ON THE ADDITIVITY OF TRISECTION GENUS

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ABSTRACT. In this paper, we explain what a trisection of a 4-manifold is and show how to construct one. A trisection naturally gives rise to a quadruple of non-negative integers $(g; g_0, g_1, g_2)$, encoding the genera g_0, g_1, and g_2 of the three 4-dimensional handlebodies and the genus g of the central surface Sigma (the common intersection of the three handle-bodies). The trisection genus of M, denoted $g(M)$, is the minimal genus of a central surface in any trisection of M. We mainly explore whether the trisection genus is additive under the connected sum.

CONTENTS

1. INTRODUCTION

In 1898, Poul Heegaard introduced the idea of a Heegaard splitting, which is a decomposition of a compact oriented 3-manifold that results from dividing it into two handle-bodies.Every closed, orientable three-manifold may be so obtained. While Heegaard splittings were studied extensively in the 1960s, it was not until a few decades later that the field was rejuvenated by Andrew Casson and Cameron Gordon (1987), primarily through their concept of strong irreducibility.

Trisection of 4-manifold is just the higher dimensional generalization of Heegaard splitting of 3-manifold. In 2016, Gay and Kirby introduced the concept of a trisection for arbitrary smooth, oriented closed 4– manifolds. A trisection naturally gives rise to a quadruple of non- negative integers $(g; g_0, g_1, g_2)$, encoding the genera g_0, g_1, and g_2 of the three 4-dimensional handlebodies and the genus g of the central surface Σ (common intersection of the three handle-bodies). The trisection genus of M, denoted $g(M)$, is the minimal genus of a central surface in any trisection of M. A trisection with $g(\Sigma) = g(M)$ is called a minimal genus trisection. In this

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paper, we are particularly interested in the problem of whether trisection genus is additive under connected sum operation.

In dimensions ≤ 4, there is a bijective correspondence between isotopy classes of smooth and piecewise linear structures. All manifolds are assumed to be piecewise linear (PL) in this paper unless stated otherwise. Our definition and results apply to any compact smooth manifold by passing to its unique piecewise linear structure.

2. Trisection and its Relationship to Heegaard Splitting

A Heegaard splitting of a closed orientable 3-manifold M is a decomposition of M into two handlebodies H_0 and H_1 which intersect exactly along their boundaries. We now set up an analogous story in dimension four:

Definition 2.1 (Trisection of closed manifold)**.** Let M be a closed, connected, piecewise linear 4–manifold. A trisection of M is a collection of three PL submanifolds $H_0, H_1, H_2 \subset M$, subject to the following four conditions:
(1) Each H_i is PL homeomorphic to a standard PL 4–dimensional 1–handlebody of genus g_i.
(2) The handle-bodies H_i have pairwise disjoint interior, and $M = \cup_i H_i$.
(3) The intersection $H_i \cap H_j$ of any two of the handle-bodies is a 3–dimensional 1–handlebody.
(4) The common intersection $\Sigma = H_0 \cap H_1 \cap H_2$ of all three handlebodies is a closed, connected surface, the central surface.
The submanifolds $H_{ij} = H_i \cap H_j$ and Σ are referred to as the trisection submanifolds.

As an example, we show the trisection of the complex projective plane CP^2 :

Consider the moment map from the complex projective plane to the standard 2–dimensional simplex, $\mu : CP^2 \to \Delta^2 \subset R^3$ defined by

$$[z_0 : z_1, z_2] \to \frac{1}{\sum |z_k|}(|z_0|, |z_1|, |z_2|).$$

Apply first barycentric subdivision to Δ^2, and then subtract the 0–skeleton of Δ^2 from the 0–skeleton of the first barycentric subdivision; we get the dual spine Π^2 in Δ^2, which is spanned by the remaining 0-skeleton. Decomposing along Π^2 gives Δ^2 a natural dual cubical structure with three 2–cubes, and the lower-dimensional cubes that we will focus on are the intersections of non-empty collections of these top-dimensional cubes, consisting of three interior 1–cubes and one interior 0–cube. The cubical structure is indicated in Figure 1, where the interior cubes are labeled with the trisection submanifolds.

