Category $\mathcal{C}$: Objects Morphisms composition identities

Source Target

$\mathcal{C}(x,y)$ = set of morphisms $x \to y$

$id_x \in \mathcal{C}(x,x)$

$I_x : * \to \mathcal{C}(x,x)$

$* \mapsto id_x$

$$\begin{tikzcd}[column sep=small]
 & \mathcal{C}(y,z)\times\mathcal{C}(x,y)\times\mathcal{C}(w,x) \arrow[dl, "c\times id"'] \arrow[dr, "id\times c"] & \\
\mathcal{C}(x,z)\times\mathcal{C}(w,x) \arrow[dr, "c"'] \arrow[rr, phantom, "\text{ass}"] & & \mathcal{C}(y,z)\times\mathcal{C}(w,y) \arrow[dl, "c"] \\
 & \mathcal{C}(w,z) &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{C}(x,y)\times\mathcal{C}(x,x) \arrow[d, "c"'] & \mathcal{C}(x,y)\times * \arrow[l, "id\times I"'] \arrow[dl, equal] \\
\mathcal{C}(x,y) &
\end{tikzcd}
\qquad
\begin{tikzcd}
* \times \mathcal{C}(x,y) \arrow[r, "I\times id"] \arrow[dr, equal] & \mathcal{C}(y,y)\times\mathcal{C}(x,y) \arrow[d, "c"] \\
 & \mathcal{C}(x,y)
\end{tikzcd}$$

$\mathcal{C}at$: Objects : categories

Morphisms=? : Functors

$F : \mathcal{C} \to \mathcal{D}$ $\quad F : Ob\,\mathcal{C} \to Ob\,\mathcal{D}$

$F : \mathcal{C}(x,y) \to \mathcal{D}(Fx, Fy)$

$$\begin{tikzcd}
 & \mathcal{C}(x,x) \arrow[d, "F"] \\
* \arrow[ur, "I_x"] \arrow[r, "I_{Fx}"'] & \mathcal{D}(Fx,Fx)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}(y,z)\times\mathcal{C}(x,y) \arrow[r, "c"] \arrow[d, "F\times F"] & \mathcal{C}(x,z) \arrow[d, "F"] \\
\mathcal{D}(Fy,Fz)\times\mathcal{D}(Fx,Fy) \arrow[r, "c"] & \mathcal{D}(Fx,Fz)
\end{tikzcd}$$

$F(id_x) = id_{Fx}$ $\qquad F(g\circ f) = Fg \circ Ff$

Sets — Rings

Monoids — Modules over a ring

Groups — Fields, vector spaces

Abelian groups

Spaces T_0 spaces

Alexandroff spaces A-space = Alexandroff and T_0

Poset = partially ordered set

$(P, \leq)$ [Transitive $x \leq y$ $y \leq z \Rightarrow x \leq z$

preorder [reflexive $x \leq x$

antisymmetric $x \leq y$ and $y \leq x \Rightarrow x = y$

Topological space $(X, \mathcal{U})$ = {open sets}

$\phi, X \in \mathcal{U}$. $\bigcap_{\text{finite}} V \in \mathcal{U}$ $\bigcup V \in \mathcal{U}$

Alexandroff: $\bigcap V \in \mathcal{U}$ (not just finite)

$\mathcal{U}_x = \bigcap \{V \mid x \in V\}$ $x \leq y$ if $x \in \mathcal{U}_y$

$x \leq y, y \leq z \Rightarrow x \leq z$ exercize

$x \leq x$ clear

T_0: Topology separates points. $\mathcal{U}_x = \mathcal{U}_y \Rightarrow x = y$

In any category, such as $\mathcal{C}at$, an isomorphism $f : \mathcal{C} \to \mathcal{D}$ is a morphism (functor in $\mathcal{C}at$) with an inverse $f^{-1} : \mathcal{D} \to \mathcal{C}$

$ff^{-1} = id$ $f^{-1}f = id$

<u>Thm.</u> The categorys of T_0 Alexandroff spaces is isomorphic to the category of posets.

So posets have homotopy groups.

<u>Ex</u>: There is a poset with 6 pts and infinitely many non-zero homotopy groups

Hasse diagram

This morning:
Combinatorial Species: functor: $F^\times \to F^\times$
($F^\times$: finite sets and isomorphisms)

$T \sim S$ if T is nat. iso to S

Will make precise shortly

Some categories have a notion of homotopy

Sets? No

Spaces? Yes $f \simeq g$ if $\exists\, h: X \times I \to Y$

$f, g: X \to Y$ $\quad \ni h(x,0) = f(x)$

$h(x,1) = g(x)$

Categories: Yes! homotopy = nat trans.

