Characteristic classes

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CHAPTER

Introduction

These notes had their genesis in a class Peter May taught in the spring (?) quarter of 1974 at the University of Chicago. Robert Bruner was assigned the task of writing them up in a coherent fashion based on his class notes and Peter's notes. They were used in this handwritten form for many years at the University of Chicago. (What is the true version of this??) In the summer of 2012 Mike Catanzaro took on the task of TEXing the notes. After that, May and Bruner undertook some reorganization and added a few items to make the notes more self contained.

The precipitating event in the decision to publish them was a question from a colleague about the cohomology of a particular homogeneous space. It became clear these basic results in algebraic topology should be available in textbook form. (???????)

Compare to Mimura and Toda???

ADAPT THE FOLLOWING INTRODUCTORY SKETCH

We develop the classical theory of characteristic classes. Our procedure is simultaneously to compute the cohomology of the relevant classifying spaces and to display the standard axiomatically determined characteristic classes.

We first compute the homology and cohomology of Stiefel varieties and classical groups and then use the latter computations to pass to classifying spaces. Along the way, we compute the cohomologies of various homogeneous spaces, such as

$$Sp(n)/U(n), \quad U(2n)/Sp(n), \quad U(n)/O(n), \quad \text{and} \quad SO(2n)/U(n).$$

We also obtain the usual intrinsic characterizations, via the Thom isomorphism, of the Stiefel-Whitney and Euler classes.

Since we shall have a plethora of explicit calculations, some generic notational conventions will help to keep order.

We shall end up with the usual characteristic classes

$w_i \in H^i(BO(n); \mathbb{F}_2)$, the *Stiefel-Whitney* classes

$c_i \in H^{2i}(BU(n); \mathbb{Z})$, the *Chern* classes

$k_i \in H^{4i}(BSp(n); \mathbb{Z})$, the *symplectic* classes

$P_i \in H^{4i}(BO(n); \mathbb{Z})$, the *Pontryagin* classes

$\chi \in H^{2n}(BSO(2n); \mathbb{Z})$, the *Euler* class.

The P_i and χ will be studied in coefficient rings containing $1/2$ before being introduced integrally. We use the same notations for integral characteristic classes and for their images in cohomology with other coefficient rings.

Prerequisites: To do. (Just say "see the next chapter"?)

CHAPTER I

Classical groups and bundle theory

We introduce the spaces we shall study and review the fundamentals of bundle theory in this chapter. Aside from a few arguments included for didactic purposes, proofs are generally sketched or omitted. However, Sections 3 and 6 contain some material either hard to find or missing from the literature, and full proofs of such statements have been supplied.

We assume once and for all that all spaces we consider are to be of the homotopy type of CW-complexes. This ensures that a weak homotopy equivalence, namely a map which induces isomorphisms of homotopy groups for all choices of basepoints, is a homotopy equivalence. By the basic results of Milnor [**14**] (see also Schon [**16**]), this is not a very restrictive assumption. We also assume that all spaces are paracompact. This ensures that all bundles are numerable (in the sense specified in Section 2). Since all metric spaces, all countable unions of compact spaces, and all CW-complexes (Miyazaki [**15**] or Fritsch and Piccinini [**7**, Thm 1.3.5]), are paracompact, this assumption is also not unduly restrictive.

1. The classical groups

All of our work will deal with the classical Lie groups and related spaces defined in this chapter. Good general references for this section are Adams [**2**] and Chevalley [**5**].

Let $\mathbb{K}$ denote any one of $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, the real numbers, complex numbers, or quaternions. For $\alpha \in \mathbb{K}$, let $\overline{\alpha}$ denote the conjugate of α. A *right inner product space* over $\mathbb{K}$ is a right $\mathbb{K}$-module W, together with a function $(\ ,\ )\colon W \times W \longrightarrow \mathbb{K}$ which satisfies the following properties.

(i) $(x, y + y') = (x, y) + (x, y')$
(ii) $(x, y\alpha) = (x, y)\alpha$ for any $\alpha \in \mathbb{K}$
(iii) $(x, y) = \overline{(y, x)}$
(iv) $(x, x) \in \mathbb{R}$, $(x, x) \geq 0$, and $(x, x) = 0$ if and only if $x = 0$.

The unmodified term *inner product space* will mean right inner product space. All inner product spaces will be finite or countably infinite dimensional; we write $\dim W = \infty$ in the latter case.

We say that a $\mathbb{K}$-linear transformation $T\colon W \longrightarrow W$ is of *finite type* if W contains a finite dimensional subspace V invariant under T such that T restricts to the identity on $V^{\perp}$.

The *classical groups* are

$$GL(W) = \{T\colon W \longrightarrow W \mid T \text{ is invertible and of finite type}\},$$

$$U(W) = \{T \mid T \in GL(W) \text{ and } T \text{ is an isometry}\},$$

and, if $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$,

$$SL(W) = \{T \mid T \in GL(W) \text{ and } \det T = 1\}$$
$$SU(W) = \{T \mid T \in U(W) \text{ and } \det T = 1\}.$$

The finite type requirement assures that the determinant is well-defined. By choice of fixed orthonormal basis for W, we can identify $GL(W)$ with the group of matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$$

where A is an invertible $n \times n$ matrix with $n < \infty$. Such a matrix is in $U(W)$ if and only if $A^{-1} = \overline{A^T}$, where $\overline{A}$ is obtained from A by conjugating each entry and $(\ )^T$ denotes the transpose.

Topologize inner product spaces as the union (or colimit) of their finite dimensional subspaces. By choice of a fixed orthonormal basis and use of matrices, the classical groups of W may be topologized as subspaces of $\mathbb{K}^{n^2}$ when $n = \dim W < \infty$. The same topology may also be specified either in terms of norms of linear transformations or as the compact open topology obtained by regarding these groups as subsets of the space of maps $W \longrightarrow W$. With this topology, $G(W)$ is a Lie group ($G = GL, U, SL$, or SU) and $U(W)$ and $SU(W)$ are compact. When $\dim W = \infty$, $G(W)$ is topologized as the union of its subgroups $G(V)$, where V runs through all finite dimensional subspaces of W or through those V in any expanding sequence with union W.

A standard theorem of linear algebra states that any element of $GL(W)$ can be written uniquely as the product of a symmetric positive definite transformation and an element of $U(W)$, and similarly with GL and U replaced by SL and SU. It follows that the inclusions $U(W) \hookrightarrow GL(W)$ and $SU(W) \hookrightarrow SL(W)$ are homotopy equivalences. For our purpose, it suffices to restrict attention to $U(W)$ and $SU(W)$.

A convenient framework in which to view the classical groups is as follows.

Definition 1.1. Let $\mathscr{I}_{\mathbb{K}}$ denote the category of finite or countably infinite dimensional inner product spaces of $\mathbb{K}$ with linear isometries as morphisms. Note that isometries need not be surjective.

Then U and SU are functors from $\mathscr{I}_{\mathbb{K}}$ to the category of topological groups. Obviously if V and W are objects in $\mathscr{I}_{\mathbb{K}}$ of the same dimension, then there is an isomorphism $V \cong W$ in $\mathscr{I}_{\mathbb{K}}$ which induces isomorphisms $U(V) \cong U(W)$ and $SU(V) \cong SU(W)$.

This formulation has the conceptual clarity common to basis free presentations and will be useful in our proof of Bott periodicity. However, for calculational purposes, it is more convenient to deal with particular representatives of the classical groups. We define examples as follows, where $\mathbb{K}^n$ has its standard inner product.

(i) $O(n) = U(\mathbb{R}^n)$ and $O = U(\mathbb{R}^\infty)$ the *orthogonal* groups
(ii) $SO(n) = SU(\mathbb{R}^n)$ and $SO = SU(\mathbb{R}^\infty)$ the *special orthogonal* groups
(iii) $U(n) = U(\mathbb{C}^n)$ and $U = U(\mathbb{C}^\infty)$ the *unitary* groups
(iv) $SU(n) = SU(\mathbb{C}^n)$ and $SU = SU(\mathbb{C}^\infty)$ the *special unitary* groups
(v) $Sp(n) = U(\mathbb{H}^n)$ and $Sp = U(\mathbb{H}^\infty)$ the *symplectic* groups

There is another family of classical groups not included in this scheme, namely the *spinor* groups $\mathrm{Spin}(n)$ for $n > 2$ and $\mathrm{Spin} = \mathrm{Spin}(\infty)$. We define $\mathrm{Spin}(n)$ to be the universal covering group of $SO(n)$. Each $\mathrm{Spin}(n)$ for $n < \infty$ is a Lie group and $\mathrm{Spin} = \bigcup_n \mathrm{Spin}(n)$. Since $\pi_1(SO(n)) = \mathbb{Z}/2\mathbb{Z}$, $\mathrm{Spin}(n)$ is a 2-fold cover of $SO(n)$.

An alternative description of the spinor groups in terms of Clifford algebras is given in Chevalley [**5**, p.65].

There are forgetful functors

$$(\quad)^{\mathbb{R}} \colon \mathscr{I}_{\mathbb{C}} \longrightarrow \mathscr{I}_{\mathbb{R}} \qquad \text{and} \qquad (\quad)^{\mathbb{C}} \colon \mathscr{I}_{\mathbb{H}} \to \mathscr{I}_{\mathbb{C}}.$$

If W is in $\mathscr{I}_{\mathbb{C}}$, then $W^{\mathbb{R}}$ is the underlying real vector space with inner product the real part of the inner product of W. This induces an inclusion $U(W) \subset SU(W^{\mathbb{R}})$. Thus

$$U(n) \subset SO(2n) \qquad \text{and} \qquad U \subset SO.$$

Similarly, for W in $\mathscr{I}_{\mathbb{H}}$, we have $U(W) \subset SU(W^{\mathbb{C}})$ and thus

$$Sp(n) \subset SU(2n) \qquad \text{and} \qquad Sp \subset SU.$$

There are also extension of scalars functors

$$(\quad)_{\mathbb{C}} \colon \mathscr{I}_{\mathbb{R}} \longrightarrow \mathscr{I}_{\mathbb{C}} \qquad \text{and} \qquad (\quad)_{\mathbb{H}} \colon \mathscr{I}_{\mathbb{C}} \longrightarrow \mathscr{I}_{\mathbb{H}}.$$

If W is in $\mathscr{I}_{\mathbb{R}}$, then $W_{\mathbb{C}} = W \otimes_{\mathbb{R}} \mathbb{C}$ with inner product

$$(v \otimes \alpha, w \otimes \beta) = \overline{\alpha}(v, w)\beta.$$

This induces inclusions $U(W) \hookrightarrow U(W_{\mathbb{C}})$ and $SU(W) \hookrightarrow SU(W_{\mathbb{C}})$ via $T \mapsto T \otimes 1$. Thus

$$O(n) \subset U(n), \qquad O \subset U, \qquad SO(n) \subset SU(n), \qquad \text{and} \qquad SO \subset SU.$$

Similarly, for W in $\mathbb{C}$, $W_{\mathbb{H}} = W \otimes_{\mathbb{C}} \mathbb{H}$ as a right H-space. In this case, the noncommutativity of $\mathbb{H}$ requires careful attention; we are forced to the formula

$$(v \otimes \alpha, w \otimes \beta) = \overline{\alpha}(v, w)\beta.$$

for the inner product. This gives $U(W) \subset U(W_{\mathbb{H}})$ and thus

$$U(n) \subset Sp(n) \qquad \text{and} \qquad U \subset Sp.$$

These inclusions are summarized in the following diagram, the vertical inclusions of which are given by extension of scalars.

$$\begin{array}{ccccccccc}
SO(n) & \longrightarrow & O(n) & & & & & & \\
\downarrow & & \downarrow & & & & & & \\
SU(n) & \longrightarrow & U(n) & \longrightarrow & SO(2n) & \longrightarrow & O(2n) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & Sp(n) & \longrightarrow & SU(2n) & \longrightarrow & U(2n) & \longrightarrow & SO(4n) \longrightarrow O(4n).
\end{array}$$

In low dimensions, we have the following identifications:

(i) $SO(1) = SU(1) = e$ and $O(1) = \mathbb{Z}/2\mathbb{Z}$
(ii) $SO(2) \cong U(1) = T^1$ (the circle group)
(iii) $\mathrm{Spin}(3) \cong SU(2) \cong Sp(1) = S^3$ (the group of norm one quaternions)
(iv) $\mathrm{Spin}(4) \cong Sp(1) \times Sp(1)$
(v) $\mathrm{Spin}(5) \cong Sp(2)$
(vi) $\mathrm{Spin}(6) \cong SU(4)$

Together with the 2-fold covers $\mathrm{Spin}(n) \longrightarrow SO(n)$, this list gives all local isomorphisms among the classical Lie groups.

The following theorem will be essential to our work. Recall that a torus is a Lie group isomorphic to $T^n = (T^1)^n$, for some n.

Theorem 1.2. *A compact connected Lie group G contains maximal tori. Any two such are conjugate, and G is the union of its maximal tori.*

Actually, we shall only use particular maximal tori in our canonical examples of classical Lie groups. In $U(n)$, the subgroup of diagonal matrices is a maximal torus T^n. In $SU(n)$, the subgroup of diagonal matrices of determinant 1 is a maximal torus T^{n-1}. In $Sp(n)$, the subgroup of diagonal matrices with complex entries is a maximal torus T^n. In $SO(2n)$ or $SO(2n+1)$, the subgroup of matrices of the form $\mathrm{diag}(A_1, A_2, \ldots, A_n)$ or $\mathrm{diag}(A_1, A_2, \ldots, A_n, 1)$ with each $A_i \in SO(2) \cong T$ is a maximal torus T^n.

The quotient N/T, where T is a maximal torus in a compact Lie group G and N is the normalizer of T in G, is a finite group called the *Weyl group* of G and denoted $W(G)$. We shall say more about these groups where they are used.

2. Fiber bundles

Although our main interest will be in vector bundles, we prefer to view them in their proper general setting as examples of fiber bundles. This section and the next will give an exposition of the more general theory. We essentially follow Steenrod [**18**], but with a number of modifications and additions reflecting more recent changes in point of view.

Recall that a cover V_j of a space B is said to be *numerable* if it is locally finite and if each V_j is $\lambda_j^{-1}([0,1))$, for some map $\lambda_j \colon B \longrightarrow I$. Since every open cover of a paracompact space has a numerable refinement, we agree to restrict attention to numerable covers throughout. One motivation for doing so is the following standard result; see for example May [**11**, Sec 3.8].

