ALGEBRAIC COBORDISM OVER THE COMPLEX NUMBERS

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Abstract. In this paper, we give an introduction to motivic homotopy theory, and then use the motivic Adams spectral sequence to compute the bi-graded homotopy groups of the algebraic cobordism spectrum over the complex numbers, up to a square-zero mutliplicative extension.

Contents

1. Introduction

Numerous classical constructions in algebraic topology, such as cohomology theories or homotopy groups of spheres, are better understood through the perspective of stable homotopy theory. One of the oldest unsolved problems in algebraic topology is the calculation of homotopy groups of spheres $\pi_n S^k$ for all $n, k \geq 0$; Freudenthal first noticed that the homotopy groups $\pi_{n+r} S^r$ stabilize under suspension for large enough r (more precisely, for $r \geq n+2$). These are called the stable stems, and are by definition the homotopy groups of the sphere spectrum:

$$\pi_n^{\mathrm{S}} := \pi_n \mathsf{S} = \varinjlim_r \pi_{n+r} S^r.$$

One of the great advantages of the stable homotopy category $\mathcal{SH}$ is that it provides a way to compute stable homotopy groups of any nice enough spectrum, namely the Adams Spectral Sequence:

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}_p^\vee}(\mathbb{F}_p, \mathsf{H}_*(\mathsf{E}; \mathbb{F}_p)) \Longrightarrow \pi_{t-s} \mathsf{E}_p^\wedge,$$

which takes as input the homology of E with $\mathbb{Z}/p\mathbb{Z}$ coefficients (as a comodule over the dual Steenrod Algebra $\mathcal{A}_p^\vee$), and converges to the homotopy groups of the p-completion of E. Doing this for all primes p allows one, at least in some cases, to completely determine the homotopy groups of E. One such fortunate example is the complex bordism spectrum MU, whose homotopy is known to be polynomial:

$$\mathsf{MU}_* \cong \mathbb{Z}[x_1, x_2, \ldots], \quad |x_i| = 2i.$$

Thanks to the recent work of Voevodsky and Morel, a similar theory has been constructed for schemes: For any perfect base field k (or even more general base schemes), one can construct a homotopy category $\mathcal{H}(k)$, endowed with natural functors $\mathcal{H} \longrightarrow \mathcal{H}(k)$ and $\mathsf{Sm}_k \longrightarrow \mathcal{H}(k)$ from the classical homotopy category $\mathcal{H}$, and from the category Sm_k of (suitably nice) schemes over k, having the property that $\mathbb{A}^1 = \operatorname{Spec} k[x]$ becomes contractible. So $\mathcal{H}(k)$ can be viewed as a generalization of classical homotopy theory to algebraic geometry. A distinctive feature of $\mathcal{H}(k)$ is that it has bigraded spheres $S^{p,q}$ for $0 \leq q \leq p$, coming from smash products of two distinct spheres: one is the "Tate circle" $S^{1,1}$ coming from $\mathbb{A}^1 - 0 \in \mathsf{Sm}_k$, and the other is the "simplicial circle" $S^{1,0}$ coming from $S^1 \in \mathcal{H}$. In a manner analogous to C_2-equivariant stable homotopy theory, one can stabilize with respect to both spheres, thus obtaining the motivic stable homotopy category $\mathcal{SH}(k)$. The homotopy groups and (co)homology theories are all bi-graded:

$$\pi_{m,n}\mathsf{E} := [\Sigma^{m,n}\mathsf{S}, \mathsf{E}] := \operatorname{Hom}_{\mathcal{SH}(k)}(\Sigma^{m,n}\mathsf{S}, \mathsf{E})$$
$$\mathsf{E}_{m,n}X := \pi_{m,n}(\mathsf{E} \wedge X_+), \quad \mathsf{E}^{m,n}X := [X_+, \Sigma^{m,n}\mathsf{E}].$$

At times, it is more convenient to use a different grading convention, which mimics the $RO(C_2)$-grading from equivariant stable homotopy theory:

$$1 = (1,0), \quad \alpha = (1,1), \quad \rho = 1 + \alpha = (2,1).$$

So, for instance, we would write $S^{3,2} = S^{1+2\alpha}$. The classical Chow and algebraic K-theory rings can be written in terms of multiples of ρ:

$$\mathrm{CH}^n(X) \cong \mathsf{H}\mathbb{Z}^{n\rho}(X), \quad \text{and} \quad \mathrm{K}^n(X) \cong \mathsf{KGL}^{n\rho}(X),$$

where $\mathsf{H}\mathbb{Z}$ and KGL are certain motivic spectra generalizing the classical Eilenberg-MacLane and K-theory spectra. When writing a statement for all gradings simultaneously, we will write $\pi_\star, \mathsf{E}_\star, \mathsf{E}^\star$, where $\star$ symbolizes an element of $RO(C_2)$.

With this notation, there exists a Motivic Adams Spectral Sequence (MASS):

$$E_2^{s,t,u} = \operatorname{Ext}^{s,t,u}_{\mathcal{A}_p^{\mathsf{mot},\vee}}(\mathsf{H}_\star(\mathsf{S};\mathbb{F}_p), \mathsf{H}_\star(\mathsf{E};\mathbb{F}_p)) \Longrightarrow \pi_{t-s,u}\mathsf{E}^\wedge_{\mathsf{H}\mathbb{F}_p},$$

where $\mathcal{A}_p^{\mathsf{mot},\vee}$ is the dual motivic Steenrod Algebra $\pi_\star(\mathsf{H}\mathbb{F}_p \wedge \mathsf{H}\mathbb{F}_p)$, and $\mathsf{E}^\wedge_{\mathsf{H}\mathbb{F}_p}$ is the $\mathsf{H}\mathbb{F}_p$-nilpotent completion, which will be defined later. We also have a motivic analogue of the complex cobordism spectrum, namely MGL, for which it so happens that the $\mathsf{H}\mathbb{F}_p$-nilpotent completion is just the p-completion. Using the MASS, we show that

Theorem 1.1. *There is an isomorphism of bi-graded rings*

$$\mathsf{MGL}_\star/\mathsf{tors} \simeq \mathsf{H}\mathbb{Z}_\star/\mathsf{tors}[b_1, b_2, \ldots], \tag{1.2}$$

with $|b_i| = (2i, i)$.[1]

Moreover we show that $\mathsf{MGL}_\star$ can be expressed as a square-zero extension of two known objects. However, the methods used so far seem too weak to give a complete description of the multiplicative structure, since the Hochschild 2-cocycle classifying this square-zero extension may not be homologous to zero.

Structure of the paper. In Section 2, we give a brief introduction to motivic homotopy theory, beginning with the unstable category, and then explaining the stabilization process. In Section 3, we describe the Hopf invariant and its higher

[1] $/\mathsf{tors}$ denotes the mod-torsion functor $A \longmapsto A/\mathsf{tors}(A)$.

analogues, leading into the construction of the Motivic Adams Spectral Sequence, and then explain why in the case of MGL it converges to its p-completion. In Section 4, we compute the completions explicitly, using the MASS, as well as the rational MGL, and then we use the homotopy fracture square to determine the structure $\pi_\star \mathsf{MGL}$ up to a square-zero extension. In the Appendix we explain a technical point which allows one to deduce the homological Thom isomorphism from the known cohomological one, at least for smooth projective base schemes.

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2. The motivic stable homotopy category $\mathcal{SH}(k)$

As advertised, in this section we introduce the construction of a homotopy category $\mathcal{H}(k)$ of smooth schemes over k, together with a stable category $\mathcal{SH}(k)$, which suitably generalize the standard unstable and stable homotopy categories $\mathcal{H}$ and $\mathcal{SH}$ from algebraic topology. Although this construction can be performed in a quite vast degree of generality (e.g. even replacing k with an arbitrary base scheme S), we will be primarily interested in the case that $k = \mathbb{C}$. Consequently, the reader should feel free to work over the complex numbers, and use the classical (schemeless) viewpoint if they choose to. Although this category was originally constructed by Morel and Voevodsky in [MV99], we will follow two more recent papers, namely Morel's introduction to the subject [Mor04], as well as the so-called Seattle lectures [Voe99] written up by Weibel.

2.1. **The unstable category.**

We want a good notion of homotopy theory, in which the affine line $\mathbb{A}^1$ is contractible. One major stumbling block is given by the scarcity of limits and colimits of schemes. Without them, we are unable to perform standard constructions in homotopy theory, such as quotient spaces, mapping cones, Thom spaces, etc., as well as building classifying spaces like $\mathbb{P}^\infty$ or BGL.

There is a standard way to "formally add" limits and colimits to a category. Indeed, for any category $\mathcal{C}$, we may consider the Yoneda embedding

$$h : \mathcal{C} \longrightarrow \mathrm{Fun}(\mathcal{C}^{\mathsf{op}}, \mathsf{Sets}),$$

which sends an object X to the contravariant Hom functor $\mathrm{Hom}_{\mathcal{C}}(-, X)$. This allows us to view $\mathcal{C}$ as a full subcategory of the larger category $\mathrm{Fun}(\mathcal{C}^{\mathsf{op}}, \mathsf{Sets})$, which indeed has all limits and colimits (computed point-wise). For simplicity, we denote the latter by $\mathrm{Psh}(\mathcal{C})$, and speak of its objects as *presheaves* on $\mathcal{C}$. The Yoneda embedding preserves limits, literally by definition:

$$\lim_{i\in I} \mathrm{Hom}_{\mathcal{C}}(Y, X_i) \cong \mathrm{Hom}_{\mathcal{C}}(Y, \lim_{i\in I} X_i).$$

However, the analogous result is false for colimits, which means that colimits in this enlarged category are probably not what we want them to be. One can check, for example, that if $U \cup V = X$ is an open cover of a scheme, then $h_U \amalg_{h_{U\cap V}} h_V$ is (almost) never isomorphic to h_X.

There is an elegant solution to this problem, namely to restrict to a subcategory of $\mathrm{Psh}(\mathcal{C})$ of well-behaved objects called *sheaves.* From now on, we will denote by Sm_k the category of separated smooth schemes of finite type over a field k of characteristic zero, following [Mor04].[2] We want a full sub-category $\mathrm{Shv}(\mathsf{Sm}_k) \subset \mathrm{Psh}(\mathsf{Sm}_k)$ of so-called sheaves, such that the left square in diagram (2.1) below is co-cartesian for all U, V.

$$\begin{tikzcd} h_{U\cap V} \arrow[r] \arrow[d] & h_V \arrow[d] \\ h_U \arrow[r] & h_{U\cup V} \end{tikzcd} \qquad \begin{tikzcd} F(U\cap V) & F(V) \arrow[l] \\ F(U) \arrow[u] & F(U\cup V). \arrow[l] \arrow[u] \end{tikzcd} \tag{2.1}$$

By the Yoneda lemma, given any presheaf F, the universal property of pushouts is equivalent to the square shown on the right of (2.1) being cartesian.

