THE IMAGINARY-TIME FEYNMAN PATH INTEGRAL

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Abstract. The Feynman path integral, which involves the integral of an oscillating complex exponential over a space of paths, is difficult to make rigorous mathematical sense of in many physical situations of interest. In this paper, we consider a special kind of path integral known as the imaginary-time path integral. To make sense of this integral, we begin by constructing the Wiener measure on an n-dimensional Wiener space of paths and then state a helpful theorem characterizing integration with respect to this measure. We then introduce basic operator semigroup theory in the context of its relevance to evolution equations which describe physical systems. Finally, we give a proof of the Trotter product formula and use it to prove a version of the Feynman-Kac formula for potential functions that are continuous and of compact support; this gives rigorous meaning to the imaginary-time Feynman path integral by connecting the idea of an integral over a space of paths to a semigroup known as the heat semigroup.

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1. Introduction

The state of a quantum mechanical particle at time t can be described by a complex-valued wave function $\psi(x,t) \in L^2(\mathbb{R}^d)$. We interpret $\psi(x,t)$ as a *probability amplitude* so that the quantity $|\psi(x,t)|^2$ gives the probability density of the particle being at a position $x \in \mathbb{R}^d$ at time t. We can then calculate the probability that the particle is in some Borel set A in $\mathbb{R}^d$ at time t with the integral

$$\int_A |\psi(x,t)|^2 \, dx.$$

If our particle starts off in some initial state $\psi(x,0)$, a natural question to ask is how the state of our particle evolves in time. Feynman's path integral formulation of quantum

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mechanics is one way to answer this question. Feynman began by first considering the following simpler question.

Question 1.1. Suppose it is given that a particle starts at position x_0 at time 0. Then what is the probability amplitude $K(x, t; x_0, 0)$ that the particle ends up at position x at time $t > 0$?

Let $\phi[\omega(s)]$ be the probability amplitude that the particle takes the path $\omega : [0, t] \to \mathbb{R}^d$ to get from x_0 to x, where the path ω is continuous and $\omega(0) = x_0$ and $\omega(t) = x$. Then we can write the probability amplitude $K(\cdot)$ as just the sum of the probability amplitudes $\phi[\cdot]$ along every path $\omega(s)$ connecting x_0 and x. Feynman showed that the probability amplitude that a particle takes a path $\omega(s)$ from x_0 to x is given by the complex exponential

$$e^{(i/\hbar)S[\omega(s)]},$$

where $\hbar$ is the reduced Planck constant and $S[\omega(s)]$ is referred to as the *action* along the path $\omega(s)$ and is given by

$$S[\omega(s)] = \int_0^t \frac{1}{2}m\left(\frac{d\omega(s)}{ds}\right)^2 - V(\omega(s))\,ds,$$

where m is the mass of the particle, the term $\frac{1}{2}m\left(\frac{d\omega(s)}{ds}\right)^2$ is the kinetic energy of the particle along the path, and $V(\cdot)$ is the potential in which the particle is moving. Hence, we can write $K(\cdot)$ heuristically as an integral over all paths $\omega(s)$,

$$K(x, t; x_0, 0) = \int_{\text{all } \omega(s) \text{ connecting } x_0 \text{ and } x} e^{(i/\hbar)S[\omega(s)]}\,\mathcal{D}\omega,$$

where the symbol $\mathcal{D}\omega$ is used to refer to an integral over all paths. Notice that the action $S[\omega(s)]$ is symmetric in the sense that if we were to instead consider the reverse path $\omega^*(s)$ where $\omega^*(0) = x$ and $\omega^*(t) = x_0$ then $S[\omega^*(s)]$ is equal to $S[\omega(s)]$. With this observation, we can describe the time evolution of the state of the particle by the (heuristic) equation

$$\psi(x, t) = \int_{\text{all } \omega \text{ with } \omega(0) = x} e^{(i/\hbar)S[\omega(s)]}\psi(\omega(t), 0)\,\mathcal{D}\omega, \tag{1.1}$$

where each $\omega : [0, t] \to \mathbb{R}^d$ is continuous and satisfies the condition $\omega(0) = x$. Notice that $x_0 \coloneqq \omega(t)$ can take on any value in $\mathbb{R}^d$. The complex exponential in the integrand of (1.1) gives us the probability amplitude that the particle goes from position x_0 at time 0 to position x at time t while the wave function $\psi(\omega(t), 0)$ gives the probability amplitude that the particle was at x_0 in the first place. Their product gives the probability amplitude that the particle is at position x at time t given that it was at position $x_0 = \omega(t)$ at time 0. The integral then sums over all possible initial positions x_0—the particle could have been anywhere at time 0—and this yields the state $\psi(x, t)$ of the particle at time t.

The more conventional approach to quantum mechanics describes the time evolution of the state of a particle by the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t}\psi(x, t) = H\psi(x, t),\ \psi(x, 0) = \psi(x), \tag{1.2}$$

where we have the operator $H \coloneqq -\frac{\hbar^2}{2m}\Delta + V(x)$ with $\Delta \coloneqq \sum_{i=1}^d \frac{\partial^2}{\partial x_j^2}$ denoting the Laplacian in $\mathbb{R}^d$. It can be shown via a physical derivation that the Schrödinger equation can be recovered from Feynman's path integral [1]. If we solve (1.2) purely formally, we can write the solution as

$$\psi(x,t) = e^{-(i/\hbar)Ht}\psi(x). \tag{1.3}$$

If we combine (1.3) with (1.1), we obtain

$$\psi(x,t) = e^{-(i/\hbar)Ht}\psi(x) = \int_{\text{over all paths } \omega} e^{(i/\hbar)S[\omega(s)]}\psi(\omega(t),0)\,\mathcal{D}\omega, \tag{1.4}$$

which connects the time evolution in the Schrödinger approach to the time evolution in the Feynman approach.

From a mathematical perspective now, we would like to determine what exactly it means to integrate over a space of paths. Specifically, we want to give rigorous meaning to the integral in (1.1). To do this, we might try to construct a measure on the space of paths and then use the theory of Lebesgue integration to make sense of the path integral. Unfortunately, it can be proved that there is no such measure that allows us to directly make sense of Feynman's path integral [2]. However, in 1947, when attending a talk by Feynman, Mark Kac realized that if instead we pass to imaginary time by making the substitution $s = -is$, then the Feynman path integral in (1.1) becomes

$$\mathrm{FP}_{\mathrm{Im}}(\omega) \coloneqq \int_{\text{all } \omega \text{ with } \omega(0)\,=\,x} \exp\left(-\int_0^t \frac{1}{2}\left(\frac{d\omega(s)}{ds}\right)^2 - V(\omega(s))\,ds\right)\psi(\omega(t),0)\,\mathcal{D}\omega,$$

where for mathematical simplicity we have set all physical constants $\hbar, m$ to unity. The idea is that we can then interpret the expression

$$\exp\left(-\int_0^t \frac{1}{2}\left(\frac{d\omega(s)}{ds}\right)^2 ds\right)\mathcal{D}\omega$$

as the *Wiener measure* on the space of continuous paths. Similarly, in the Schrödinger approach, when we pass to imaginary time and set all physical constants to unity, the Schrödinger equation becomes a heat equation with interaction term

$$\frac{\partial}{\partial t}\psi(x,t) = -H\psi(x,t),\ \psi(x,0) = \psi(x).$$

This equation has the formal solution $\psi(x,t) = e^{-tH}\psi(x)$ and so we can establish the imaginary-time analogue of (1.4) as

$$\psi(x,t) = e^{-tH}\psi(x) = \mathrm{FP}_{\mathrm{Im}}(\omega). \tag{1.5}$$

This is the Feynman-Kac formula and establishing it is the main result of this paper. We begin in Section 2 by constructing the Wiener measure and then giving a helpful theorem that describes what integration with respect to this measure looks like. Next, in Section 3, we introduce the basics of operator semigroup theory so that we have the tools to make rigorous sense of the expression e^{-tH} which characterizes the solution to the heat equation (1.5). In Section 4, we make use of operator semigroup theory to prove the Trotter product formula which plays a crucial role in our proof of the Feynman-Kac formula. Finally, in Section 5, we prove the Feynman-Kac formula and briefly mention how our approach in the imaginary-time case can be rigorously extended to the "real-time" Feynman path integral in (1.1). In this paper, we assume that the reader is familiar with basic measure theory and functional analysis.

2. Construction of the Wiener Measure

We want to make rigorous sense of an integral over a space of paths. To accomplish this, we will first construct a measure on the space of all paths known as the Wiener measure and then define integration with respect to this measure.

We begin by defining precisely what we mean by the space of all paths.

Definition 2.1. The d-dimensional *Wiener space* $\mathcal{C}_0^{t,d}$ is the space of all continuous paths $\omega : [0,t] \to \mathbb{R}^d$ such that $\omega(0) = 0$, endowed with the supremum norm given by

$$\|\omega\|_{\mathcal{C}_0^{t,d}} = \sup_{s\in[0,t]} \|\omega(s)\|$$

where $\|\omega(s)\|$ is the standard Euclidean norm on $\mathbb{R}^d$. To simplify notation, we shall write $\mathcal{C}_0^t$ to refer to one-dimensional Wiener space.

Proposition 2.2. *Wiener space is a separable Banach space.*

Proof. We prove separability only and refer the reader to [3, Chapter 3] for the fact that Wiener space is a Banach space.

Each $\omega \in \mathcal{C}_0^{t,d}$ is a function from $\mathbb{R}$ to $\mathbb{R}^d$ so we can write it as

$$\omega(s) = (\omega_1(s), \omega_2(s), \ldots, \omega_d(s)).$$

The Weierstrass approximation theorem [4, Theorem 20.41] tells us that polynomials are dense in $\mathcal{C}_0^t$ since $[0,t]$ is a compact interval. Hence, the functions of the form

$$p(s) = (p_1(s), p_2(s), \ldots, p_n(s)),$$

where each p_j is a polynomial, are dense in $\mathcal{C}_0^{t,d}$. But the polynomials with rational coefficients are a countable dense subset of the polynomials, and so this proves separability. $\square$

Our goal now is to construct a measure on the measurable space $\big(\mathcal{C}_0^{t,d}, \mathcal{B}(\mathcal{C}_0^{t,d})\big)$, where $\mathcal{B}(\mathcal{C}_0^{t,d})$ is the Borel σ-algebra of Wiener space. We shall do this by defining the measure on a special class of sets in $\mathcal{C}_0^{t,d}$ known as the *cylinder sets* and then using the Carathéodory extension theorem to extend the measure to $\mathcal{B}(\mathcal{C}_0^{t,d})$. Our overall approach is guided by that of Johnson and Lapidus's construction of the Wiener measure on one-dimensional Wiener space $\mathcal{C}_0^t$ [3]. However, we define the cylinder sets in a slightly different manner which allows us to untroublesomely extend the construction to d-dimensional Wiener space $\mathcal{C}_0^{t,d}$.