Theorem 2.1. *A map $p\colon E \longrightarrow B$ is a fibration if it restricts to a fibration $p^{-1}(U) \longrightarrow U$, for all U in a numerable cover of B.*

Here, by a *fibration*, we understand a map $p\colon E \longrightarrow B$ which satisfies the covering homotopy property: for any map $f\colon X \longrightarrow E$ and homotopy $h\colon X \times I \longrightarrow B$ of pf, there is a homotopy H of f with $pH = h$. It follows that, for any basepoint in any fiber $F = p^{-1}(b)$, there is a long exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_n F \longrightarrow \pi_n E \longrightarrow \pi_n B \longrightarrow \pi_{n-1} F \longrightarrow \cdots$$

A fiber bundle is a locally trivial fibration with coordinate patches glued together continuously by means of some specified group. To be precise, recall that a (left) action by a topological group G on a space F is a map $G \times F \longrightarrow F$ such that $g \cdot (g'f) = (gg') \cdot f$ and $e \cdot f = f$, where e is the identity element of G. The group G is said to act *effectively* on F if $g \cdot f = g' \cdot f$ for all $f \in F$ implies $g = g'$; equivalently, the only element of G which acts trivially on F is e. The reader may want to think in terms of $G = U(W)$ and $F = W$ for some inner product space W.

Definition 2.2. A *coordinate bundle* $\xi = (E, p, B, F, G, \{V_j, \phi_j\})$ is a map $p\colon E \longrightarrow B$, an effective transformation group G of F, a numerable cover $\{V_j\}$ of B, and homeomorphisms $\phi_j \colon V_j \times F \longrightarrow p^{-1}(V_j)$ such that the following properties hold.

(i) $p \circ \phi_j \colon V_j \times F \longrightarrow V_j$ is the projection onto the first variable.
(ii) If $\phi_{j,x} \colon F \longrightarrow p^{-1}(x)$ is defined by $\phi_{j,x}(f) = \phi_j(x, f)$, then, for each $x \in V_i \cap V_j$, $\phi_{j,x}^{-1} \circ \phi_{i,x} \colon F \longrightarrow F$ coincides with operation by a (necessarily unique) element $g_{ji}(x) \in G$.
(iii) The function $g_{ji} \colon V_i \cap V_j \longrightarrow G$ is continuous.

Two coordinate bundles are *strictly equivalent* if they have the same *base space* B, *total space* E, *projection* p, *fiber* F, and *group* G and if the union of their *atlases* $\{V_j, \phi_j\}$ and $\{V'_k, \phi'_k\}$ is again the atlas of a coordinate bundle. A *fiber bundle*, or *G-bundle with fiber F*, is a strict equivalence class of coordinate bundles.

Definition 2.3. A map $(\widetilde{f}, f)$ of coordinate bundles is a pair of maps $f \colon B \longrightarrow B'$ and $\widetilde{f} \colon E \longrightarrow E'$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\widetilde{f}} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow[f]{} & B' \end{array}$$

commutes and the following properties hold.
(i) For each $x \in V_j \cap f^{-1}(V'_k)$, $(\phi'_{k,x})^{-1} \widetilde{f}_x \phi_{j,x} \colon F \longrightarrow F$ coincides with operation by a (necessarily unique) element $\overline{g}_{kj}(x) \in G$.
(ii) The function $\overline{g}_{kj} \colon V_j \cap f^{-1}(V'_k) \longrightarrow G$ is continuous.

Note that $\widetilde{f}$ is determined by f and the $\overline{g}_{kj}$ via the formula

$$\widetilde{f}(y) = \phi'_k(f(x), \overline{g}_{kj}(x)\phi_{j,x}^{-1}(y)) \text{ for } x \in V_j \cap f^{-1}(V'_k) \text{ and } y \in p^{-1}(x).$$

If f is a homeomorphism, then so is $\widetilde{f}$ and $(\widetilde{f}^{-1}, f^{-1})$ is again a bundle map. Two coordinate bundles with the same base space, fiber, and group are said to be *equivalent* if there is a bundle map between them which is the identity on the base space. Two fiber bundles are said to be equivalent if they have equivalent representative coordinate bundles.

These notions can all be described directly in terms of *systems of transition functions* $\{V_j, g_{ji}\}$, namely a numerable cover $\{V_j\}$ of B together with maps

$$g_{ji} \colon V_i \cap V_j \longrightarrow G$$

which satisfy the cocyle condition

$$g_{kj}(x) g_{ji}(x) = g_{ki}(x) \text{ for } x \in V_i \cap V_j \cap V_k$$

(from which $g_{ii}(x) = e$ and $g_{ij}(x) = g_{ji}(x)^{-1}$ follow). The maps g_{ji} of Definition 2.2 certainly satisfy this condition.

Theorem 2.4. *If G is an effective transformation group of F, then there exists one and, up to equivalence, only one G-bundle with fiber F, base space B and a given system $\{V_j, g_{ji}\}$ of transition functions. If ξ and ξ' are G-bundles with fiber F over B and B' determined by $\{V_j, g_{ji}\}$ and $\{V'_j, g'_{ji}\}$ and if $f \colon B \longrightarrow B'$ is any map, then a bundle map $(\widetilde{f}, f) \colon \xi \longrightarrow \xi'$ determines and is determined by maps $\overline{g}_{kj} \colon V_j \cap f^{-1}(V'_k) \longrightarrow G$ such that*

$$\overline{g}_{kj}(x) g_{ji}(x) = \overline{g}_{ki}(x) \ \textit{for}\ x \in V_i \cap V_j \cap f^{-1}(V'_k)$$

and

$$g'_{hk}(f(x)\overline{g}_{kj}(x) = \overline{g}_{hj}(x) \text{ for } x \in V_j \cap f^{-1}(V'_k \cap V'_h).$$

When $B = B'$ and f is the identity, these conditions on $\{\overline{g}_{kj}\}$ prescribe equivalence. When, further, ξ and ξ' have the same coordinate neighborhoods (as can always be arranged up to strict equivalence by use of intersections), ξ and ξ' are equivalent if and only if there exist maps $\psi_j \colon V_j \longrightarrow G$ such that

$$g'_{ji}(x) = \psi_j^{-1}(x)g_{ji}(x)\psi_i(x) \text{ for } x \in V_i \cap V_j.$$

For the first statement, E can be constructed from $\coprod V_j \times F$ by identifying $(x, y) \in V_i \times F$ with $(x, g_{ji}(x)y) \in V_j \times F$ whenever $x \in V_i \cap V_j$. For the second statement, $\widetilde{f}$ can and must be specified by the formula in Definition 2.3. For the last statement, set $\psi_j = (\overline{g}_{jj})^{-1}$ and $\overline{g}_{kj}(x) = \psi_j(x)^{-1}g_{kj}(x)$ to construct $\{\psi_j\}$ from $\{\overline{g}_{kj}\}$ and conversely. The requisite verifications are straightforward; see Steenrod [**18**, Sec. 2-3].

Fiber bundles are often just called G-bundles since Theorem 2.4 makes clear that the fiber plays an auxiliary role. In particular, we have described equivalences independently of F, and the set of equivalence classes of G-bundles is thus the same for all choices of F. We shall return to this point in the next section, where we consider the canonical choice $F = G$. Note too that the effectiveness of the action of G on F is not essential to the construction. In other words, if in Definitions 2.2 and 2.3 we assume given maps g_{ji} and $\overline{g}_{kj}$ with the prescribed properties, then we may drop the effectiveness since we no longer need the clauses (necessarily unique) in parts (ii).

The basic operations on fiber bundles can be described conveniently directly in terms of transition functions. The product $\xi_1 \times \cdots \times \xi_n$ of G-bundles ξ_q with fibers F_q and systems of transition functions $\{(V_j)_q, (g_{ij})_q\}$ is the $G_1 \times \cdots \times G_n$-bundle with fiber $F_1 \times \cdots \times F_n$ and system of transition functions given by the evident n-fold products of neighborhoods and maps. Its total space, base space, and projection are also the obvious products.

For a G-bundle ξ with fiber F, base space B, and system of transition functions $\{V_j, g_{ij}\}$ and for a map $f \colon A \longrightarrow B$, $\{f^{-1}(V_j), g_{ji} \circ f\}$ is a system of transition functions for the *induced G-bundle* $f^*\xi$ with fiber F over A. The total space of $f^*\xi$ is the pullback of f along the projection $p \colon E \longrightarrow B$. If $(\widetilde{f}, f) \colon \xi' \longrightarrow \xi$ is any bundle map, then ξ' is equivalent to $f^*\xi$. A crucially important fact is that homotopic maps induce equivalent G-bundles; see Steenrod [**18**, p.53] or Dold [**6**]. This is the second place where numerability plays a role.

For our last construction, we suppose given a continuous group homomorphism $\gamma \colon G \longrightarrow G'$, a specified G-space F and a specified G'-space F'. If ξ is a G-bundle with fiber F, base space B, and a system of transition functions $\{V_j, g_{ji}\}$, then $\{V_j, \gamma g_{ji}\}$ is a system of transition functions for the *coinduced G'-bundle* $\gamma_*\xi$ with fiber F' over B. As one special case, suppose that $F = F'$ and G acts on F through γ, $g \cdot f = (\gamma g) \cdot f$. We then say that $\gamma_*\xi$ is obtained from ξ by *extending its group* to G'. We say that the group of a G'-bundle ξ' with fiber F is *reducible* to G if ξ' is equivalent as a G'-bundle to some extended bundle $\gamma_*\xi$. Such an equivalence is called a *reduction* of the structural group. This language is generally only used when γ is the inclusion of a closed subgroup, in which case the last statement of Theorem 2.4 has the following immediate consequence.

Corollary 2.5. *Let H be a closed subgroup of G. A G-bundle ξ specified by a system of transition functions $\{V_j, g_{ji}\}$ has a reduction to H if and only if there exist maps $\psi_j\colon V_j \longrightarrow G$ such that*

$$\psi_j(x)^{-1} g_{ji}(x)\psi_i(x) \in H \text{ for all } x \in V_i \cap V_j.$$

A G-bundle is said to be *trivial* if it is equivalent to the G-bundle given by the projection $B \times F \longrightarrow F$ or, what amounts to the same thing, if its group can be reduced to the trivial group.

3. Principal bundles and homogeneous spaces

The key reason for viewing vector bundles in the context of fiber bundles is that the general theory allows the clearer understanding of the global structure of vector bundles that comes from the comparison of general fiber bundles to principal bundles.

Definition 3.1. A *principal G-bundle* is a G-bundle with fiber G regarded as a left G-space under multiplication. The principal G-bundle specified by the same system of transition functions as a given G-bundle ξ is called its *associated principal bundle* and denoted $\mathrm{Prin}\,\xi$. It is immediate from Theorem 2.4 that two G-bundles with same fiber are equivalent if and only if their associated principal bundles are equivalent. Two G-bundles with possibly different fibers are said to be *associated* if their associated principal bundles are equivalent.

If $\pi\colon Y \longrightarrow B$ is a principal G-bundle, then G acts from the right on Y in such a way that the coordinate functions $\phi_j\colon V_j \times G \longrightarrow Y$ are G-maps, where G acts on $V_j \times G$ by right translation of the second factor. Moreover, B may be identified with the orbit space of Y with respect to this action. The following description of the construction of general fiber bundles from principal bundles is immediate from the proof of Theorem 2.4.

Lemma 3.2. *Let $\pi\colon Y \longrightarrow B$ be a principal G-bundle. The associated G-bundle $p\colon E \longrightarrow B$ with fiber F has total space*

$$E = Y \times_G F = (Y \times F)/\sim, \text{ where } (yg, f) \sim (y, gf).$$

The map p is induced by passage to orbits from the projection $Y \times F \longrightarrow Y$.

The construction of $\mathrm{Prin}\,\xi$ from ξ is less transparent and will not be used in our work. We motivate it with the following categorical digression.[1]

Remark 3.3. We assume the reader knows about adjoint functors and the usual mapping space adjunction

$$Map(X \times Y, Z) \cong Map(X, Map(Y, Z))$$

for spaces X, Y, Z. We specialize to

$$\mathrm{Map}(Y \times F, E) \cong \mathrm{Map}(Y, \mathrm{Map}(F, E))$$

for a principal G-bundle $\pi\colon Y \longrightarrow B$, a left G-space F, and a G-bundle $\xi\colon E \longrightarrow B$ with fiber F. The definition of $\mathrm{Prin}\,\xi$ is designed to give an induced adjoint equivalence

$$\mathrm{Map}_B(G, F)(Y \times_G F, E) \cong \mathrm{Map}_B(G)(Y, \mathrm{Prin}\,\xi)$$

[1] We have not seen this observation in the literature.

Here $\mathrm{Map}_B(G, F)$ denotes maps of G-bundles with fiber F over B and $\mathrm{Map}_B(G)$ denotes maps of principal G-bundles over B.

Explicitly, if ξ is given by $p\colon E \longrightarrow B$ and has fiber F, call a map $\psi\colon F \longrightarrow p^{-1}(x)$ *admissible* if $\phi_{j,x}^{-1} \circ \psi\colon F \longrightarrow F$ coincides with action by an element of G, where $x \in V_j$, and note that admissibility is independent of the choice of coordinate neighborhood V_j. The total space Y of $\mathrm{Prin}\,\xi$ is the set of admissible maps $F \longrightarrow E$. Its projection to B is induced by p and its right G-action is given by composition of maps. Provided that the topology on G coincides with that obtained by regarding it as a subspace of the space of maps $F \longrightarrow F$ (with the compact open topology), Y is topologized as a subspace of the space of maps $F \longrightarrow E$. This proviso is satisfied in all of our examples.

The following consequence of Corollary 2.5 is often useful.

Proposition 3.4. *A principal G-bundle $\pi\colon Y \longrightarrow B$ is trivial if and only if it admits a cross section $\sigma\colon B \longrightarrow Y$.*

Proof. Necessity is obvious. Given a cross section σ and an atlas $\{V_j, \phi_j\}$, the maps $\psi_j\colon V_j \longrightarrow G$ given by $\psi_j(x) = \phi_{j,x}^{-1} \circ \sigma(x)$ satisfy

$$\psi_j(x)^{-1} g_{ji}(x)\psi_i(x) = e \text{ for } x \in V_i \cap V_j. \qquad \square$$

Another useful fact is that the local continuity conditions (iii) in Definitions 2.2 and 2.3 can be replaced by a single global continuity condition in the case of principal bundles.