We recognize this to be exactly the classical sheaf condition for a cover by two open subsets; note, though, that here the triples $(U, V, U\cup V)$ run through *the whole* category of schemes, not just open subsets of a given scheme X. It turns out that we want the sheaf condition to be satisfied for more than just open covers by open immersions:

Definition 2.2. [Mor04, 2.1.4] A *basic Nisnevich covering* is a pull-back square in Sm_k of the form

$$\begin{tikzcd} W \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "i"] & X \end{tikzcd} \tag{2.3}$$

where p is an étale map and i is an open immersion, satisfying the condition that

$$p^{-1}((X-U)_{\mathsf{red}}) \longrightarrow (X-U)_{\mathsf{red}}$$

is an isomorphism.[3]

Remark 2.4. If p is an open immersion as well, (2.3) becomes a Zariski covering with two open subsets. So the notion of a Nisnevich covering generalizes that of a Zariski covering.

Definition 2.5. The category of sheaves $\mathrm{Shv}(\mathsf{Sm}_k)$ is the full subcategory of $\mathrm{Psh}(\mathsf{Sm}_k)$ consisting of objects F that satisfy the sheaf condition with respect to basic Nisnevich coverings, i.e. such that F applied to every square of the form (2.3), is a pull-back square.

This category has many of the desired properties from classical sheaf theory. In particular, there exists a sheafification functor, and this can be used to compute colimits in the standard way by abstract nonsense:[4]

Proposition 2.6. [Mor04, 2.1.7] *The inclusion* $i : \mathrm{Shv}(\mathsf{Sm}_k) \hookrightarrow \mathrm{Psh}(\mathsf{Sm}_k)$ *has a left adjoint* $a : \mathrm{Psh}(\mathsf{Sm}_k) \longrightarrow \mathrm{Shv}(\mathsf{Sm}_k)$.

[2] There are many different choices that we can make concerning the specific category of schemes that we want our theory to be based on. For instance, in the Seattle lectures [Voe99], the category of *affine* schemes is used.

[3] The subscript red signifies the reduced-induced subscheme.

[4] Whenever $\mathcal{C} \subset \mathcal{D}$ has a left adjoint–let's call it $\mathcal{C}$-ification–one can show that limits in $\mathcal{C}$ are the same as limits in $\mathcal{D}$ (if they exist in $\mathcal{D}$), and similarly that colimits in $\mathcal{C}$ are the $\mathcal{C}$-ification applied to the same colimits in $\mathcal{D}$ (if they exist in $\mathcal{D}$).

One also finds that representable presheaves are sheaves [Mor04, 2.1.3], so that we may view Sm_k as a full subcategory of $\mathrm{Shv}(\mathsf{Sm}_k)$. This fact is quite trivial for the Zariski topology (it is literally the gluing property of morphisms), but requires some care to prove for Nisnevich coverings.

Remark 2.7. In Grothendieck's language, one can talk about the notion of "topology" on an arbitrary category, which broadly speaking consists of a given family of collections of morphisms with the same target, called *coverings*; these are required to satisfy a list of axioms, see [MLM94] for more details. In this language, $\mathrm{Shv}(\mathsf{Sm}_k)$ is exactly the category of sheaves for the Grothendieck topology constructed by Nisnevich. A Grothendieck topology with the property that all representable presheaves are sheaves (which is the case for Nisnevich) is sometimes referred to as "subcanonical."

As is customary in algebraic topology, we also consider the pointed version: we call $h_{\operatorname{Spec} k}$ the point (and sometimes denote it by $*$ for brevity), and we define a pointed sheaf to be a sheaf F, together with a map $* \longrightarrow F$. Note also that $h_{\operatorname{Spec} k}$ is the terminal object in both $\mathrm{Psh}(\mathsf{Sm}_k)$ and $\mathrm{Shv}(\mathsf{Sm}_k)$. We denote the category of pointed sheaves by $\mathrm{Shv}_\bullet(\mathsf{Sm}_k)$. Thanks to the existence of limits and colimits, we can perform the following standard constructions from algebraic topology:

(1) *Quotients*. Given an inclusion of sheaves $X \hookrightarrow Y$, we may form the quotient Y/X, defined to be the pushout $Y \amalg_X *$.
(2) *Wedge sums*. Given two pointed sheaves X and Y, we may define $X \vee Y$ to be the pushout $X \amalg_* Y$. This is in fact the coproduct in the category of pointed sheaves.
(3) *Smash products*. From the universal property of the pushout, we get an inclusion $X \vee Y \hookrightarrow X \times Y$, whose quotient we define to be the smash product:
$$X \wedge Y = X \times Y / X \vee Y.$$
(4) *Thom spaces*. If $E \longrightarrow X$ is a geometric vector bundle,[5] we define the Thom space $\mathrm{Th}(E)$ to be $E/(E - X)$, where $X \hookrightarrow E$ is the zero-section.

Now, we can address the issue of homotopy. The modern way of thinking about homotopy theory in an arbitrary category is given by the theory of *model categories*, which was first introduced by Quillen in [Qui67] (with small differences from the modern definition). For a modern account, one can consult [Hov99] or [MP12]. Following [Voe99], we can already put a model structure on $\mathrm{Shv}_\bullet(\mathsf{Sm}_k)$, determined by the classes of weak equivalences and cofibrations

$\mathcal{C}$ Cofibrations are the monomorphisms
$\mathcal{W}$ The class of weak equivalence is the smallest class of arrows such that
  (a) $\mathcal{W}$ contains all isomorphisms;
  (b) $\mathcal{W}$ has the 2-out-of-3 property;
  (c) If $X_\alpha \xrightarrow{f_{\alpha\beta}} X_\beta$ is a filtered diagram of arrows in $\mathcal{C} \cap \mathcal{W}$, then each transfinite composite $X_\alpha \longrightarrow \varinjlim X_\beta$ is in $\mathcal{W}$;

[5] The majority of introductory algebraic geometry courses define vector bundles as locally free sheaves, which a priori seems to have little to do with the geometric intuition from algebraic topology. However, there is an appropriate notion of a geometric vector bundle as well; to construct one from a locally free sheaf $\mathcal{F}$, one takes the relative spec construction $\mathrm{Spec}_X(\mathrm{Sym}^* \mathcal{F})$ of the symmetric sheaf of algebras on $\mathcal{F}$.

(d) The pushout of a weak equivalence along a cofibration is also a weak equivalence
(e) The pushout of a map in $\mathcal{C} \cap \mathcal{W}$ along any map is still a weak equivalence.

Definition 2.8. The motivic homotopy category $\mathcal{H}(k)$ is the homotopy category of the model category above.

There is also a model structure for simplicial sheaves, which is described in [Mor04], and which leads to the same homotopy category. In $\mathrm{Shv}_\bullet(\mathsf{Sm_k})$, we have a distinguished cosimplicial object given by

$$[n] \longmapsto \operatorname{Spec} k[x_0, \dots, x_n]/(x_0 + \cdots + x_n - 1),$$

just like how in classical topology we have $[n] \longmapsto \{(x_i) \in \mathbb{R}^{n+1} : \Sigma x_i = 1, x_i \geq 0\}$. This allows us to "geometrically realize" simplicial sets (viewed as constant simplicial sheaves). In particular, we get a functor $\mathcal{H} \longrightarrow \mathcal{H}(k)$ As an illustration, in $\mathcal{H}(k)$ we can define the *simplicial* 1*-sphere* $S^1_{\mathsf{s}} = \Delta^1/\partial\Delta^1$. If we use the non-simplicial model category, then we must first apply geometric realization, so that $S^1_{\mathsf{s}} = \mathbb{A}^1/\{0 \sim 1\}$. One also has another potential candidate for a sphere, namely $S^1_{\mathsf{t}} = \mathbb{A}^1 - 0$. Indeed, if we work over the complex numbers, this would have the homotopy type of the 1-sphere. For $p \geq q$, define the (p, q)-sphere to be

$$S^{p,q} = \underbrace{S^1_{\mathsf{s}} \wedge \cdots \wedge S^1_{\mathsf{s}}}_{p-q \text{ times}} \wedge \underbrace{S^1_{\mathsf{t}} \wedge \cdots \wedge S^1_{\mathsf{t}}}_{q \text{ times}},$$

and write $\Sigma^{p,q} X = S^{p,q} \wedge X$. There is a functor $\mathcal{H}(\mathbb{C}) \longrightarrow \mathcal{H}$ called Betti realization, which is a generalization of the complex-points functor; through this functor, $S^{p,q}$ goes to S^p, which explains our choice of grading. Lastly, we mention that $\mathbb{P}^1$ is homotopy-equivalent to $S^{2,1}$, since $\mathbb{P}^1 \simeq \mathbb{P}^1/\mathbb{A}^1 \simeq \mathbb{A}^1/(\mathbb{A}^1 - 0)$.

We end this subsection with the following useful fact called Homotopy Purity, which mimics the tubular neighborhood theorem from topology. It is for this reason that the Nisnevich topology is used.

Theorem 2.9. [Voe99, 1.15] *Let $Z \subset X$ be a smooth pair, with normal bundle $N_X Z$. Then, we have a weak equivalence*

$$X/(X - Z) \simeq \mathrm{Th}(N_X Z).$$

2.2. **The stable category.** Now that we have a good homotopy category of smooth schemes $\mathcal{H}(k)$, we may try to create a stable homotopy category by inverting the suspension. In fact, we can make the same definition of spectra as in classical topology: an S^1-spectrum E is a sequence of pointed simplicial sheaves E_n $(n \geq 0)$, together with morphisms $\sigma_n : \mathsf{E}_n \wedge S^1_s \longrightarrow \mathsf{E}_{n+1}$. Morphisms are defined in the usual way, to yield a category $\mathsf{Sp}^{S^1}(k)$.

However, it turns out that this notion is not too useful. Since we have bi-graded spheres $S^{p,q}$, we would like all of these to be reflected in the spectrum structure. In this sense, the true construction of the motivic stable homotopy category will mimic the construction from equivariant stable homotopy theory, where we invert suspensions coming from all representation spheres.

Since $\mathbb{P}^1$ contains both types of spheres, we first define $\mathbb{P}^1$-spectra, following [Mor04, 5.1.1]:

Definition 2.10. A $\mathbb{P}^1$-spectrum E is a sequence of pointed simplicial sheaves E_n, together with structure morphisms

$$\sigma_n : \mathsf{E}_n \wedge \mathbb{P}^1 \longrightarrow \mathsf{E}_{n+1}.$$

Morphisms of $\mathbb{P}^1$-spectra $\mathsf{E} \longrightarrow \mathsf{F}$ are sequences of morphisms $f_n : \mathsf{E}_n \longrightarrow \mathsf{F}_n$ that are compatible with the structure maps in the usual sense. We thus have a category $\mathsf{Sp}^{\mathbb{P}^1}(k)$ of $\mathbb{P}^1$-spectra.