Under the moment map, the 2–cubes pull back to 4–balls

$$\{[z_0 : z_1 : z_2] | z_i = 1, |z_j| \leq 1, |z_k| \leq 1\};$$

the interior 1–cubes pull back to solid tori $S^1 \times D^2$ defined by

$$\{[z_0 : z_1 : z_2] | z_i = 1, |z_j| = 1, |z_k| \leq 1\};$$

and the interior 0–cube pulls back to a 2–torus $\Sigma = S^1 \times S^1$ defined by $\{[z0 : z1 : z2] | z_0 = 1, |z_1| = 1, |z_2| = 1\}$.

The central surface is thus a Heegaard surface for the 3–sphere boundary of each 4–ball. This shows that the cubical structure pulls back to a trisection with central surface a torus. This is shown schematically in Figure 1 below.

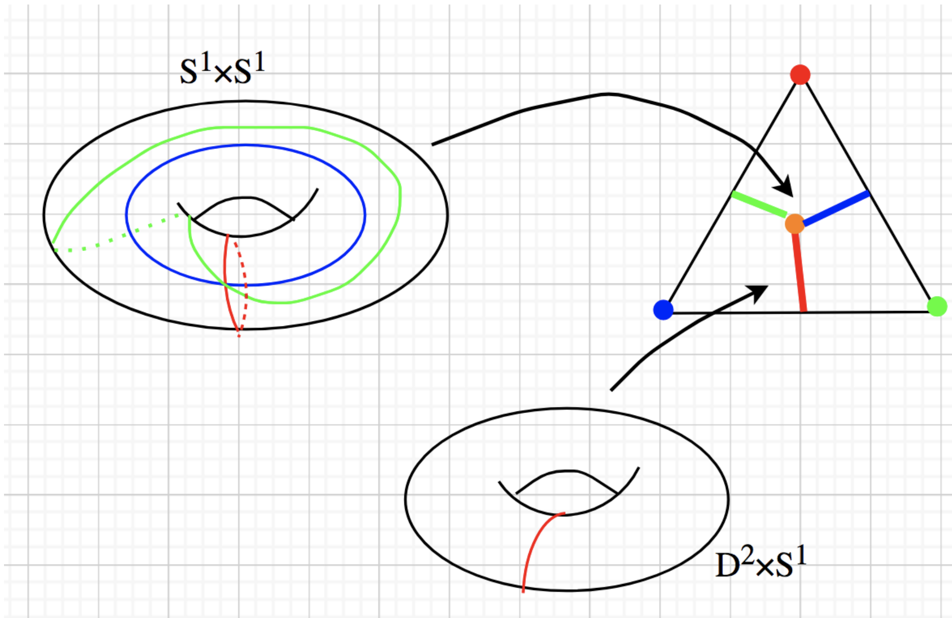

FIGURE 1. Trisection Diagram for CP^2.

3. ALGORITHM TO COMPUTE TRISECTION

In [3], Bell et al. introduces a method of inducing a trisection on M is from tricolourings of the triangulation. This is now summarised; the reader is referred to [3] for a more detailed discussion.

Let M be a closed, connected 4–manifold with (not necessarily simplicial, but possibly singular) triangulation T. A partition P_0, P_1, P_2 of the set of all vertices of T is a tricolouring if every 4–simplex meets two of the partition sets in two vertices and the remaining partition set in a single vertex. In this case, we also say that the triangulation is tricoloured.

Denote the vertices of the standard 2–simplex Δ^2 by v_0, v_1, and v_2 . When there is a tricolouring, a natural map $\mu : M \to \Delta^2$ can be defined by sending the vertices in P_k to v_k and extending this map linearly over each simplex. More explicitly, for every simplex formed by vertices $v_1, v_2, \cdots v_i$ in M, map the simplex to the geometric center of $\mu(v_1), \mu(v_2), \cdots, \mu(v_i)$ in Δ^2.