Definition 2.3. Let $A_1, A_2, \ldots, A_n$ be Borel sets in $\mathbb{R}^d$, that is, sets in the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$ of d-dimensional Euclidean space. Also, let $s_1, s_2, \ldots, s_n$ be a sequence of times satisfying the inequality

$$0 < s_1 < s_2 < \cdots < s_n \leq t.$$

Then the *cylinder set* $I(s_1, \ldots, s_n; A_1, \ldots, A_n)$ is the set consisting of all paths ω in $\mathcal{C}_0^{t,d}$ such that $\omega(s_j) \in A_j$ for all $j = 1, 2, \ldots, n$. We shall refer to the times s_j as *restriction times* and the Borel sets A_j as their corresponding *restriction windows*. Figure 1 provides an illustration of what a cylinder set looks like.

Notation 2.4. The cylinder sets are of the form

$$\{\omega \in \mathcal{C}_0^{t,d} : \omega(s_j) \in A_j \text{ for } j = 1, 2, \dots n\}.$$

We denote the collection of sets of this form by $\mathcal{I}$. That is, $\mathcal{I}$ is the collection of all cylinder sets.

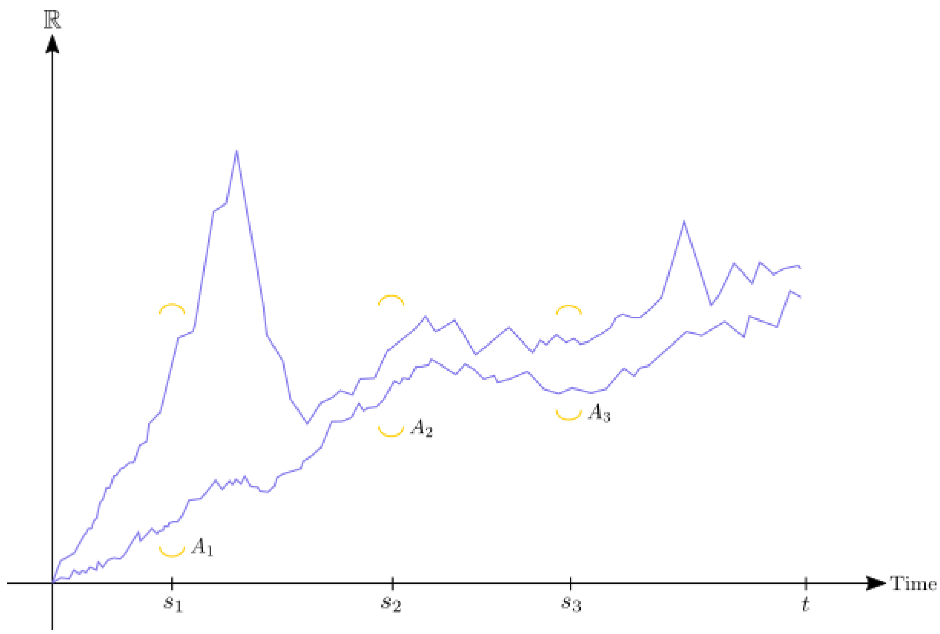

FIGURE 1. Paths in $\mathcal{C}_0^t$ that are a part of the cylinder set $I(s_1, s_2, s_3; A_1, A_2, A_3)$ passing through the Borel sets A_1, A_2, and A_3 at times s_1, s_2, and s_3 respectively.

We now define the Wiener measure for the cylinder sets.

Definition 2.5. The Wiener measure of a cylinder set is defined by the set function

(2.6)

$$\begin{aligned}
&\mathfrak{m}(I(s_1, \dots, s_n; A_1, \dots, A_n)) \\
&\quad := \int_{A_n} \cdots \int_{A_2} \int_{A_1} \left(\frac{1}{2\pi(s_1 - 0)}\right)^{d/2} e^{\frac{-\|u_1\|^2}{2(s_1-0)}} \times \left(\frac{1}{2\pi(s_2 - s_1)}\right)^{d/2} e^{\frac{-\|u_2-u_1\|^2}{2(s_2-s_1)}} \times \cdots \\
&\quad \times \left(\frac{1}{2\pi(s_n - s_1)}\right)^{d/2} e^{\frac{-\|u_n-u_{n-1}\|^2}{2(s_n-s_1)}} \, du_1 \, du_2 \, \dots \, du_n \\
&\quad = \int_{A_n} \cdots \int_{A_2} \int_{A_1} \prod_{k=1}^{n} G_d(u_k - u_{k-1}, s_k - s_{k-1}) \, d\vec{U}
\end{aligned}$$

where in the final equality we let $u_0 \coloneqq 0$ and $s_0 \coloneqq 0$ and employ the simplification of notation

$$G_d(u,s) \coloneqq \left(\frac{1}{2\pi s}\right)^{d/2} e^{\frac{-\|u\|^2}{2s}}$$

which we shall use frequently in this paper.

Remark 2.7. One can think of the product of Gaussians in the integrand of (2.6) as a product of probability densities. The first factor gives the probability density of a particle being at $u_1 \in \mathbb{R}^d$ at time s_1 given that it started at the origin at time 0. The second factor gives the probability density of a particle being at $u_2 \in \mathbb{R}^d$ at time s_2 given that it was at u_1 at time s_1 and so on and so forth. Hence, the integral over the entire product of Gaussians gives the probability that a particle starting at the origin passes through all of the restriction windows at the corresponding restriction times.

We now want to show that we can extend the set function $\mathfrak{m}$ defined on the cylinder sets to a measure on the measurable space $\big(\mathcal{C}_0^{t,d}, \mathcal{B}(\mathcal{C}_0^{t,d})\big)$. The Carathéodory extension theorem will allow us to do this provided that we satisfy its assumptions. Before stating Carathéodory's theorem, we introduce two definitions which are necessary to its understanding.

Definition 2.8. Let $\mathcal{S}$ be a collection of subsets of a set X. Then $\mathcal{S}$ is a *semi-algebra* if and only if the following conditions are satisfied:

(i) The empty set and X are in $\mathcal{S}$.
(ii) If A and B are two sets in $\mathcal{S}$, then $A \cap B$ is in $\mathcal{S}$. (Closed under intersection)
(iii) If A is in $\mathcal{S}$, then its complement $X \setminus A$ can be written as a finite union of pairwise disjoint sets in $\mathcal{S}$.

Definition 2.9. Let X be a non-empty set and let $\mathcal{R}$ be a collection of subsets of X such that $\emptyset \in \mathcal{R}$. Then the set function $\rho : \mathcal{R} \to [0, \infty]$ is called a *pre-measure* on $\mathcal{R}$ if it satisfies the following two properties:

(i) $\rho(\emptyset) = 0$.
(ii) The set function ρ is countably additive. That is if $\{R_k\}_{k=1}^{\infty}$ is a sequence of pairwise disjoint sets in $\mathcal{R}$ and $\bigcup_{k=1}^{\infty} R_k$ is also in $\mathcal{R}$, then

$$\rho\left(\bigcup_{k=1}^{\infty} R_k\right) = \sum_{k=1}^{\infty} \rho(R_k).$$

Theorem 2.10 (Carathéodory's extension theorem for semi-algebras)**.** *Let $\mathcal{S}$ be a semi-algebra of a nonempty set X and let ρ be a pre-measure defined on $\mathcal{S}$. Then there exists a measure μ which extends ρ to $\sigma(\mathcal{S})$, the σ-algebra generated by $\mathcal{S}$, such that $\mu(S) = \rho(S)$ for all $S \in \mathcal{S}$. If μ is σ-finite, that is if X can be written as a countable union of sets of finite μ measure, then μ is the unique extension of ρ.*

Proof. Refer to [5]. □

In order to be able to apply Carathéodory's extension theorem to extend the set function $\mathfrak{m}$ defined on $\mathcal{I}$ to the measurable space $\big(\mathcal{C}_0^{t,d}, \mathcal{B}(\mathcal{C}_0^{t,d})\big)$, we need to prove four things:

(1) $\mathcal{I}$ forms a semi-algebra of $\mathcal{C}_0^{t,d}$.
(2) $\mathfrak{m}$ is a pre-measure.

(3) $\mathfrak{m}$ is σ-finite.
(4) The σ-algebra $\sigma(\mathcal{I})$ generated by the cylinder sets is equal to the Borel σ-algebra $\mathcal{B}(\mathcal{C}_0^{t,d})$ on Wiener space.

We shall prove them in the order listed above and in doing so will arrive at the main result of this section, Theorem 2.35.

Proposition 2.11. *The collection of all cylinder sets $\mathcal{I}$ forms a semi-algebra of $\mathcal{C}_0^{t,d}$.*

Proof. We will check that each of the conditions in Definition 2.8 is satisfied:

(i) The empty set in $\mathcal{C}_0^{t,d}$ is a cylinder set since it can be written as
$$\{\omega \in \mathcal{C}_0^{t,d} : \omega(t) \in \emptyset_{\mathbb{R}^d}\}$$
because the empty set $\emptyset_{\mathbb{R}^d}$ in $\mathbb{R}^d$ is a Borel set. The entire Wiener space $\mathcal{C}_0^{t,d}$ is also a cylinder set since it can be written as
$$\{\omega \in \mathcal{C}_0^{t,d} : \omega(t) \in \mathbb{R}^d\}$$
because the entire n-dimensional Euclidean space $\mathbb{R}^d$ is also a Borel set.

(ii) Let I and J be cylinder sets. Then $I \cap J$ consists of all paths that are in both I and J. This can be expressed as a cylinder set. The restriction times of $I \cap J$ are given by the union of the restriction times of I and J. Furthermore, for a particular restriction time in $I \cap J$ that is is common to both I and J the corresponding restriction window is given by the intersection of the restriction windows in I and J. Otherwise, if the restriction time belongs to only one of I or J then the corresponding restriction window is just the same restriction window as in I or J.

(iii) Consider a general cylinder set $C = I(s_1, \ldots, s_n; A_1, \ldots, A_n)$. We need to show that we can write $\mathcal{C}_0^{t,d} \setminus C$ as a finite union of disjoint cylinder sets. There is a direct way to show this, but we shall break up the proof into two cases because it will illustrate something important about how we have defined the cylinder sets.