In order to put a model structure on this, we make the following definition of homotopy presheaves $\tilde{\pi}_n(\mathsf{E})_m$ for all $m, n \in \mathbb{Z}$ (note that they are not just groups, like in classical topology):

$$\tilde{\pi}_n(\mathsf{E})_m(U) = \varinjlim_r [S_{\mathsf{s}}^{n+m} \wedge U_+ \wedge (\mathbb{P}^1)^{r-m}, \mathsf{E}_r].$$

The usual proof shows that these are abelian group presheaves. Now we may define the following model structure:

- $\mathcal{W}$ Weak equivalences are maps which induce isomorphisms on all $\tilde{\pi}_n(-)_m$.
- $\mathcal{C}$ Cofibrations are maps $f : \mathsf{E} \longrightarrow \mathsf{F}$ such that $f_0 : \mathsf{E}_0 \longrightarrow \mathsf{F}_0$ is a cofibration, and the morphisms

  $$\mathsf{E}_{n+1} \vee_{\mathsf{E}_n \wedge \mathbb{P}^1} (\mathsf{F}_n \wedge \mathbb{P}^1) \longrightarrow \mathsf{F}_{n+1}$$

  are all cofibrations ($n \geq 0$).

One may indeed check that this is a model structure, so it gives rise to a stable homotopy category $\mathcal{SH}^{\mathbb{P}^1}(k)$. This has all of the nice properties one would expect, such as a triangulated structure given by $\Sigma^{1,0}$, a compatible smash product and an internal Hom, as well as tensoring and cotensoring over $\mathcal{H}(k)$.

Because of the equivalences $\mathbb{P}^1 \simeq \mathrm{Th}(1) \simeq S^{2,1}$, one may form analogous notions of spectra for the other two spaces. One can show that this in fact leads to equivalent homotopy categories:

$$\mathcal{SH}^{\mathbb{P}^1}(k) \simeq \mathcal{SH}^{\mathrm{Th}(1)}(k) \simeq \mathcal{SH}^{S^{2,1}}(k).$$

We hence denote all of these by $\mathcal{SH}(k)$, for short. Different constructions are better suited for different situations, as we can see in the following examples of spectra:

(1) *Suspension spectra.* For any pointed simplicial sheaf X, one can form the suspension spectrum $\Sigma^\infty_{\mathbb{P}^1}(X)$ with n-th space $X \wedge (\mathbb{P}^1)^{\wedge n}$, and the structure maps being the identity. This yields a functor $\mathcal{H}(k) \longrightarrow \mathcal{SH}(k)$.

(2) *Thom spectrum.* Just like in classical topology, one can construct a classifying space $\mathrm{Gr}(n) = \mathrm{BGL}_n$ of n-dimensional planes, with a universal bundle γ_n over it. We also have maps $i_n : \mathrm{Gr}(n) \longrightarrow \mathrm{Gr}(n+1)$ that classify the inclusion $\mathrm{GL}(n) \hookrightarrow \mathrm{GL}(n+1)$; and there is an isomorphism of bundles $i_n^*\gamma_{n+1} \cong \gamma_n \oplus 1$. This yields an isomorphism

$$\mathrm{Th}(i_n^*\gamma_{n+1}) \cong \mathrm{Th}(\gamma_n) \wedge \mathrm{Th}(1),$$

which gives a structure map

$$\mathrm{Th}(\gamma_n) \wedge \mathrm{Th}(1) \cong \mathrm{Th}(i_n^*\gamma_{n+1}) \longrightarrow \mathrm{Th}(\gamma_{n+1}).$$

These together give us a $\mathrm{Th}(1)$-spectrum, denoted MGL, and called the algebraic cobordism spectrum.

(3) *Eilenberg-MacLane spectra.* The naive definition of an Eilenberg-MacLane space in terms of its homotopy groups unfortunately does not work here. The right idea comes instead from the Dold-Thom construction from classical topology. Recall that for a pointed space X, we may consider the free abelian group $\mathbb{Z}[X]$ (appropriately topologized, with zero as the basepoint), and one can show that $\pi_*\mathbb{Z}[X] \cong \tilde{H}_* X$. So $\mathbb{Z}[S^n]$ is a model for $K(\mathbb{Z}, n)$.

The analogous notion here is given by the construction $L[X]$, defined to be the presheaf which sends a connected scheme U to the free abelian group on the set of transfers from U to X, i.e. the set of irreducible closed $W \subset U \times X$ such that the projection $W \longrightarrow U$ is finite and surjective. It can be proven that these are Nisnevich sheaves, see [Voe99, 3.1]. Now we define $K(\mathbb{Z}(n), 2n) = L(\mathbb{A}^n)/L(\mathbb{A}^n - 0)$. We can also define multiplication maps

$$K(\mathbb{Z}(n), 2n) \wedge K(\mathbb{Z}(m), 2m) \longrightarrow K(\mathbb{Z}(n+m), 2n+2m)$$

given by the external product of cycles; and using the standard inclusion $(\mathbb{P}^1)^{\wedge n} \simeq \mathbb{A}^n/(\mathbb{A}^n - 0) \longrightarrow L(\mathbb{A}^n)/L(\mathbb{A}^n - 0)$, we get a ring spectrum $H\mathbb{Z}$. One can also generalize this to have coefficients in a ring, by working with free R-modules instead of free abelian groups; see [Voe99] for more details.

(4) *K-theory spectrum.* Just like in classical topology, one can construct a Bott map $(\mathbb{Z} \times \mathrm{BGL}) \wedge \mathbb{P}^1 \longrightarrow \mathbb{Z} \times \mathrm{BGL}$. Iterating this, one obtains a K-theory spectrum called KGL, which represents algebraic K-theory.

3. The Motivic Adams Spectral Sequence

Recall that the Steenrod Algebra $\mathcal{A}_p$ is the ring $\mathsf{H}\mathbb{F}_p^* \mathsf{H}\mathbb{F}_p$ of operations in mod-p-cohomology, and that its dual $\mathcal{A}_p$ is the ring $(\mathsf{H}\mathbb{F}_p)_* \mathsf{H}\mathbb{F}_p$ of "co-operations." In the classical case, the structure of the dual is originally due to Milnor [Mil58]. His computation has been generalized to the motivic case as follows.

For brevity, we use the following notations for various (co)homology rings; we make no distinction between homological and cohomological notation, since it will be clear from the context.

Notation 3.1.

- M: the motivic (co)homology of a point over $\mathbb{Z}$
- $\mathrm{M}_n, \mathrm{M}_{\mathbb{Q}}$: the motivic (co)homology of a point over $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Q}$ respectively.
- $\mathrm{M}_p^{\wedge}$: the homotopy of the completion of $\mathsf{H}\mathbb{Z}$ at p.

Theorem 3.2. [MVW06, Introduction] *For a strictly Henselian local scheme X over k and an integer n prime to* $\mathrm{char}(k)$*, we have*

$$H^{p,q}(X, \mathbb{Z}/n) = \begin{cases} \mu_n^{\otimes q}(X) & \text{for } p = 0 \\ 0 & \text{for } p \neq 0, \end{cases}$$

with $\mu_n(X)$ the group of n-th roots of unity in X. In particular, $\mathrm{M}_n = \mathbb{Z}/n[\tau]$*, where $|\tau| = (0, -1)$ is some generator of* $\mu_n(\mathrm{Spec}\,\mathbb{C})$.

With this language, we have the following description of the Steenrod Algebra over $\mathbb{C}$.

Theorem 3.3. [Hoy15, Theorem 5.12] *At $p = 2$, the dual motivic Steenrod Algebra takes the form*

$$\mathcal{A}_2^{\mathsf{mot},\vee} = \mathrm{M}_2[\tau_i, \xi_j]/(\tau_i^2 - \tau\xi_{i+1})$$

with $i \geq 0, j \geq 1$, and degrees $|\tau_i| = (2^{i+1}-1, 2^i - 1)$ and $|\xi_j| = (2^{j+1}-2, 2^j-1)$. The comultiplication laws are

$$\Delta(\tau) = \tau \otimes 1$$

$$\Delta(\tau_i) = \tau_i \otimes 1 + 1 \otimes \tau_i + \sum_{k=0}^{i-1} \xi_{i-k}^{2^k} \otimes \tau_k$$

$$\Delta(\xi_i) = \xi_i \otimes 1 + 1 \otimes \xi_i + \sum_{k=1}^{i-1} \xi_{i-k}^{2^k} \otimes \xi_k$$

where $\mathrm{M}_2 = \mathbb{F}_2[\tau]$, with bidegree $|\tau| = (0,-1)$.

At $p > 2$, the dual motivic Steenrod Algebra takes the form

$$\mathcal{A}_p^{\mathsf{mot},\vee} = \mathrm{M}_p[\tau_i, \xi_j]/(\tau_i^2)$$

with $i \geq 0, j \geq 1$, and degrees $|\tau_i| = (2p^i - 1, p^i - 1)$ and $|\xi_j| = (2p^j - 2, p^j - 1)$. The comultiplication laws are the same as above.

There is a Betti realization map from the motivic Steenrod algebra to the classical one, and the symbols ξ_i and τ_j map to the symbols with the same names. The element τ maps to 1 in the case that $p = 2$. From now on, we will drop the mot superscript from the notation, since everything will be assumed to be motivic, unless otherwise specified.

Now that we know what the Steenrod Algebra is, we describe the convergence of the Motivic Adams Spectral Sequence (MASS) in certain good cases. To this end, we must first introduce the notion of cellularity. In classical topology, everything is cellular, in the sense that all spaces that we work with can be given the weak homotopy type of a CW complex. However, in the motivic world, this is not the case anymore (even if we allow bi-graded spheres $S^{p,q}$).

To make this precise, here is the following definition, according to Dugger and Isaksen [DI05] and [DI10]:

Definition 3.4. The class of cellular spectra is the smallest class of spectra that contains $S^{p,q}$ for all $p, q \in \mathbb{Z}$, and is closed under weak equivalences and homotopy colimits.

It turns out that most of the common motivic spectra, such as $\mathsf{H}\mathbb{Z}, \mathsf{KGL}, \mathsf{MGL}$ are cellular; this is proven for instance in [DI05]. Under this assumption, they prove in [DI10] that

Theorem 3.5. [DI10, 7.10] *For a cellular spectrum E, we have a tri-graded spectral sequence*

$$E_2^{s,t,u} = \mathrm{Ext}^{s,t,u}_{\mathcal{A}_p^\vee}(\mathrm{M}_p, \mathsf{H}_\star(X; \mathbb{F}_p)) \Longrightarrow \pi_{t-s,u} X^\wedge_{\mathsf{H}\mathbb{F}_p},$$

with differentials d_r shifting degree by $(r, r-1, 0)$.

Here, the symbol $X^\wedge_{\mathsf{H}\mathbb{F}_p}$ denotes the nilpotent completion, i.e. the homotopy limit of the cosimplicial spectrum

$$E \wedge X \rightrightarrows E^{\wedge 2} \wedge X \overset{\longrightarrow}{\rightrightarrows} E^{\wedge 3} \wedge X \cdots$$

where coface maps are induced by the unit, and codegeneracies are induced by multiplication. For connective spectra, the nilpotent completion has been shown ([Man18, Remark 7.3.5]) to admit a simpler description. Namely, if $\eta : \Sigma^{1,1}\mathsf{S} \longrightarrow \mathsf{S}$

is the motivic Hopf map, coming from the projection $\mathbb{A}^2 - 0 \longrightarrow \mathbb{P}^1$ in the unstable category, then the nilpotent completion coincides with the (p,η)-completion $X^\wedge_{p,\eta} = (X^\wedge_\eta)^\wedge_p$. See [Man18] for a thorough introduction to completions at various elements of $\pi_\star \mathsf{S}$.

