

Category  $\mathcal{C}$ : Objects Morphisms  
 composition identities

Source Target  
 $\mathcal{C}(x, y)$  = set of morphisms  $x \rightarrow y$

$id_x \in \mathcal{C}(x, x)$       $\mathcal{C}(y, z) \times \mathcal{C}(x, y) \times \mathcal{C}(w, x)$

$I_x: * \rightarrow \mathcal{C}(x, x)$   
 $\mathbb{F} \rightarrow id_x$

$\mathcal{C}(x, z) \times \mathcal{C}(w, x)$       $\mathcal{C}(y, z) \times \mathcal{C}(w, y)$   
 $\downarrow \mathcal{C} \times id_y$       $\downarrow id_x \times \mathcal{C}$   
 $\mathcal{C}(x, z) \times \mathcal{C}(w, x)$       $\mathcal{C}(y, z) \times \mathcal{C}(w, y)$   
 $\downarrow \mathcal{C}$       $\downarrow \mathcal{C}$   
 $\mathcal{C}(w, z)$

$\mathcal{C}(x, y) \times \mathcal{C}(x, x) \xleftarrow{id_x \times I} \mathcal{C}(x, y) \times *$       $* \times \mathcal{C}(x, y) \xrightarrow{I \times id} \mathcal{C}(y, y) \times \mathcal{C}(x, y)$   
 $\downarrow \mathcal{C}$       $\downarrow \mathcal{C}$   
 $\mathcal{C}(x, y)$       $\mathcal{C}(x, y)$

Cat: Objects: categories  
 Morphisms = ? : Functors

$F: \mathcal{C} \rightarrow \mathcal{D}$       $F: Ob \mathcal{C} \rightarrow Ob \mathcal{D}$

$F: \mathcal{C}(x, y) \rightarrow \mathcal{D}(Fx, Fy)$

$*$   $\xrightarrow{I_x} \mathcal{C}(x, x)$   
 $\downarrow F$   
 $I_{Fx} \rightarrow \mathcal{D}(Fx, Fx)$

$\mathcal{C}(y, z) \times \mathcal{C}(x, y) \xrightarrow{\mathcal{C}} \mathcal{C}(x, z)$   
 $\downarrow F \times F$       $\downarrow F$

$\mathcal{D}(Fy, Fz) \times \mathcal{D}(Fx, Fy) \xrightarrow{\mathcal{C}} \mathcal{D}(Fx, Fz)$

$F(id_x) = id_{Fx}$

$F(g \circ f) = Fg \circ Ff$

Sets

Monoids

Groups

Abelian groups

Spaces  $T_0$  spaces

Alexandroff spaces A-space = Alexandroff and  $T_0$

Rings

Modules over a ring

Fields, vector spaces

(2)

Poset = partially ordered set

$(P, \leq)$  preorder  $\left\{ \begin{array}{l} \text{Transitive} \quad x \leq y, y \leq z \Rightarrow x \leq z \\ \text{reflexive} \quad x \leq x \\ \text{antisymmetric} \quad x \leq y \text{ and } y \leq x \Rightarrow x = y \end{array} \right.$

Topological space  $(X, \mathcal{U}) = \{\text{open sets}\}$

$\emptyset, X \in \mathcal{U}$ .  $\bigcap_{\text{finite}} V \in \mathcal{U}$   $\cup V \in \mathcal{U}$

Alexandroff:  $\bigcap V \in \mathcal{U}$  (not just finite)

$U_x = \bigcap \{V \mid x \in V\}$

$x \leq y$  if  $x \in U_y$

$x \leq y, y \leq z \Rightarrow x \leq z$

exercise

$x \leq x$  clear

$T_0$ : Topology separates points.  $U_x = U_y \Rightarrow x = y$

In any category, such as  $\text{Cat}$ , an

isomorphism  $f: \mathcal{C} \rightarrow \mathcal{D}$  is a morphism

(functor in  $\text{Cat}$ ) with an inverse  $f^{-1}: \mathcal{D} \rightarrow \mathcal{C}$

$ff^{-1} = \text{id}$   $f^{-1}f = \text{id}$

Thm. The category of T. Alexandroff<sup>3</sup> spaces is isomorphic to the category of posets.