Case 1: None of the restriction windows of C are all of $\mathbb{R}^d$. A path that fails to go through even one of the restriction windows of C is a path that belongs to $\mathcal{C}_0^{t,d} \setminus C$. And if a path does not go through a particular restriction window A, then it must go through its complement $A^c := \mathbb{R}^d \setminus A$. Since the complement of a Borel set is a Borel set, we can write $\mathcal{C}_0^{t,d} \setminus C$ as a finite union of n cylinder sets as follows:

$$\begin{aligned}\mathcal{C}_0^{t,d} \setminus C = I(s_1; A_1^c) \cup I(s_1, s_2; A_1, A_2^c) \cup I(s_1, s_2, s_3; A_1, A_2, A_3^c) \cup \cdots \\ \cup I(s_1, \ldots, s_n; A_1, A_2, \ldots, A_{n-1}^c) \cup I(s_1, \ldots, s_n; A_1, A_2, \ldots, A_n^c).\end{aligned} \tag{2.12}$$

Case 2: One or more of the restriction windows are all of $\mathbb{R}^d$. These restriction windows are just extraneous conditions since all paths must pass through some point in $\mathbb{R}^d$ at any time. Hence, we can simply delete these restriction windows and the corresponding restriction times from C and then proceed in the same way as we did in Case 1.

□

Case 2 in the above proof of Proposition 2.11 raises the question of whether $\mathfrak{m}$ is well-defined on $\mathcal{I}$ since there are multiple ways to specify the same cylinder set. Given any cylinder set we can always add more restriction times with corresponding restriction windows all of $\mathbb{R}^d$. These restriction times, which we shall refer to as *artificial restriction times*, do not change the cylinder set nor its associated Wiener measure.

Definition 2.13. Let I be a cylinder set that is not the entire space $\mathcal{C}_0^{t,d}$. Then the *minimal representation* of I is such that I has no artificial restriction times.

Proposition 2.14. *The Wiener measure* $\mathfrak{m}$ *is well-defined on* $\mathcal{I}$ *in the sense that the Wiener measure of the minimal representation of a cylinder set agrees with all other representations which contain artificial restriction times.*

To prove $\mathfrak{m}$ is well-defined, we first need the following lemma.

Lemma 2.15 (Chapman-Kolmogorov Equation)**.** *Let r, s, t, be real numbers such that $r < s < t$. Then the following equation holds:*

$$\begin{aligned}(2.16)\qquad \int_{\mathbb{R}^d} \left(\frac{1}{2\pi(t-s)}\right)^{\frac{1}{2}} \exp\left(\frac{-\|w-v\|^2}{2(t-s)}\right) \cdot \left(\frac{1}{2\pi(s-r)}\right)^{\frac{1}{2}} \exp\left(\frac{-\|v-u\|^2}{2(s-r)}\right) dv \\ = \left(\frac{1}{2\pi(t-r)}\right)^{\frac{1}{2}} \exp\left(\frac{-\|w-u\|^2}{2(t-r)}\right).\end{aligned}$$

Proof. The proof is a straightforward but lengthy computation that is outlined in [3, Proposition 3.2.3] for $\mathbb{R}$. After following the computation in Lapidus, we can recover (2.16) by a simple application of Fubini's theorem. □

Proof of Proposition 2.14. Suppose we have an arbitrary cylinder set given by $A = I(s_1, \ldots, s_n; A_1, \ldots, A_n)$. Then $\mathfrak{m}(A)$ is given by (2.6). Now let B be a cylinder set that is equivalent to A but has an artificial restriction time s_b. Then there are two cases to consider.

Case 1: $0 < s_1 < \cdots < s_n < s_b < t$. Referring to Definition 2.5, we calculate

$$(2.17)\qquad \mathfrak{m}(B) = \int_{\mathbb{R}^d} \left(\frac{1}{2\pi(s_b - s_n)}\right)^{d/2} e^{\frac{-\|u_a\|^2}{2(s_b - s_n)}} \int_{A_n} \cdots \int_{A_1} \prod_{k=1}^{n} G_d(u_k - u_{k-1}, s_k - s_{k-1})\, d\vec{U}\, du_a$$

Using Fubini's theorem and recognizing

$$\int_{\mathbb{R}^d} \left(\frac{1}{2\pi(s_b - s_n)}\right)^{d/2} e^{\frac{-\|u_a\|^2}{2(s_b - s_n)}}\, du_a = 1$$

as a standard Gaussian integral, we recover (2.6) from (2.17).

Case 2: $s_k < s_b < s_{k+1}$ *where* $k \in \{0, 1, \ldots, n-1\}$ *and* $s_0 \coloneqq 0$. This case is much like Case 1 except we end up with two more Gaussians in the integrand instead of just one as in (2.17). We then proceed in a similar fashion except we use Lemma 2.15 to recover (2.6).

Finally, if B has multiple artificial restriction times instead of just one, then we apply the above process multiple times—once for each artificial restriction time—until we recover (2.6). □

We have shown that the collection $\mathcal{I}$ of cylinder sets forms a semi-algebra and that $\mathfrak{m}$ is well-defined on this semi-algebra. We will now first prove two useful facts about $\mathfrak{m}$ and then proceed to show that $\mathfrak{m}$ is a pre-measure and is σ-finite.

Proposition 2.18. *We have* $\mathfrak{m}(\mathcal{C}_0^{t,d}) = 1$ *and* $\mathfrak{m}(\emptyset) = 0$.

Proof. We can write $\mathcal{C}_0^{t,d}$ as the cylinder set $\{\omega \in \mathcal{C}_0^{t,d} : \omega(t) \in \mathbb{R}^d\}$. Hence, by Definition 2.5 we have

$$\mathfrak{m}(\mathcal{C}_0^{t,d}) = \int_{\mathbb{R}^d} \left(\frac{1}{2\pi t}\right)^{d/2} e^{-\frac{\|u\|^2}{2t}}\, du = 1, \tag{2.19}$$

where we have recognized the integral in (2.19) as a standard Gaussian integral. Similarly, since we can write the empty set as the cylinder set $\{\omega \in \mathcal{C}_0^{t,d} : \omega(t) \in \emptyset_{\mathbb{R}^d}\}$, it follows directly from Definition 2.5 that $\mathfrak{m}(\emptyset) = 0$. $\square$

Proposition 2.20. *The set function* $\mathfrak{m}$ *defined on* $\mathcal{I}$ *is a pre-measure.*

Lemma 2.21. *The set function* $\mathfrak{m}$ *is countably additive on* $\mathcal{I}$ *meaning that if* $\{I_k\}_{n=1}^{\infty}$ *is a sequence of pairwise disjoint cylinder sets and* $\bigcup_{k=1}^{\infty} I_k$ *is also a cylinder set, then the following equality holds*

$$\mathfrak{m}\left(\bigcup_{k=1}^{\infty} I_k\right) = \sum_{k=1}^{\infty} \mathfrak{m}(I_k)$$

where the I_k *are cylinder sets.*

Proof. Refer to [6, Chapter 2]. $\square$

Proof of Proposition 2.20. By Proposition 2.18, we have that $\mathfrak{m}(\emptyset) = 0$, and by Lemma 2.21 we know that $\mathfrak{m}$ is countably additive. It then follows directly from Definition 2.9 that $\mathfrak{m}$ is a pre-measure. $\square$

Proposition 2.22. *We have that* $\mathfrak{m}$ *is* σ*-finite.*

Proof. Wiener space $\mathcal{C}_0^{t,d}$ can be written as the cylinder set

$$\{\omega \in \mathcal{C}_0^{t,d} : \omega(t) \in \mathbb{R}^d\}$$

and by Proposition 2.18, we have that $\mathfrak{m}(\mathcal{C}_0^{t,d}) = 1$. $\square$

We have now satisfied the conditions of Carathéodory's extension theorem and so we know that we can extend the pre-measure $\mathfrak{m}$ to a measure on the measurable space $\left(\mathcal{C}_0^{t,d}, \sigma(\mathcal{I})\right)$. To complete the construction, all that is left is to prove the following proposition.

Proposition 2.23. *The sigma algebra* $\sigma(\mathcal{I})$ *generated by the cylinder sets is the Borel* σ*-algebra of Wiener space* $\mathcal{B}(\mathcal{C}_0^{t,d})$.

To prove this proposition, we first need to establish two lemmas and a result known as the Dynkin π-λ theorem.

Lemma 2.24. *The function* $F_s : \mathcal{C}_0^{t,d} \to \mathbb{R}^d$ *defined as*

$$F_s(\omega) \coloneqq \omega(s),$$

where $0 < s \leq t$, *is continuous.*

Proof. We want to show that F_s is continuous at any $\omega \in \mathcal{C}_0^{t,d}$. Suppose we are given $\varepsilon > 0$ and $\omega \in \mathcal{C}_0^{t,d}$. Then choose $\delta = \varepsilon$. It follows that for all $\gamma \in \mathcal{C}_0^t$, if $\|\gamma - \omega\|_{\mathcal{C}_0^{t,d}} < \delta$ then we have

$$\|F_s(\gamma) - F_s(\omega)\| = \|\gamma(s) - \omega(s)\| \leq \sup_{s' \in [0,t]} \|\gamma(s') - \omega(s')\| = \|\gamma - \omega\|_{\mathcal{C}_0^{t,d}} < \delta = \varepsilon.$$

□

Lemma 2.25. *Let X and Y be two metric spaces, and let $f : X \to Y$ be a continuous function. Then f is Borel measurable meaning that for any $U \in \mathcal{B}(Y)$, we have $f^{-1}(U) \in \mathcal{B}(X)$.*

Proof. The proof follows directly from the discussion in [7, Chapter 2]. □

Notation 2.26. Let M be a metric space with metric d. We denote an open ball of radius δ centered at $x \in M$ as

$$B_\delta(x) \coloneqq \{y : d(x, y) < \delta\}$$

and similarly a closed ball as

$$\bar{B}_\delta(x) \coloneqq \{y : d(x, y) \leq \delta\}.$$

Definition 2.27. Let X be a set and P be a non-empty a collection of subsets of X such that if A and B are in P then $A \cap B \in P$. Then P is a *π-system*.

Remark 2.28. Comparing Definitions 2.8 and 2.27, it is clear that any semi-algebra is a π-system, and hence $\mathcal{I}$ is a π-system.

Definition 2.29. A *λ-system* is a collection L of subsets of a set X satisfying the following conditions:

(i) $\emptyset \in L$ and $X \in L$.
(ii) If $A \in L$, then $X \setminus A$ is in L (Closed under complements).
(iii) If $A_1, A_2, \ldots$ are a sequence of pairwise disjoint sets in L, then $\bigcup_{i=1}^\infty A_i \in L$.

Remark 2.30. From Definition 2.29, it is clear that any σ-algebra is a λ-system and hence $\mathcal{B}(\mathcal{C}_0^{t,d})$ is a λ-system.

Theorem 2.31 (Dynkin π-λ theorem)**.** *If P is a π-system and L is a λ-system, then $\sigma(P) \subseteq L$ provided that $P \subseteq L$.*

Proof. See [8, Theorem 2.4]. □

Harnessing the power of Lemmas 2.24 and 2.25 and the Dynkin π-λ theorem, we can now finally prove Proposition 2.23.