Moreover, in the case of MGL, it is known that η acts trivially, i.e. that the unit map $\mathsf{S} \longrightarrow \mathsf{MGL}$ factors through $\mathsf{C}\eta$; see, for instance, [Hoy15, Theorem 3.8]. This means that MGL is already η-local, so we obtain the following result:

Corollary 3.6. *There is a tri-graded spectral sequence*

$$E_2^{s,t,u} = \mathrm{Ext}^{s,t,u}_{\mathcal{A}_p^\vee}(\mathrm{M}_p, \mathsf{H}_\star(\mathsf{MGL}; \mathbb{F}_p)) \Longrightarrow \pi_{t-s,u} \mathsf{MGL}^\wedge_p,$$

with differentials d_r shifting degree by $(r, r-1, 0)$.

Lastly, we briefly explain the motivation for this spectral sequence; the construction is exactly the same as in classical algebraic topology, and is presented in [DI10]. To classify maps $f : \mathsf{E} \longrightarrow \mathsf{F}$ up to homotopy, the most basic idea is to apply the homology functor to it. That is, we have an invariant, which we denote by e_0:

$$[\Sigma^\star \mathsf{E}, \mathsf{F}] \xrightarrow{e_0} \mathrm{Ext}^{0,\star}_{\mathcal{A}_p}(\mathsf{H}_\star \mathsf{E}, \mathsf{H}_\star \mathsf{F}), \tag{3.7}$$

where $\mathsf{H}_\star$ is short-hand notation for $\mathsf{H}_\star(-;\mathbb{F}_p)$ for a fixed prime p. When this vanishes, the long-exact sequence associated to the cofiber sequence $\mathsf{E} \xrightarrow{f} \mathsf{F} \longrightarrow \mathsf{C}f$ expresses $\mathsf{H}_\star \mathsf{C}f$ as an extension of $\mathsf{H}_\star \mathsf{E}[1]$ by $\mathsf{H}_\star \mathsf{F}$. This is called the Hopf invariant, which we denote by e_1:

$$\operatorname{Ker} e_0 \xrightarrow{e_1} \mathrm{Ext}^{1,\star+1}_{\mathcal{A}_p}(\mathsf{H}^\star \mathsf{E}, \mathsf{H}^\star \mathsf{F}). \tag{3.8}$$

These e_0 and e_1 can be generalized to invariants e_s for arbitrary $s \geq 0$, each defined on the kernel of the previous, and with images in a *quotient* of $\mathrm{Ext}^{n,\star+n}$. Indeed, the Adams spectral sequences gives a descending filtration $(A^s)_{s\geq 0}$ on $[\Sigma^\star \mathsf{E}, \mathsf{F}]$, together with maps $e_s : A^s \longrightarrow E^\infty_{s,\star+s}$ such that A^0 is the whole group, and $A^s = \operatorname{Ker} e_{s-1}$ for all $s \geq 1$. So in some sense the Adams spectral sequence generalizes the more primitive invariants e_0 and e_1.

4. Application of MASS to $\mathsf{MGL}_\star$

Our goal, in this section, is to determine as much information about $\mathsf{MGL}_\star$ as possible, using the Motivic Adams Spectral Sequence. We will first prove the following local result:

Theorem 4.1. *The p-complete MGL spectrum has homotopy*

$$\pi_\star \mathsf{MGL}^\wedge_p \cong \mathrm{M}^\wedge_p[b_1, b_2, \ldots]$$

with $|b_i| = (2i, i)$.

This is proven in [OØ14, 2.5] by different means (namely, the slice spectral sequence). We will instead prove this using the Adams spectral sequence. To this end, we must first determine the motivic homology $\mathsf{H}_\star \mathsf{MGL}$ as a comodule over the dual Motivic Steenrod Algebra $\mathcal{A}_p^\vee = \mathsf{H}_\star \mathsf{H}$, where H is short-hand notation for $\mathsf{H}\mathbb{F}_p$, for a fixed prime p.

This has already been done in [Hoy15], but we include a summary of the proof here for the sake of completeness. The result is what one would expect, given the classical case:

Theorem 4.2. [Hoy15, Theorem 6.5] *Let $\mathcal{P} \subset \mathcal{A}_p^\vee$ be the Hopf subalgebra which is multiplicatively generated by the ξ_i over M_p. Then $\mathsf{H}_\star \mathsf{MGL}$ is a co-free $\mathcal{P}$-comodule. More explicitly, there is an isomorphism*

$$\mathsf{H}_\star \mathsf{MGL} \xrightarrow{\sim} \mathcal{P} \otimes \mathbb{F}_p[b_i']$$

of left $\mathcal{A}_p^\vee$-comodule algebras, where $|b_i| = (2i, i)$ and i runs through all positive integers not of the form $p^k - 1$.

The proof mirrors the classical case. First, we must understand the homology of MGL simply as a ring; and to do this, we will use the motivic version of the Thom isomorphism. The cohomological version is a classical result of Voevodsky [Voe03, Proposition 4.3], which comes from the projective bundle formula:

Theorem 4.3. *For any $X \in \mathsf{Sm}_\mathbb{C}$, and any vector bundle ε on X of dimension d, there is an isomorphism*

$$\mathsf{H}^\star X \xrightarrow{\sim} \widetilde{\mathsf{H}}^{\star + d\rho}(\mathrm{Th}\, \varepsilon)$$

given by cupping with a Thom class.

For a homological version of the Thom isomorphism, see the Appendix, Theorem A.8. Applying the Thom isomorphism to each individual Thom space $\mathsf{MGL}_{(n)} = \mathrm{Th}(\gamma_n \longrightarrow \mathrm{BGL}_n)$, and passing to the limit, we obtain an isomorphism

$$\mathsf{H}_\star \mathsf{MGL} \xrightarrow{\sim} \mathsf{H}_\star \mathrm{BGL}.$$

This can be proven to be an isomorphism of algebras, due to the compatibility of the Thom isomorphism with the Whitney sum; see Theorem A.8.

So we have reduced the problem to computing $\mathsf{H}_\star\mathrm{BGL}$. This is covered in detail in [NSØ09, Proposition 6.1], and the result is basically the same as in classical topology. In short, the idea is to inductively use the Gysin sequence associated to the closed immersion $\mathrm{Gr}(n, d) \hookrightarrow \mathrm{Gr}(n+1, d+1)$, and the fact that the complement of this inclusion is a bundle that deformation retracts onto $\mathrm{Gr}(n, d + 1)$. The end result is the following:

Theorem 4.4. [NSØ09, 6.2]

(1) $\mathsf{H}^\star\mathrm{BGL}_n = \mathsf{H}^\star[\![c_1, c_2, \ldots, c_n]\!]$*, with c_i the i-th tautological Chern class*
(2) $\mathsf{H}^\star\mathrm{BGL} = \mathsf{H}^\star[\![c_1, c_2, \ldots]\!]$*, with c_i the i-th tautological Chern class*
(3) $\mathsf{H}_\star\mathrm{BGL} = \mathsf{H}_\star[b_1, b_2, \ldots]$ *with $b_i \in \mathsf{H}_{2i,i}$ dual to c_1^i (by convention, $b_0 = 1$).*

$$\mathbb{P}^\infty_+ \hookrightarrow \mathsf{MGL}_{(1)} \longrightarrow \Sigma^{2,1}\, \mathsf{MGL}, \tag{4.5}$$

will completely determine the coaction on the latter. It follows from Theorem 4.4 above (with $n = 1$) that $\mathsf{H}^\star\mathbb{P}^\infty$ is polynomial over M_p, generated by one element (a universal Chern class $c_1 \in \mathsf{H}^{2,1}\mathbb{P}^\infty$). By duality, $\mathsf{H}_\star\mathbb{P}^\infty$ is generated by dual elements β_i (i.e. β_i is dual to c_1^i).

The Cartan formula allows one to compute the Steenrod operations on c_1^i from the operations on c_1; after dualization, one finds

$$\Delta(\beta_n) = \sum_{m+|\xi^R|=n} a_{m,R}\xi^R \otimes \beta_m, \tag{4.6}$$

where R is any finite sequence $(r_1, r_2, \ldots, r_N)$ of non-negative numbers of sum at most n, and $a_{m,R}$ are the multinomial coefficients

$$a_{n,R} = \frac{n!}{(r_1! \cdot r_2! \cdot \cdots \cdot r_N!) \cdot (n - \Sigma r_i)!}. \tag{4.7}$$

The sum (4.6) is finite, because one must have $|\xi^R| \leq n$, and there are only finitely many monomials $|\xi^R|$ with this property.

See [Hoy15] for more details about this computation. Returning to MGL, we note that the map (4.5) sends each β_i to b_{i+1}. Thus,

$$\Delta(b_n) = \sum_{m+|\xi^R|=n} a_{m+1,R}\xi^R \otimes b_m, \tag{4.8}$$

Since we know Δ on the polynomial generators b_n, we have in fact uniquely determined the comodule algebra structure.

Now, we will show how this is in fact a cofree $\mathcal{P}$-comodule, in the precise sense of Theorem 4.2. First, we see that the coaction factors through the subalgebra $\mathcal{P} \subset \mathcal{A}_p^\vee$; indeed, this is a direct consequence of the formula (4.8), because only ξ's are involved. Also, we note that in (4.8), all of the coefficients involved lie inside $\mathbb{F}_p \subset \mathrm{M}_p$ (so we are not using any elements of M_p of nonzero degrees). So we get a factorization

$$\mathsf{H}_\star \, \mathsf{MGL} \xrightarrow{\psi} \mathcal{P} \otimes \mathbb{F}_p[b_i] \longrightarrow \mathcal{A} \otimes \mathsf{H}_\star \, \mathsf{MGL}\,.$$

This map ψ is not yet an isomorphism, but we claim that it is after we post-compose with the map that kills the b_i with $i = p^k - 1$:

$$\mathsf{H}_\star \, \mathsf{MGL} \xrightarrow{\psi} \mathcal{P} \otimes \mathbb{F}_p[b_i] \longrightarrow\!\!\!\!\rightarrow \mathcal{P} \otimes \mathbb{F}_p[b_i]_{i \neq p^k-1}.$$

One can see that the domain and codomain of this composite map are vector spaces of the same dimension, so it suffices to see that it is surjective. For $i \neq p^k - 1$, the element $1 \otimes b_i$ in the codomain is hit (mod decomposables), by counitality of the coaction:

$$\Delta(b_i) = 1 \otimes b_i + \cdots\,.$$

A bit more subtly, $\xi_k \otimes 1$ is hit (mod decomposables) by b_{p^k-1}. This is due to a number-theoretic fact, which tells us precisely when the multinomial coefficients $a_{m,R}$ vanish or not mod p. One can consult [Hoy15] for more details on this result. Thus, the homology is indeed cofree over $\mathcal{P}$.