Definition 3.5. Let Y be a right G-space and let $\mathrm{Orb}\,Y$ denote the subspace of $Y \times Y$ consisting of all pairs of points in the same orbit under the action of G. The space Y is said to be a *principal G-space* if $yg = y$ for any one $y \in Y$ implies $g = e$ and if $\tau\colon \mathrm{Orb}\,Y \longrightarrow G$ specified by $\tau(y, yg) = g$ is continuous. Let $B = Y/G$ with projection $\pi\colon Y \longrightarrow B$. Then Y is said to be *locally trivial* if B has a numerable cover $\{V_j\}$ together with homeomorphisms $\phi_j\colon V_j \times G \longrightarrow \pi^{-1}(V_j)$ such that $\pi\phi_j$ is the projection on V_j and ϕ_j is a right G-map.

Proposition 3.6. *A map $\pi\colon Y \longrightarrow B$ is a principal G-bundle if and only if Y is a locally trivial principal G-space, $B = Y/G$, and π is the projection onto orbits. If $\pi\colon Y \longrightarrow B$ and $\pi'\colon Y' \longrightarrow B'$ are principal G-bundles, then maps $\widetilde{f}\colon Y \longrightarrow Y'$ and $f\colon B \longrightarrow B'$ specify a bundle map $\pi \longrightarrow \pi'$ if and only if $\widetilde{f}$ is a right G-map and f is obtained from $\widetilde{f}$ by passage to orbits.*

Proof. Since any right G-map $G \longrightarrow G$ is left multiplication by an element of G, conditions (i) and (ii) of Definition 2.2 and 2.3 certainly hold for $\{V_j, \phi_j\}$ as in the previous definition. It is only necessary to relate the continuity conditions (iii) to the continuity of τ. Since

$$\phi_i(x, e) = \phi_{i,x}(e) = \phi_{j,x}(g_{ji}(x)) = \phi_j(x, e)g_{ji}(x) \text{ for } x \in V_i \cap V_j,$$

the following diagram commutes, where $\omega(h, g) = g^{-1}hg$.

$$\begin{array}{ccc} (V_i \cap V_j) \times G & \xrightarrow{(\phi_i, \phi_j)} & \mathrm{orb}\,\pi^{-1}(V_i \cap V_j) \\ {\scriptstyle g_{ij} \times 1}\downarrow & & \downarrow{\scriptstyle \tau} \\ G \times G & \xrightarrow{\quad\omega\quad} & G \end{array}$$

Moreover, (ϕ_i, ϕ_j) is a homeomorphism. It follows that τ is continuous if and only if all g_{ji} are so. Similarly, with the notations of Definition 2.3 (iii), the following diagram commutes.

$$\begin{array}{ccc}
\big(V_j \cap f^{-1}(V'_k)\big) \times G & \xrightarrow{(\widetilde{f}\phi_j, \phi'_k f)} & \mathrm{orb}(\pi')^{-1}(f(V_j) \cap V'_k) \\
\Big\downarrow{\scriptstyle \overline{g}_{kj} \times 1} & & \Big\downarrow{\scriptstyle \tau} \\
G \times G & \xrightarrow{\quad\omega\quad} & G
\end{array}$$

Therefore, the $\overline{g}_{kj}$ are continuous if τ is so. □

If H is a closed subgroup of a topological group G, we denote by G/H the space of left cosets gH in G with the quotient topology. Such a coset space is called a *homogeneous space*. The basic method in our study of the cohomology of classical groups will be the inductive analysis of various bundles relating such spaces. We need some preliminary observations in order to state the results which provide the requisite bundles. Subgroups are understood to be closed throughout.

We let G act on G/H by left translation. Let $H_0 \subset G$ be the subset of those elements g which act trivially on G/H. Explicitly, H_0 is a closed normal subgroup of G contained in H and is the largest subgroup of H which is normal in G. The factor group G/H_0 acts effectively on G/H.

Note that G, and thus also G/H_0, acts *transitively* on G/H. That is, for every pair of cosets x, x', there exists g such that $g \cdot x = x'$. Conversely if G acts transitively on a space X and if H is the *isotropy group* of a chosen basepoint $x \in X$, namely the subgroup of elements which fix x, then H is a closed subgroup of G and the map $p\colon G \longrightarrow X$ specified by $p(g) = gx$ induces a continuous bijection $q\colon G/H \longrightarrow X$. By the definition of the quotient topology, q^{-1} is continuous if and only if p is an open map. This is certainly the case when G is compact Hausdorff. We shall make frequent use of such homeomorphisms q and shall generally regard them as identifications.

In most cases of interest to us, the group H_0 is trivial by virtue of the following observation.

Lemma 3.7. *For $\mathbb{K} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, the largest subgroup of $U(\mathbb{K}^{n-1})$ which is normal in $U(\mathbb{K}^n)$ is the trivial group.*

Proof. $U(\mathbb{K}^n)/U(\mathbb{K}^{n-1})$ is homeomorphic to the unit sphere S^{dn-1}, $d = \dim_{\mathbb{R}} \mathbb{K}$, on which $U(\mathbb{K}^n)$ itself acts effectively. □

We need one other concept. Let $p\colon G \longrightarrow G/H$ be the quotient map. A *local cross section* for H in G is a neighborhood U of the basepoint eH in G/H together with a map $f\colon U \longrightarrow G$ such that $pf = \mathrm{id}$ on U. When G is a Lie group, a local cross section always exists by Chevally [**5**, p.110]. By Cartan, Moore, et al [**4**, p.5.10], the infinite classical groups are enough like Lie groups that essentially the same argument works for such G and reasonable H. The idea is that if G has a Lie algebra $\mathfrak{G}$ and H has a Lie algebra $\mathfrak{H} \subset \mathfrak{G}$ with $\mathfrak{G} = \mathfrak{H} \oplus \mathfrak{H}^{\perp}$, then the exponential local homeomorphism $\exp\colon (\mathfrak{G}, \mathfrak{H}) \longrightarrow (G, H)$ can be used to show that there is a homeomorphism $\phi\colon V \times H \longrightarrow W$ specified by $\phi(v, h) = \exp(v)h$, where V is a suitably small open neighborhood of 0 in $\mathfrak{H}^{\perp}$ and W is an open neighborhood of e in G. Then $U = p(W)$ and $f(u) = \exp(v)$ if $p\phi(v, h) = u$ specify the required local cross section.

Proposition 3.8. *If H has a local cross section in G, then $p\colon G \longrightarrow G/H$ is a principal H-bundle.*

PROOF. If $f\colon U \longrightarrow G$ is a local cross section, then $\{g(U) \mid g \in G\}$ is an open cover of G/H and the right H-maps $\phi_h\colon g(U) \times H \longrightarrow p^{-1}(g(U))$ specified by $\phi_g(gu, h) = gf(u)h$ are homeomorphisms. The continuity of $\tau\colon \mathrm{Orb}\, G \longrightarrow H$ is clear, hence the conclusion is immediate from Proposition 3.6. □

The proposition admits the following useful generalization.

Proposition 3.9. *If H has a local cross section in G and $\pi\colon Y \longrightarrow B$ is a principal G-bundle, then the projection $q\colon Y \longrightarrow Y/H$ is a principal H-bundle.*

PROOF. Let $\{V_j, \phi_j\}$ be an atlas for π. With the notations of the previous proof, let $W_{j,g} = q\phi_j(V_j \times p^{-1}(gU)) \subset Y/H$. Then the right H-maps

$$\omega_{j,g}\colon W_{j,g} \times H \longrightarrow q^{-1}(W_{j,g})$$

given by

$$\omega_{j,g}(\phi_j(x, \phi_g(gu, e))H, h) = \phi_j(x, gf(u))h$$

are homeomorphisms, where $q(y) = yH$. The conclusion is again immediate from Proposition 3.6. □

These principal bundles appear in conjunction with associated bundles with homogeneous spaces as fibers.

Lemma 3.10. *If $\pi\colon Y \longrightarrow B$ is a principal G-bundle, then passage to orbits yields a G/H_0 bundle $Y/H \longrightarrow B$ with fiber G/H.*

PROOF. This is immediate by passage to orbits on the level of coordinate functions. □

This lends to the following generalization of Proposition 3.4.

Proposition 3.11. *If H has a local cross section in G and $\pi\colon Y \longrightarrow B$ is a principal G-bundle, then π admits a reduction of its structure group to H if and only if the orbit bundle $Y/H \longrightarrow B$ admits a cross section.*

PROOF. We use Corollary 2.5 and the notations of the previous two proofs. Given $\psi_j\colon V_j \longrightarrow G$ such that $\psi_j(x)^{-1}g_{ji}(x)\psi_i(x) \in H$ for $x \in V_i \cap V_j$, the formula $\sigma(x) = \phi_j(x, \psi_j(x))H$ for $x \in V_j$ specifies a well-defined global cross section $\sigma\colon B \longrightarrow Y/H$. Conversely, given σ, the maps $\psi_j\colon V_j \longrightarrow G$ specified by $\psi_j(x) = gf(u)$ if $\sigma(x) = \phi_j(x, gf(u))H$ satisfy the cited condition. □

Finally, we note that Proposition 3.8 and Lemma 3.10 together imply most of the following generalization of the former.

Proposition 3.12. *Let $J \subset H \subset G$ and let H admit a local cross section in G. Then the inclusion of cosets $G/H \hookrightarrow G/J$ is an H/J_0-bundle with fiber H/J, where J_0 is the largest subgroup of J which is normal in H. Moreover, left translation by elements of G specifies self maps of this bundle, and its associated principal bundle is $G/J_0 \longrightarrow G/H$.*

4. Vector bundles, Stiefel and Grassmann manifolds

By a *vector bundle*, we understand a $U(W)$-bundle ξ with fiber W, where W is any finite dimensional inner product space over $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. When $W = \mathbb{K}^n$, we refer to ξ as a (real, complex, or quaternionic) n-plane bundle. Taking the group to be $U(W)$ rather than $GL(W)$ implies that we can give ξ an inner product metric. That is, we can transport the inner product of W onto fibers $p^{-1}(x)$ by means of the coordinate functions, and the positive definite quadratic forms given by the real numbers (y, y) for $y \in p^{-1}(x)$ then specify a continuous function $\mu\colon E \longrightarrow \mathbb{R}$. The map μ is generally called a *Euclidean metric* in the real case and a *Hermitian metric* in the complex case.

Vector bundles are our basic objects of study, and we need various operations on them. We give a generic construction. Recall that $\mathscr{I}_{\mathbb{K}}$ denotes the category of finite or countably infinite dimensional inner product spaces over $\mathbb{K}$ with linear isometries as morphisms (Section 1). Let $\mathscr{I}_{\mathbb{K}}^* \subset \mathscr{I}_{\mathbb{K}}$ denote the subcategory containing all objects and the linear isometric isomorphisms. This is a topological category, by which we understand a category enriched over topological spaces. That is, its hom sets are topological spaces and composition is continuous. A functor between topological categories is said to be *continuous* if it induces continuous maps on hom sets. Suppose we're given such a functor

$$T\colon \mathscr{I}_{\mathbb{K}_1}{}^* \times \cdots \times \mathscr{I}_{\mathbb{K}_n}{}^* \longrightarrow \mathscr{I}_{\mathbb{K}}^*$$

(where $\mathbb{K}$ and each $\mathbb{K}_q$ is one of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$). We allow contravariance; that is, some of the $\mathscr{I}_{\mathbb{K}}{}^*$ may be replaced by their opposites, and then $U(W_q)$ is to be replaced by its opposite below. Then T gives continuous homomorphisms

$$T\colon U(W_1) \times \cdots \times U(W_n) \longrightarrow UT(W_1, \ldots, W_n) \text{ for } W_q \in \mathscr{I}_{\mathbb{K}_q}{}^*.$$

Suppose given vector bundles ξ_q with fiber W_q over base spaces B_q and suppose that ξ_q is specified by a system of transition functions $\{V_{j,q}, (g_{ij})_q\}$. We obtain a vector bundle with fiber $T(W_1, \ldots, W_n)$ over $B_1 \times \cdots \times B_n$ by virtue of the system of transition functions obtained by composing products of maps $(g_{ij})_q\colon V_{i,q} \cap V_{j,q} \longrightarrow U(W_q)$, or $(g_{ji})_q^{-1}$ if T is contravariant in the q^{th} variable, with the above maps T. This defines the external operation on bundles determined by T. With $B_1 = \cdots = B_n = B$, the pullback of this external operation along the diagonal $\Delta\colon B \longrightarrow B^n$ gives the corresponding internal operation. The notation $T(\xi_1, \ldots, \xi_n)$ will be reserved for the internal operation.

The most important example is the *Whitney sum* $\xi \oplus \xi'$, which is obtained from $\oplus\colon \mathscr{I}_{\mathbb{K}}^* \times \mathscr{I}_{\mathbb{K}}^* \longrightarrow \mathscr{I}_{\mathbb{K}}^*$. Note that if η is a sub-bundle of a vector bundle ξ, so that each fiber of η is a sub-inner product space of the corresponding fiber of ξ, then η has a complement $\eta^{\perp}$ such that $\xi = \eta \oplus \eta^{\perp}$. An n-plane bundle ξ is *trivial* if and only if it admits n orthonormal cross sections and, by fiberwise Gram-Schmidt orthonormalization, this holds if and only if ξ admits n linearly independent cross sections. Indeed, a nowhere zero cross section prescribes a $\mathbb{K}$-line sub-bundle, and the conclusion follows by induction.

Other examples are given by such functors as Hom, the tensor product, and exterior powers. Note in particular that our general context includes such operations as

$$\otimes_R\colon \mathscr{I}_{\mathbb{H}}^* \times \mathscr{I}_{\mathbb{H}}^* \longrightarrow \mathscr{I}_{\mathbb{R}}^* \text{ and } \otimes_R\colon \mathscr{I}_{\mathbb{H}}^* \times \mathscr{I}_{\mathbb{R}}^* \longrightarrow \mathscr{I}_{\mathbb{H}}^*.$$

These will be useful in the study of Bott periodicity.

We next recall some of the classical examples of homogeneous spaces.