Proof of Proposition 2.23. We need to show that $\sigma(\mathcal{I}) = \mathcal{B}(\mathcal{C}_0^{t,d})$. Our strategy will be to show that $\sigma(\mathcal{I}) \subseteq \mathcal{B}(\mathcal{C}_0^t)$ and $\mathcal{B}(\mathcal{C}_0^t) \subseteq \sigma(\mathcal{I})$.

We first show that $\sigma(\mathcal{I}) \subseteq \mathcal{B}(\mathcal{C}_0^t)$. Consider the function $F_s : \mathcal{C}_0^{t,d} \to \mathbb{R}^d$ such that $F_s(\omega) = \omega(s)$. By Lemma 2.24, F_s is continuous and hence by Lemma 2.25 it is Borel measurable. Now notice that any arbitrary cylinder set can be written as follows:

$$I(s_1, \ldots, s_n; A_1, \ldots, A_n) = F_{s_1}^{-1}(A_1) \cap F_{s_2}^{-1}(A_2) \cap \cdots \cap F_{s_n}^{-1}(A_n).$$

Then it follows from the Borel measurability of each F_{s_j} that their countable intersection is Borel measurable and hence I is clearly in $\mathcal{B}(\mathcal{C}_0^{t,d})$. Now since $\mathcal{I}$ is a π-system and $\mathcal{B}(\mathcal{C}_0^{t,d})$ is a λ-system, by Theorem 2.31 it follows that $\sigma(I) = \mathcal{B}(\mathcal{C}_0^{t,d})$.

It now remains to show that $\mathcal{B}(\mathcal{C}_0^{t,d}) \subseteq \sigma(\mathcal{I})$. Since the collection of open subsets of $\mathcal{C}_0^{t,d}$ forms a π-system (the intersection of two open sets is an open set) by Theorem 2.31 it suffices to show that the open subsets of $\mathcal{C}_0^{t,d}$ are in $\sigma(\mathcal{I})$. However, since $\mathcal{C}_0^{t,d}$ is a Banach space, every open subset of $\mathcal{C}_0^{t,d}$ can be written as a countable union of open balls. Hence, we only need to show that every open ball is in $\sigma(\mathcal{I})$ since by the definition of a σ-algebra if the open balls are in $\sigma(\mathcal{I})$ then their countable union (open subsets) is in $\sigma(\mathcal{I})$. But every open ball can be written as a countable union of closed balls as follows:

$$B_r(\omega) \;=\; \bigcup_{\{n\,:\,0<\frac{1}{n}<r\}} \bar{B}_{r-\frac{1}{n}}(\omega).$$

Since a countable union of countable unions is countable, to complete the proof we only need to show that every closed ball is in $\sigma(\mathcal{I})$.

Let $\bar{B}_\delta(\omega_0)$ be an arbitrary closed ball in $\mathcal{C}_0^{t,d}$. Now consider the set $S \coloneqq \{s_1, s_2, s_3, \ldots\}$ consisting of a sequence of all the rational times in $[0,t]$. This set is countable and everywhere dense in $[0,t]$. Now for each positive integer N we define

$$K_N \coloneqq \{\omega \in \mathcal{C}_0^{t,d} : \|\omega(s_j) - \omega_0(s_j)\| \le \delta \text{ for } j = 1, 2, \ldots, N\}$$

where $\|\cdot\|$ is the standard Euclidean norm in $\mathbb{R}^d$. The condition $\|\omega(s_j) - \omega_0(s_j)\| \le \delta$ specifies all paths which pass through a closed ball in $\mathbb{R}^d$ at a specific time. A closed ball is a Borel set, and so it is clear that every K_N is a cylinder set. This means that every K_N is in $\sigma(\mathcal{I})$ and hence $\bigcap_{N=1}^\infty K_N$ is in $\sigma(\mathcal{I})$. We can complete the proof then by showing that $\bar{B}_\delta(\omega_0) = \bigcap_{N=1}^\infty K_N$.

Since $\bar{B}_\delta(\omega_0)$ consists of paths that are at most δ away from ω_0 at any time whereas K_N consists of paths that are at most δ away from ω_0 at only N times, it is clear that $\bar{B}_\delta(\omega_0) \subseteq K_N$ for any N. Hence, $\bar{B}_\delta(\omega_0) \subseteq \bigcap_{N=1}^\infty K_N$.

To finish the proof, we now just need to show that $\bigcap_{N=1}^\infty K_N \subseteq \bar{B}_\delta(\omega_0)$. We shall do this by proving the contrapositive: if a path ω is not in $\bar{B}_\delta(x_0)$ then it is not in $\bigcap_{N=1}^\infty K_N$. If ω is not in $\bar{B}_\delta(x_0)$ then there exists a time s' in $[0,t]$ and a positive real number δ_1 such that

$$\|\omega(s') - \omega_0(s')\| = \delta + \delta_1.$$

Now since S is everywhere dense in $[0,t]$, we can choose a subsequence $\{s_k\}$ from S such that $s_k \to s'$ as $k \to \infty$, and hence, since all paths in $\mathcal{C}_0^{t,d}$ are continuous, $\omega(s_k) \to \omega(s')$ and $\omega_0(s_k) \to \omega_0(s)$ as $k \to \infty$. By the definition of a limit, it follows that there exists some k such that the following two inequalities hold:

$$\|\omega(s_k) - \omega(s')\| < \frac{\delta_1}{4} \tag{2.32}$$

$$\|\omega_0(s') - \omega_0(s_k)\| < \frac{\delta_1}{4} \tag{2.33}$$

Combining (2.32) and (2.33) and using the triangle equality yields

$$\|\omega(s_k) - \omega_0(s_k) - (\omega(s') - \omega_0(s'))\| < \frac{\delta_1}{2}. \tag{2.34}$$

Finally, after applying the reverse triangle inequality to (2.34) and recognizing that $\|\omega(s') - \omega_0(s')\| = \delta + \delta_1$ we obtain

$$||\|\omega(s_k) - \omega_0(s_k)\| - (\delta + \delta_1)| < \frac{\delta_1}{2}$$

which implies that there exists some k such that

$$\|\omega(s_k) - \omega_0(s_k)\| > \delta + \frac{\delta_1}{2}.$$

This means that ω is not in any K_N which has s_k as a restriction time and hence ω is not in $\bigcap_{N=1}^{\infty} K_N$. □

Hence, we have shown that the Wiener measure $\mathfrak{m}$ that we defined on the semi-algebra $\mathcal{I}$ of cylinder sets can be extended to the Borel σ-algebra $\mathcal{B}(\mathcal{C}_0^{t,d})$ of Wiener space. The following theorem summarizes everything we have proved about the Wiener measure thus far.

Theorem 2.35. *There exists a unique, countably additive, probability measure $\mathfrak{m}$ on the measurable space $\left(\mathcal{C}_0^{t,d}, \mathcal{B}(\mathcal{C}_0^{t,d}), \mathfrak{m}\right)$ such that if I is a cylinder set (Definition 2.3) then $\mathfrak{m}(I)$ is given by* (2.6)*.*

We now conclude this section by stating a helpful theorem regarding integration with respect to the Wiener measure.

Theorem 2.36 (Wiener's Integration Formula)**.** *Suppose $f : \mathbb{R}^{dn} \to \mathbb{R}$ is a function which depends on the values of paths in $\mathcal{C}_0^{t,d}$ at only finitely many times. Then the following formula holds:*

$$\int_{\mathcal{C}_0^{t,d}} f(\omega(s_1), \omega(s_2), \ldots, \omega(s_n))\, d\mathfrak{m}(\omega) = \int_{\mathbb{R}^{dn}} f(u_1, u_2, \ldots, u_n) \prod_{k=1}^{n} G_d(u_k - u_{k-1}, s_k - s_{k-1})\, d\vec{U}.$$

Proof. The proof for one-dimensional Wiener space $\mathcal{C}_0^t$ is given in [3, Theorem 3.3.5] but the theorem holds for $\mathcal{C}_0^{t,d}$ [2, Theorem 20.2]. □

Corollary 2.37. *Theorem 2.36 holds for complex-valued functions $f : \mathbb{R}^{dn} \to \mathbb{C}$.*

Proof. The proof follows directly by breaking up f into its real and imaginary parts $f = \mathrm{Re}(f) + i \cdot \mathrm{Im}(f)$. □

Theorem 2.36 and Corollary 2.37 are extremely important because they allow us to convert an integral over a space of paths into a familiar integral over Euclidean space. They will play a crucial role in our proof of the Feynman-Kac formula.

3. Operator Semigroups and the Free Heat Equation

We now turn to the basic theory behind operator semigroups with two goals in mind:

(1) to understand the free heat semigroup which characterizes the solution to the heat equation and

(2) to provide the necessary background for the proof of the Trotter product formula in the next section.

For the remainder of this paper we shall let X denote a Banach space, and we will write $D(\cdot)$ to denote the domain of an operator.

Definition 3.1. A *strongly continuous semigroup* (or C_0 semigroup) is a family of bounded linear operators $\{T(t) : t \in \mathbb{R}^+ := [0, \infty)\}$ from X to X that satisfies the following three properties:

(i) $T(s + t) = T(s)T(t)$ for all s and t in $\mathbb{R}^+$. (Semigroup Property)
(ii) $T(0) = I$ where I is the identity operator. (Identity Property)
(iii) For all $\phi \in X$, if $t \to 0^+$, then $T(t)\phi \to \phi$ with respect to the norm on X. (Continuity Property)

Definition 3.2. A *contraction semigroup* is a C_0 semigroup such that $\|T(t)\| \leq 1$ for all $t \in \mathbb{R}^+$ where $\|\cdot\|$ is the operator norm on the Banach space X.

Definition 3.3. The *generator* $A : D(A) \subseteq X \to X$ of a C_0 semigroup $\{T(t)\}$ is the linear operator defined by the equation

$$A\phi = \lim_{t\to 0^+} \frac{T(t)\phi - \phi}{t} = \lim_{t\to 0^+} \frac{T(t)\phi - T(0)\phi}{t} = \frac{d}{dt}T(t)\phi\bigg|_{t=0} \tag{3.4}$$

where $D(A)$ is all $\phi \in X$ such that the limit in (3.4) exists.

Remark 3.5. It is a well known fact that a C_0 semigroup is determined uniquely by its generator. That is, if $\{T(t)\}$ and $\{S(t)\}$ are two semigroups that have the same generator, then $T(t) = S(t)$ for all $t \in \mathbb{R}^+$. A concise proof of this can be found in [9, Theorem II.1.4].

Although Definition 3.3 seems straightforward enough, in practice it can often be quite tricky to determine exactly what the domain of the generator looks like. For this reason, it is often helpful to be able to work on a smaller subspace of the generator's domain.

Definition 3.6. A *core* $\mathcal{D}$ of a linear operator A is a subspace that is dense in $D(A)$ with respect to the graph norm

$$\|\phi\|_A := \|\phi\| + \|A\phi\|.$$

Although for the purposes of this paper Definition 3.6 is all we need, there are a number of fascinating theorems relating properties of an operator A to subspaces which are cores of A; the interested reader is referred to [9, Chapter 1].