Remark 4.9. We also note here that the exact same method applies for the computation of the classical $\mathsf{H}_*\mathsf{MU}$, and that moreover the Betti realization map

$$\mathsf{H}_\star\mathsf{MGL} \longrightarrow \mathsf{H}_*\mathsf{MU}$$

sends the b_i's above to the classical b_i's.

We can now proceed with the Adams spectral sequence. The E_2-term takes the form

$$\begin{aligned} E_2 &= \mathrm{Ext}_{\mathcal{A}_p^\vee}(\mathrm{M}_p, \mathcal{P} \otimes \mathbb{F}_p[x_i]_{i \neq p^k-1}) \\ &= \mathrm{Ext}_{\mathcal{A}_p^\vee}(\mathrm{M}_p, \mathcal{P}) \otimes \mathbb{F}_p[x_i]_{i \neq p^k-1}. \end{aligned}$$

To compute this, we use the base-change theorem for Ext:

Theorem 4.10. [Rav86, A1.1.19] *Let $f : \Gamma \longrightarrow \Sigma$ be a surjective map of graded connected Hopf algebras (with base ring A), such that the inclusion $\Gamma \,\square_\Sigma A \hookrightarrow \Gamma$ is split in the category of A-modules. If M is a right Γ-comodule and N is a left Σ-comodule which is flat over A, then*

$$\mathrm{Ext}_\Gamma(A, \Gamma \,\square_\Sigma N) = \mathrm{Ext}_\Sigma(A, N).$$

We want $\mathcal{P}$ to be $\mathcal{A}_p^\vee \,\square_\Sigma \mathrm{M}_p$, so we define

$$\Sigma := \mathcal{A}_p^\vee /\!\!/ \mathcal{P} = \Lambda_{\mathrm{M}_p}(\tau_0, \tau_1, \ldots),$$

where τ_i turn out to be primitive elements. We now wish to apply the base-change theorem for Ext, with $A = N = \mathrm{M}_p$. To do this, one has to check that $\mathcal{P} \hookrightarrow \mathcal{A}_p^\vee$ is a split monomorphism in the category of M_p-modules. For $p > 2$, this is true because because the relations we are quotienting by to obtain $\mathcal{A}_p^\vee$ have coefficients in $\mathbb{F}_p \subset \mathrm{M}_p$. For $p = 2$, more care is needed: one can show that elements of the form $\tau^I \xi^J$ give an M_2-basis of $\mathcal{A}_p^\vee$, where J is any sequence, and I is any sequence with entries in $\{0, 1\}$. Then $\mathcal{P}$ is spanned by a sub-basis, which proves the claim. Thus,

$$\begin{aligned}\mathrm{Ext}_{\mathcal{A}_p^\vee}(\mathrm{M}_p, \mathcal{P}) &\cong \mathrm{Ext}_\Sigma(\mathrm{M}_p, \mathrm{M}_p)\\ &\cong \mathrm{Ext}_{\Lambda(\tau_i)}(\mathbb{F}_p, \mathbb{F}_p) \otimes \mathrm{M}_p.\end{aligned}$$

It is a standard exercise in homological algebra to shows that Ext over an exterior algebra on primitive generators τ_i is polynomial in generators of the same internal degrees, and of homological degree 1 (one can use, for instance, the Koszul complex). In other words,

$$E_2 = \mathrm{M}_p[b_i]_{i=p^k-1} \otimes_{\mathrm{M}_p} \mathrm{M}_p[b_i]_{i \neq p^k-1} = \mathrm{M}_p[b_i],$$

with $(t-s)$-degrees $|b_i| = (2i, i)$, where $i \geq 0$ (note that, in particular, $i = 0$ is allowed).

We would now like this to collapse, just like in the classical case. For this, we need to show that the E_2-page is concentrated in even $(t-s)$-degrees, which follows at once from the following property that $M_n \simeq \mathbb{Z}/n[\tau]$, $|\tau| = (0, -1)$. So $E_2 = E_\infty$. We must now handle the extension problems, and potential multiplicative problems. For the former, we use the standard trick of looking at how multiplication by p is reflected in the spectral sequence. We claim that if $x \in E_\infty$ is a permanent cycle, representing an element $\tilde{x} \in \mathsf{MGL}^\wedge_{p,\star}$, then $b_0 x$ represents $p\tilde{x}$. This will follow at once from the multiplicative structure of the spectral sequence if we can show that it is true for $\tilde{x} = 1$. To show this, we need

Theorem 4.11. [OØ14, 7.5] *The map $\mathsf{MGL}_{\leq 0} \longrightarrow \mathsf{H}\mathbb{Z}_{\leq 0}$ is an equivalence, under the classical t-structure of Morel.*

In particular, this tells us that $\mathsf{MGL}_{0,0} = \mathbb{Z}$. So there is a filtration on $\mathbb{Z}$ such that the graded quotient is isomorphic to $\mathbb{F}_p$ in each degree. It follows, by induction, that the only possible filtration is the p-adic one. In particular, the element b_0, which lies in $(t-s, u, s)$-grading $(0, 0, 1)$ represents the element p, as desired. Using this now we can solve the extension problems, because at each step we know whether an element is killed by p or not. Once we have done this, we now know that $\mathsf{MGL}^\wedge_{p,\star}$ is a free $\mathbb{Z}_p$-module. Also, since the E_∞ term is free as an M_p-algebra, there are no multiplicative extensions either, and our theorem is proven.

Rationally, the homotopy of MGL is even easier to compute, thanks to the Cisinski-Deglise Theorem [OØ14, Theorem 6.2], which says that the Moore spectrum $\mathsf{S}\mathbb{Q}$ is weakly equivalent to $\mathsf{H}\mathbb{Q}$. In other words, $\pi_\star \mathsf{MGL}_\mathbb{Q}$ is isomorphic to $\mathsf{H}_\star(\mathsf{MGL};\mathbb{Q})$. Just as before, one can use the Thom isomorphism to reduce to BGL, whose homology is known to be polynomial (like in the classical case). So

Theorem 4.12. *The rational* MGL *spectrum has homotopy*

$$\pi_\star \mathsf{MGL}_\mathbb{Q} \cong \mathrm{M}_\mathbb{Q}[b_1, b_2, \ldots]$$

with $|b_i| = (2i, i)$.

In order to compute $\pi_\star \mathsf{MGL}$ globally, we use the arithmetic fracture square of [OØ14, Appendix]:

$$\begin{tikzcd} \mathsf{E} \arrow[r] \arrow[d] & \prod_p \mathsf{E}_p^\wedge \arrow[d, "r"] \\ \mathsf{E}_\mathbb{Q} \arrow[r, "s"] & \left(\prod_p \mathsf{E}_p^\wedge\right)_\mathbb{Q} \end{tikzcd} \tag{4.13}$$

which is a homotopy pullback square associated to any motivic spectrum E. In particular, we can apply this for $\mathsf{H}\mathbb{Z}$ and MGL, and try to compare the two associated long-exact sequences:

$$\cdots \longrightarrow \mathrm{M} \longrightarrow \mathrm{M}_\mathbb{Q} \times \prod_p \mathrm{M}_p^\wedge \xrightarrow{\rho} \left(\prod_p \mathrm{M}_p^\wedge\right) \otimes \mathbb{Q} \longrightarrow \cdots \tag{4.14}$$

$$\cdots \longrightarrow \mathsf{MGL}_\star \longrightarrow \mathrm{M}_\mathbb{Q}[b_i] \times \prod_p \mathrm{M}_p^\wedge[b_i] \xrightarrow{\rho} \left(\prod_p \mathrm{M}_p^\wedge[b_i]\right) \otimes \mathbb{Q} \longrightarrow \cdots \tag{4.15}$$

At a first glance, it would seem plausible that (4.15) is just (4.14) tensored with $\mathbb{Z}[b_i]$; however there are a couple of issues: first of all, infinite products do not commute with taking polynomial rings, and secondly, there is no guarantee that the b_i from $\prod_p \mathsf{MGL}_p^\wedge$ map to the same elements as the b_i from $\mathsf{MGL}_\mathbb{Q}$. In fact, these b_i are not even well-defined; recall that in the MASS we are allowed to pick any lift b_i that represents the corresponding element in the E_∞-page of the spectral sequence.

To solve this problem, we will use formal group laws. As proven by Levine in [LM07], $\mathsf{MGL}^{\rho *}$ is naturally isomorphic to the "geometric" algebraic cobordism, which is a Borel-Moore cohomology functor; in particular, it has a theory of Chern classes, i.e. there exists a universal Chern class $c_1 \in \mathsf{MGL}^\rho\, \mathbb{P}^\infty$, which is represented by the map (4.5). Just like in classical topology, the tensor product of two line bundles is classified by a formal group law F_{MGL}:

$$c_1(\mathcal{L} \otimes \mathcal{M}) = F_{\mathsf{MGL}}(c_1\mathcal{L}, c_1\mathcal{M}).$$

This formal group law in cohomology gives a formal group law in homology, just by reversing the grading ($\mathsf{MGL}_\star = \mathsf{MGL}^{-\star}$). So there is a map $\mathrm{L}_* \longrightarrow \mathsf{MGL}_{\rho *}$ from the Lazard ring that classifies it. By applying Betti realization to the map (4.5), we find that the universal Chern class $c_1 \in \mathsf{MGL}^\rho$ maps to the usual Chern class in MU^2, and hence that the two formal group laws are compatible. Thus, we get a commutative diagram

$$\begin{tikzcd} & \mathsf{MGL}_{\rho *} \arrow[dd, "\sim"] \\ \mathrm{L}_* \arrow[ur, "\sim"] \arrow[dr, "\sim"'] & \\ & \mathsf{MU}_{2*} \end{tikzcd}$$

The bottom map is an isomorphism by Quillen's famous result. In fact, Levine showed in [Lev09] that all maps are isomorphisms, which will be important later in dealing with the e_0-invariant. In [NSØ09], Naumann et. al. show that

Theorem 4.16. *If X is a smooth scheme, the map*

$$\mathsf{H}\mathbb{Q}^\star(X) \otimes \mathrm{L}^* \longrightarrow \mathsf{MGL}^\star_{\mathbb{Q}}(X)$$

is an isomorphism.