So posets have homotopy groups.

Ex: There is a poset with 6 pts and infinitely many non-zero homotopy groups



Hasse diagram

This morning:

Combinatorial Species: functor;  $F^X \rightarrow F^X$

( $F^X$ : finite sets and

$T \sim S$  if  $T$  is natl iso to  $S$  (isomorphisms))

Will make precise shortly

Some categories have a notion of homotopy

Sets? No

Spaces? Yes  $f \approx g$  if  $\exists h: X \times I \rightarrow Y$

$f, g: X \rightarrow Y$   $\exists h(x, 0) = f(x)$

$h(x, 1) = g(x)$

Categories: Yes! homotopy = Nat Trans.

$\mathcal{Q}$  : objects  $0, 1$  morphisms 4

$$0 \xrightarrow{I} 1$$

$\text{id} \quad \text{id}$

homotopy  $H: F \rightarrow G$

$$F, G: \mathcal{C} \rightarrow \mathcal{Q}$$

$$H: \mathcal{C} \times I \rightarrow \mathcal{Q}$$

$$H(x, 0) = F(x) \quad H(f, 0) = F(f)$$

$$H(x, 1) = G(x) \quad H(f, 1) = G(f)$$

$$\eta_x \equiv H(\text{id}_x, I): F(x) \rightarrow G(x)$$

$\forall f: x \rightarrow y$ , The following diagram commutes

$$\begin{array}{ccc}
 H(x, 0) & \xrightarrow{H(f, \text{id}_0)} & H(y, 0) \\
 \downarrow H(\text{id}_x, I) & \searrow F(f) & \downarrow \eta_y \\
 F(x) & \xrightarrow{F(f)} & F(y) \\
 \downarrow \eta_x & \searrow & \downarrow \eta_y \\
 G(x) & \xrightarrow{G(f)} & G(y) \\
 \downarrow H(\text{id}_x, I) & \searrow & \downarrow H(\text{id}_y, I) \\
 H(x, 1) & \xrightarrow{H(f, \text{id}_1)} & H(y, 1)
 \end{array}$$

$$\begin{array}{c}
 (\text{id}_y, I) \circ (f, \text{id}_0) \\
 \parallel \\
 (f, I) \\
 \parallel \\
 (f, \text{id}_1) \circ (\text{id}_x, I)
 \end{array}$$

Natural transformation  
 $\eta: F \rightarrow G$

Coambiguous names :

→  
5

homotopy = natural transformation  
in  $\text{Cat}$  of functors

homotopy in spaces is an equivalence relation  
 $h: f \simeq g \quad h^{-1}: g \simeq f \quad h(x, z) = h(x, 1-z)$

homotopy in categories is NOT,  $F \simeq G \not\Rightarrow G \simeq F$

homotopy equivalence of spaces

$$f: X \rightarrow Y \quad g: Y \rightarrow X$$

$$g \circ f \simeq \text{id}_X \quad f \circ g \simeq \text{id}_Y$$

Natural isomorphism  $\eta_X$  an isomorphism  
for all  $X$

Equivalence of categories

$$F: \mathcal{C} \rightarrow \mathcal{D} \quad G: \mathcal{D} \rightarrow \mathcal{C}$$

$$G \circ F \simeq \text{Id}_{\mathcal{C}} \quad F \circ G \simeq \text{Id}_{\mathcal{D}}$$

natural isomorphisms

Finite sets  $\approx \{n = \{1, 2, \dots, n\} \mid n \geq 0\}$  6

category is connected if there is a "path" connecting any two objects.

a category is a groupoid if every morphism is an isomorphism

Connected groupoids  $\approx$  groups.