Now that we are better acquainted with C_0 semigroups and their generators, a natural question to ask is what sorts of operators generate C_0 semigroups. We devote the next two definitions and subsequent proposition to answering this question.

Definition 3.7. The *graph* of an operator $A : D(A) \subseteq X \to X$ is defined as the following set:

$$\mathcal{G}(A) := \{(\phi, A\phi) \in X \times X : \phi \in D(A)\}.$$

Definition 3.8. An operator $A : D(A) \subseteq X \to X$ is a *closed operator* if its graph $\mathcal{G}(A)$ is a closed subset of $X \times X$. In more concrete terms, this means that if $\{\phi_n\}_{n=1}^{\infty}$ is a sequence of functions from $D(A)$ such that $\lim_{n\to\infty} \|\phi_n - \phi\|_X = 0$ and $\lim_{n\to\infty} \|A\phi_n - y\|_X = 0$, then $\phi \in D(A)$ and $y = A\phi$.

Definition 3.9. An operator A is said to be *closable* if whenever $\{\phi_n\}_{n=1}^{\infty}$ is a sequence of operators from $D(A)$ such that $\lim_{n\to\infty} \|\phi_n\| = 0$, then $\lim_{n\to\infty} \|A\phi_n\| = 0$. The closure $\overline{A}$ of an operator A is the operator associated with the closure $\overline{\mathcal{G}(A)}$ of the graph of A.

Proposition 3.10. *To be the generator of a C_0 semigroup, an operator A must satisfy the following two necessary but not sufficient conditions:*

(1) A must be closed.
(2) $D(A)$ must be dense in X.

Proof. Refer to [9, Theorem II.1.4] □

With some of the basics of operator semigroup theory in hand, we can now apply it to study a certain class of problems known as evolution equations. In general, a physical system can often be described by its initial state whose evolution in time is governed by a differential equation. The following definition makes this concrete.

Definition 3.11. An *evolution equation* is a differential equation

$$\frac{d}{dt}u(t) = Au(t) \tag{3.12}$$

along with initial state

$$u(0) = \phi$$

where $A : D(A) \subseteq X \to X$ is an operator, $u(t) \in D(A)$ for all $t \geq 0$, and the equality in (3.12) is understood to mean that the following limit holds:

$$\lim_{h\to 0} \left\| \frac{u(t+h) - u(t)}{h} - Au(t) \right\| = 0.$$

Semigroups turn out to be a powerful tool for understanding solutions to evolution equations as the following theorem demonstrates.

Theorem 3.13. *Suppose the operator A in* (3.12) *is the generator of a C_0 semigroup $\{T(t)\}$. Then $u(t) \coloneqq T(t)\phi$ is differentiable with continuous derivative (continuously differentiable) for all $t \in [0, \infty)$ and is the unique solution to the evolution equation problem $\frac{d}{dt}u(t) = Au(t),\ u(0) = \phi \in D(A)$.*

Proof. Refer to [10, Chapter II] for a thorough proof or to [3, Theorem 9.1.4] for a simpler proof that omits uniqueness. □

Theorem 3.13 is of extreme significance from both a physical and a mathematical perspective. Using the tools of operator semigroup theory, it tells us two important things about the solutions of an evolution equation. First, it gives a condition for a solution to an evolution equation to exist and be unique. From a physical perspective this is very convenient as uniqueness means that there is no ambiguity regarding how the state of the system evolves in time. And secondly, since $u(t) \coloneqq T(t)\phi$ is continuously differentiable, Theorem 3.13 tells us that the evolution of a physical system depends continuously on its initial state.

Now that we are familiar with how operator semigroup theory can be applied to evolution equations, we examine a specific example which is of especial interest to us—the free heat equation. We first introduce the following definition and theorem which will be of use to us.

Definition 3.14. Let $f(x)$ and $g(x)$ be two functions defined on $\mathbb{R}^d$. Then their *convolution* is defined as

$$(f * g)(x) \coloneqq \int_{\mathbb{R}^d} f(x-y)g(y)\,dy.$$

Theorem 3.15 (Young's Inequality)**.** *Let f and g be functions in $L^p(\mathbb{R}^d)$ and $L^q(\mathbb{R}^d)$ respectively such that*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1.$$

with $1 \leq p, q, \leq r \leq \infty$. Then the following inequality holds:

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

Proof. Refer to [7, Theorem 3.9.4] □

We now proceed with our analysis of the free heat equation.

Example 3.16. Let us begin with the free Schrödinger equation which is obtained by getting rid of the potential function $V(\cdot)$ in (1.2). Then, when we pass to imaginary time and set all physical constants to unity, the free Schrödinger equation turns into the free heat equation

$$\frac{\partial}{\partial t}\psi(x,t) = \frac{1}{2}\Delta\psi(x,t),\ \psi(x,0) = \psi_0(x) \tag{3.17}$$

where $x \in \mathbb{R}^d$, $\Delta = \sum_{i=0}^{d} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian in $\mathbb{R}^d$, $\psi_0(x)$ is a function describing the initial state of the system, and the Banach space we are working in is $L^2(\mathbb{R}^d)$ the space of all square integrable complex-valued functions on $\mathbb{R}^d$. We define the free Hamiltonian as the operator $H_0 \coloneqq -\frac{1}{2}\Delta$ with domain

$$D(H_0) = \{\psi \in L^2(\mathbb{R}^d) : \Delta\psi \text{ is in } L^2(\mathbb{R}^d) \text{ in the distributional sense}\}.^1$$

We can then rewrite (3.17) as

$$\frac{\partial}{\partial t}\psi(x,t) = -H_0\psi(x,t),\ \psi(x,0) = \psi_0(x). \tag{3.18}$$

It is a well known fact [12] that the solution to this equation is given by

$$\psi(x,t) = \int_{\mathbb{R}^d} (2\pi t)^{-d/2} e^{\frac{-\|x-y\|^2}{2t}} \psi_0(y)dy = (G_d(x,t) * \psi_0)(x). \tag{3.19}$$

where we remind ourselves that $G_d(x,t) \coloneqq (2\pi t)^{-d/2} e^{\frac{-\|x\|^2}{2t}}$. Now we can write this solution in semigroup notation as

$$\psi(x,t) = T(t)\psi_0(x) = (G_d(x,t) * \psi_0)(x)$$

where we define $T(0) \coloneqq I$.

The family of operators $\{T(t)\}$ forms a C_0 contraction semigroup. The semigroup property $T(s+t) = T(s)T(t)$ can be easily verified by applying Lemma 2.15. Also, the continuity property that $T(t)\psi_0(x) \to \psi_0(x)$ with respect to the L^2 norm as $n \to \infty$ can be seen conceptually (and with some effort proved rigorously) by noting that the Gaussian $G_d(x,t)$ becomes like a Dirac δ function as $t \to 0^+$. Finally, we can prove that $\{T(t)\}$ is

[1] For a quick refresher on distributional derivatives and the distributional Laplacian see [11].

a contraction semigroup by showing that $\|T(t)\psi_0\|_{L^2} \leq \|\psi_0\|_{L^2}$ for all $\psi_0 \in L^2(\mathbb{R}^d)$ and all $t \in \mathbb{R}^+$:

$$\begin{aligned}\|T(t)\psi_0\|_{L^2} &= \|(G_d(x,t) * \psi_0)(x)\|_{L^2} \\ &\leq \|G_d(x,t)\|_{L^1} \|\psi_0\|_{L^2} && \text{by Theorem 3.15} \\ &= \|\psi_0\|_{L^2},\end{aligned}$$

where in the last equality we have recognized $\|G_d(x,t)\|_{L^1}$ as a standard Gaussian integral. It can be shown through a fairly lengthy computation that $-H_0$ is the generator of the contraction semigroup $\{T(t)\}$ [3, Chapter 10]. Hence, by Theorem 3.13 we know that the solution $T(t)\psi_0(x)$ is unique and continuously differentiable.

Finally, we notice that if we were to solve (3.18) purely formally we would obtain the solution

$$\psi(x,t) = e^{-tH_0}\psi_0(x).$$

Hence, we often write the semigroup $T(t)$ as e^{-tH_0}. This notation emphasizes the fact that the semigroup has generator $-H_0$ and provides the solution to the heat equation. We should be careful with this notation however because semigroups do not always possess all the analogous properties of the exponential function.

Remark 3.20. The Gaussian in (3.19) that shows up in the solution to the heat equation is reminiscent of the Gaussians that showed up when we were constructing the Wiener measure in Section 2. In fact, using Theorem 2.36, we could have written the solution as

$$\psi(x,t) = e^{-tH_0}\psi_0(x) = \int_{\mathcal{C}_0^{t,d}} \psi_0(\omega(t) + x)\, d\mathfrak{m}(\omega).$$

This gives us a flavor for how semigroups related to heat equations might be represented as integrals over Wiener space.

We now conclude this section by talking about a special kind of semigroup which will be of use in the next section. Suppose A is a bounded operator from X to X. Then for any $t \in \mathbb{R}$, we can write

$$T(t) := e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}.$$

This sum converges absolutely since $\|A\|$ is finite because A is bounded. The semigroup $\{T(t)\}$ is called a *uniformly continuous semigroup* and its generator is the bounded operator A. In this case, it is conventional to use the notation e^{tA} instead of $T(t)$ to refer to the semigroup.

4. The Trotter Product

We will now use some of the ideas that we developed in the previous section to prove the Trotter product formula which plays a crucial role in our proof of the Feynman-Kac formula. Our approach in this section will be to prove a related theorem known as the Chernoff product formula and then show that the Trotter product follows as a direct application.

To motivate the importance of the Trotter product, let us first consider a special case where we deal with uniformly continuous semigroups which are generated by bounded operators A, B, and $A + B$.

Proposition 4.1. *Suppose A and B are bounded operators on a Banach space X such that A and B commute, that is $AB = BA$. Then $e^{tA}e^{tB} = e^{t(A+B)}$.*

Proof Sketch. The proof follows by writing out $e^{t(A+B)}$ as the power series

$$\sum_{k=0}^{\infty} t^k \frac{(A+B)^k}{k!},$$

using the binomial formula and the commutativity of A and B to expand $A+B$, and then using the Cauchy product rule to recover $e^A e^B$. The Cauchy product rule can be used because each of the power series corresponding to e^A, e^B, and e^{A+B} converge because A, B, and hence $A+B$ are bounded. □

Proposition 4.1 is unfortunately quite restrictive; not only do the operators in question A, B have to be bounded, they also have to commute. We hope then to find a formula that is more broadly applicable even if it is only loosely similar to the multiplicative property in Proposition 4.1.

We first state a theorem that will be of considerable use in the remainder of this section.