In particular, it follows that any set of polynomial generators $x_1, x_2, \dots$ for the Lazard ring will also yield generators of $\mathsf{MGL}_{\mathbb{Q},\star}$ over $\mathrm{M}_{\mathbb{Q}}$. We denote their images in $\mathsf{MGL}_\star$ by the same symbols. With this in mind, we prove the following:

Proposition 4.17. *The elements $x_1, x_2, \dots$ in $\mathsf{MGL}_\star$, of bidegrees $|x_i| = (2i, i)$ have the property that their images in $\pi_\star \mathsf{MGL}_{\mathbb{Q}}$ and $\pi_\star \mathsf{MGL}^\wedge_p$ are polynomial generators (i.e. we may choose b_i in Theorems 4.1 and 4.12 to be the images of x_i).*

Proof. We already know that the x_i map to polynomial generators in $\pi_\star \mathsf{MGL}_{\mathbb{Q}}$ by Theorem 4.16; so we are left with showing the analogous statement for $\pi_\star \mathsf{MGL}^\wedge_p$. To show that x_i plays the role of b_i (modulo decomposables), we must show that x_i lies in the appropriate Adams filtration level (level $s = 0$ if $i \neq p^k - 1$, and level $s = 1$ if $i = p^k - 1$), and that it maps to the right element in the E_∞ page. For the case that $i \neq p^k - 1$, this is straightforward: the map $\mathsf{MGL} \longrightarrow \mathsf{MU}$ induces a homomorphism on the Adams associated graded; and we already know that the x_i in $\mathsf{MGL}_\star$ map to polynomial generators of MU_*. In particular, the e_0-invariant of x_i has a nonzero b_i-coefficient for the motivic spectral sequence, because it does so for the classical spectral sequence as well. This shows that x_i is (up to decomposables, i.e. products of x_j's with $j < i$) a good representative of the permanent cycle b_i.

We now treat the more delicate case $i = p^k - 1$. We know that these have zero e_0-invariant for the classical spectral sequence, and we must first show that the same is true motivically. Concretely, the e_0-invariant is the map $\mathsf{H}_\star \mathsf{S}^{i\rho} \longrightarrow \mathsf{H}_\star \mathsf{MGL}$, which is uniquely determined by the image of the "fundamental class" of the shifted sphere spectrum in $\mathsf{H}_{i\rho} \mathsf{MGL}$. We know that this element maps to zero in $\mathsf{H}_{2i}\mathsf{MU}$, and we must show that it is itself zero. But we know from Levine's theorem that the Betti realization map $\mathsf{H}_{i\rho} \mathsf{MGL} \longrightarrow \mathsf{H}_{2i}\mathsf{MU}$ is an isomorphism, so this implies that the motivic e_0-invariant is zero as well. Now that we know this fact, the e_1-invariant is well-defined; and by the same argument as before, the fact that the classical e_1-invariant has nontrivial b_i-coefficient implies that the motivic e_1-invariant has nontrivial b_i-coefficient as well; so up to decomposables, x_i is a good choice of b_i. □

Now, we may proceed with the arithmetic fracture square. The long-exact sequences (4.14) and (4.15) can be broken up into short-exact sequences

$$0 \longrightarrow \mathrm{Cok}\,\rho_{\mathsf{H}\mathbb{Z}}[1] \longrightarrow \mathsf{H}\mathbb{Z}_\star \longrightarrow \mathrm{Ker}\,\rho_{\mathsf{H}\mathbb{Z}} \longrightarrow 0 \tag{4.18}$$

$$0 \longrightarrow \mathrm{Cok}\,\rho_{\mathsf{MGL}}[1] \longrightarrow \mathsf{MGL}_\star \longrightarrow \mathrm{Ker}\,\rho_{\mathsf{MGL}} \longrightarrow 0, \tag{4.19}$$

where $\rho_{\mathsf{E}} = r - s$ is the difference of the two maps r and s from (4.13):

$$\rho_E : \pi_\star \mathsf{E}_\mathbb{Q} \times \prod_p \pi_\star \mathsf{E}_p^\wedge \longrightarrow \left(\prod_p \pi_\star \mathsf{E}_p^\wedge\right) \otimes \mathbb{Q}.$$

We first note a few properties of the kernel and cokernel of ρ. The first observation is that the cokernel of ρ is both torsion, and divisible. Indeed, it is the quotient of a $\mathbb{Q}$-vector space, so it is divisible, and it is torsion because any element of $\left(\prod_p \pi_\star \mathsf{E}_p^\wedge\right) \otimes \mathbb{Q}$ can be multiplied by an integer to lie in $\prod_p \pi_\star \mathsf{E}_p^\wedge$. Furthermore, divisibility implies injectivity for abelian groups, so the extension splits:

$$\mathsf{H}\mathbb{Z}_\star \simeq \mathrm{Ker}\,\rho_{\mathsf{H}\mathbb{Z}} \oplus \mathrm{Cok}\,\rho_{\mathsf{H}\mathbb{Z}}[1]$$
$$\mathsf{MGL}_\star \simeq \mathrm{Ker}\,\rho_{\mathsf{MGL}} \oplus \mathrm{Cok}\,\rho_{\mathsf{MGL}}[1].$$

If we further use the fact that $\mathrm{Cok}\,\rho$ is torsion, we get the following equalities mod torsion:

$$\mathsf{H}\mathbb{Z}_\star / \mathsf{tors} \simeq \mathrm{Ker}\,\rho_{\mathsf{H}\mathbb{Z}} / \mathsf{tors} \tag{4.20}$$

$$\mathsf{MGL}_\star / \mathsf{tors} \simeq \mathrm{Ker}\,\rho_{\mathsf{MGL}} / \mathsf{tors} \tag{4.21}$$

The kernel of ρ also has a nicer interpretation; namely, it is the fiber-product

$$\begin{tikzcd} \mathrm{Ker}\,\rho_{\mathsf{E}} \arrow[r, dashed] \arrow[d, dashed] & \prod_p \pi_\star \mathsf{E}_p^\wedge \arrow[d] \\ \pi_\star \mathsf{E}_\mathbb{Q} \arrow[r] & \left(\prod_p \pi_\star \mathsf{E}_p^\wedge\right) \otimes \mathbb{Q} \end{tikzcd} \tag{4.22}$$

In particular, it has a ring structure, and in fact the projection maps in (4.18) are ring homomorphisms. So the isomorphisms indicated in (4.20) are ring isomorphisms. Our first goal is to prove the following theorem:

Theorem 4.23. *There is an isomorphism of bi-graded rings*

$$\mathsf{MGL}_\star / \mathsf{tors} \simeq \mathsf{H}\mathbb{Z}_\star / \mathsf{tors}[b_1, b_2, \ldots], \tag{4.24}$$

with $|b_i| = (2i, i)$.

Proof. By applying the mod-torsion functor to (4.22), we get another square, which can be seen to be cartesian as well:

$$\begin{tikzcd} \mathrm{Ker}\,\rho_{\mathsf{E}}/\mathsf{tors} \arrow[r, dashed] \arrow[d, dashed] & \left(\prod_p \pi_\star \mathsf{E}_p^\wedge\right) / \mathsf{tors} \arrow[d] \\ \pi_\star \mathsf{E}_\mathbb{Q} \arrow[r] & \left(\prod_p \pi_\star \mathsf{E}_p^\wedge\right) \otimes \mathbb{Q} \end{tikzcd} \tag{4.25}$$

In short, the argument is as follows: pull-backs are particular cases of kernels, so it suffices to show that if $0 \longrightarrow A \longrightarrow B \longrightarrow C$ is a left-exact sequence with C torsion-free, then mod-torsion preserves its exactness. Because the torsion functor

$\mathsf{tors}(-) := \mathrm{Tor}_1(\mathbb{Q}/\mathbb{Z}, -)$ is left-exact, and C is torsion-free, we get an exact sequence $0 \longrightarrow \mathsf{tors}(A) \longrightarrow \mathsf{tors}(B) \longrightarrow 0$. The claim follows by applying the Snake Lemma to

$$\begin{tikzcd} & \mathsf{tors}(A) \arrow[r] \arrow[d] & \mathsf{tors}(B) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C & \end{tikzcd}$$

As mentioned earlier, we would like the square for MGL to be isomorphic to the square for $\mathsf{H}\mathbb{Z}$, tensored with the polynomial ring $\mathbb{Z}[b_1, b_2, \ldots]$; however, this is not necessarily true, because infinite products do not commute with tensors. The idea is to look at the following composite of two squares:

$$\begin{tikzcd} \mathsf{MGL}_\star/\mathsf{tors} \arrow[r, dashed, "?"'] \arrow[rr, bend left] \arrow[d] & \left(\prod_p \mathrm{M}_p^\wedge\right)[b_i]/\mathsf{tors} \arrow[r] \arrow[d] & \left(\prod_p \mathrm{M}_p^\wedge[b_i]\right)/\mathsf{tors} \arrow[d] \\ \mathrm{M}_{\mathbb{Q}}[b_i] \arrow[r] & \left(\prod_p \mathrm{M}_p^\wedge\right) \otimes \mathbb{Q}[b_i] \arrow[r] & \left(\prod_p \mathrm{M}_p^\wedge[b_i]\right) \otimes \mathbb{Q} \end{tikzcd}$$

The left-most square is the square we wish to prove to be cartesian, and the outer square is the square which we already know to be cartesian. If we show that the right-most square is cartesian, this implies that the left-most one exists and is cartesian as well, by abstract nonsense.

One can check, more generally, that if $A \xrightarrow{i} B$ is any inclusion of torsion-free abelian groups, then

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A \otimes \mathbb{Q} \arrow[r] & B \otimes \mathbb{Q} \end{tikzcd}$$

is cartesian. Briefly, $a/n \in A \otimes \mathbb{Q}$ and $b \in B$ map to the same element in $B \otimes \mathbb{Q}$ if and only if $a/n = b/1$, i.e. $a = nb$. So both a/n and b come from the same element $b = a/n$ in A. This concludes the proof. $\square$

Remark 4.26. In our case, the kernel $\operatorname{Ker} \rho_{\mathsf{MGL}}$ is already torsion-free. This is because the sequence

$$\cdots \longrightarrow \mathrm{M}_{p^3} \longrightarrow \mathrm{M}_{p^2} \longrightarrow \mathrm{M}_p$$

has zero $\lim^1$-term, and the inverse limit $\mathrm{M}_p^\wedge$ is just $\mathbb{Z}_p[\tau]$, which is torsion-free. So, really, we have proven

$$\operatorname{Ker} \rho_{\mathsf{MGL}} \cong \mathsf{H}\mathbb{Z}_\star/\mathsf{tors}[b_1, b_2, \ldots].$$

We would like to further investigate the full structure of $\mathsf{MGL}_\star$. To this end, we introduce the following concept:

Definition 4.27. Let R be an algebra over a commutative ring k, and let I be a k-balanced[6] (R, R)-bimodule. A square-zero extension is a split-exact sequence of k-modules

$$0 \longrightarrow I \longrightarrow A \xrightarrow{\pi} R \longrightarrow 0$$

[6]We recall that if A and B are k-algebras, a k-balanced (A, B)-bimodule M is a module over $A \otimes_k B^{\mathsf{op}}$. In other words, being k-balanced means that $ux = xu$ for all $u \in k, m \in M$.

where A is a k-algebra, π is a map of rings, $I^2 = 0$ in A, and the R-bimodule structure that descends on $I \subset A$ is the one given.

