we can choose an element $C \in \Sigma$ such that $c_{11} = x^{d_1}$ where d_1 is maximal.

We may also assume that the polynomials c_{1i} only have terms of degree $> d_1$ because we can do the row operation consisting of subtracting $k[x^{-1}]$-multiples of x^{d_1} from c_{1i} to eliminate all terms of c_{1i} of degree $\leq d_1$. We can similarly assume $\deg(c_{1i}) < d_i$ by doing the column operation consisting of subtracting $k[x]$-multiples of x^{d_i} from c_{1i}. Therefore, we can show that $c_{i1} = 0$ for $i \neq 1$ by showing $d_1 \geq d_i$

Assume that $d_1 < d_i$ for some i. Then swap the first and ith row to obtain a new matrix C_1' and note that the gcd of the elements in the first row of C_1' is $x^{d_1'}$ with $d_1' > d_1$ so after adding $k[x]$-multiples of columns to each other, we may assume the element in the top corner of C' is $x^{d_1'}$ giving an element in Σ with $d_1' > d_1$ contradicting the assumption that d_1 is maximal.

Therefore, $c_{1i} = 0$ for all $i \neq 1$, so $C = D(d_1, ..., d_n)$ with each $d_i \geq 0$, and we can do a swap of the ith and jth row followed by a swap of the ith and jth column to swap the location of the d_i and d_j appearing in $D(d_1, ..., d_n)$, so by doing suitable permutations we may assume $d_1 \geq ... \geq d_n$ which completes the proof in the case where A has entries only in $k[x]$.

To prove the general case where $A \in \mathrm{GL}_n(k[x, x^{-1}])$ and A does not necessarily have entries in $k[x]$, choose a large enough N such that $x^N A$ has entries only in $k[x]$. Then there exists $U \in \mathrm{GL}_n(k[x])$ and $V \in \mathrm{GL}_n(k[x^{-1}])$ such that $x^N A = VD(d_1, ..., d_n)U$ with $d_1 \geq ... \geq d_n$ so $A = D(d_1 - x^N, ..., d_n - x^N)$. □

Theorem 7.3. *Every rank n vector bundle $\mathcal{V}$ on $\mathbb{P}^1$ can be written as $\mathcal{O}(d_1) \oplus ... \oplus \mathcal{O}(d_n)$ where $d_1 \geq ... \geq d_n$ Moreover the multiset of elements $\{d_1, ..., d_n\}$ is unique.*

Proof. By Lemma 7.2 afnd Remark 5.12, we may assume that the transition functions are given by a matrix of the form $D(d_1, ..., d_n)$ with $d_1 \geq .... \geq d_n$. We see that the transition matrix of $\mathcal{O}(d_1) \oplus ... \oplus \mathcal{O}(d_n)$ is given by $D(d_1, ..., d_n)$ which proves that every rank n vector bundle is of the form $\mathcal{O}(d_1) \oplus ... \oplus \mathcal{O}(d_n)$ with $d_1 \geq ... \geq d_n$. To show uniqueness, assume $\mathcal{O}(d_1)^{r_1} \oplus ... \oplus \mathcal{O}(d_k)^{r_k} \cong \mathcal{O}(\widetilde{d}_1)^{s_1} \oplus ... \oplus \mathcal{O}(\widetilde{d}_l)^{s_l}$ with $d_1 > ... > d_k$ and $\widetilde{d}_1 > ... > \widetilde{d}_s$. Without loss of generality, assume $d_1 \geq \widetilde{d}_1$. Then $\mathcal{O}(-d_1) \otimes (\mathcal{O}(d_1)^{r_1} \oplus ... \oplus \mathcal{O}(d_k)^{r_k}) \cong \mathcal{O}(-d_1) \otimes (\mathcal{O}(\widetilde{d}_1)^{s_1} \oplus ... \oplus \mathcal{O}(\widetilde{d}_l)^{s_l})$, so by Proposition 5.12, $\mathcal{O}(0)^{r_1} \oplus \mathcal{O}(d_2 - d_1)^{r_2} \oplus ... \oplus \mathcal{O}(d_k - d_1)^{r_k}) \cong \mathcal{O}(\widetilde{d}_1 - d_1)^{s_1} \oplus ... \oplus \mathcal{O}(\widetilde{d}_l - d_1)^{s_l}$. In our proof of injectivity in Theorem 7.1, we proved that as a vector space over k, $\mathcal{O}(n)(X)$ has dimension $n - 1$ when $n \geq 0$ and dimension 0 otherwise. In particular, since $\mathcal{O}(0)^{r_1} \oplus \mathcal{O}(d_2 - d_1) \oplus ... \oplus \mathcal{O}(d_k - d_1)^{r_k}$ has dimension r_1, it follows that $\mathcal{O}(\widetilde{d}_1 - d_1)^{s_1} \oplus ... \oplus \mathcal{O}(\widetilde{d}_l - d_1)^{s_l}$ has dimension r_1, and thus, $s_1 = r_1$ and $\widetilde{d}_1 = d_1$. Similarly, if we assume without loss of generality that $d_2 \geq \widetilde{d}_2$, then $\mathcal{O}(-d_2) \otimes (\mathcal{O}(d_1)^{r_1} \oplus ... \oplus \mathcal{O}(d_k)^{r_k}) \cong \mathcal{O}(-d_2) \otimes (\mathcal{O}(\widetilde{d}_1)^{s_1} \oplus ... \oplus \mathcal{O}(\widetilde{d}_l)^{s_l})$ so $\mathcal{O}(d_1 - d_2)^{r_1} \oplus ... \oplus \mathcal{O}(d_k - d_2)^{r_k} \cong \mathcal{O}(d_1 - d_2)^{r_1} \oplus ... \oplus \mathcal{O}(\widetilde{d}_l - d_2)^{s_l}$. Therefore, by a similar dimensionality argument, $d_2 = \widetilde{d}_2$ and $r_2 = s_2$. Continuing inductively we can see that $k = l$, and for all i, $d_i = \widetilde{d}_i$, and $r_i = s_i$ concluding the proof of the uniqueness of the multiset $\{d_1, ..., d_n\}$. □

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