Note that the pre-image of v_k is a graph Γ_k in the 1–skeleton of M spanned by the vertices in P_k.

The strategy in section 3.2 of [3] is to use μ to pull back the cubical structure of the simplex to a trisection of M. The dual spine Π^n in an n–simplex Δ^n is the $(n-1)$-dimensional subcomplex of the first barycentric subdivision of Δ^n spanned by those vertices of the first barycentric subdivision which are not vertices of Δ^n itself. This is shown for $n = 2$ and $n = 3$ in Figure 2 below.

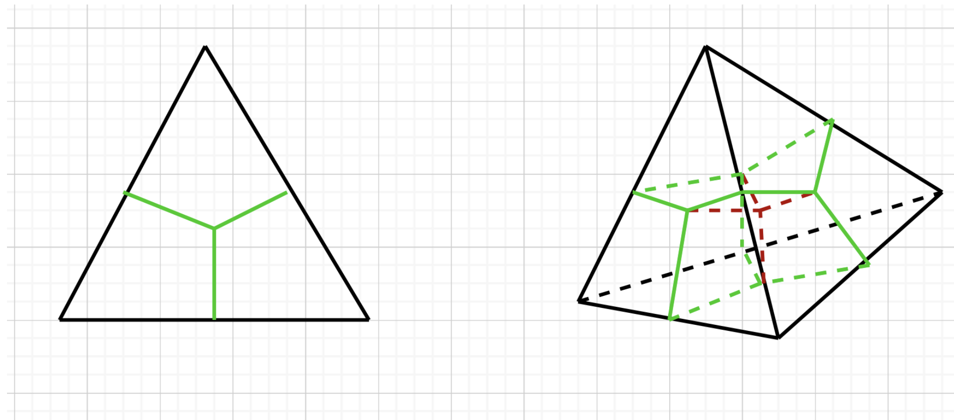

FIGURE 2. Dual Spine. Left: $\Pi^2 \subset \Delta^2$; Right: $\Pi^3 \subset \Delta^3$

Decomposing along Π^n gives Δ^n a natural cubical structure with $n+1$ cubes of dimension n, and the lower-dimensional cubes that we focus on are the intersections of non-empty collections of these top-dimensional cubes.

Recall that a compact subpolyhedron P in the interior of a manifold M is called a (PL) spine of M if M collapses onto P. If P is a spine of M, then $M \backslash P$ is PL homeomorphic with $\partial M \times [0, 1)$.

The pre-images under μ of the dual cubes of $\Pi^2 \subset \Delta^2$ have very simple combinatorics, as illustrated by Figure 3 below.
For each pentachoron (i.e. a 4-simplex) of M, without loss of generality suppose one of its vertex is in P_0, two of its vertices are in P_1, and two of its vertices are in P_2. If we choose 3 vertices from the 5 vertices of the pentachoron, such that they are from P_0, P_1, and P_2 respectively, then the 2-simplex formed by the 3 vertices would be mapped to the barycenter of Δ^2. There are totally 4 ways to choose such 3 vertices, which gives four 2-simplices. The four 2-simplices can be viewed as the 4 vertices of a dual 2-cube. Hence, the barycenter of Δ^2 pulls back to exactly one 2–cube in each pentachoron of M, and these glue together to form a surface Σ in M.

Σ is the common boundary of each of the three 3–manifolds obtained as pre-images of an interior 1–cube (edge) of Δ^2, whose endpoints are the barycenter O of Δ^2 and the barycenter O' of an edge a of Δ^2. If $a = v_0v_1$:

For each pentachoron, if we choose 2 vertices from the 5 vertices of the pentachoron, such that they are from P_0 and P_1 respectively, then the 1-simplex formed by the 2 vertices would be mapped to O'. There are totally 2 ways to choose such 2 vertices, which gives two 1-simplices. The two 1-simplices can be viewed as the 2 vertices of a dual 1-cube. Thus O' pulls back to an 1-cube and O pulls back to a 2-cube; jointing the 1-cube and 2-cube together, we see that OO' pulls back to a prism in M.