Remark 4.28. Given such a square-zero extension, fix a splitting $A \simeq I \oplus R$ of it (as k-modules), and define $\psi : R \times R \longrightarrow I$ by the identity

$$(0, r_1) \cdot (0, r_2) = (\psi(r_1, r_2), r_1 \cdot r_2)$$

in R. Then the general multiplication formula for two elements of A is given by

$$(a_1, r_1) \cdot (a_2, r_2) = (a_1 r_2 + r_1 a_2 + \psi(r_1, r_2), r_1 r_2).$$

The associativity condition can be translated into the following 2-cocycle condition:

$$\psi(r_1, r_2) \cdot r_3 + \psi(r_1 r_2, r_3) = r_1 \cdot \psi(r_2, r_3) + \psi(r_1, r_2 r_3).$$

One can check that the set of equivalence classes of such extensions is in bijection with the Hochschild cohomology group $\mathrm{HH}^2_k(R, I)$.

We claim that the extensions (4.18) are both square-zero extensions. In this sense, we prove an even more general result, which allows one to interpret the long-exact sequences associated to certain homotopy-cartesian squares as square-zero extensions. For this, we will use the stronger notion of a highly structured ring spectrum, i.e. a monoid in a good category of motivic spectra. For a thorough development of such a category, we refer the reader to Jardine's category of motivic symmetric spectra [Jar00]. It has been shown in [PPR08] that MGL can be given the structure of a motivic symmetric ring spectrum; so, in particular, there is a well-behaved category of MGL-modules.

Proposition 4.29. *Let* $\mathsf{E} \xrightarrow{\phi} \mathsf{F}$ *be a map of motivic ring spectra, whose cofiber in* $\mathcal{SH}(\mathbb{C})$ *is* $\mathbb{Q}$*-local, or equivalently, so that the square*

$$\begin{array}{ccc} \mathsf{E} & \longrightarrow & \mathsf{F} \\ \downarrow & & \downarrow \\ \mathsf{E}_{\mathbb{Q}} & \longrightarrow & \mathsf{F}_{\mathbb{Q}} \end{array} \tag{4.30}$$

is homotopy cartesian. Then $\mathsf{E}_\star$ *is a square-zero extension*

$$0 \longrightarrow I \longrightarrow \mathsf{E}_\star \longrightarrow R \longrightarrow 0,$$

with $R = \pi_\star \mathsf{E}_{\mathbb{Q}} \times_{\pi_\star \mathsf{F}_{\mathbb{Q}}} \pi_\star \mathsf{F}_\star$ *as a ring and* $I = (\mathbb{Q}/\mathbb{Z}) \otimes \operatorname{Cok} \phi_\star$*, with the left and right action coming from left and right multiplication on the module* $\operatorname{Cok} \phi_\star$*.*

Proof. There is an associated long-exact sequence associated to the homotopy cartesian square (4.30):

$$\cdots \longrightarrow \pi_\star \mathsf{E} \longrightarrow \pi_\star \mathsf{F} \times \pi_\star \mathsf{E}_{\mathbb{Q}} \xrightarrow{\rho} \pi_\star \mathsf{F}_{\mathbb{Q}} \longrightarrow \pi_\star \mathsf{E}[1] \longrightarrow \cdots .$$

The map ρ is in fact a map of rings, and the connecting homomorphism we claim is a map of $\mathsf{E}_\star$-modules. To see this, recall that a homotopy cartesian square gives a diagram with exact rows

$$\begin{array}{ccccccc} \mathsf{C}[-1] & \longrightarrow & \mathsf{E} & \longrightarrow & \mathsf{F} & \longrightarrow & \mathsf{C} \\ \| & & \downarrow & & \downarrow & & \| \\ \mathsf{C}[-1] & \longrightarrow & \mathsf{E}_{\mathbb{Q}} & \longrightarrow & \mathsf{F}_{\mathbb{Q}} & \longrightarrow & \mathsf{C} \end{array}$$

where $\mathsf{F}_{\mathbb{Q}} \longrightarrow \mathsf{C} \longrightarrow \mathsf{E}[1]$ is the connecting homomorphism. Since $\mathsf{E}, \mathsf{F}, \mathsf{E}_{\mathbb{Q}}$ and $\mathsf{F}_{\mathbb{Q}}$ are all E-bimodules in a good category of spectra, then C is also an E-bimodule, with all maps above being bimodule maps. Thus, the connecting homomorphism is a bimodule homomorphism as well. So we have an extension of rings

$$0 \longrightarrow \operatorname{Cok}\rho[1] \longrightarrow \mathsf{E}_\star \longrightarrow \operatorname{Ker}\rho \longrightarrow 0.$$

Here, $\operatorname{Ker}\rho$ can be interpreted as the fiber-product $R = \mathsf{E}_{\mathbb{Q},\star} \times_{\mathsf{F}_{\mathbb{Q},\star}} \mathsf{F}_\star$; and the surjection is in fact a ring map. The ideal $\operatorname{Cok}\rho[1]$ can be rewritten as $(\mathbb{Q}/\mathbb{Z}) \otimes \operatorname{Cok}\phi_\star$. Indeed, we can apply the following algebraic fact:

$$\frac{A \otimes B}{A' \otimes B + A \otimes B'} \simeq A/A' \otimes B/B', \quad \forall A' \subset A, B' \subset B \text{ abelian groups},$$

with $A = F_\star, A' = \operatorname{Im}\phi_\star, B = \mathbb{Q}, B' = \mathbb{Z}$. So it remains to prove that this ideal is square-zero, and that the left and right actions are correct.

That this is square-zero comes from the fact that $\mathbb{Q}/\mathbb{Z}$ is both torsion and divisible. So if $a, b \in (\mathbb{Q}/\mathbb{Z}) \otimes \operatorname{Cok}\phi_\star$, then there exists $n \in \mathbb{Z} - 0$ such that $na = 0$, and so $ab = a \cdot n \cdot \frac{b}{n} = 0$. The left and right actions come from the fact that the connecting homomorphism is an E-module map. □

Corollary 4.31. *The extensions (4.18) are square-zero extensions, where the action of* $\operatorname{Ker}\rho$ *on* $\operatorname{Cok}\rho[1]$ *is the natural one coming from the map of rings* $\operatorname{Ker}\rho \longrightarrow \pi_\star\left(\prod_p \mathsf{E}\right) \otimes \mathbb{Q}$ *(where* $\mathsf{E} = \mathsf{H}\mathbb{Z}$ *or* MGL *respectively).*

Remark 4.32. We can reduce the search for the Hochschild 2-cocycle to some extent in our case, since we are dealing with polynomial rings. Indeed, consider the slightly more general scenario of a square-zero extension

$$0 \longrightarrow I \longrightarrow E \xrightarrow{\pi} R[x_1, x_2, \ldots] \longrightarrow 0.$$

We claim that the equivalence class of this extension is uniquely determined by the equivalence class of the smaller extension

$$0 \longrightarrow I \longrightarrow \pi^{-1}R \longrightarrow R \longrightarrow 0.$$

Indeed, choose elements $x_i \in E$ that map to the x_i in R, and let $s : R[x_i]_{i \geq 1} \longrightarrow E$ be a section of π in the category of abelian groups. We may construct a new section

$$\tilde{s}\left(\sum r_I x^I\right) := \sum s(r_I) x^I,$$

which is in fact a $\mathbb{Z}[x_1, x_2, \ldots]$-module map. The cocycle ψ induced by this new splitting is uniquely determined by its values on R, because:

$$\begin{aligned} \psi(r_I x^I, r_J x^J) &= \tilde{s}(r_I r_J x^{I+J}) - \tilde{s}(r_I x^I)\tilde{s}(r_J x^J) \\ &= [s(r_I r_J) - s(r_I)s(r_J)]\, x^{I+J} \\ &= \psi(r_I, r_J) x^{I+J}. \end{aligned}$$

Thus, we have restricted our search to a cocycle

$$\psi \in \mathrm{HH}^2(R, I).$$

We do have some nontrivial information due to the fact that $\mathsf{H}\mathbb{Z} = \mathsf{MGL}/(x_1, x_2, \ldots)$ proven in [Hu05]. That is, it follows that the image of the cocycle above in $\mathrm{HH}^2(\mathrm{M}/\mathsf{tors}, \operatorname{Cok}\rho_{\mathsf{H}\mathbb{Z}}[1])$ represents the extension $\mathsf{H}\mathbb{Z}$. However, we have not succeeded in finding any further information on this cocycle.

Appendix A. Homological Thom Isomorphism

In this section, we will show how to use Spanier-Whitehead duality (i.e. categorical duality in $(\mathcal{SH}, \wedge, \mathsf{S})$) to deduce the homological version of the Thom isomorphism, from the known cohomological version. To do this, we will need to have some control over what happens to Thom spaces under Spanier-Whitehead duality. If the base is projective, this has been done in [Hu05].

Before we introduce the results we need from [Hu05], we will introduce the necessary terminology. For any *projective* scheme $X \in \mathsf{Sm}_k$, there exists a canonical "affinization" procedure (see Section 3 of [Hu05]) which produces an affine scheme $U(X)$, together with an affine n-space bundle map $U(X) \longrightarrow X$,[7] which in particular is a weak equivalence. If ξ is a vector bundle on a scheme X, we will use the short-hand notation X^ξ for the Thom space of ξ. If ξ is a *virtual* vector bundle on an *affine* X, then one can find large enough n such that $\xi \oplus 1^n \cong \zeta$ is an actual bundle, and we may define

$$X^\xi := \Sigma^{-n\rho} X^\zeta$$

in the stable homotopy category. This visibly does not depend on the choice of n, since $X^{\zeta \oplus 1} \simeq \Sigma^\rho X^\zeta$. In case X is not affine, we may pass to the affinization and apply the same construction as above. In what follows, we will define the virtual normal bundle ν_X of a projective scheme X as $-\tau_X$, where τ_X is the tangent bundle. We drop the subscripts X when the base is understood from context. With this language, we have

Theorem A.1. [Hu05, Remark 1] *If X is a projective smooth scheme in Sm_k, the Spanier-Whitehead dual of X_+ is X^ν. More generally, if ξ is a vector bundle on X, then the Spanier-Whitehead dual of X^ξ is $X^{\nu-\xi}$.*

The idea for constructing a homological Thom isomorphism is that if we have an oriented ring spectrum E,[8] we let n be the dimension of X, and d be the dimension of ξ, then we have a chain of isomorphisms

$$\widetilde{\mathsf{E}}_\star X^\xi \overset{\mathrm{SW}}{\cong} \widetilde{\mathsf{E}}^{-\star} X^{\nu-\xi} \overset{\mathrm{Th}}{\cong} \widetilde{\mathsf{E}}^{-\star+d\rho} X^\nu \overset{\mathrm{SW}}{\cong} \mathsf{E}_{\star-d\rho} X, \tag{A.2}$$

where SW means Spanier-Whitehead duality, and Th represents the relative version of the cohomological Thom isomorphism, which can itself be written as the composite of two classical Thom isomorphisms:

$$\widetilde{\mathsf{E}}^{-\star} X^{\nu-\xi} \cong \widetilde{\mathsf{E}}^{-\star+(d+n)\rho} X_+ \cong \widetilde{\mathsf{E}}^{-\star+d\rho} X^\nu$$

While this description does give *some* isomorphism that looks like the classical Thom isomorphism, it is not sufficient for our purposes. Indeed, one must explain how an explicit choice of Thom class for ξ leads to a choice of Thom isomorphism, how this isomorphism behaves under Whitney sums of bundles (i.e. the multiplicative structure), and also why it coincides with the usual Thom isomorphism when we pass to Betti realization.