Definition 4.1. Let W be an inner product space over $\mathbb{K}$. A q-frame in W is an ordered q-tuple of orthonormal vectors. A q-plane in W is a sub inner product space of dimension q over $\mathbb{K}$. Let $V_q(W)$ be the set of q-frames in W topologized as a subspace of W^q. Let $G_q(W)$ be the set of q-planes in W, let $\pi\colon V_q(W) \longrightarrow G_q(W)$ be the map which sends a q-frame to the q-plane it spans, and give $G_q(W)$ the resulting quotient topology. The spaces $V_q(W)$ and $G_q(W)$ are called the *Stiefel manifolds* and *Grassmann manifolds* of W. Clearly $U(W)$ acts transitively on these spaces. If $x \in V_q(W)$ spans $X \in G_q(W)$, then $U(X^\perp)$ fixes x. There result homeomorphisms

$$V_q(W) \cong U(W)/U(X^\perp) \text{ and } G_q(W) \cong U(W)/U(X) \times U(X^\perp),$$

and $\pi\colon V_q(W) \longrightarrow G_q(W)$ is a principal $U(X)$-bundle. The associated bundle with fiber X has total space

$$\{(Y, w) \mid Y \text{ is a } q\text{-plane in } W,\ w \text{ is a vector in } Y\}$$

topologized as a subspace of $G_q(W)\times W$ and given the evident projection to $G_q(W)$. These are the *classical universal vector bundles.*

We also need the oriented variants of these spaces.

Definition 4.2. An *orientation* of an inner product space Y of dimension q over $\mathbb{R}$ or $\mathbb{C}$ is an equivalence class of q-frames, where q-frames y and y' are equivalent if the element $g \in U(Y)$ such that $gy = y'$ has determinant one. Let $\widetilde{G}_q(W)$ be the set of oriented q-planes in W, let $\pi\colon V_q(W) \longrightarrow \widetilde{G}_q(W)$ send a q-frame to the oriented q-plane it determines, and give $\widetilde{G}_q(W)$ the resulting quotient topology. Let $p\colon \widetilde{G}_q(W) \longrightarrow G_q(W)$ be given by neglect of orientation. The space $\widetilde{G}_q(W)$ is called the *oriented Grassmann manifold* of W. For any oriented $X \in \widetilde{G}_q(W)$, there is a homeomorphism

$$\widetilde{G}_q(W) \cong U(W)/\left(SU(X) \times U(X^\perp)\right);$$

$\pi\colon V_q(W) \longrightarrow \widetilde{G}_q(W)$ is a principal $SU(X)$-bundle and $p\colon \widetilde{G}_q(W) \longrightarrow G_q(W)$ is a principal S^{d-1}-bundle, where S^{d-1} is $\mathbb{Z}/2\mathbb{Z}$ in the real case and the circle group in the complex case. The associated bundle of π with fiber X has total space

$$\{(Y, w) \mid Y \text{ is an oriented } q\text{-plane in } W,\ w \text{ is a vector in } Y.\}$$

The following lemma will imply that, when $\dim W = \infty$, the bundles $V_q(W) \longrightarrow G_q(W)$ and $V_q(W) \longrightarrow \widetilde{G}_q(W)$ are in fact "universal" in the sense to be discussed in the following section.

Lemma 4.3. *If* $\dim W = \infty$*, then* $V_q(W)$ *is contractible for all* q.

FIX

Proof. By May [**12**, I.1.3], the space of linear isometries $Y \longrightarrow W$ is contractible for all inner product spaces Y. Let $X \subset W$ have dimension q. Since $\dim X^\perp = \infty$, the inclusion $X^\perp \hookrightarrow W$ is homotopic through isometries to an isomorphism, hence the inclusion $U(X^\perp) \hookrightarrow U(W)$ is homotopic to a homeomorphism and is thus a homotopy equivalence. Therefore, $\pi_* V_q(W) = 0$ by the long exact sequence of homotopy groups of the fibration sequence $U(X^\perp) \longrightarrow U(W) \longrightarrow V_q(W)$. The conclusion follows. $\square$

Remark 4.4. For n finite or $n = \infty$, $G_1(\mathbb{K}^{n+1})$ is the projective space $\mathbb{K}P^n$ of lines through the origin in $\mathbb{K}^{n+1}$ and $V_1(\mathbb{K}^{n+1})$ is the unit sphere $S^{d(n+1)-1}$ in $\mathbb{K}^{n+1}$, $d = \dim_{\mathbb{R}} \mathbb{K}$. The principal S^{d-1}-bundles $S^{d(n+1)-1} \longrightarrow \mathbb{K}P^n$ obtained by sending a point to the line it determines are called *Hopf bundles.* The associated line bundles are called the *canonical line bundles* over projective spaces. However, the reader should be warned that, in the complex case, some authors take the conjugates of these line bundle to be "canonical".

For calculational purposes, we record the canonical examples of Stiefel manifolds and the various bundles relating them.

Definition 4.5. For $0 < q \leq n$, the Stiefel manifold $V_q(\mathbb{K}^n)$ is the homogeneous space $U(\mathbb{K}^n)/U(\mathbb{K}^{n-q})$. That is,

$$\begin{array}{rcl} V_q(\mathbb{R}^n) & \cong & O(n)/O(n-q) \\ V_q(\mathbb{C}^n) & \cong & U(n)/U(n-q) \\ V_q(\mathbb{H}^n) & \cong & Sp(n)/Sp(n-q). \end{array}$$

Lemma 4.6. *For $q < n$, the natural maps*

$$SO(n)/SO(n-q) \longrightarrow V_q(\mathbb{R}^n)$$

and

$$SU(n)/SU(n-q) \longrightarrow V_q(\mathbb{C}^n)$$

are homeomorphisms. There are canonical homeomorphisms

$$\begin{array}{rclcrclcrcl} V_n(\mathbb{R}^n) & \cong & O(n) & \quad & V_n(\mathbb{C}^n) & \cong & U(n) & \quad & V_n(\mathbb{H}^n) & \cong & Sp(n) \\ V_{n-1}(\mathbb{R}^n) & \cong & SO(n) & & V_{n-1}(\mathbb{C}^n) & \cong & SU(n) & & & & \\ V_1(\mathbb{R}^n) & \cong & S^{n-1} & & V_1(\mathbb{C}^n) & \cong & S^{2n-1} & & V_1(\mathbb{H}^n) & \cong & S^{4n-1}. \end{array}$$

The results of the previous section have the following immediate consequence.

Proposition 4.7. *For $0 < p < q \leq n$, there is a commutative diagram*

$$\begin{array}{ccccc} U(\mathbb{K}^{n-q}) & = & U(\mathbb{K}^{n-q}) & & \\ \downarrow & & \downarrow & & \\ U(\mathbb{K}^{n-p}) & \longrightarrow & U(\mathbb{K}^n) & \longrightarrow & V_p(\mathbb{K}^n) \\ \downarrow & & \downarrow & & \| \\ V_{q-p}(\mathbb{K}^{n-p}) & \longrightarrow & V_q(\mathbb{K}^n) & \longrightarrow & V_p(\mathbb{K}^n) \end{array}$$

in which the left two columns are principal $U(\mathbb{K}^{n-q})$-bundles, the map between these columns is a bundle map, the middle row is a principal $U(\mathbb{K}^{n-p})$-bundle, and the bottom row is its associated bundle with fiber $V_{q-p}(\mathbb{K}^{n-p})$.

The case $p = q - 1$ is of particular interest since it allows inductive study of these spaces for fixed n.

5. The classification theorem and characteristic classes

While vector bundles are our ultimate objects of study (although they themselves are of greatest interest as a tool for the study of manifolds), our calculational focus will be on classifying spaces. It is our belief that this focus yields the most efficient proofs and the greatest insight, at least from the viewpoint of algebraic topology.

Definition 5.1. A *universal bundle* for a topological group G is a principal G-bundle $\pi\colon EG \longrightarrow BG$ such that EG is contractible. Any such base space BG is called a *classifying space* for G.

Proposition 3.12 and Lemma 4.3 imply that if G is a closed subgroup of $U(\mathbb{K}^q)$, then the principal G-bundle

$$V_q(\mathbb{K}^q \oplus \mathbb{K}^\infty) = U(\mathbb{K}^q \oplus \mathbb{K}^\infty)/U(\mathbb{K}^\infty) \longrightarrow U(\mathbb{K}^q \oplus \mathbb{K}^\infty)/G \times U(\mathbb{K}^\infty)$$

is universal. This is especially pertinent, since any compact Lie group embeds in some $U(\mathbb{C}^q)$ by the Peter-Weyl Theorem (e.g. [**9**, Thm. 1.15]).

Theorem 5.2 (The Classification Theorem)**.** *If G acts effectively on a space F, then equivalence classes of G-bundles over X with fiber F are in natural one-to-one correspondence with homotopy classes of maps $X \longrightarrow BG$.*

The correspondence assigns to a map $f\colon X \longrightarrow BG$ the G-bundle with fiber F associated to the induced principal G-bundle $f^*\pi$. Every topological group has a universal bundle π, and any two universal bundles are canonically equivalent. In particular, any two classifying spaces for G are canonically homotopy equivalent.

The classification theorem admits various proofs, such as those of Steenrod [**18**], Dold [**6**][OB is this the right Dold?] [[MC: Don't know]], and tom Dieck [**19**], the last being particularly elegant. The proof given in May [**11**] has the advantage that it applies equally well to the classification of fibrations and to the classification of bundles and fibrations with various kinds of additional structure, such as an orientation with respect to a generalized cohomology theory.

Let $[X, Y]$ denote the set of homotopy classes of maps $X \longrightarrow Y$. When X and Y have basepoints (denoted $*$), let $[X, Y]_0$ denote the set of based homotopy classes of based maps $X \longrightarrow Y$. If $* \longrightarrow X$ is a cofibration (as always holds if X is a CW-complex), then $\pi_q Y$ acts on $[X, Y]_0$ since evaluation at $*$ is a fibration $Y^X \longrightarrow Y$. The action is trivial if Y is an H-space by Whitehead [**20**, p. 119]. Neglect of basepoints defines a bijection from the orbit set $[X, Y]_0/\pi_1(Y)$ to $[X, Y]$. We shall see in the next section that BG is simply connected if G is connected. Thus $[X, BG] = [X, BG]_0$ if G is connected or if BG is an H-space.

Two familiar examples are

$$B\mathbb{Z}/2\mathbb{Z} = BO(1) = \mathbb{R}\mathrm{P}^\infty = \mathbb{K}(\mathbb{Z}/2\mathbb{Z}, 1)$$

and

$$BT^1 = BU(1) = \mathbb{C}\mathrm{P}^\infty = \mathbb{K}(\mathbb{Z}, 2),$$

where $\mathbb{K}(\pi, n)$ denotes a space with n^{th} homotopy group π and remaining homotopy groups zero. Such Eilenberg-Maclane spaces represent cohomology,

$$H^n(X; \pi) = [X, \mathbb{K}(\pi, n)]_0$$

and we conclude from the classification theorem that $O(1)$-bundles and $U(1)$-bundles over X are in natural one-to-one correspondence with elements of the cohomology groups $H^1(X;\mathbb{Z}/2\mathbb{Z})$ and $H^2(X;\mathbb{Z})$ respectively.

Many important properties of classifying spaces can be deduced directly from the classification theorem. Some details of proofs may help clarify the translation back and forth between bundle theory and homotopy theory.

Proposition 5.3. *Up to homotopy, passage to classifying spaces specifies a product-preserving functor from topological spaces to topological spaces.*

Proof. If $\pi_i\colon EG_i \longrightarrow BG_i$ is a universal bundle for $G_i, i = 1, 2$, then $\pi_1 \times \pi_2\colon EG_1 \times EG_2 \longrightarrow BG_1 \times BG_2$ is clearly a universal bundle for $G_1 \times G_2$. Therefore $BG_1 \times BG_2$ is a classifying space for $G_1 \times G_2$. If $\gamma\colon G \longrightarrow G'$ is a continuous homomorphism, coinduction assigns a principal G'-bundle $\gamma_*\xi$ to a principal G-bundle ξ (see section 2). Any map $B\gamma\colon BG \longrightarrow BG'$ also converts principal G-bundles to principal G'-bundles, via composition with classifying maps. Since these constructions are both natural with respect to the operation of pulling a bundle back along a map, it suffices to specify $B\gamma$ to be the classifying map of the principal G'-bundle coinduced from the universal G-bundle to ensure that $B\gamma$ induces γ_*. □

While the proposition suffices for our purposes and will lead to the classification of particular maps $B\gamma$ in the next section, the precise construction of classifying spaces and universal bundles by use of the "geometric bar construction" yields much sharper results. Indeed, with this construction, E and B are functors before passage to homotopy, $EG_1 \times EG_2$ and $BG_1 \times BG_2$ are naturally homeomorphic to $E(G_1 \times G_2)$ and $B(G_1 \times G_2)$, and, B is homotopy preserving in the sense that if γ and γ' are homotopic through homomorphisms, then $B\gamma$ is homotopic to $B\gamma'$. Much more is true, and the reader is referred to for an exposition.

It is immediate from the proposition and the generic construction of operations on vector bundles in the previous section that if T is a continuous functor of the sort considered there, then

$$BU(W_1) \times \cdots \times BU(W_n) \simeq B\left(U(W_1) \times \cdots \times U(W_n)\right) \xrightarrow{BT} BU(T(W_1, \ldots, W_n))$$

induces the corresponding external operation on bundles via composition with the products of classifying maps. If $f_q\colon X \longrightarrow BU(W_q)$ classifies ξ_q, then the following composite classifies the internal operation $T(\xi_1, \ldots \xi_n)$:

$$X \xrightarrow{\Delta} X^n \xrightarrow{f_1\times\cdots\times f_n} BU(W_1) \times \cdots \times BU(W_n) \xrightarrow{BT} BU(T(W_1, \ldots, W_n)).$$

If T is contravariant in the q^{th} variable, then, to arrange that $BU(W_q)^{\text{op}}$ rather than $BU(W_q)$ appears as the q^{th} space in the domain of BT, we must precompose with $B\chi\colon BU(W_q) \longrightarrow BU(W_q)^{\text{op}}$, where $\chi\colon G \longrightarrow G^{\text{op}}$ is the anti-isomorphism $\chi(g) = g^{-1}$.