The case is similar when $a = v_0v_2$. When $a = v_1v_2$, since there are 4 ways to chose one vertex from P_1 and P_2 respectively, O' pulls back to a dual 2-cube instead, and thus OO' pulls back to a 3-cube.

Hence, we see that each such 3–manifold is made up of cubes and triangular prisms.

It is shown in [3] that the above construction gives a trisection if:

1. the graph Γ_k is connected for each k; and
2. the pre-image of an interior 1–cube of Δ^2 has a 1–dimensional spine.

A tricolouring is a c-tricolouring if Γ_k is connected for each k. A c-tricolouring is a ts-tricolouring if the pre-image of each interior 1–cube collapses onto a 1–dimensional spine. In this case, the dual cubical structure of Δ^2 pulls back to a trisection of M, as illustrated by Figure 3 below.

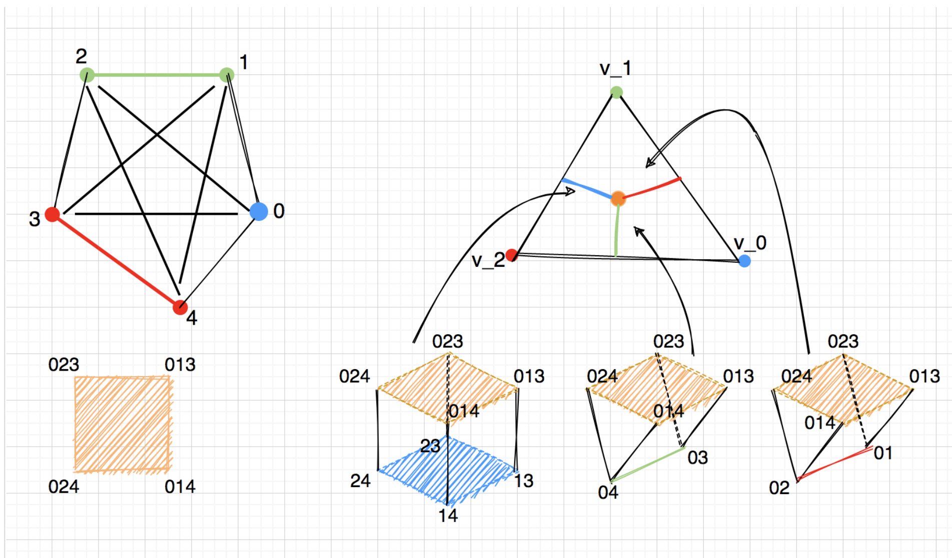

FIGURE 3. Pieces of the trisection submanifolds. The vertices of the pieces are barycenters of faces and labelled with the corresponding vertex labels. The central surface meets the pentachoron in a square. Two of the 3–dimensional trisection submanifolds meet the pentachoron in triangular prisms and the third meets it in a cube.

Definition 3.1 (Trisection supported by triangulation)**.** We say that a trisection of M is supported by the triangulation T of M, if T is ts-tricolourable and the trisection is isotopic to the pull-back of the dual cubical structure of Δ^2. In this case, the trisection is said to be dual to the corresponding ts-tricolouring.

An algorithm to construct a ts-tricolouring from an arbitrary initial triangulation is given in [6]. It uses subdivisions and bistellar moves in order to justify that the resulting triangulation has a ts-tricolouring. As a result, the number of pentachora of the final triangulation is larger by a factor of 120 compared to the number of pentachora of the initial one (see [6, Theorem 4].)

4. Upper-bounds for Trisection Genus under Connected Sum

A connected sum of two m-dimensional manifolds is a manifold formed by deleting a ball inside each manifold and gluing together the resulting boundary spheres. This operation is the most important method of constructing complex manifolds from simple ones, and plays a key role in the classification of closed surfaces. The classification theorem of closed surfaces states that any connected closed surface is homeomorphic to some member of one of these three families: the sphere, the connected sum of g tori for $g \geq 1$, and the connected sum of k real projective planes for $k \geq 1$.