Theorem 4.2 (Limit Theorem)**.** *Suppose A is an operator that generates a contraction semigroup $\{T_A(t)\}$ on a Banach space X, and let $\mathcal{D}$ be a core of A. Additionally, let $\{A_n\}_{n=1}^{\infty}$ be a sequence of operators that generate contraction semigroups $\{T_{A_n}(t)\}$ such that for each $\phi \in \mathcal{D}$, we have that ϕ is in the domain of A_n for all $n > N \in \mathbb{N}$ where N might depend on the specific ϕ and that*

$$A_n\phi \to A\phi \ \text{ as } \ n \to \infty.$$

Then for each $\psi \in X$,

$$T_{A_n}(t)\psi \to T_A(t)\psi \ \text{ as } \ n \to \infty$$

and the convergence is uniform on all compact intervals of t in $[0, \infty)$.

Proof. See [13]. □

We now state and prove the Chernoff product formula.

Theorem 4.3 (Chernoff Product)**.** *Let X be a Banach space and let $\mathcal{L}(X)$ be the set of all bounded operators from X to X. Let $F : [0, \infty) \to \mathcal{L}(X)$ be a continuous map such that $\|F(t)\| \leq 1$ for all $t \in [0, \infty)$ and $F(0) = I$ where I is the identity operator. Furthermore, let A be an operator on X that generates a contraction semigroup $\{T_A(t)\}$ and let $\mathcal{D}$ be a core of A. Then if*

$$\lim_{h \to 0^+} \frac{F(h) - I}{h}\phi = A\phi \tag{4.4}$$

for each $\phi \in \mathcal{D}$, then for each $\psi \in X$,

$$\lim_{n \to \infty} \left(F(t/n)\right)^n \psi = T_A(t)\psi \tag{4.5}$$

where the convergence in (4.5) is uniform on compact intervals of t in $[0, \infty)$.

To prove the Chernoff product formula we first introduce the following useful notation.

Notation 4.6. For fixed $t > 0$ let

$$Q_n := \frac{F(t/n) - I}{t/n}$$

for any $n \in \mathbb{N}$.

Next, we prove the following two lemmas.

Lemma 4.7. *For any $n \in \mathbb{N}$, the semigroup $\{T_{Q_n}(t)\}$ generated by Q_n is a contraction semigroup.*

Proof. To prove $\{T_{Q_n}(t)\}$ is a contraction semigroup we just need to show that $\|T_{Q_n}(t)\| \leq 1$ for all $t \in \mathbb{R}^+$. Each Q_n is clearly a bounded operator since both $F(t/n)$ and I are bounded. Hence, we can use the exponential notation and write the semigroup operators $T_{Q_n}(t)$ as e^{tQ_n}. Now since the identity operator I commutes with any operator, by Proposition 4.1 we have

$$\left\|e^{tQ_n}\right\| = \left\|e^{n(F(t/n)-I)}\right\| = \left\|e^{-nI}e^{nF(t/n)}\right\|.$$

Now by power series expansion of e^{-nI} we can clearly see that $e^{-nI} = e^{-n}$. Using this, the triangle inequality, and the fact that $\|F(t)\| \leq 1$ for all $t \geq 0$ we obtain

$$\begin{aligned} \left\|e^{tQ_n}\right\| &= \left\|e^{-nI}e^{nF(t/n)}\right\| = e^{-n}\left\|e^{nF(t/n)}\right\| \\ &\leq e^{-n}e^{n\|F(t/n)\|} \leq e^{-n}e^{n} = 1. \end{aligned}$$

□

Lemma 4.8. *Let X be a Banach space and let $B : X \to X$ be a bounded operator such that $\|B\| \leq 1$. Then the following inequality holds for all $\psi \in X$ and for all $n \in \mathbb{N}$:*

$$\left\|\left(e^{n(B-I)} - B^n\right)\psi\right\| \leq \sqrt{n}\,\|(B-I)\,\psi\|.$$

Proof. Refer to [10, Lemma 8.5]. □

With the above two lemmas in hand, we can now prove the Chernoff product formula.

Proof of Theorem 4.3. By change of variable, we have for all $\phi \in \mathcal{D}$

$$\begin{aligned} \lim_{h\to 0^+} \frac{F(h)-I}{h}\phi &= \lim_{m\to\infty} m\,(F\,(1/m) - I)\,\phi \\ &= \lim_{n\to\infty} \frac{n}{t}\,(F\,(t/n) - I)\,\phi \qquad \text{by letting } m = \frac{n}{t} \text{ since } t \text{ is fixed} \\ &= \lim_{n\to\infty} Q_n\phi = A\phi. \qquad \text{by (4.4)} \end{aligned} \tag{4.9}$$

Note that when we made the change of variable from h to n we went from a $\mathbb{R}$-valued variable to a $\mathbb{N}$-valued variable. In (4.9) we have shown that for all $\phi \in \mathcal{D}$,

$$Q_n\phi \to A\phi \text{ as } n \to \infty,$$

and so it follows from Theorem 4.2 that for all $\psi \in X$

$$T_{Q_n}(t)\psi \to T_A(t)\psi \text{ as } n \to \infty \tag{4.10}$$

uniformly on compact intervals of t in $[0,\infty)$. Now since $\|F(t)\| \leq 1$ for all $t \geq 0$, by Lemma 4.8 we obtain the following inequality:

$$\begin{aligned}
\|(T_{Q_n}(t) - (F(t/n))^n)\phi\| &= \left\|\left(e^{tQ_n} - (F(t/n))^n\right)\phi\right\| \\
&= \left\|\left(e^{n(F(t/n)-I)} - (F(t/n))^n\right)\phi\right\| \\
&\leq \sqrt{n}\,\|(F(t/n) - I)\phi\| \\
&= \frac{t}{\sqrt{n}}\left\|\left(\frac{F(t/n) - I}{t/n}\right)\phi\right\| \\
&= \frac{t}{\sqrt{n}}\,\|Q_n\phi\|\,.
\end{aligned}$$

Since $Q_n\phi \to A\phi$ as $n \to \infty$, it follows that the entire expression $\frac{t}{\sqrt{n}}\|Q_n\phi\|$ goes to 0 uniformly on compact intervals of $t \in [0,\infty)$ as n goes to infinity. Hence, by the inequality established above, it follows that

$$\|(T_{Q_n}(t) - (F(t/n))^n)\phi\| \to 0 \ \text{ as } \ n \to \infty \tag{4.11}$$

uniformly on compact intervals of t. Combining (4.10) and (4.11), we conclude that

$$\|(T_A(t) - (F(t/n))^n)\phi\| \to 0 \ \text{ as } \ n \to \infty$$

uniformly on compact intervals of t for all $\phi \in \mathcal{D}$. But now $\mathcal{D}$ is dense in $D(A)$, and by Proposition 3.10, $D(A)$ is dense in X. Therefore, $\mathcal{D}$ is dense in X, and since both $T_A(t)$ and $(F(t/n))^n$ are bounded and hence continuous operators, it follows that

$$\|(T_A(t) - (F(t/n))^n)\psi)\| \to 0 \ \text{ as } \ n \to \infty$$

for all $\psi \in X$ uniformly on compact intervals of t. □

We now state the Trotter product formula and show that its proof follows from a direct application of the Chernoff product.

Theorem 4.12 (Trotter Product)**.** *Let A, B, and the closure of their sum $\overline{A+B}$ be operators on X that generate contraction semigroups $\{S_A(t)\}$, $\{S_B(t)\}$, and $\{T(t)\}$ respectively. Then for all ψ in X, the following limit holds uniformly on compact intervals of $t \in \mathbb{R}^+$:*

$$\lim_{n\to\infty}\left(S_A\left(\frac{t}{n}\right)S_b\left(\frac{t}{n}\right)\right)^n\psi = T(t)\psi,$$

where the convergence is in the norm of X.

Proof. We let

$$F(t) := S_A(t)S_B(t).$$

Then the proof is simply a matter of verifying that the assumptions of Theorem 4.3 are satisfied. Since both $S_A(t)$ and $S_B(t)$ are contraction semigroups, it is clear that for all $t \in \mathbb{R}^+$,

$$\|F(t)\| = \|S_A(t)S_B(t)\| \leq \|S_A(t)\|\,\|S_B(t)\| \leq 1.$$

Also, by the identity property of C_0 semigroups

$$F(0) = S_A(0)\cdot S_B(0) = I\cdot I = I.$$

Finally, we recognize that $\mathcal{D} \coloneqq D(A) \cap D(B)$ is a core of $\overline{A+B}$. We can show through a computation [3, Theorem 11.1.4] that for all $\phi \in \mathcal{D}$, we have

$$\lim_{h\to 0^+} \frac{F(h)-I}{h}\phi = (A+B)\phi = (\overline{A+B})\phi,$$

where in the last equality we use the fact that the operator $A+B$ is equivalent to its closure $\overline{A+B}$ on $\mathcal{D}$. Hence, all the assumptions of Theorem 4.3 are satisfied, and so the proof is complete. □

The Trotter product gives us a relationship between the contraction semigroups generated by two operators and the contraction semigroup generated by their sum. This will play a crucial role in our proof of the Feynman-Kac formula in which we deal with the Hamiltonian operator H formed by the operator sum $-\frac{1}{2}\Delta + V$.

5. The Feynman-Kac Formula

We are now prepared to prove the Feynman-Kac formula. We begin by introducing some of the necessary background which will be required for the proof. First, we state a standard result from measure theory which we will need.

Theorem 5.1 (Dominated Convergence Theorem)**.** *Let $\{f_n\}_{k=1}^{\infty}$ be a sequence of complex-valued measurable functions on the measure space (X, Σ, μ) that converges pointwise to a function f. If for all $x \in X$ this sequence of functions satisfies the inequality*

$$|f_n(x)| \leq g(x),$$

where g is a real-valued measurable function such that $\int_X |g|\, d\mu < \infty$, then

$$\lim_{n\to\infty} \int_X f_n\, d\mu = \int_X f d\mu$$

Proof. Refer to [14, Theorem 1.34]. □

Next, we introduce a definition and prove a result regarding multiplication operators.

Definition 5.2. Suppose f is a real-valued function on $\mathbb{R}^d$. Then the operator M_f on $L^2(\mathbb{R}^d)$ of multiplication by f is the operator such that

$$M_f\phi = f\phi$$

for all $\phi \in D(M_f) = \{\phi \in L^2(\mathbb{R}^d) : f\phi \in L^2(\mathbb{R}^d)\}$.