Skeleton of a category

Full subcategory, one object chosen in each isomorphism class

Full subcategory  $\mathcal{B} \subset \mathcal{C}$   $f: x \rightarrow y$   $x, y \in \mathcal{B}$   
 $\Rightarrow f \in \mathcal{B}$   
 $\mathcal{B}(x, y) = \mathcal{C}(x, y)$

Exercise  $\text{Skel } \mathcal{C} \hookrightarrow \mathcal{C}$  ( $\mathcal{L}$  small set of objects; Kittycony)  
is an equivalence of categories

Generalization

$F: \mathcal{C} \rightarrow \mathcal{D}$  is  $\left\{ \begin{array}{l} \text{faithful, } F \text{ if } F: \mathcal{C}(x, y) \xrightarrow{\text{inj}} \mathcal{D}(F(x), F(y)) \\ \text{full, and } F \text{ if } F: \mathcal{C}(x, y) \xrightarrow{\text{surj}} \mathcal{D}(F(x), F(y)) \\ \text{essentially} \\ \text{surjective} \\ \text{ON objects} \end{array} \right. \forall y \in \mathcal{D} \exists x \in \mathcal{C} \exists F(x) \cong y$

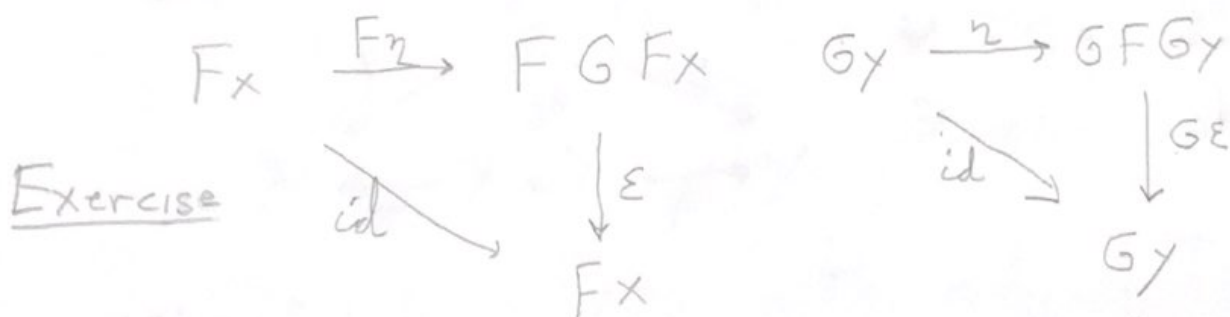
if and only if  $F$  is an equivalence

$F: \mathcal{C} \rightarrow \mathcal{D} \quad G: \mathcal{D} \rightarrow \mathcal{C}$  adjoint  $\uparrow$

$$\mathcal{D}(Fx, y) \cong \mathcal{C}(x, Gy) \quad \text{Natural in both variables}$$

$$\varepsilon: FGy \rightarrow y \iff \text{id}_{Gy}$$

$$\text{id}_{Fx} \iff \eta: x \rightarrow GFx$$



triangle identities  $\iff$  adjunction

(free, forgetful) adjunction  
 $F$   $U =$  underlying

$$\text{Groups}(FS, T) \cong \text{Sets}(S, UT)$$

$$\text{Rings}(FA, R) \cong \text{Ab}(A, UR)$$

$B: \text{Cat} \xrightarrow{\text{sets}} \text{Spaces}$  "classifying space functor"  
 Will come back to this

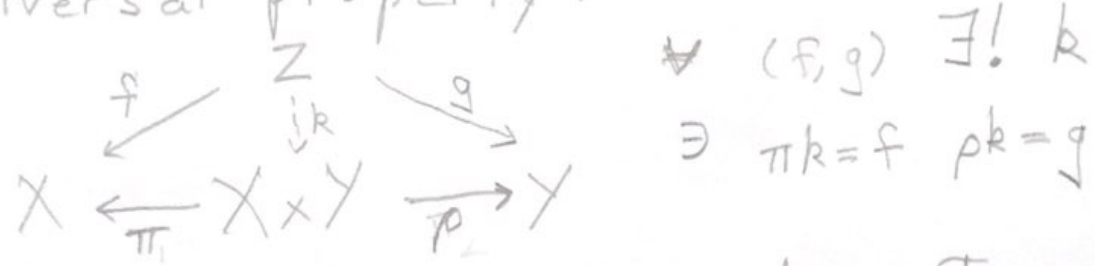
Is  $\text{Cat}(\mathcal{C}, \mathcal{D})$  a category? YES

$$f(x,y) = g(x)(y) \quad 8$$

sets;  $\text{Set}_f(X,Y)$  is a set  $\text{Set}_g(X \times Y, Z) \cong \text{Set}_g(X, \text{Set}_g(Y,Z))$   
 spaces;  $\text{Space}(X,Y) \equiv \text{Map}(X,Y)$  is a space.