We shall study such operations via canonical examples. In particular, for any classical group G ($G = O$, U, etc.), Whitney sums are induced by the maps

$$\rho_{mn}\colon BG(m) \times BG(n) \longrightarrow BG(m+n)$$

obtained from the block sum of matrix homomorphisms

$$G(m) \times G(n) \longrightarrow G(m+n).$$

The inclusion $G(n) \hookrightarrow G(n+1)$ is block sum with $I \in G(1)$, hence the corresponding map

$$i_n \colon BG(n) \longrightarrow BG(n+1)$$

induces addition of trivial line bundles.

Say that two vector bundles ξ and ξ' are *stably equivalent* if $\xi \oplus \epsilon$ is equivalent to $\xi' \oplus \epsilon'$ for some trivial bundles ϵ and ϵ'. With our explicit Grassmann manifold construction of classifying spaces, we have that i_n is an inclusion and the union of the $BG(n)$ is a classifying space for the infinite classical group G.

The natural map $BG(n) \longrightarrow BG$ induces the transformation which sends an n-plane bundle to its stable equivalence class. For a finite dimensional CW-complex X, a map $X \longrightarrow BG$ necessarily factors through some $BG(n)$. For such spaces, $[X, BG]$ is in natural one-to-one correspondence with the set of stable equivalence classes of $G(n)$-bundles over X.

The map $BO(n) \longrightarrow BU(n)$ induced by the inclusion $O(n) \longrightarrow U(n)$ represents complexification of real vector bundles, and similarly for our other forgetful maps and extension of scalars maps between classical groups. In sum, we may think of the classification theorem as providing an equivalence between the theory of vector bundles and the study of classifying spaces of classical groups.

We shall exploit this equivalence for the study of characteristic classes.

Definition 5.4. Let G be a topological group. A *characteristic class* c for G-bundles associates to each G-bundle ξ over X a cohomology class $c(\xi) \in H^*(X)$ (for a given cohomology theory H^*) naturally with respect to G-bundle maps; that is, if $(\widetilde{f}, f) \colon \xi \longrightarrow \xi'$ is a map of G-bundles, then $f^*c(\xi') = c(\xi)$.

We have not mentioned a fiber. While the definition implicitly assumes a fixed choice, any choice will do. We adopt the convention that associated bundles have the same characteristic classes since they have the same classifying maps.

Lemma 5.5. *Characteristic classes for G-bundles are in one-to-one correspondence with elements of H^*BG.*

PROOF. We may restrict attention to principal G-bundles. If $\pi \colon EG \longrightarrow BG$ is universal, c is a characteristic class, and $f \colon X \longrightarrow BG$ classifies ξ, then $c(f) = f^*c(\pi)$. Thus c is completely determined by $c(\pi) \in H^*BG$. Conversely, $\gamma \in H^*BG$ determines a characteristic class c by $c(\pi) = \gamma$ and naturality. □

Categorically, this is a special case of the Yoneda lemma: natural transformations $[?, Y] \longrightarrow F(?)$ are in one-to-one correspondence with elements of $F(Y)$ for any set-valued contravariant homotopy functor F.

The lemma is the philosophical basis for our calculations. Observe that we may study the effect of operations on vector bundles on characteristic classes by calculating the induced map on the cohomology of the relevant classifying spaces. For example, the effect of Whitney sum on characteristic classes can be deduced from the map p^*_{mn}.

6. Some homotopical properties of classifying spaces

We collect a few miscellaneous facts about classifying spaces for later use.

Let $\pi \colon EG \longrightarrow BG$ be a universal G-bundle. If $h \colon EG \times I \longrightarrow EG$ is a contracting homotopy, $h(y, 0) = *$ and $h(y, 1) = y$, let $\widetilde{h} \colon EG \longrightarrow PBG$ by the map specified by $\widetilde{h}(y)(t) = \pi h(y,t)$, where PBG is the space of paths in BG which start at $\pi(*)$. Let $p\colon PBG \longrightarrow BG$ be the end-point projection, so that $p^{-1}\pi(*)$ is the loop space ΩBG. Then $p\widetilde{h} = \pi$ and $\widetilde{h}$ restricts to a map $\zeta\colon G \longrightarrow \Omega BG$, where G is identified with the fiber $\pi^{-1}(\pi(*))$. Thus the following diagram commutes.

$$\begin{array}{ccccc} G & \longrightarrow & EG & \xrightarrow{\pi} & BG \\ {\scriptstyle\zeta}\downarrow & & \downarrow{\scriptstyle\widetilde{h}} & & \| \\ \Omega BG & \longrightarrow & PBG & \xrightarrow{p} & BG\,. \end{array}$$

By composition of long exact homotopy sequences, this yields the following result.

Proposition 6.1. *BG is a connected space and $\pi_{n+1}BG$ is naturally isomorphic to $\pi_n G$ for $n \geq 0$. The map $\zeta\colon G \longrightarrow \Omega BG$ is a homotopy equivalence.*

In particular, BG is simply connected if G is connected.

We have observed that BG is a functor of G. We need several results about the behavior of this functor on particular kinds of maps. These will all be consequences of the following criterion for recognizing when a map of classifying spaces is $B\gamma$ for some homomorphism γ.

Lemma 6.2. *Let $\gamma\colon G \longrightarrow G'$ be a continuous homomorphism and let $\pi\colon EG \longrightarrow BG$ and $\pi'\colon EG' \longrightarrow BG'$ be universal bundles. If $f\colon EG \longrightarrow EG'$ is any map such that $f(yg) = f(y)\gamma(g)$ for all $y \in EG$ and $g \in G$, then the map $BG \longrightarrow BG'$ obtained from f by passage to orbits is in the homotopy class $B\gamma$.*

Proof. Regarding G' as a left G-space via γ, we see by inspection of definitions that $\gamma_*\pi$ is the principal G'-bundle $EG \times_G G' \longrightarrow BG$. By Proposition 3.6, the G'-map

$$f \times 1\colon EG \times_G G' \longrightarrow EG' \times_{G'} G' = EG'$$

gives a bundle map $\gamma_*(\pi) \longrightarrow \pi'$. The conclusion follows. □

As a first example, we have the following observation.

Lemma 6.3. *If $g \in G$ and $\gamma_g\colon G \longrightarrow G$ is given by conjugation, $\gamma_g(h) = g^{-1}hg$, then $B\gamma_g$ is the identity map of BG.*

Proof. The map $EG \longrightarrow EG$ given by right multiplication by g satisfies the prescribed equivariance property and induces the identity map on BG. □

Henceforward in this section, let H be a closed subgroup with a local cross section in G and let $i\colon H \longrightarrow G$ denote the inclusion. By Proposition 3.9, $EG \longrightarrow EG/H$ is a universal H-bundle. Applying Lemma **??** to the identity map of EG, we obtain the following observation.

Lemma 6.4. *The map $Bi\colon BH \longrightarrow BG$ is the bundle*

$$BH = EG/H \longrightarrow EG/G = BG$$

with fiber G/H.

Now assume further that H is normal in G with quotient $\mathbb{K} = G/H$; that is, assume given an extension

$$1 \longrightarrow H \xrightarrow{i} G \xrightarrow{j} \mathbb{K} \longrightarrow 1.$$

Let $EG \longrightarrow BG$ and $E\mathbb{K} \longrightarrow B\mathbb{K}$ be universal bundles for G and $\mathbb{K}$ and take $EG \longrightarrow EG/H = BH$ to be the universal bundle for H. We may assume given a map $Ej\colon EG \longrightarrow E\mathbb{K}$ such that $(Ej)(yg) = (Ej)(y)j(g)$. Since $j(h) = e$ for $h \in H$, Ej factors through BH and we obtain a bundle map

$$\begin{array}{ccc} BH & \longrightarrow & E\mathbb{K} \\ {\scriptstyle Bi}\downarrow & & \downarrow \\ BG & \xrightarrow{Bj} & B\mathbb{K}. \end{array}$$

In particular, this square is a pullback.

Recall that the homotopy fiber $F(f)$ of a (based) map $f\colon X \longrightarrow Y$ is defined by the pullback diagram

$$\begin{array}{ccc} F(f) & \longrightarrow & PY \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Equivalently, $F(f)$ is the actual fiber over the basepoint of the map $NX \longrightarrow Y$ obtained by turning f into a fibration via the standard mapping path fibration construction. There is thus a long exact homotopy sequence

$$\cdots \longrightarrow \pi_{n+1}Y \longrightarrow \pi_n F(f) \longrightarrow \pi_n X \longrightarrow \pi_n Y \longrightarrow \cdots .$$

Using the diagram above Proposition 6.1, we see by the universal property of pullbacks that there is a map $\theta\colon BH \longrightarrow F(Bj)$ such that the following diagram commutes.

$$\begin{array}{ccccc} \mathbb{K} & \longrightarrow & BH & \xrightarrow{Bi} & BG \\ {\scriptstyle \zeta}\downarrow & & \downarrow{\scriptstyle \theta} & & \| \\ \Omega B\mathbb{K} & \longrightarrow & F(Bj) & \longrightarrow & BG \end{array}$$

By the five lemma, θ induces an isomorphism on homotopy groups and is thus a homotopy equivalence. This conclusion may be restated as follows.

Proposition 6.5. *Up to homotopy, the sequence*

$$BH \xrightarrow{Bi} BG \xrightarrow{Bj} B\mathbb{K}$$

is a fiber sequence.

Finally, retaining the hypotheses above, assume further that $\mathbb{K}$ is discrete. For $g \in G$, we have the conjugation homomorphism $\gamma_g\colon H \longrightarrow H$. On the other hand, $Bi\colon BH \longrightarrow BG$ is a principal $\mathbb{K}$-bundle, hence we have a right action of $\mathbb{K}$ on BH. In fact, Bi is a regular cover and $\mathbb{K}$ is its group of covering transformations. We have the following generalization of Lemma 6.3.

Lemma 6.6. *If $k = gH \in \mathbb{K} = G/H$, then the covering transformation $k\colon BH \longrightarrow BH$ is homotopic to $B\gamma_g\colon BH \longrightarrow BH$.*

Proof. The action of $\mathbb{K}$ on $BH = EG/H$ is induced from the action of G on EG, and right translation by g gives a map $EG \longrightarrow EG$ with the equivariance property prescribed for γ_g in Lemma 6.2. $\square$

We shall later apply this to the extension

$$1 \longrightarrow SO(n) \longrightarrow O(n) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1.$$

It also applies to the extensions

$$1 \longrightarrow T \longrightarrow N \longrightarrow W(G) \longrightarrow 1,$$

where T is a maximal torus in a compact Lie group G, N is the normalizer of T in G, and $W(G)$ is the Weyl group. The covering transformations of BT given by the elements of $W(G)$ are of fundamental importance of the following result.

Lemma 6.7. *For $\sigma \in W(G)$, the following diagram is homotopy commutative.*

$$\begin{CD} BT @>{Bi}>> BG \\ @V{\sigma}VV @| \\ BT @>{Bi}>> BG \end{CD}$$

PROOF. If $g \in N \subset G$ has image σ, then $\sigma \simeq B\gamma_g$. Since $\gamma_g i \simeq i\gamma_g$ and $B\gamma_g \simeq 1$ on BG by Lemma 6.3, the conclusion follows. $\square$

CHAPTER II

Algebraic preliminaries and spectral sequences

1. The algebras, coalgebras, and Hopf algebras of interest

Exterior algebras
Algebras with simple systems of generators
Polynomial algebras
duality, coalgebras, and Hopf algebras
exterior coalgebras and Hopf algebras
Divided polynomial coalgebras and Hopf algebras
Filtrations of algebras

Lemma 1.1. *Suppose that A is a graded commutative ring with a multiplicative exhaustive filtration $A = F_0 \supset F_1 \supset \cdots$. If the associated graded E^0A is a free commutative algebra on elements $\{x_\alpha + F_{s_\alpha}\}$ then A is a free commutative algebra on $\{x_\alpha\}$.*

Define simple system of generators and state the analogous theorem for them.

2. Spectral sequences

Generalities

3. The Serre spectral sequence

Include facts about transgression and the edge homs and maybe interaction with Steenrod operations. Universally transgressive elements.

4. The Rothenberg-Steenrod spectral sequence

Simplicial spaces, the bar construction $B(Y, G, X)$
The bar construction spectral sequence
The universal principal G-bundle this way, $\Sigma G \longrightarrow BG$

5. The Eilenberg-Moore spectral sequence

CHAPTER III

Cohomology of the Classical Groups and Stiefel Manifolds

With the exception of characteristic classes, which have traditional indexings we shall respect, *we shall index homology and cohomology classes by their degrees*. That is, given a sequence of classes in degrees $2i-1$, say, we shall label them x_{2i-1} rather than x_i. Moreover, when we have a canonical map $X \longrightarrow Y$ and given homology classes of X or cohomology classes of Y, we shall generally use the same notations for the images of these classes in the homology of Y or the cohomology of X.

We fix the canonical fundamental classes

$$i_n \in H_n(S^n;\mathbb{Z}) \cong \mathbb{Z} \quad \text{and} \quad \iota_n \in H^n(S^n;\mathbb{Z}) \cong \mathbb{Z}$$

and use the same notations for their images in homology or cohomology with other coefficients. Explicitly, i_n is the image of the identity map $S^n \longrightarrow S^n$ under the Hurewicz homomorphism $\pi_n(S^n) \longrightarrow H_n(S^n;\mathbb{Z})$, and ι_n is the dual generator characterized by $\iota_n(i_n) = 1$ under the evaluation pairing $H^n(S^n;\mathbb{Z}) \otimes H_n(S^n;\mathbb{Z}) \longrightarrow \mathbb{Z}$. We often write i or ι when the dimension n is clear from the context. *We do not assume prior knowledge of any further explicit calculations of homology or cohomology groups.* We do assume that the reader is familiar with the Künneth and universal coefficient theorems.

1. The complex and quaternionic Stiefel manifolds

In this section, we compute the homology and cohomology of the complex and quaternionic Stiefel manifolds inductively, ending with the homology and cohomology of the complex and symplectic classical groups. All homology and cohomology groups are to be taken with integer coefficients. The only tool we shall need is the Serre spectral sequence in the special cases (with fiber and base space a sphere, respectively) which give the Gysin and Wang exact sequences.