Proposition 5.3. *Suppose $V : \mathbb{R}^d \to \mathbb{R}$ is a continuous function of compact support. Then the multiplication operator M_V is a bounded linear operator with operator norm $\|M_V\| = \|V\|_{L^\infty}$.*

Proof. Since V is a continuous function of compact support, it is in $L^\infty(\mathbb{R}^d)$ and $|V|$ has a least upper bound $M = \|V\|_{L^\infty}$. Hence,

$$\|M_V\phi\|_{L^2} = \|V\phi\|_{L^2} \leq \|M\phi\|_{L^2} = M\,\|\phi\|_{L^2}$$

for all $\phi \in D(M_V)$. The linearity of M_V follows directly from the linearity of multiplication by functions and so M_V is a bounded linear operator.

It remains to show that $\|M_V\| = \|V\|_{L^\infty}$. It is clear that

$$\|M_V\| = \sup_{\phi \in D(M_V)} \frac{\|M_V \phi\|_{L^2}}{\|\phi\|_{L^2}} \leq \sup_{\phi \in D(M_V)} \frac{\|M\phi\|_{L^2}}{\|\phi\|_{L^2}} = M.$$

To complete the proof then, we just need to show that $\|M_V\| \geq M - \varepsilon$ for all ε satisfying $M > \epsilon > 0$. Suppose we are given such an ε. Let

$$E_\varepsilon = \{x \in \mathbb{R}^d : |V(x)| \geq M - \varepsilon\}.$$

Each E_ε is Lebesgue measurable since V and hence $|V|$ is continuous. Now define

$$\phi_\varepsilon(x) := \begin{cases} 1 & x \in E_\varepsilon \\ 0 & \text{otherwise} \end{cases}$$

Since V is of compact support it follows that $\phi_\varepsilon(x)$ is in $L^2(\mathbb{R}^d)$. Moreover, it is clear from the definition of $\phi_\varepsilon(x)$ that the inequality $|V(x)\phi_\varepsilon(x)| \geq (M - \varepsilon)|\phi_\varepsilon(x)|$ is satisfied. Hence,

$$\begin{aligned} \|M_V\| = \sup_{\phi \in D(M_V)} \frac{\|M_V \phi\|_{L^2}}{\|\phi\|_{L^2}} \geq \frac{\|V\phi_\varepsilon\|_{L^2}}{\|\phi_\epsilon\|_{L^2}} &= \frac{\left(\int_{\mathbb{R}^d} |V(x)\phi_\varepsilon(x)|^2 \, dx\right)^{1/2}}{\left(\int_{\mathbb{R}^d} |\phi_\varepsilon(x)|^2 \, dx\right)^{1/2}} \\ &= \frac{(M - \varepsilon) \, \|\phi_\varepsilon(x)\|_{L^2}}{\|\phi_\varepsilon(x)\|_{L^2}} = M - \varepsilon. \end{aligned}$$

We can make ε arbitrarily small, and so it follows that $\|M_V\| = M = \|V\|_{L^\infty}$. □

Notation 5.4. Henceforth, we will just write V instead of M_V to denote the operator of multiplication since $M_V \phi = V\phi$. It should be clear from the context whether we are referring to the function by itself or as a multiplication operator. For clarity, if we think there might be any ambiguity then we will write $V(\cdot)$ to emphasize that we are only talking about the function.

We now state the Feynman-Kac formula which provides a mathematically rigorous connection between an integral over paths and the heat semigroup which characterizes the time evolution of an imaginary-time quantum particle.

Theorem 5.5 (Feynman-Kac Formula)**.** *Let the potential function $V : \mathbb{R}^d \to \mathbb{R}$ be a continuous function of compact support. Then the heat semigroup generated by the operator $-H = -(-\frac{1}{2}\Delta + V)$ is given by the formula*

$$e^{-tH}\psi(x) = \int_{\mathcal{C}_0^{t,d}} \exp\left(-\int_0^t V(\omega(s) + x) \, ds\right) \psi(\omega(t) + x) \, d\mathfrak{m}(\omega), \tag{5.6}$$

where $\psi \in L^2(\mathbb{R}^d)$ is a complex-valued function with domain all of $\mathbb{R}^d$.

We justify the assumption that ψ has domain all of $\mathbb{R}^d$ by the fact that a particle should have some probability amplitude of being at any given point in space. Also, notice that the way we state the Feynman-Kac formula here is slightly different from how we stated it in the introduction in (1.5). In (1.5), we considered an integral over a space of paths ω with the condition $\omega(0) = x$. However, in (5.6) we are integrating with respect to the Wiener measure on Wiener space where the condition on the paths is that $\omega(0) = 0$. Hence, we must translate all the paths in Wiener space by x. The $+x$ term in the argument of $V(\cdot)$ and $\psi(\cdot)$ takes care of this.

Before proceeding in earnest to the proof of the Feynman-Kac formula, we will first require four lemmas.

Lemma 5.7. *The sum of operators $H = -\frac{1}{2}\Delta + V$ is a closed operator and hence equal to its closure $\overline{H} = \overline{-\frac{1}{2}\Delta + V}$.*

Proof. Refer to [3, p. 278]. □

Lemma 5.8. *The semigroup e^{-tH} generated by the operator $-H$ is a contraction semigroup.*

Proof. Refer to [3, Chapter 12]. □

Lemma 5.9. *Suppose a sequence of complex-valued functions $\{f_n\}_{n=1}^{\infty}$ on $\mathbb{R}^d$ converges to a function f in the L^p sense where $1 \leq p \leq \infty$. Then there exists a subsequence $\{f_{n_j}\}_{j=1}^{\infty}$ that converges pointwise to f everywhere in $\mathbb{R}^d$ except perhaps on a set of Lebesgue measure zero.*

Proof. See [14, Theorem 3.12]. □

Lemma 5.10. *Let our potential function $V(\cdot)$ satisfy the inequality $V(x) \geq 0$ for all x in its domain. Also, let $C \in \mathbb{R}$ be a constant. Then the semigroup generated by $-(H+C)$ satisfies (in the L^2 sense) the equality $e^{-t(H+C)}\psi = e^{-tC}e^{-tH}\psi$ for all $\psi \in L^2(\mathbb{R}^d)$.*

Proof. From Lemma 5.8 it is clear that $e^{-t(H+C)}$ is a contraction semigroup. It is a natural strategy then to apply the Trotter product formula (Theorem 4.12).

First assume $C \geq 0$. Then $\left\|e^{-tC}\right\| \leq 1$. Using the Trotter product formula, the semigroup property of e^{-tH}, and the fact that e^{-tC} commutes with everything, we obtain

$$\begin{aligned} e^{-t(H+C)}\psi &= \lim_{n\to\infty} \left(e^{-(t/n)H}e^{-(t/n)C}\right)^n \psi = \lim_{n\to\infty} e^{-tC}\left(e^{-(t/n)H}\right)^n \psi \\ &= \lim_{n\to\infty} e^{-tC}e^{-tH}\psi = e^{-tC}e^{-tH}\psi, \end{aligned} \tag{5.11}$$

as desired.

Now consider the case where $C < 0$. Then the Trotter product formula is not directly applicable since $\left\|e^{-tC}\right\| > 1$. However, $e^{tC}e^{-tC}$ is a contraction semigroup since by Proposition 4.1,

$$\left\|e^{tC}e^{-tC}\right\| = \left\|e^{-t(C-C)}\right\| = \|1\| = \|I\| = 1.$$

The operator $e^{-t(C-C)}$ commutes with everything, and so using similar reasoning as before, we obtain

$$\begin{aligned} e^{tC}e^{-t(H+C)}\psi = e^{-t(H+C-C)}\psi &= \lim_{n\to\infty}\left(e^{-(t/n)(C-C)}e^{-(t/n)H}\right)^n \psi \\ &= \lim_{n\to\infty} e^{-t(C-C)}\left(e^{-(t/n)H}\right)^n \psi = \lim_{n\to\infty} e^{-t(C-C)}e^{-tH}\psi \\ &= e^{tC}e^{-tC}e^{-tH}\psi, \end{aligned} \tag{5.12}$$

where the first equality is justified using the exact same reasoning as in (5.11) since e^{tC} is a contraction semigroup. It immediately follows from (5.12) that $e^{-t(H+C)} = e^{-tC}e^{-tH}$. □

Proof of Theorem 5.5. Notice that if the Feynman-Kac formula holds for a given potential function $V(\cdot)$ then it holds for any $V(\cdot)+C$ where $C \in \mathbb{R}$ is a constant. This is because on the right hand side of (5.6) the integral in the exponent becomes $-\int_0^t V(\cdot)+C\,ds$. We can then integrate this constant term to get e^{-tC} which we can then pull out of the entire Wiener integral since it does not depend on the path ω. Similarly, by Lemma 5.10, we get a factor of e^{-tC} the left hand side of 5.5 as well. Hence, we can assume without loss of generality that $V(x) \geq 0$ for all x in its domain.

Since $V(\cdot)$ is a continuous function of compact support, by Proposition 5.3, the operator V is a bounded linear operator. Hence, the semigroup e^{-tV} generated by the operator $-V$ is uniformly continuous and has the familiar power series representation

$$e^{-tV} = \sum_{k=0}^{\infty} \frac{(-tV)^k}{k!}.$$

Also, since V is a multiplication operator it is clear that for all $t \in \mathbb{R}^+$, e^{-tV} is a multiplication operator as well.

Now we know from Example 3.16 that the free heat semigroup is a contraction semigroup given by

$$e^{-tH_0}\psi(x) = \int_{\mathbb{R}^d} (2\pi t)^{-d/2} e^{\frac{-\|x-y\|^2}{2t}} \psi(y)\,dy. \tag{5.13}$$

From (5.13) it is clear that

$$e^{-tH_0}e^{-tV}\psi(x) = (2\pi t)^{-d/2} \int_{\mathbb{R}^d} e^{\frac{-\|x-y\|^2}{2t}}\, e^{-tV(y)}\psi(y)dy. \tag{5.14}$$