What is a "good" category of spaces?

Have product, characterized by a "universal property":



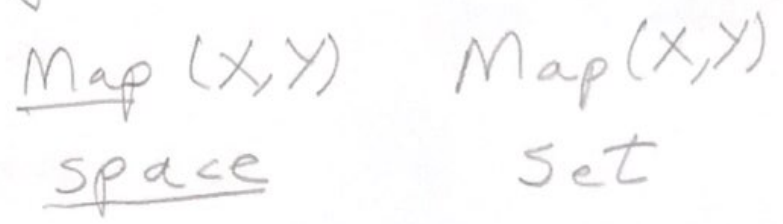
Want:  $\text{Map}(X,Y)$  with adjunction (for each fixed Y)

$$\text{Map}(X \times Y, Z) \cong \text{Map}(X, \text{Map}(Y, Z))$$

$f \qquad \qquad \qquad g$

$$f(x,y) = g(x)(y)$$

"cartesian closed category" Top  
 categorical notation:





$\text{Cat}(\mathcal{C}, \mathcal{D}) = \text{set of functors } \mathcal{C} \rightarrow \mathcal{D}$  9

$\text{Cat}(\mathcal{C}, \mathcal{D})$  = category of functors  $\mathcal{C} \rightarrow \mathcal{D}$   
(objects)

maps are natural transformations

$\eta: F \rightarrow G, \zeta: G \rightarrow H$

Composition  $\zeta \circ \eta: F \rightarrow H$

$(\zeta \circ \eta)_x: F(x) \rightarrow H(x)$   
 $\eta_x \rightarrow G(x) \xrightarrow{\zeta_x}$

Algebra? Ab category of Abelian groups  
and homomorphisms

$\text{Ab}(A, B) = \text{set of homomorphisms } A \rightarrow B$

$\text{Ab}(A, B)$  = Abelian group of homomorphisms

$\text{Hom}(A, B)$   $(f+g)(a) = f(a) + g(a)$   $0(a) = 0$   
 $(-f)(a) = -f(a)$   $\forall a$  the zero homo.

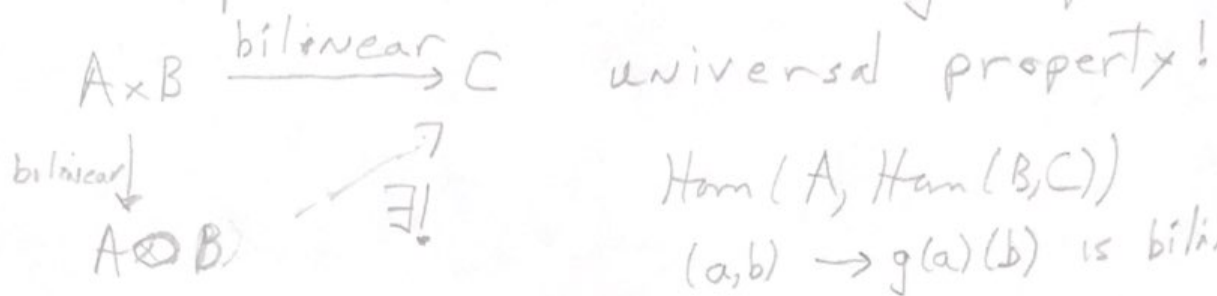
Not cartesian closed

$\text{Hom}(A \otimes B, C) \cong \text{Hom}(A, \text{Hom}(B, C))$

Exercise:  $\text{Cat}(\mathcal{B} \times \mathcal{D}, \mathcal{E}) \cong \text{Cat}(\mathcal{B}, \text{Cat}(\mathcal{D}, \mathcal{E}))$

cartesian closed

# Tensor product of Abelian groups 10



$\text{Hom}(A, \text{Hom}(B, C))$   
 $(a, b) \rightarrow g(a)(b)$  is bilinear

Tensor product  $M \otimes_R N$  of modules over a commutative ring works the same way.

$$\text{Hom}(A \otimes B, C) \cong \text{Hom}(A, \text{Hom}(B, C))$$