Theorem 1.1. *Let $1 \le q \le n$. As algebras,[BOB deg symbol?]*

$$H^*(V_q(\mathbb{C}^n)) = E\{y_{2i-1} \mid n-q < i \le n\},\ \deg\ y_{2i-1} = 2i-1,$$

and

$$H^*(V_q(\mathbb{H}^n)) = E\{z_{4i-1} \mid n-q < i \le n\},\ |z_{4i-1}| = 4i-1.$$

Proof. We treat the complex case. The symplectic case is entirely similar. Abbreviate $V_q = V_q(\mathbb{C}^n)$. We proceed by induction on q. We begin with $V_1 = S^{2n-1}$, taking $y_{2n-1} = \iota_{2n-1}$. This completes the case $n = 1$, so we assume that $n \ge 2$. Assume the result for V_{q-1} and consider the Serre spectral sequence $\{E_r\}$ of the bundle $V_q \longrightarrow V_{q-1}$ with fiber $V_1(\mathbb{C}^{n-q+1}) = S^{2(n-q+1)-1}$. It is clear from the long exact homotopy sequences of such bundles that the V_q are simply connected. As algebras, we have

$$E_2 = H^*(V_{q-1}) \otimes H^*(S^{2(n-q+1)-1}).$$

All differentials on the x axis (or base axis) E_2^{*0} are zero since they land in the fourth quadrant, which consists of zero groups. The rows E_2^{*q} are zero for $0 < q < 2(n-q+1)-1$. Therefore the only generator that might possibly support a non-trivial differential is the fundamental class ι, and it can only "transgress", that is, map from the y axis (or fiber axis) to the x-axis via $d_{2(n-q+1)}$. By the induction hypothesis, $H^p(V_{q-1}) = 0$ for $0 < p < 2(n-q+2)-1$. In particular, $H^{2(n-q+1)}(V_{q-1}) = 0$, so that $d_{2(n-q+1)}(\iota)$ lands in a zero group. Therefore $E_2 = E_\infty$ and $H^*(V_q)$ is as claimed. □

Hereafter, we shall summarize arguments like the one above by asserting that $E_2 = E_\infty$ (or the spectral sequence collapses at E_2) for dimensional reasons.

Remark 1.2. As suggested by the notation, we may choose the generators y_{2i-1} and z_{4i-1} consistently as q and n vary. To be precise about this, consider the commutative diagram

$$\begin{array}{ccccc}
S^{2(n-q)-1} & = & S^{2(n-q)-1} & & \\
\downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota} & & \\
V_q(\mathbb{C}^{n-1}) & \xrightarrow{\iota} & V_{q+1}(\mathbb{C}^n) & \xrightarrow{\pi} & S^{2n-1} \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \| \\
V_{q-1}(\mathbb{C}^{n-1}) & \xrightarrow[\iota]{} & V_q(\mathbb{C}^n) & \xrightarrow[\pi]{} & S^{2n-1}.
\end{array}$$

Here the π are projections of bundles and the ι are inclusions of fibers. We assume the y_{2i-1} have been chosen compatibly in $H^*(V_q(\mathbb{C}^m))$ for $m-q < i \le m < n$. We have seen that the spectral sequences of the two columns satisfy $E_2 = E_\infty$ for dimensional reasons, and a similar argument shows that the Serre spectral sequences of the two rows also satisfy $E_2 = E_\infty$. By consideration of the edge homomorphism, $\iota^* : H^j(V_q(\mathbb{C}^n)) \longrightarrow H^j(V_{q-1}(\mathbb{C}^{n-1}))$ is an isomorphism for $j < 2n-1$. We specify generators y_{2i-1} for $H^*(V_q(\mathbb{C}^n))$ by requiring $\iota^*(y_{2i-1}) = y_{2i-1}$ for $i < n$ and taking $y_{2n-1} = \pi^*(\iota_{2n-1})$. It follows inductively from the diagram that $\pi^*(y_{2i-1}) = y_{2i-1}$ in $H^*(V_{q+1}(\mathbb{C}^n))$ for $n-q < i \le n$. It also follows that $\iota^*(y_{2n-1}) = 0$ in $H^*(V_q(\mathbb{C}^{n-1}))$.

Theorem 1.3. *As Hopf algebras,*

$$H^*(U(n)) = E\{y_{2i-1} \mid 1 \le i \le n\} \quad \text{and} \quad H^*(U) = E\{y_{2i-1} \mid i \ge 1\},$$
$$H^*(SU(n)) = E\{y_{2i-1} \mid 2 \le i \le n\} \quad \text{and} \quad H^*(SU) = E\{y_{2i-1} \mid i \ge 2\},$$
$$H^*(Sp(n)) = E\{z_{4i-1} \mid 1 \le i \le n\} \quad \text{and} \quad H^*(Sp) = E\{z_{4i-1} \mid i \ge 1\}.$$

In each case, the corresponding homology Hopf algebra is the exterior Hopf algebra on the dual generators.

PROOF. First consider the case of finite n. As algebras, the stated cohomologies are immediate from Theorem 1.1. We must show that the generators are primitive. We proceed by induction on n, the case $n = 1$ being trivial. Thus assume the result for $n-1$ and let $\iota : U(n-1) \longrightarrow U(n)$ be the standard inclusion. By Remark 1.2,

ι^* is an isomorphism in degrees less than $2n-1$. By the induction hypothesis, this implies that y_{2i-1} is primitive for $i<n$. Write

$$\psi(y_{2n-1}) = y_{2n-1}\otimes 1 + \sum y'\otimes y'' + 1\otimes y_{2n-1}, \deg\, y' > 0 \ \text{ and } \ \deg\, y'' > 0.$$

Since $\iota^*(y_{2n-1}) = 0$ and $(\iota^*\otimes\iota^*)\psi = \psi\iota^*$, we must have that $(\iota^*\otimes\iota^*)(\sum y'\otimes y'') = 0$. Since $\iota^*\otimes\iota^*$ is an isomorphism in the pairs of degrees in which the elements $y'\otimes y''$ occur, it follows that $\sum y'\otimes y'' = 0$. The y_{2i-1} in $H^*(SU(n))$ are the images of the y_{2n-1} in $H^*(U(n))$ and are thus also primitive. The proof that the z_{4i-1} are primitive is similar.

By use of the Poincaré duality theorem, the universal coefficients theorem, or the homology version of the calculation of Theorem 1.1, we see that $H_*(U(n))$ is a free graded $\mathbb{Z}$-module of finite type and that $H^*(U(n))$ and $H_*(U(n))$ are dual to one another as $\mathbb{Z}$-modules. Since locally finite exterior Hopf algebras are self-dual, this implies the homology statement.

Finally, for the infinite classical groups, the homology statement is immediate since homology commutes with colimits, and the cohomology statements follow by duality. □

2. The real Stiefel manifolds

The cohomology of the real Stiefel manifolds is complicated by the failure of the dimensional argument used to prove Theorem 1.1 and by the presence of 2-torsion. Accordingly, we will calculate separately the cohomology at the prime 2 (that is with $\mathbb{F}_2$ coefficients) and away from the prime 2 (that is, with coefficients in a ring in which 2 is invertible). This is a standard technique which will occur over and over in our work.

We fix n, and we often write V_q for $V_q(\mathbb{R}^n)$ in this section. We need a little homotopical input, namely a calculation of the homotopy groups of the V_q in the "Hurewicz dimension", the largest dimension j below which all homotopy groups vanish, so that π_j maps isomorphically to H_j (if $j>1$ or π_1 is Abelian).

Lemma 2.1. *For $1\le q\le n$, $V_q(\mathbb{R}^n)$ is $(n-q-1)$-connected and*

$$\pi_{n-q}(V_q(\mathbb{R}^n)) = \begin{cases} \mathbb{Z} & \text{if } q<n \text{ and } q=1 \text{ or } n-q \text{ is even} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } q=n \text{ or } n-q \text{ is odd.} \end{cases}$$

Proof. Since $V_1 = S^{n-1}$ and $V_n = O(n)$, we may assume that $1<q<n$ and thus $n\ge 3$. The main work is in the case $q=2$. Here the fibration

$$S^{n-2} \xrightarrow{\iota} V_2 \xrightarrow{\pi} V_1 = S^{n-1}$$

yields the exact sequence

$$\pi_{n-1}(S^{n-1}) \xrightarrow{\partial} \pi_{n-2}(S^{n-2}) \xrightarrow{\iota_*} \pi_{n-2}(V_2) \longrightarrow 0,$$

and we must compute ∂. For any r, let $E^r\subset S^r$ be the upper hemisphere,

$$E^r = \Big\{x = (x_1,\cdots,x_{r+1}) \mid \sum x_i^2 = 1 \ \text{ and } \ x_1\ge 0\Big\},$$

and let S^{r-1} be the equator, $S^{r-1} = \{x \mid x\in E^r \ \text{ and } \ x_1 = 0\}$. Give S^r the basepoint $* = (1,0,\cdots,0)$. Represent elements of V_2 by $2\times n$ matrices A such that AA^t is the 2×2 identity matrix. This makes sense since such matrices specify pairs of orthonormal vectors in $\mathbb{R}^n$. (We note parenthetically that this identifies $V_2(\mathbb{R}^n)$ with the unit sphere bundle of the tangent bundle of S^{n-1}.)

Define $\mu : E^{n-1} \longrightarrow V_2$ by

$$\mu(x) = \begin{bmatrix} e_1 - 2x_1 x \\ e_2 - 2x_2 x \end{bmatrix}$$

where the e_i are the standard basis vectors. Represent $\pi : V_2 \longrightarrow S^{n-1}$ by $\pi(a_{ij}) = (a_{1j})$. The composite $\pi\mu$ sends S^{n-2} to the basepoint $* \in S^{n-1}$ and restricts to a homeomorphism $E^{n-1} - S^{n-2} \longrightarrow S^{n-1} - *$. Therefore

$$\pi\mu : (E^{n-1}, S^{n-2}) \longrightarrow (S^{n-1}, *)$$

represents a generator of $\pi_{n-1}(S^{n-1})$ and the restriction

$$\tau : S^{n-2} \longrightarrow \pi^{-1}(*) = S^{n-2}$$

of μ represents a generator of $\mathrm{Im}(\partial) \subset \pi_{n-2}(S^{n-2})$. Write $y_j = x_{j+1}$, so that

$$\tau(y_1, \cdots, y_{n-1}) = (1 - 2y_1^2, -2y_1y_2, \cdots, -2y_1y_{n-1}).$$

Then τ sends S^{n-3} to the basepoint $* \in S^{n-2}$, restricts to a homeomorphism $E^{n-2} - S^{n-3} \longrightarrow S^{n-2} - *$, and satisfies $\tau(-y) = \tau(y)$. Therefore τ represents the sum of the identity map and the antipodal map (or the negative of this sum depending on orientations). The antipodal map on S^{n-2} has degree $(-1)^{n-1}$, hence τ has degree $\pm(1 + (-1)^{n-1})$. Thus $\partial = 0$ if n is even and $\partial i_{n-1} = \pm 2 i_{n-2}$ if n is odd. This proves the lemma for $q = 2$.

For $q = 3$, the commutative diagram

$$\begin{array}{ccccc}
S^{n-3} = V_1(\mathbb{R}^{n-2}) & \longrightarrow & V_2(\mathbb{R}^{n-1}) & \longrightarrow & V_1(\mathbb{R}^{n-1}) = S^{n-2} \\
\| & & \downarrow & & \downarrow \iota \\
S^{n-3} = V_1(\mathbb{R}^{n-2}) & \longrightarrow & V_3(\mathbb{R}^n) & \longrightarrow & V_2(\mathbb{R}^n) = V_2
\end{array}$$

gives rise to the commutative diagram

$$\begin{array}{ccccccc}
\pi_{n-2}(S^{n-2}) & \overset{\partial}{\searrow} & & & & & \\
\downarrow \iota_* & & & & & & \\
\pi_{n-2}(V_2) & \xrightarrow{\partial'} & \pi_{n-3}(S^{n-3}) & \longrightarrow & \pi_{n-3}(V_3) & \longrightarrow & 0.
\end{array}$$

Here $\mathrm{Im}(\partial) = \mathrm{Im}(\partial')$ since ι_* is an epimorphism. We know ∂ from the case $q = 2$, and the result for $q = 3$ follows.

Finally, for $q > 3$, the long exact homotopy sequence of the fibration

$$V_{q-2}(\mathbb{R}^{n-2}) \longrightarrow V_q \longrightarrow V_2$$

and the fact that $\pi_i(V_2) = 0$ for $i < n - 2$ show that $\pi_i(V_{q-2}(\mathbb{R}^{n-2})) \cong \pi_i(V_q)$ for $i < n - 3$. The result follows from known cases by induction on n. $\square$

We begin our homological calculations by considering V_2. Since $V_2(\mathbb{R}^2) = O(2)$, which is homeomorphic to $S^0 \times S^1$, we already know $H^*(V_2(\mathbb{R}^2); \mathbb{Z})$.

Lemma 2.2. *Assume that $n \geq 3$. If n is even, then*

$$H^*(V_2(\mathbb{R}^n); \mathbb{Z}) = E\{x_{n-2}, x_{n-1}\}.$$

If n is odd, then

$$H^*(V_2(\mathbb{R}^n); \mathbb{Z}) = E\{x_{n-1}, x_{2n-3}\}/(2x_{n-1}, x_{n-1}x_{2n-3}).$$

PROOF. In the Serre spectral sequence of the fibration $S^{n-2} \longrightarrow V_2 \longrightarrow S^{n-1}$, we have $E_2 = H^*(S^{n-1}) \otimes H^*(S^{n-2})$. For dimensional reasons, the only possible non-zero differential is the transgression $d_{n-1}(\iota_{n-2})$, which must be a multiple of ι_{n-1}. If n is even, we have

$$\mathbb{Z} = \pi_{n-2}(V_2) \cong H_{n-2}(V_2;\mathbb{Z}) \cong H^{n-2}(V_2;\mathbb{Z})$$

by the Hurewicz and universal coefficient theorems. Therefore $d_{n-1}(\iota_{n-2}) = 0$ and $E_2 = E_\infty$. The algebra structure of $H^*(V_2;\mathbb{Z})$ is as stated since the squares of the generators lie in zero groups. If n is odd,

$$\mathbb{Z}/2\mathbb{Z} = \pi_{n-2}(V_2) \cong H_{n-2}(V_2;\mathbb{Z})$$

and

$$H^{n-1}(V_2;\mathbb{Z}) \cong \operatorname{Hom}\left(H_{n-1}(V_2;\mathbb{Z}),\mathbb{Z}\right) \oplus \operatorname{Ext}\left(H_{n-2}(V_2;\mathbb{Z}),\mathbb{Z}\right).$$

Since $\operatorname{Ext}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, we must have $d_{n-1}(\iota_{n-2}) = \pm 2\iota_{n-1}$. Then $E_n = E_\infty$ has the stated form, the class ι_{n-1} giving rise to the class x_{n-1} and the class $\iota_{n-1} \otimes \iota_{n-2}$ giving rise to the class x_{2n-3}, while $x_{n-1}x_{2n-3} = 0$ since this product lies in a zero group. □

Now we consider mod 2 cohomology. Here separate consideration of cases as in the previous lemma is no longer necessary.