With (5.14) in hand we can now calculate the n-th Trotter product:

$$\begin{aligned}
&\left(e^{-(t/n)H_0}e^{-(t/n)V}\right)^n \psi(x) \\
&= \left(\frac{2\pi t}{n}\right)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{\frac{-\|x-y_1\|^2}{2(t/n)}}\, e^{-(t/n)V(y_1)} \left(e^{-(t/n)H_0}e^{-(t/n)V}\right)^{n-1} \psi(y_1)\,dy_1 \\
&= \left(\frac{2\pi t}{n}\right)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{\frac{-\|x-y_1\|^2}{2(t/n)}}\, e^{-(t/n)V(y_1)} \times \\
&\quad \left(\frac{2\pi t}{n}\right)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{\frac{-\|y_1-y_2\|^2}{2(t/n)}}\, e^{-(t/n)V(y_2)} \left(e^{-(t/n)H_0}e^{-(t/n)V}\right)^{n-2} \psi(y_2)\,dy_2 dy_1 \\
&= \cdots \qquad\qquad \text{(observing the general pattern)} \\
&= \left(\frac{2\pi t}{n}\right)^{-\frac{dn}{2}} \underbrace{\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d}}_{\text{n times}} e^{-\sum_{k=1}^{n} \frac{-\|y_{k-1}-y_k\|^2}{2(t/n)}}\, e^{-(t/n)\sum_{k=1}^{n} V(y_k)} \psi(y_n)\,dy_n \cdots dy_1,
\end{aligned}$$

where in the last equality we set $y_0 = x$ so that the notation works out. Now let us make the substitution $u_k = y_k - x$ for all $k = 1, 2, \dots n$. We obtain

$$\left(e^{-(t/n)H_0}e^{-(t/n)V}\right)^n \psi(x) \tag{5.15}$$
$$= \left(\frac{2\pi t}{n}\right)^{-\frac{dn}{2}} \int_{\mathbb{R}^{nd}} e^{-\sum_{k=1}^{n} \frac{-\|u_{k-1}-u_k\|^2}{2(t/n)}} e^{-(t/n)\sum_{k=1}^{n} V(u_k+x)} \psi(u_n + x)\, du_n \cdots du_1$$
$$= \int_{\mathbb{R}^{nd}} e^{-(t/n)\sum_{k=1}^{n} V(u_k+x)} \psi(u_n + x) \prod_{k=1}^{n} G_d\left(u_{k-1} - u_k, \frac{t}{n}\right) d\vec{U},$$

where we let $u_0 = y_0 - x = 0$ and employ the standard simplification of notation by writing $\int_{\mathbb{R}^{nd}}$ instead of $\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d}$. We would like to write the integral in (5.15) as an integral over Wiener space. Since $\psi(\cdot)$ is complex-valued, we can do this by applying Corollary 2.37. This yields

$$\left(e^{-(t/n)H_0}e^{-(t/n)V}\right)^n \psi(x) = \int_{\mathcal{C}_0^{t,d}} e^{-(t/n)\sum_{k=1}^{n} V(\omega(kt/n)+x)} \psi(\omega(t) + x)\, d\mathfrak{m}(\omega) \tag{5.16}$$

To complete the proof, we need to be able to apply the Trotter product formula (Theorem 4.12) to the left hand side of (5.16). We already know that e^{-tH_0} and e^{-tH} are contraction semigroups from Example 3.16 and Lemma 5.8 respectively. Also, e^{-tV} is a multiplication operator and so by Proposition 5.3 its operator norm $\left\|e^{-tV}\right\|$ is equal to $\left\|e^{-tV}\right\|_{L^\infty}$. Hence, e^{-tV} is also a contraction semigroup since for all $t \in \mathbb{R}^+$:

$$\begin{aligned}
\left\|e^{-tV}\right\| = \left\|e^{-tV}\right\|_{L^\infty} &= \left\|\sum_{k=1}^{\infty} \frac{(-t)^n}{k!} V(\cdot)^k\right\|_{L^\infty} \\
&= \sup_{x\in\mathbb{R}^d} \left|\sum_{k=1}^{\infty} (-1)^k \frac{t^k}{k!} (V(x))^k\right| \\
&= \sup_{x\in\mathbb{R}^d} \sum_{k=1}^{\infty} (-1)^k \frac{t^k}{k!} (V(x))^k \\
&= \sup_{x\in\mathbb{R}^d} \sum_{k=1}^{\infty} (-1)^k \frac{t^k}{k!} |V(x)|^k & \text{since } V(\cdot) \geq 0 \\
&= \sum_{k=1}^{\infty} \frac{(-t)^k}{k!} \sup_{x\in\mathbb{R}^d} |V(x)|^k \\
&= \sum_{k=1}^{\infty} \frac{(-t)^k}{k!} \sup_{x\in\mathbb{R}^d} \|V\|_{L^\infty}^k = e^{-t\|V\|_{L^\infty}} \leq 1,
\end{aligned}$$

where we were able to write the L^∞ norm as a supremum instead of as an essential supremum since $V(\cdot)$ and hence $e^{-tV(\cdot)}$ are continuous functions.

Having satisfied the assumptions of Theorem 4.12, we can apply the Trotter product formula. This yields the following the limit which holds in the L^2 norm for all $\psi \in L^2(\mathbb{R}^d)$:

$$\lim_{n\to\infty} \left(e^{-(t/n)H_0}e^{-(t/n)V}\right)^n \psi = e^{-tH}\psi. \tag{5.17}$$

By Lemma 5.9 it follows that there exists a subsequence $\{n_j\}$ such that the convergence in 5.17 holds pointwise for all $x \in \mathbb{R}^d$ except perhaps on a set of Lebesgue measure zero. The function $V(\omega(\cdot) + x)$ is continuous since it is a composition of continuous functions and so consequently the exponent

$$-\frac{t}{n_j} \sum_{k=1}^{n_j} V\left(\omega\left(\frac{kt}{n_j}\right) + x\right)$$

in the integrand of (5.16) is a Riemann sum that converges to the Riemann integral

$$-\int_0^t V(\omega(s) + x)\, ds$$

as $j \to \infty$ for any $\omega \in \mathcal{C}_0^{t,d}$ and $x \in \mathbb{R}^d$. Hence,

$$e^{-\frac{t}{n_j}\sum_{k=1}^{n_j} V(\omega(kt/n_j)+x)} \psi(\omega(t) + x) \to e^{-\int_0^t V(\omega(s)+x)\,ds} \psi(\omega(t) + x) \text{ as } j \to \infty$$

for any $x \in \mathbb{R}^d$ and any $\omega \in \mathcal{C}_0^{t,d}$. We now rewrite (5.16) using the subsequence $\{n_j\}$ and take the limit as $j \to \infty$ on both sides. This yields

$$\lim_{j\to\infty} \left(e^{-(t/n_j)H_0} e^{-(t/n_j)V}\right)^{n_j} \psi(x) = \lim_{j\to\infty} \int_{\mathcal{C}_0^{t,d}} e^{-(t/n_j)\sum_{k=1}^{n_j} V(\omega(kt/n_j)+x)} \psi(\omega(t) + x)\, d\mathfrak{m}(\omega)$$

$$\Longrightarrow\ e^{-tH}\psi(x) = \lim_{j\to\infty} \int_{\mathcal{C}_0^{t,d}} e^{-(t/n_j)\sum_{k=1}^{n_j} V(\omega(kt/n_j)+x)} \psi(\omega(t) + x)\, d\mathfrak{m}(\omega) \tag{5.18}$$

Now if can pull the limit inside the integral in 5.18 then the proof is complete. Fortunately, the integrand satisfies the inequality

$$\begin{aligned}\left| e^{-(t/n_j)\sum_{k=1}^{n_j} V(\omega(kt/n_j)+x)} \psi(\omega(t) + x) \right| &\le e^{(t/n_j)\sum_{k=1}^{n_j} \|V(\cdot)\|_{L^\infty}} |\psi(\omega(t) + x| \\ &= e^{t\|V\|_{L^\infty}} |\psi(\omega(t) + x)|,\end{aligned}$$

and since $e^{t\|V\|_{L^\infty}} |\psi(\omega(t) + x)|$ depends on the path ω at only one time t, we can use Theorem 2.36 to conclude

$$\int_{\mathcal{C}_0^{t,d}} e^{t\|V\|_{L^\infty}} |\psi(\omega(t) + x)|\, d\mathfrak{m}(\omega) = \int_{\mathbb{R}^d} e^{t\|V\|_{L^\infty}} |\psi(y + x)| \cdot G_d(y, t)\, dy < \infty,$$

where the inequality holds because ψ is in $L^2(\mathbb{R}^d)$ and hence $|\psi(\cdot)|$ is bounded except perhaps on a set of Lebesgue measure zero. Hence, by the dominated convergence theorem

$$\begin{aligned}&\lim_{j\to\infty} \int_{\mathcal{C}_0^{t,d}} e^{-(t/n_j)\sum_{k=1}^{n_j} V(\omega(kt/n_j)+x)} \psi(\omega(t) + x)\, d\mathfrak{m}(\omega) \\ &\qquad = \int_{\mathcal{C}_0^{t,d}} \exp\left(-\int_0^t V(\omega(s) + x)\, ds\right) \psi(\omega(t) + x)\, d\mathfrak{m}(\omega)\end{aligned}$$

and the proof is complete. □

Remark 5.19. Through a limiting argument, it is possible to extend the proof above to the more general case where the potential function $V(\cdot)$ is only required to be bounded [3, Chapter 12].

By proving the Feynman-Kac formula, we have provided a rigorous mathematical underpinning for the imaginary-time Feynman path integral. In certain cases, we can extend this approach back to the "real-time" Feynman path integral via methods of analytic continuation. The precise details of how this is accomplished are well beyond the scope of this paper; we refer the interested reader to [3].

Resources

The figure in this paper was made using Inkscape and the TexText package.

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References

[1] R. Shankar. *Principles of Quantum Mechanics* 2e. Springer Science and Business Media. 1994.
[2] Brian C. Hall. *Quantum Theory for Mathematicians.* Springer Science and Business Media. 2013.
[3] Gerald W. Johnson and Michel L. Lapidus. *The Feynman Integral and Feynman's Operational Calculus.* Oxford University Press. 2000
[4] Richard F. Bass. *Real analysis for graduate students* 2e. 2014.
[5] Matthew Aldridge. Proof of Carathéodory's extension theorem. https://mpaldridge.github.io/teaching/ma40042-notes-07.pdf
[6] T. Hida. *Brownian Motion.* Springer-Verlag. 1980.
[7] Vladimir I. Bogachev. *Measure Theory.* Springer-Verlag. 2007.
[8] Allan Gut. *Probability: A Graduate Course.* Springer Science and Business Media. 2005.
[9] Klaus-Jochen Engel and Rainer Nagel. *A Short Course on Operator Semigroups.* Springer Science and Business Media. 2006.
[10] Jerome A. Goldstein. *Semigroups of Linear Operators and Applications.* Oxford University Press. 1985.
[11] Karl Glasner. Distributions and Distributional Derivatives. https://www.math.arizona.edu/~kglasner/math456/GREENS1.pdf.
[12] Lawrence C. Evans. *Partial Differential Equations* 2e. American Mathematical Society. 2010.
[13] Shlomo Sternberg. *Convergence of Semi-groups. The Trotter Product Formula. Feynman Path Integrals.* https://people.math.harvard.edu/~shlomo/212a/17.pdf. 2014.
[14] Walter Rudin. *Real and Complex Analysis* 3e. McGraw-Hill. 1987.
[15] James Glimm and Arthur Jaffe. *Quantum Physics: A Functional Integral Point of View* 2e. Springer-Verlag. 1987.
[16] Richard P. Feynman and Albert R. Hibbs. *Quantum Mechanics and Path Integrals.* McGraw-Hill Companies. 1965.
[17] Sheree L. LeVarge *Semigroups of Linear Operators.* https://www.math.arizona.edu/~flaschka/Topmatter/527files/termpapers/levarge.pdf. 2003.