In a certain sense, we will show that this really is *the* Thom isomorphism, but without knowing a priori that is an isomorphism. Indeed, recall from classical topology that there is a purely spectral construction of the Thom isomorphism as follows:

[7] This is not necessarily a vector bundle, but only a fiber bundle with contractible fiber

[8] An orientation on a motivic ring spectrum E is a class $x \in \mathsf{E}^\rho \mathbb{P}^\infty$ which restricts to the canonical generator of $\widetilde{\mathsf{E}}^\rho \mathbb{P}^1 \simeq \mathsf{E}^0$.

if $X^\xi \xrightarrow{\theta} \Sigma^n \mathsf{E}$ is a Thom class, then the image under the Thom isomorphism of a cohomology class $X_+ \xrightarrow{x} \Sigma^* \mathsf{E}$ is given by

$$X^\xi \xrightarrow{\mathrm{Th}\Delta} X^\xi \wedge X_+ \xrightarrow{\theta \wedge x} \Sigma^n \mathsf{E} \wedge \Sigma^* \mathsf{E} \longrightarrow \Sigma^{*+n} \mathsf{E},$$

and similarly if $\Sigma^* \mathsf{S} \longrightarrow X^\xi \wedge \mathsf{E}$ is a homology class, then its image under the Thom isomorphism is

$$\Sigma^* \mathsf{S} \longrightarrow X^\xi \wedge \mathsf{E} \xrightarrow{\mathrm{Th}\Delta} X_+ \wedge X^\xi \wedge \mathsf{E} \longrightarrow X_+ \wedge \Sigma^n \mathsf{E} \wedge \mathsf{E} \longrightarrow X_+ \wedge \Sigma^n \mathsf{E}. \tag{A.3}$$

These definitions make sense motivically as well, even if we can't show that they give isomorphisms. We will show that (A.2) and (A.3) are equal, and in particular are isomorphisms.

We begin by giving a better description of the relative Thom isomorphism used in (A.2). So far, this seems to rely on the choice of two Thom classes, one for ν and one for $\nu - \xi$. We show that one can in fact make this more canonical, so that it only depends on a Thom class for ξ.

Definition A.4. If ε, η are vector bundles on X, then the generalized Thom diagonal

$$X^{\varepsilon \oplus \eta} \xrightarrow{\mathrm{Th}\Delta} X^\varepsilon \wedge X^\eta$$

is defined as the Thom-ification of the following map of bundles

$$\begin{tikzcd} E(\varepsilon \oplus \eta) \arrow[r] \arrow[d] & E(\varepsilon) \times E(\eta) \arrow[d] \\ X \arrow[r, "\Delta"] & X \times X \end{tikzcd}$$

If one of the two bundles is trivial, we recover the usual Thom diagonal.

Proposition A.5. *(1) The Thom diagonal is associative:*

$$\begin{tikzcd} X^{\varepsilon \oplus \eta \oplus \chi} \arrow[r] \arrow[d] & X^{\varepsilon \oplus \eta} \wedge X^\chi \arrow[d] \\ X^\varepsilon \wedge X^{\eta \oplus \chi} \arrow[r] & X^\varepsilon \wedge X^\eta \wedge X^\chi \end{tikzcd}$$

(2) If $X^\varepsilon \longrightarrow \Sigma^{n\rho} \mathsf{E}$ and $X^\eta \longrightarrow \Sigma^{m\rho} \mathsf{E}$ are Thom classes, then the "external sum"

$$X^{\varepsilon \oplus \eta} \xrightarrow{\mathrm{Th}\Delta} X^\varepsilon \wedge X^\eta \longrightarrow \Sigma^{n\rho} \mathsf{E} \wedge \Sigma^{m\rho} \mathsf{E} \xrightarrow{\mu} \Sigma^{(m+n)\rho} \mathsf{E}$$

is also a Thom class.

Proof. Similar to classical algebraic topology. □

Proposition A.6. *Given Thom classes on $X^{\nu - \xi}$ and X^ξ, (2) above also yields a Thom class on X^ν. The relative Thom isomorphism*

$$\widetilde{\mathsf{E}}^\star X^{\nu - \xi} \cong \widetilde{\mathsf{E}}^{\star + (d+n)\rho} X_+ \cong \widetilde{\mathsf{E}}^{\star + d\rho} X^\nu$$

does not depend on the choice of Thom class for $\nu - \xi$. Moreover, it can be expressed as follows: given any map $X^{\nu - \xi} \longrightarrow \Sigma^\star \mathsf{E}$, its image under the composite isomorphism above is given by

$$X^\nu \xrightarrow{\mathrm{Th}\Delta} X^{\nu - \xi} \wedge X^\xi \longrightarrow \Sigma^\star \mathsf{E} \wedge \Sigma^{d\rho} \mathsf{E} \xrightarrow{\mu} \Sigma^{\star + d\rho} \mathsf{E}.$$

Proof. Follows from the associativity of the Thom diagonal, and of the multiplication on E. □

Corollary A.7. *The homological Thom isomorphism defined in (A.2) depends only on a choice of Thom class for ξ.*

Now, we are ready to show the equivalence between the classical Thom isomorphism (A.3), and the new proposed Thom isomorphism (A.2). More precisely, we prove the commutativity of the diagram

$$\begin{tikzcd} \widetilde{\mathsf{E}}^{-\star}X^{\nu-\xi} \arrow[r, "\mathrm{SW}"] \arrow[d, "\mathrm{Th}"] & \widetilde{\mathsf{E}}_{\star}X^{\xi} \arrow[d, "(A.3)"] \\ \widetilde{\mathsf{E}}^{-\star+d\rho}X^{\nu} \arrow[r, "\mathrm{SW}"] & \widetilde{\mathsf{E}}_{\star-d\rho}X_{+} \end{tikzcd}$$

This will prove our claim, because all arrows above with the exception of (A.3) are known to be isomorphisms.

For clarity, we will assume grading to be implicit to the maps we write down; so instead of writing $\Sigma^n X \longrightarrow Y$, we will just write $X \longrightarrow Y$ and implicitly understand that this maps decreases grading by n. Given an element $x : X^{\nu-\xi} \longrightarrow \mathsf{E}$ from the top-left corner of the diagram, we must compute its images under the two composites, and show they are equal. If we first go right, and then down, we get the map

$$\mathsf{S} \xrightarrow{\eta} X^{\nu-\xi} \wedge X^{\xi} \xrightarrow{x\wedge\mathrm{Th}\Delta} \mathsf{E} \wedge X^{\xi} \wedge X_{+} \xrightarrow{\mathsf{E}\wedge\theta\wedge X_{+}} \mathsf{E} \wedge \mathsf{E} \wedge X_{+} \xrightarrow{\mu\wedge X_{+}} \mathsf{E} \wedge X_{+}$$

and if we first go down, and then right, we get

$$\mathsf{S} \xrightarrow{\eta} X^{\nu} \wedge X_{+} \xrightarrow{\mathrm{Th}\Delta\wedge X_{+}} X^{\nu-\xi} \wedge X^{\xi} \wedge X_{+} \xrightarrow{x\wedge\theta\wedge X_{+}} \mathsf{E} \wedge \mathsf{E} \wedge X_{+} \xrightarrow{\mu\wedge X_{+}} \mathsf{E} \wedge X_{+}$$

where η represents the unit of the Spanier-Whitehead duality, and θ is the chosen Thom class. A moment's thought shows that this reduces to showing that the following square is commutative:

$$\begin{tikzcd} \mathsf{S} \arrow[r, "\eta"] \arrow[d, "\eta"] & X^{\nu-\xi} \wedge X^{\xi} \arrow[d, "\mathrm{Th}\Delta"] \\ X^{\nu} \wedge X_{+} \arrow[r, "\mathrm{Th}\Delta"] & X^{\nu-\xi} \wedge X^{\xi} \wedge X_{+} \end{tikzcd}$$

This follows at once from the associativity of the Thom diagonal, and an explicit description of η from [Hu05, Theorem A.1], i.e. that η is the composite of the collapse map $\mathsf{S} \longrightarrow X^{\nu}$ dual to $X_{+} \twoheadrightarrow S^0$, and the Thom diagonal $X^{\nu} \longrightarrow X^{\nu-\xi} \wedge X^{\xi}$.

The desired properties of the homological Thom isomorphism now follow just as in classical algebraic topology, since we have expressed it on the level of spectra in the same way. To sum up our conclusions, we have the following theorem:

Theorem A.8. *Let E be an oriented ring spectrum, let X be a smooth projective scheme, and let ξ be a vector bundle of rank d on it. Then any choice of Thom class $X^{\xi} \longrightarrow \Sigma^{d\rho}\mathsf{E}$ gives rise to a Thom isomorphism map*

$$\widetilde{\mathsf{E}}_{\star+d\rho}X^{\xi} \xrightarrow{\sim} \mathsf{E}_{\star}X$$

obtained via the usual construction (A.3), or by (A.2). Moreover, this satisfies the following properties:

(1) The Betti realization of this is the usual Thom isomorphism.

(2) (Naturality) If $X \xrightarrow{f} Y$ *is a map and* ξ *is a bundle on* Y*, then there is a commutative square*

$$\begin{tikzcd} \widetilde{\mathsf{E}}_{\star+d\rho}X^{f^*\xi} \arrow[r] \arrow[d, "\mathrm{Th}\sim"] & \widetilde{\mathsf{E}}_{\star+d\rho}Y^{\xi} \arrow[d, "\mathrm{Th}\sim"] \\ \mathsf{E}_\star X \arrow[r] & \mathsf{E}_\star Y \end{tikzcd}$$

(3) (Products) If ε *and* η *are bundles on* X, Y *of ranks* n *and* m*, and* $\varepsilon \times \eta$ *is the exterior Whitney sum over* $X \times Y$*, then there is a commutative square*

$$\begin{tikzcd} \widetilde{\mathsf{E}}_{\star+n\rho}X^{\varepsilon} \otimes \widetilde{\mathsf{E}}_{\star+m\rho}Y^{\eta} \arrow[r, "\smile"] \arrow[d, "\mathrm{Th}\sim"] & \widetilde{\mathsf{E}}_{\star+(m+n)\rho}(X \times Y)^{\varepsilon\times\eta} \arrow[d, "\mathrm{Th}\sim"] \\ \mathsf{E}_\star X \otimes \mathsf{E}_\star Y \arrow[r, "\smile"] & \mathsf{E}_\star(X \times Y) \end{tikzcd}$$

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