Since connected sum operation is so significant, we shall study what are some additive invariants under connected sum. A trisection naturally gives rise to a quadruple of non- negative integers $(g; g_0, g_1, g_2)$, encoding the genera g_0, g_1, and g_2 of the three 4-dimensional handlebodies and the genus g of the central surface Σ (common intersection of the three handle-bodies). The trisection genus of $g(M)$ is the minimal genus of a central surface in any trisection of M. It is known that genus is additive under connected sum, so a natural question is whether trisection genus is also additive under connected sum.

Suppose M_0 has a trisection of genus g_0 and M_1 has a trisection of genus g_1. In [4], Gay and Kirby constructed a trisection of $M_0 \# M_1$, the connected sum of M_0 and M_1, with trisection genus $g_0 + g_1$ by choosing two standardly trisected 4–balls in both 4–manifolds. Hence, we have:

Theorem 4.1. *If M_0 has a trisection of genus g_0 and M_1 has a trisection of genus g_1, then the connected sum of M_0 and M_1 has a trisection of genus $g_0 + g_1$. In particular, $g(M_0 \# M_1) \leq g(M_0) + g(M_1)$.*

Now, it suffices to show that $g(M_0 \# M_1) \geq g(M_0) + g(M_1)$, and we prove that trisection genus is additive under connected sum.

5. Cases for Some Special 4-manifold

A very special 4-manifold is the $K3$ surface, a 4–manifold that can be described as a quartic in CP^3 given by the equation $z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0$.

In [5], it is shown that the trisection genera of CP^2 and $S^2 \times S^2$ are equal to their respective second Betti numbers. Moreover, the second Betti number is additive, and the trisection genus is subadditive under taking connected sums. Hence, the trisection genus of every 4–manifold which is a connected sum of arbitrarily many (PL standard) copies of CP^2, $S^2 \times S^2$ and the K_3 surface must be additive under connected sum.

6. Thoughts on how to Solve Additivity of Trisection Genus

We first make a conjecture. The analogous of trisection genus in dimension 3 is Heegaard genus. It is already shown that Heegaard genus is additive under connected sum, so we conject that trisection genus is indeed additive under connected sum.

The most direct idea about solving this problem is to extend the method that has been used to proof the additivity of Heegaard genus. In [4], Casson and Gordon shows that Heegaard genus is additive under connected sum. They first shows that for any 3-manifolds M_0 and M_1, we can construct a Heegaard splitting of $M_0 \# M_1$ with Heegaard genus $g(M_0) + g(M_1)$, so $g(M_0 \# M_1) \leq g(M_0) + g(M_1)$; then, they shows that for any 3-manifolds M such that $M = M_0 \# M_1$, we can construct a Heegard spliting of M_0 and M_1 with genus g_1 and g_2 respectively, from a minimal spliting of M, such that $g_1 + g_2 = g(M)$, which means $g(M_0 \# M_1) \geq g(M_0) + g(M_1)$. Combining the two results, we have $g(M_0 \# M_1) = g(M_0) + g(M_1)$.

The first step is very simple, and we have already done analogous work in 4-dimension. In order to prove the second step, $g(M_0 \# M_1) \geq g(M_0) + g(M_1)$, Casson and Gordon uses the result from [9], which is the lemma below:

Lemma 6.1. *Let (W, W') be a Heegaard splitting of $(M; B, B')$. Let $(S, \partial S) \subset (M, B \amalg B')$ be a disjoint union of essential 2-spheres and disks. Then there exists a disjoint union of essential 2-spheres and disks S^* in M such that*
(i) S^* is obtained from S by ambient 1-surgery and isotopy;
(ii) each component of S^* meets F in a single circle;
(iii) there exist complete disk systems D, D' for W, W' respectively such that $D \cap S^* = D' \cap S^* = \emptyset$.