$\mathcal{Q}$: objects 0, 1 morphisms

$$\begin{tikzcd} 0 \arrow[r, "I"] \arrow[loop below, "id"] & 1 \arrow[loop below, "id"] \end{tikzcd}$$

<u>homotopy</u> $H : F \to G$

$$F, G : \mathcal{C} \to \mathcal{Q}$$

$$H : \mathcal{C} \times \mathcal{Q} \longrightarrow \mathcal{Q}$$

$$H(x, 0) = F(x) \quad H(f, id_0) = F(f)$$

$$H(x, 1) = G(x) \quad H(f, id_1) = G(f)$$

$$\eta_x \equiv H(id_x, I) : F(x) \longrightarrow G(x)$$

$\forall f : x \to y$, The following diagram commutes

$$\begin{tikzcd}
H(x,0) \arrow[rrr, "H(f, id_0)"] \arrow[dr, equal] \arrow[ddd, "H(id_x, I)"'] & & & H(y,0) \arrow[ddd, "H(id_y, I)"] \\
& F(x) \arrow[r, "F(f)"] \arrow[d, "\eta_x"'] \arrow[dr] & F(y) \arrow[ur, equal] \arrow[d, "\eta_y"] & \\
& G(x) \arrow[r, "G(f)"'] \arrow[dl, equal] & G(y) \arrow[dr, equal] & \\
H(x,1) \arrow[rrr, "H(f, id_1)"'] & & & H(y,1)
\end{tikzcd}$$

$$(id_y, I) \circ (f, id_0)$$
$$\|$$
$$(f, I)$$
$$\|$$
$$(f, id_1) \circ (id_x, I)$$

<u>Natural transformation</u>

$$\eta : F \to G$$

Coambiguous names:

homotopy = natural transformation
in $\mathcal{C}at$ of functors

homotopy in spaces is an equivalence relation
$h: f \simeq g \quad h^{-1}: g \simeq f \quad h^{-1}(x,t) = h(x, 1-t)$
homotopy in categories is NOT. $F \simeq G \not\Rightarrow G \simeq F$

homotopy equivalence of spaces
$$f: X \to Y \quad g: Y \to X$$
$$g \circ f \simeq id_X \quad f \circ g \simeq id_Y$$

Natural isomorphism η_X an isomorphism for all X

Equivalence of categories
$$F: \mathcal{C} \to \mathcal{D} \quad G: \mathcal{D} \to \mathcal{C}$$
$$G \circ F \cong Id_{\mathcal{C}} \quad F \circ G \cong Id_{\mathcal{D}}$$
natural isomorphisms

Finite sets $\approx \{ n = \{1, 2, \ldots, n\} \}$ $n \geq 0$

category is connected if there is a "path" connecting any two objects.

a category is a groupoid if every morphism is an isomorphism

connected groupoids $\approx$ groups.

Skeleton of a category

Full subcategory, one object chosen in each isomorphism class

full subcategory $\mathcal{B} \subset \mathcal{C}$ $f : x \to y$ $x, y \in \mathcal{B}$ $\Rightarrow f \in \mathcal{B}$

$\mathcal{B}(x, y) = \mathcal{C}(x, y)$

<u>Exercise</u> $\mathrm{Skel}\, \mathcal{C} \hookrightarrow \mathcal{C}$ ($\mathcal{C}$ small : set of objects ; Kittycory)

is an equivalence of categories

<u>Generalization</u>

$F : \mathcal{C} \to \mathcal{D}$ is
- faithful, if $F : \mathcal{C}(X,Y) \xrightarrow{\text{inj}} \mathcal{D}(FX, FY)$
- full, and if $F : \mathcal{C}(X,Y) \xrightarrow{\text{surj}} \mathcal{D}(FX, FY)$
- essentially surjective on objects $\forall\, y \in \mathcal{D}\ \exists\, x \in \mathcal{C}\ \ni$ $F(X) \cong Y$