Theorem 2.3. *The cohomology $H^*(V_q(\mathbb{R}^n);\mathbb{F}_2)$ has a simple system of generators $\{x_i \mid n-q \leq i < n\}$.*

PROOF. We proceed by induction on q, beginning with $V_1 = S^{n-1}$. Assume the result for V_{q-1} and consider the Serre spectral sequence in mod 2 cohomology of the bundle $V_q \longrightarrow V_{q-1}$ with fiber S^{n-q}. We have $E_2 = H^*(V_{q-1};\mathbb{F}_2) \otimes H^*(S^{n-q};\mathbb{F}_2)$ since $\pi_1(V_{q-1}) = 0$ if $q < n$ and since $V_n \longrightarrow V_{n-1}$ is the trivial double cover $O(n) \longrightarrow SO(n)$. All differentials except d_{n-q+1} are zero for dimensional reasons. If $d_{n-q+1}(\iota_{n-q})$ were non-zero, we would have $H^{n-q}(V_q;\mathbb{F}_2) = 0$. However, $H^{n-q}(V_q;\mathbb{F}_2) = \mathbb{F}_2$ by Lemma 2.1 and the Hurewicz and universal coefficient theorems. Therefore $E_2 = E_\infty$ and the conclusion follows. □

Here we cannot conclude that the squares of the generators x_i are zero, since they land in non-zero groups in general. In fact, we shall see later that these squares are non-zero whenever they land in non-zero groups.

Remark 2.4. Exactly as in Remark 1.2, the x_i may be chosen consistently as q and n vary. To be precise, consider the canonical maps

$$\iota : V_{q-1}(\mathbb{R}^{n-1}) \longrightarrow V_q(\mathbb{R}^n) \quad \text{and} \quad \pi : V_{q+1}(\mathbb{R}^n) \longrightarrow V_q(\mathbb{R}^n).$$

Then $\iota^*(x_i) = x_i$ for $i < n-1$, $\pi^*(x_i) = x_i$ for $i > n-q$, and $\pi^*(x_{n-q}) = 0$. The key point is that $E_2 = E_\infty$ in the Serre spectral sequence of the fibration $V_{q-1}(\mathbb{R}^{n-1}) \xrightarrow{\iota} V_q(\mathbb{R}^n) \xrightarrow{\pi} S^{n-1}$, by dimensional reasons, so that ι^* is an isomorphism in degrees less than n.

Theorem 2.5. *The cohomology Hopf algebra $H^*(SO(n);\mathbb{F}_2)$ has a simple system of primitive generators $\{x_i \mid 1 \leq i < n\}$, and $H_*(SO(n);\mathbb{F}_2)$ is an exterior algebra on the dual generators.*

PROOF. We need only check that the x_i are primitive, and the argument is exactly the same as in the proof of Theorem 1.3. □

Since $O(n)$ is the disjoint union of $SO(n)$ and $t\cdot SO(n)$ for any element $t \in O(n)$ of determinant -1 which we can choose to satisfy $t^2 = 1$, there is no problem in reading off the homology and cohomology of $O(n)$ from that of $SO(n)$.

Lemma 2.6. *With coefficients in any commutative ring R,*

$$H_*(O(n)) = H_*(SO(n)) \oplus \chi_0 H_*(SO(n))$$

and

$$H^*(O(n)) = H^*(SO(n)) \oplus x_0 H^*(SO(n))$$

where $\chi_0 \in H_0(SO(n))$ satisfies $\chi_0^2 = 1$ and $x_0 \in H^0(O(n))$ satisfies $x_0(\chi_0) = 1$ and $x_0^2 = x_0$.

Proof. For any space X, $H_0(X)$ is the free R-module generated by $\pi_0(X)$, and we take χ_0 to be the class of t. In the cohomology of any space, $x^2 = x$ if $x \in H^0(X)$ is dual to an element of $\pi_0(X) \subset H_0(X)$. [[Better, if x is the characteristic function of any set of components.]] □

We now turn to cohomology away from the prime 2. In the following proofs, we often abbreviate $H^*(X) = H^*(X; R)$ when the ring R of coefficients is clear from the context.

Theorem 2.7. *The integral homology and cohomology of $V_q(\mathbb{R}^n)$ has no odd torsion. If R is a commutative ring in which 2 is invertible, then*

$$H^*(V_q(\mathbb{R}^n); R) = E\{x_{4i-1} \mid n-q < 2i < n\} \otimes E\{y_{n-1} \mid n \text{ even}\} \otimes E\{x_{n-q} \mid n-q \text{ even}\}.$$

Proof. The first statement follows from the second and the universal coefficient theorem: if $H_*(V_q;\mathbb{Z})$ had any odd torsion, then $H^*(V_q;\mathbb{Z}[1/2])$ would have odd torsion. The second statement is clear for $V_1 = S^{n-1}$ and follows from Lemma 2.2 and the universal coefficient theorem for V_2. For the general case, assume that $q > 2$ and proceed by induction on q.

If $n-q$ is even, then the E_2-term $H^*(V_{q-1}) \otimes H^*(S^{n-q})$ of the Serre spectral sequence of the fibration $S^{n-q} \longrightarrow V_q \longrightarrow V_{q-1}$ has the form claimed. By a check of dimensions, the only differential on generators that might conceivably be non-zero is $d_{n-q+1}(\iota_{n-q})$, but this lands in the group $H^{n-q+1}(V_{q-1})$, which is zero by the induction hypothesis. Thus $E_2 = E_\infty$. There is no multiplicative extension problem since the square of x_{n-q} lies in a zero group.

If $n-q$ is odd, then the E_2-term $H^*(V_{q-2}) \otimes H^*(V_2(\mathbb{R}^{n-q+2}))$ of the Serre spectral sequence of the fibration $V_2(\mathbb{R}^{n-q+2}) \longrightarrow V_q \longrightarrow V_{q-2}$ has the form claimed. Since $H^*(V_2(\mathbb{R}^{n-q+2})) = E\{x_{2n-2q+1}\}$, the only differential that might conceivably be non-zero is $d_{2n-2q+2}(x_{2n-2q+1})$, but this lands in the group $H^{2n-2q+2}(V_{q-2})$, which is zero by the induction hypothesis. Thus $E_2 = E_\infty$, and the conclusion follows. □

Remark 2.8. Again, the generators x_{4i-1}, y_{n-1} (n even), and x_{n-q} ($n-q$ even) can be chosen consistently as q and n vary. However, the proof in this case is more subtle than in the previous cases. To make this precise, consider the canonical maps

$$\iota : V_{q-1}(\mathbb{R}^{n-1}) \longrightarrow V_q(\mathbb{R}^n) \quad \text{and} \quad \pi : V_{q+1}(\mathbb{R}^n) \longrightarrow V_q(\mathbb{R}^n).$$

We claim that generators can be so chosen that

(a) $\iota^*(x_{4n-1}) = x_{4i-1}$ for $n-q < 2i < n-1$, $\iota^*(x_{2n-3}) = 0$ if n is odd, $\iota^*(y_{n-1}) = 0$ if n is even, and $\iota^*(x_{n-q}) = 0$ if $n-q$ is even.
(b) $\pi^*(x_{4i-1}) = x_{4i-1}$ for $n-q < 2i < n$, $\pi^*(y_{n-1}) = y_{n-1}$ if n is even, and $\pi^*(x_{n-q}) = 0$ if $n-q$ is even.

We assume inductively that such generators have been chosen for $V_q(\mathbb{R}^m)$ for $m < n$ and all q, and we consider the diagram

$$\begin{array}{ccccc}
S^{n-q-1} & = & S^{n-q-1} & & \\
\downarrow \iota & & \downarrow \iota & & \\
V_q(\mathbb{R}^{n-1}) & \xrightarrow{\iota} & V_{q+1}(\mathbb{R}^n) & \xrightarrow{\pi} & S^{n-1} \\
\downarrow \pi & & \downarrow \pi & & \| \\
V_{q-1}(\mathbb{R}^{n-1}) & \xrightarrow[\iota]{} & V_q(\mathbb{R}^n) & \xrightarrow[\pi]{} & S^{n-1}.
\end{array}$$

We prove (a) by studying the Serre spectral sequence of the bottom row in two cases. In each case, (b) can be derived from (a) by inspection of the induced diagram on cohomology and use of the induction hypothesis.

For the first case, assume that n is even. Here

$$E_2 = E\{\iota_{n-1}\} \otimes (E\{x_{4i-1} \mid n-q < 2i < n\} \otimes E\{x_{n-q} \mid q \text{ even}\})$$

By the calculation of $H^*(V_q))$ in Theorem 2.7, we must have $E_2 = E_\infty$. By the description of ι^* and π^* in terms of edge homomorphisms, we can choose x_{4i-1} and x_{n-q} in $H^*(V_q(\mathbb{R}^n))$ that map under ι^* to the elements with the same name in $H^*(V_{q-1}(\mathbb{R}^{n-1}))$, and we can take $y_{n-1} = \pi^*(\iota_{n-1})$. This ensures that $\iota^*(y_{n-1}) = 0$.

For the second case, assume that n is odd. Here

$$E_2 = E\{\iota_{n-1}\} \otimes (E\{x_{4i-1} \mid n-q < 2i < n\} \otimes E\{y_{n-1}\} \otimes E\{x_{n-q} \mid q \text{ odd}\}).$$

The only non-zero differential is d_{n-1}. Consider the map of fibrations

$$\begin{array}{ccccc}
V_{q-1}(\mathbb{R}^{n-1}) & \xrightarrow[\iota]{} & V_q(\mathbb{R}^n) & \xrightarrow[\pi]{} & S^{n-1} \\
\downarrow \pi & & \downarrow \pi & & \| \\
S^{n-2} & \xrightarrow[\iota]{} & V_2(\mathbb{R}^n) & \xrightarrow[\pi]{} & S^{n-1}.
\end{array}$$

By the proof of Lemma 2.2, $d_{n-1}(\iota_{n-2}) = \pm 2\iota_{n-1}$ in the spectral sequence of the bottom row. Since $\pi^*(\iota_{n-2}) = y_{n-2}$ by the first case, it follows by naturality that $d_{n-1}(y_{n-2}) = \pm 2\iota_{n-1}$ in the spectral sequence of the top row. If $n \equiv 1 \bmod 4$, we have a second generator x_{n-2} in $H^{n-2}(V_q(\mathbb{R}^{n-1}))$. By dimensional reasons, $d_{n-1}(x_{n-2}) = r\iota_{n-1}$ for some $r \in R$. We agree to replace our previously chosen generator x_{n-2} by $x_{n-2} + sy_{n-2}$, where $s = \mp 2^{-1}r \in R$. By the consistency of the x_{n-2} as q varies in (b), we have the same constant s for each q and so retain the consistency in (b) after the change. We also retain the consistency in (a) since $\iota^*(y_{n-1}) = 0$. Now $d_{n-1}(x_{n-2}) = 0$ and, for any q,

$$E_\infty = E\{x_{4i-1} \mid n-q < 2i < n-1\} \otimes E\{\iota_{n-1} \otimes y_{n-2}\} \otimes E\{x_{n-q} \mid q \text{ odd}\}.$$

We can choose x_{4i-1} and x_{n-q} in $H^*(V_q(\mathbb{R}^n))$ that map under ι^* to the elements with the same name in $H^*(V_{q-1}(\mathbb{R}^{n-1}))$ and can choose x_{2n-3} in filtration $n-1$ that projects to $\iota_{n-1} \otimes y_{n-2}$ in E_∞. By the edge homomorphism description of ι^*, $\iota^*(x_{2n-3}) = 0$.

Theorem 2.9. *If R is a commutative ring in which 2 is invertible, then, as Hopf algebras,*

$$H^*(SO(n); R) = E\{x_{4i-1} \mid 0 < 2i < n\} \otimes E\{y_{n-1} \mid n \text{ even}\}$$

and $H^*(SO; R) = E\{x_{4i-1} \mid i \geq 1\}$. In both cases, the corresponding homology Hopf algebra is the exterior algebra on the dual generators.

Proof. We must show that the generators x_{4i-1} and y_{n-1} are primitive. We proceed by induction on n. The result is clear if $n = 2$ or $n = 3$, when there is only one generator. When n is even, y_{n-1} is primitive since $\iota^*(y_{n-1}) = 0$ and ι^* is a monomorphism in degrees less than $n - 1$. In general, $x_{4i-1} \in H^*(SO(n))$ comes from $x_{4i-1} \in H^*(SO(n+1))$ and maps to a primitive element (or zero) in $H^*(SO(n-1))$. If we write

$$\psi(x_{4i-1}) = x_{4i-1} \otimes 1 + \sum x' \otimes x'' + 1 \otimes x_{4i-1},$$

then $\sum x' \otimes x''$ comes from an element of $H^*(SO(n+1)) \otimes H^*(SO(n+1))$ and maps to zero in $H^*(SO(n-1)) \otimes H^*(SO(n-1))$. Since $\iota : SO(n-1) \longrightarrow SO(n+1)$ induces a monomorphism in cohomology in degrees less than $4i-1$, $\sum x' \otimes x'' = 0$.