Here, "essential" means that the 2-sphere does not bound a 3-ball in M. B and B' are disjoint surfaces in ∂M with $\partial B \cong \partial B'$, such that $\partial M = B \cup B' \cup \partial B \times I$. $F = \partial_+ W = \partial_+ W'$. A complete system of pair-wise disjoint, polyhedral 2-spheres $S_1^2, \cdots, S_t^2$ has the following property: if S_{t+1}^2 is a polyhedral 2-sphere in M^3, disjoint from $S_1^2, \cdots, S_t^2$, then there is a connected component K^3 of $M^3 - (S_1^2 + \cdots + S_{t+1}^2)$ such that $\overline{K}^3$ is a 3-sphere with holes, and $K^3 = (\overline{K}^3)^\circ$, and $S_{t+1}^2 \subset \partial K^3$. Intuitively speaking, completeness means that the space bounded by $S_1^2, \cdots, S_t^2$ has covered the interior of M.

If we can prove that analogous lemma holds for closed, connected, piece-wise linear 4-manifold, we have essentially shows that trisection genus is additive. The analogous lemma is:

Lemma 6.2. *Let (H_0, H_1, H_2) be a trisection of 4-manifold M. Let $S \subset M$ be a disjoint union of essential 3-spheres and balls. Then there exists a disjoint union of essential 3-spheres and balls S^* in M such that*
(i) S^* is obtained from S by ambient 1-surgery and isotopy;
(ii) each component of S_{ij} meets the central surface Σ $(H_0 \cap H_1 \cap H_2)$ in a single circle;
(iii) there exist complete disk systems D_{01}, D_{02}, and D_{12} for $H_0 1$ $(H_0 \cap H_1)$, H_{02} and H_{12} respectively such that $B_{01} \cap S^* = D_{02} \cap S^* = D_{12} \cap S^* = \emptyset$.
(iv) there exist complete ball systems B_0, B_1, and B_2 for H_0, H_1 and H_2 respectively such that $B_0 \cap S^* = B_1 \cap S^* = B_2 \cap S^* = \emptyset$.

If this holds, then we may proof that trisection genus is additive under connected sum:

Proof. Suppose that $M = M_0 \# M_1$. Clearly $g(M) \leq g(M_0) + g(M_1)$. We shall show that $g(M) \geq g(M_0) + g(M_1)$. We assume without loss of generality that M is connected.

First assume that M is irreducible. In other words M has no essential 3-sphere. Let B be the ball in M that realizes the decomposition $M = M_1 \# M_2$. If ∂B is inessential in M, then M_2 (say) is a 4-sphere and the result is trivial. So suppose that ∂B is essential in ∂M, and let (H_0, H_1, H_2) be a trisection of $(M; \emptyset, \partial M)$ of genus $g(M)$. Since M is irreducible, Lemma 6.2 implies that B may be isotoped so that $B \cap H_i$ is a disk (which necessarily separates H_i into two 4-dimensional 1-handlebodies $[H_i]_0$ and $[H_i]_1$, say). Then $([H_0]_i, [H_1]_i, [H_2]_i)$ is a trisection of M_i of genus g_i, $i = 0, 1$, where $g_0 + g_1 = g(M)$. This shows that $g(M_0) + g(M_1) \leq g(M)$.

For the general case, suppose $M = M_1 \# M_2$, and let

$$M_1 = \#_{i=1}^{m} N_1^{(i)}, \quad M_2 = \#_{j=1}^{m} N_2^{(i)}$$

be prime connected sum decompositions of M_1, and M_2. By renumbering the N's if necessary, we may assume that the connected sum $M_1 \# M_2$ takes place over boundary components of $N_1^{(1)}$ and $N_2^{(1)}$. Thus

$$M = (N_1^{(1)} \# N_2^{(1)}) \# (\#_{i=2}^{m} N_1^{(i)}) \# (\#_{j=2}^{m} N_2^{(i)}).$$

Since $N_1^{(1)}$ and $N_2^{(1)}$ are prime and have non-empty boundary, they are irreducible. Hence $N_1^{(1)} \# N_2^{(1)}$ is irreducible and so $g(N_1^{(1)} \# N_2^{(1)}) = g(N_1^{(1)}) + g(N_2^{(1)})$ by the previous paragraph. The fact that $g(M) = g(M_1) + g(M_2)$ now follows from the additivity of genus with respect to connected sum. □

Hence, the problem can be solved as long as we prove lemma 6.2, which we will work on later.