if and only if F is an equivalence

$F: \mathcal{C} \to \mathcal{D}$ $\quad G: \mathcal{D} \to \mathcal{C}$ adjoint

$$\mathcal{D}(Fx, y) \cong \mathcal{C}(x, Gy)$$

<u>natural in both variables</u>

$$\varepsilon : FGy \to y \longleftrightarrow id_{Gy}$$

$$id_{Fx} \longmapsto \eta : x \to GFx$$

<u>Exercise</u>

$$\begin{tikzcd} Fx \arrow[r, "F\eta"] \arrow[dr, "id"'] & FGFx \arrow[d, "\varepsilon"] \\ & Fx \end{tikzcd} \qquad \begin{tikzcd} Gy \arrow[r, "\eta"] \arrow[dr, "id"'] & GFGy \arrow[d, "G\varepsilon"] \\ & Gy \end{tikzcd}$$

triangle identities $\Leftrightarrow$ adjunction

(free, forgetful) adjunction
F $\quad$ U = underlying

$$\text{Groups}(FS, T) \cong \text{Sets}(S, UT)$$

$$\text{Rings}(FA, R) \cong \text{Ab}(A, UR)$$

$$\begin{tikzcd} & s\text{-Sets} \arrow[dr] & \\ B : \text{Cat} \arrow[ur] \arrow[rr] & & \text{Spaces} \end{tikzcd}$$

"classifying space functor"

Will come back to this

Is $\text{Cat}(\mathcal{C}, \mathcal{D})$ a category? YES

$f(x,y) = g(x)(y)$

sets; $Set(X,Y)$ is a set $\quad \underset{f}{Set(X \times Y, Z)} \cong \underset{g}{Set(X, Set(Y,Z))}$

$FX = X \times Y \quad GZ = Set(Y,Z)$

spaces: $Space(X,Y) \equiv Map(X,Y)$ is a space.

What is a "good" category of spaces?

Have product, characterized by a "universal property":

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[d, "k"] \arrow[dr, "g"] & \\ X & X \times Y \arrow[l, "\pi_1"] \arrow[r, "p_2"'] & Y \end{tikzcd}$$

$\forall (f,g) \; \exists! \, k$

$\ni \pi k = f \quad \rho k = g$

Want: $Map(X,Y)$ with adjunction
(for each fixed Y)

$$\underline{Map}(X \times Y, Z) \cong \underline{Map}(X, \underline{Map}(Y, Z))$$

$$f \qquad\qquad\qquad g$$

$$f(x,y) = g(x)(y)$$

"cartesian closed category" Top

categorical notation:

$\underline{Map}(X,Y) \quad Map(X,Y)$

$\underline{space} \qquad Set$

$\mathrm{Cat}(\mathcal{C}, \mathcal{D})$ = set of functors $\mathcal{C} \to \mathcal{D}$

$\underline{\mathrm{Cat}}(\mathcal{C}, \mathcal{D})$ = category of functors $\mathcal{C} \to \mathcal{D}$ (objects)

maps are natural transformations
$\eta : F \to G$, $\zeta : G \to H$

Composition $\zeta \circ \eta : F \to H$

$$\begin{tikzcd}[column sep=small]
F(x) \arrow[rr, "(\zeta\circ\eta)_x"] \arrow[dr, "\eta_x"'] & & H(x) \\
& G(x) \arrow[ur, "\zeta_x"'] &
\end{tikzcd}$$

Algebra? Ab category of Abelian groups and homomorphisms

$\mathrm{Ab}(A, B)$ = set of homomorphisms $A \to B$

$\underline{\mathrm{Ab}}(A, B)$ = Abelian group of homomorphisms
$= \mathrm{Hom}(A, B)$

$(f+g)(a) = f(a) + g(a)$ $\quad 0(a) = 0 \;\; \forall a$ the zero homo.

$(-f)(a) = -f(a)$

<u>Not cartesian closed</u>

$\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(A, \mathrm{Hom}(B, C))$

Exercise. $\underline{\mathrm{Cat}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \cong \underline{\mathrm{Cat}}(\mathcal{C}, \underline{\mathrm{Cat}}(\mathcal{D}, \mathcal{E}))$

<u>cartesian closed</u>

Tensor product of Abèlian groups

$$\begin{tikzcd} A \times B \arrow[r, "\text{bilinear}"] \arrow[d, "\text{bilinear}"'] & C \\ A \otimes B \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

universal property!

Hom (A, Hom (B, C))

$(a,b) \to g(a)(b)$ is bilinear

Tensor product $M \otimes_R N$ of modules over a commutative ring works the same way.

$$\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(A, \mathrm{Hom}(B, C))$$