Of course, there is also a probability that lemma 6.2 cannot be proved, so it is better if we have a back-up method.

The classification theorem shows that every closed surface is essentially just the connected sum of finite types of simple surfaces. Initially, I wonder if we can also do this for closed 4-manifold? If we can categorizing all closed, connected, piecewise linear 4-manifold as connected sum of finite types of simple structures, this problem becomes easy. We could, as in the case for $K3$ surface, give an expression for trisection genus of those structures in terms of quantities that are known to be additive under connected sum, such as the second Betti number. Nonetheless, Markove [1 3] in 1958 has proven that it is impossible to classify 4-manifold, because For every $n > 3$ there is an n-manifold M^n such that the problem of homeomorphy of manifolds to M^n is undecidable. Hence, this method would not work.

The primary difficulty of solving this problem is that, it is hard to verify whether a trisection is minimal. So far, I found only one theorem which can help proving that a trisection is minimal:

Theorem 6.3. *If M has a trisection in which all 4–dimensional handlebodies are 4–balls, then the trisection is a minimal genus trisection.*

In order to prove the theorem, we first introduce a notion called stabilization:

Definition 6.4 (Stabilization)**.** Given a 4–manifold X with a trisection (X_1, X_2, X_3), we construct a new trisection (X_1', X_2', X_3'), as follows: For each $i, j \in \{1, 2, 3\}$, let H_{ij} be the handle-body $X_i \cap X_j$, with boundary $F = X_1 \cap X_2 \cap X_3$.Let a_{ij} be a properly embedded boundary parallel arc in each H_{ij}, such that the end points of a_{12} , a_{23} and a_{31} are disjoint in F . Let N_{ij} be a closed 4–dimensional regular neighborhood of a_{ij} in X (thus diffeomorphic to B^4), with N_{12} , N_{23} and N_{31} disjoint. Then we define
$X_1' = (X_1 \cup N_{23}) \backslash (\mathring{N}_{31} \cup \mathring{N}_{12})$,
$X_2' = (X_2 \cup N_{31}) \backslash (\mathring{N}_{12} \cup \mathring{N}_{23})$,
$X_3' = (X_3 \cup N_{12}) \backslash (\mathring{N}_{23} \cup \mathring{N}_{31})$,
The operation of replacing (X_1, X_2, X_3) with (X_1', X_2', X_3') is called stabilization.

The operation is illustrated by Figure 4 below.

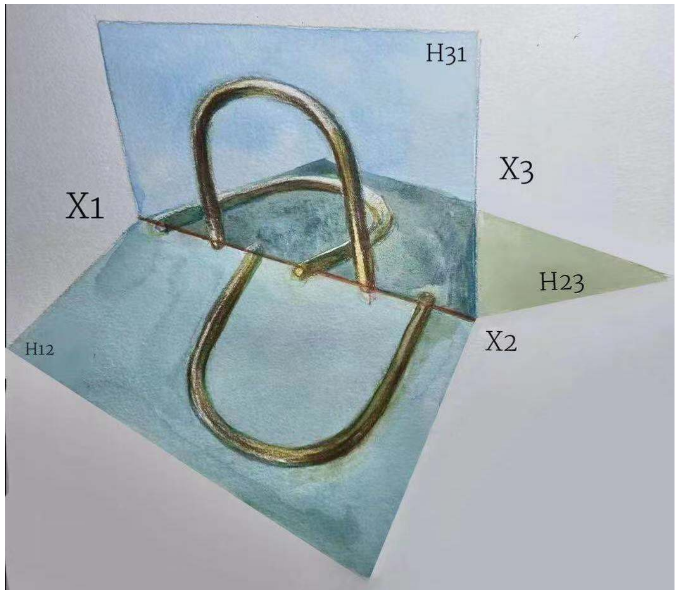

FIGURE 4. Illustration of Stabilization.

In [4], Gay and Kirby shows that

Lemma 6.5. *Given two trisections (X_1, X_2, X_3) and (X_1', X_2', X_3') of X, after stabilizing each trisection some number of times there is a diffeomorphism $h : X \to X$ isotopic to the identity with the property that $h(X_i) = h(X_i')$ for each i. In particular, $h(X_i \cap X_j) = X_i' \cap X_j'$ for $i \neq j$ in $\{1, 2, 3\}$, and $h(X_1 \cap X_2 \cap X_3) = X_1' \cap X_2' \cap X_3'$.*

With this lemma, we can prove the above theorem:

Proof. Suppose M has a $(g'; 0, 0, 0)$–trisection and a $(g; g_1, g_2, g_3)$–trisection. Moreover, suppose that $g = g(M)$. By [4], these have a common stabilisation. Suppose this is a $(g''; k_1, k_2, k_3)$–trisection. Each elementary stabilisation increases the genus of one handle-body and the genus of the central surface by one. Hence $g'' = g' + k_1 + k_2 + k_3$ and $g'' = g + (k_1 - g_1) + (k_2 - g_2) + (k_3 - g_3)$. This gives $g' \geq g(M) = g = g' + g_1 + g_2 + g_3$. This forces $g_1 = g_2 = g_3 = 0$ and $g = g$. □

All in all, if we can show that the 4-manifold $M_0 \# M_1$ has a trisection in which all 4–dimensional handlebodies are 4–balls, and has genus at least $g(M_0) + g(M_1)$, we prove that $g(M_0 \# M_1) = g(M_0) + g(M_1)$.

In fact, for an arbitrary (g, g_1, g_2, g_3) trisection of $M_0 \# M_1$, is it possible that we reduce it to a trisection of it in which all 4–dimensional handlebodies are 4–balls by repeatedly applying the reverse operation of stabilization? If so, this should make our proof simplier.

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7. Bibliography

References

[1] A.A. Markov, On unsolvability of certain problems of topology, Dokl. Akad. Nauk SSSR 123 (6) (1958) 978–980 (in Russian).

[2] A.A. Markov, The unsolvability of the homeomorphy problem, Dokl. Akad. Nauk SSSR 121 (2) (1958) 218–220 (in Russian). A.V. Chernavsky, V.P. Leksine / Annals of Pure and Applied Logic 141 (2006) 325–335 335

[3] A.A. Markov, Unsolvability of homeomorphy problem, in: J.A. Todd (Ed.), Proc. ICM, Edinburgh, 1958, Cambridge Univ. Press, 1960, pp. 300–306.

[4] A.J. Casson, C.McA. Gordon, Reducing heegaard splittings, Topology and its Applications, Volume 27, Issue 3, 1987, Pages 275-283.

[5] David Gay and Robion Kirby. Trisecting 4-manifolds. *Geom. Topol.*, 20(6):3097–3132, 2016.

[6] Jonathan Spreer and Stephan Tillmann. Determining the trisection genus of orientable and non-orientable PL 4-manifolds through triangulations. *Experimental Mathematics*, pages 1–11, 2020.

[7] M. Bell, J. Hass, J. Hyam Rubinstein, and S. Tillmann. Computing trisections of 4- manifolds. ArXiv e-prints, nov 2017. arXiv:1711.02763.

[8] M. Chu - S. Tillmann, Reflections on trisection genus, Rev. Roumaine Math. *Pures Appl.* (Proceedings of JARCS 2017), 64(4) (2019), 395-402.

[9] W. Waken, Some results on surfaces in 3-manifolds, Studies in Modern Topology (Math. Assoc. Amer., distributed by: Prentice-Hall, 1968) 34-98.