The homology statement follows essentially as in Theorem 1.3. Since R is a module over the PID $\mathbb{Z}[1/2]$, it suffices by universal coefficients to assume that $R = \mathbb{Z}[1/2]$. Since each $H_q(SO(n); R)$ is a finitely generated R-module and

$$H^q(SO(n); R) \cong \mathrm{Hom}_R(H_q(SO(n); R), R) \oplus \mathrm{Ext}^1_R(H_{q-1}(SO(n); R), R)$$

is a free R-module, the Ext term must be zero and each $H_q(SO(n); R)$ must be a free R-module. Therefore $H_*(SO(n); R)$ is dual to $H^*(SO(n); R)$. The statements about SO follow in homology by passage to colimits and in cohomology by dualization. □

CHAPTER IV

The classical characteristic classes

1. The Chern classes and $H^*(BU(n))$

Take homology and cohomology with integer coefficients in this section. From the Serre spectral sequence of the universal bundle of $S^1 = U(1)$, we see that the transgression $\tau : H^1(S^1) \longrightarrow H^2(BU(1))$ is an isomorphism, hence so is the suspension $\sigma : H^2(BU(1)) \longrightarrow H^1(S^1)$. The *canonical generator* of $H^2(BU(1))$ is the class that suspends to the canonical generator $\iota_1 \in H^1(S^1)$.

Let

$$i_n : U(n-1) \longrightarrow U(n),$$
$$j_n : SU(n) \longrightarrow U(n), \text{ and}$$
$$p_{i,j} : U(i) \times U(j) \longrightarrow U(i+j)$$

be the canonical maps. Let i_n, j_n, and $p_{i,j}$ also denote the induced maps on classifing spaces. We shall prove the following basic result in this chapter.

Theorem 1.1. *There are unique classes $c_i \in H^{2i}(BU(n))$, called the* Chern classes, *which satisfy the following four axioms.*

(i) $c_0 = 1$ *and* $c_i = 0$ *if* $i > n$.
(ii) $c_1 \in H^2(BU(1))$ *is the canonical class.*
(iii) $i_n^*(c_i) = c_i$ *(hence* $i_n^*(c_n) = 0$*).*
(iv) $p_{i,j}^*(c_k) = \sum_{a+b=k} c_a \otimes c_b$.

Moreover, $H^*(BU(n)) = P\{c_1, \ldots, c_n\}$. *In* $H^*(BSU(n))$, *define* $c_i = j_n^*(c_i)$. *Then, with* $c_1 = 0$, *(i), (iii), and (iv) again hold and* $H^*(BSU(n)) = P\{c_2, \ldots, c_n\}$.

The uniqueness implies that, proceeding inductively on n, if there are classes d_i satisfying (i)-(iv), where the right sides of the formulas in (iii) and (iv) involve Chern classes already known to be uniquely characterized, then $d_i = c_i$. In particular, this makes no reference to any particular construction of the Chern classes.

The following is an immediate reinterpretation of the theorem in the language of characteristic classes of complex vector bundles.

Theorem 1.2. *All characteristic classes of $U(n)$ and $SU(n)$ bundles can be expressed uniquely as polynomials in the Chern classes. Let $\xi : E \longrightarrow B$ be a $U(n)$ bundle. The Chern classes $c_i(\xi) \in H^{2i}(B)$ satisfy:*

(i) $c_0(\xi) = 1$ *and* $c_i(\xi) = 0$ *if* $i > n$.
(ii) $c_1(\xi) \in H^2(BU(1))$ *is the canonical class when ξ is the universal line bundle.*
(iii) $c_i(\eta \oplus \varepsilon) = c_i(\eta)$ *if η is an $(n-1)$-plane bundle and ε is the trivial line bundle.*
(iv) $c_k(\eta \oplus \xi) = \sum_{a+b=k} c_a(\eta) \otimes c_b(\xi)$ *if η is an i-plane bundle and ξ is a j-plane bundle, where $i + j = n$.*

If the structure group of ξ *can be reduced to* $SU(n)$, $c_1(\xi) = 0$.

With the Grassmannian construction of classifying spaces, we have $BU(1) = \mathbb{C}P^\infty$. The universal bundle is the evident orbit projection of S^∞ onto $\mathbb{C}P^\infty$. Its restriction over $\mathbb{C}P^n$ is the Hopf bundle $\xi_n : S^{2n+1} \longrightarrow \mathbb{C}P^n$. We have defined the Chern classes so that $c_1(\xi_n)$ is the canonical generator of $H^2(\mathbb{C}P^n)$.

Theorem 1.2.(iv) directly implies its analogue for the external direct sum $\eta \times \xi$ of bundles η over X and ξ over Y, namely

$$c_k(\eta \times \xi) = \sum_{a+b=k} c_a(\eta) \otimes c_b(\xi),$$

where we implicitly use the canonical map $H^*(X) \otimes H^*(Y) \longrightarrow H^*(X \times Y)$.

The original statement of Theorem 1.2.(iv) can be put in a convenient alternative form by writing $H^{**}(X)$ for the product over $q \geq 0$ of the groups $H^q(X)$. The elements with 0th term 1 form a group under the product induced by the cup product. We define the *total Chern class* $c(\xi) \in H^{**}(X)$ of a bundle ξ over X to be the class whose even components are the $c_i(\xi)$ and odd components 0. The equations in 1.2.(iv) then take the pleasing form

$$c(\eta \oplus \xi) = c(\eta)c(\xi).$$

This implies the *Whitney duality theorem* relating the Chern classes of the tangent bundle τ and normal bundle ν over a complex manifold M of (complex) dimension n embedded in $\mathbb{C}^q$ for some q. The sum $\tau \oplus \nu$ is the trivial q-plane bundle ε, and $c_i(\varepsilon) = 0$ for $i > 0$. This follows both from the axioms and from the fact that the classifying map of ε is null homotopic. Thus the following result is immediate.

Theorem 1.3 (Whitney duality)**.** *Let* M *be a complex manifold with tangent bundle* τ *and normal bundle* ν. *Then*

$$c(\tau)c(\nu) = 1.$$

Writing $c_i = c_i(\tau)$ and $\bar{c}_i = c_i(\nu)$, we can inductively solve for $\bar{c}_i$ in terms of c_i. Thus

$$\bar{c}_1 = -c_1,$$
$$\bar{c}_2 = -c_2 - c_1\bar{c}_1 = c_1^2 - c_2,$$
$$\bar{c}_3 = -c_3 - c_2\bar{c}_1 - c_1\bar{c}_2 = -c_1^3 + 2c_1c_2 - c_3,$$

and so on. This can be used, for example, to prove non-embedding theorems. If we find that $\bar{c}_{q+1} \neq 0$, then M cannot embed in $\mathbb{C}^q$.

Since the classifying space functor converts extensions of groups to fibrations of spaces, we have the fibration sequence

$$BSU(n) \xrightarrow{j_n} BU(n) \xrightarrow{det} BU(1),$$

where det denotes the classifying map of the determinant homomorphism $\det : U(n) \longrightarrow U(1)$. Since $BU(1) = K(\mathbb{Z}, 2)$, we may think of the classifying map det as an element of the group $H^2(BU(n))$. The map is the identity map when $n = 1$, and this means that the element is the canonical class c_1. For $n > 1$, $\det^* : H^2(BU(1)) \longrightarrow H^2(BU(n))$ is an isomorphism, by Theorem 1.1, and therefore the classifying map det may be identified with the first Chern class in general.

Let ξ be a complex n-plane bundle over a space X and also write ξ for its classifying map $X \longrightarrow BU(n)$. When $n = 1$, this homotopy class of maps may be regarded as an element of $H^2(X)$, namely the Chern class $c_1(\xi)$. Thus equivalence classes of line bundles are determined by their first Chern classes. When $n > 1$, the homotopy class $\det \circ \xi : X \longrightarrow BU(1)$ regarded as an element of $H^2(X)$ is again the Chern class $c_1(\xi)$, and thus $c_1(\xi) = 0$ if and only if the composite $\det \circ \xi$ is null homotopic. This holds if and only if there is a map $\tilde{\xi} : X \longrightarrow BSU(n)$ such that $j_n \circ \tilde{\xi}$ is homotopic to ξ. In turn, this holds if and only if the structure group of the bundle ξ can be reduced to $SU(n)$. This gives a bundle theoretic interpretation of $c_1(\xi)$.

Corollary 1.4. *The first Chern class $c_1(\xi)$ is the obstruction to reducing the structure group of a complex n-plane bundle ξ to $SU(n)$.*

We will prove Theorem 1.1 by showing that $H^*BU(n)$ can be identified with symmetric polynomials in H^*BT^n. Let $\phi_n : T^n \longrightarrow U(n)$ be the inclusion of the canonical maximal torus of diagonal matrices in $U(n)$ and also write ϕ_n for the induced map of classifying spaces $BT^n \longrightarrow BU(n)$. The Weyl group of $U(n)$ is the symmetric group Σ_n which we may represent by permutation matrices. These act on T^n by permuting the entries of diagonal matrices. Since the classifying space functor commutes with products, the Künneth theorem gives an isomorphism

$$H^*(BT^n) \cong H^*(BU(1)) \otimes \cdots \otimes H^*(BU(1)) = P\{x_1, \ldots, x_n\},$$

where there are n copies of $BU(1)$ and we write x_i for the element c_1 in the ith copy. The action of Σ_n on T^n induces an action on BT^n and hence on $H^*(BU(n))$. The action of $\sigma \in \Sigma_n$ is the map of polynomial rings given on generators by $\sigma(x_i) = x_{\sigma(i)}$.

By Lemma 6.7, for $\sigma \in \Sigma_n$ regarded as a map $BT^n \longrightarrow BT^n$, we have $\phi_n \circ \sigma \simeq \phi_n$. Passing to cohomology, this implies that the elements of the image of $\phi_n^* : H^*(BU(n)) \longrightarrow H^*(BT_n)$ are fixed under the action of Σ_n. This accounts for the appearance of the symmetric polynomials in this situation.

Theorem 1.5. *The ring homomorphism*

$$\phi_n^* : H^*BU(n) \longrightarrow (H^*BT^n)^{\Sigma_n}$$

is an isomorphism.

In fact, quite generally, if G is a compact, connected, Lie group with maximal torus T and Weyl group W and if $H_*(G)$ has no torsion for any prime p that divides the order of W, then

$$H^*(BG) \cong H^*(BT)^W.$$

For this approach to the computation of H^*BG for classical groups G, see **??**. However, we will not pursue such a general approach here.

The next result will be fundamental in our proof of Theorem 1.1 and later theorems. An element $x \in H^*G$ is said to be *universally transgressive* if it is transgressive in the universal bundle $G \to EG \to BG$. By the naturality of transgression, this implies that it is transgressive in every principal G-bundle.

Theorem 1.6. *Let R be a commutative ring. If G is a topological group such that $H^*(G; R) = E\{x_i\}$ with the x_i universally transgressive when char $R = 2$, then $H^*(BG; R) = P\{y_i : |y_i| = 1 + |x_i|\}$ with y_i the image under the transgression of x_i. Equivalently, y_i suspends to x_i*

Proof. Since primitively generated exterior Hopf algebras are self-dual, we have $H_*G \cong H^*G$. A simple calculation shows the E_2 term of the Rothenberg-Steenrod spectral sequence is

$$\mathrm{Ext}_{E\{x_i\}}(R,R) \cong P\{\tau x_i\} \quad \text{with} \quad |\tau x_i| = (1, |x_i|).$$

The element x_i transgresses to y_i because, in general, if x transgresses, then $\tau x \in E_2^{1,*}$ survive to the transgression of x. If char $R \neq 2$, then all elements of E_2 lie in even degrees, and hence $E_2 \cong E_\infty$. If char $R = 2$, then the generators τx_i must survive to E_∞ since x_i transgresses. In either case, we have $H^*(BG;R) \cong P\{y_i\}$ with $|y_i| = 1 + |x_i|$.

□

Before proving Theorem 1.5, we show how it implies Theorem 1.1.

Proof of Theorem 1.1. Let us write $S = H^*(BT^n)^{\Sigma_n}$ for the subring of elements of $H^*(BT^n)$ fixed under the action of Σ_n. Then S is the polynomial ring on the n elementary symmetric functions $\sigma_i = \sigma_i(x_1, \ldots, x_n)$. Thus

$$\begin{gathered}\sigma_1 = x_1 + \cdots + x_n,\\ \sigma_2 = x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n,\\ \vdots\\ \sigma_n = x_1 \cdots x_n.\end{gathered}$$

It is convenient to set $\sigma_0 = 1$ and $\sigma_i = 0$ if $i > n$. We define the Chern classes $c_i \in H^{2i}(BU(n))$ by

$$c_i = (\phi_n^*)^{-1}(\sigma_i).$$

Since $\phi_n^* : H^*(BU(n)) \longrightarrow S$ is an isomorphism, it is obvious that

$$H^*(BU(n)) = P\{c_1, \ldots, c_n\}.$$

It is also obvious that $c_0 = 1$, c_1 is the canonical class when $n = 1$, and $c_i = 0$ when $i > n$. We have the following commutative diagrams:

$$\begin{array}{ccc} BT^{n-1} & \xrightarrow{\phi_{n-1}} & BU^{n-1} \\ \Big\downarrow{\scriptstyle h_n} & & \Big\downarrow{\scriptstyle j_n} \\ BT^n & \xrightarrow[\phi_n]{} & BU(n) \end{array} \qquad \begin{array}{ccc} BT^i \times BT^j & \xrightarrow{\phi_i \times \phi_j} & BU(i) \times BU(j) \\ \Big\| & & \Big\downarrow{\scriptstyle p_{ij}} \\ BT^{i+j} & \xrightarrow[\phi_{i+j}]{} & BU(i+j). \end{array}$$

Here $h_n : BT^{n-1} \longrightarrow BT^n$ is induced by the inclusion $T^{n-1} \longrightarrow T^n$ of the first $n-1$ factors. Clearly $h_n^*(x_i) = x_i$ for $i < n$ and $h_n^*(x_n) = 0$. Therefore $h_n^*(\sigma_i) = \sigma_i$. This implies that $i_n^*(c_i) = c_i$ by the diagram on the left above. Since the ϕ_k^* are all monomorphisms, the diagram on the right shows that p_{ij}^* is a monomorphism and that

$$(\phi_i^* \otimes \phi_j^*)(p_{ij}^*(c_k)) = \phi_{i+j}^*(c_k) = \sigma_k(x_1, \cdots, x_{i+j}).$$

As a matter of algebra, it is easy to verify that

$$\sigma_k(x_1, \cdots, x_{i+k}) = \sum_{a+b=k} \sigma_a(x_1, \cdots, x_i)\sigma_b(x_{i+1}, \cdots, x_{i+j})$$

Therefore

$$(\phi_i^* \otimes \phi_j^*)(p_{ij}^*(c_k)) = \sum_{a+b=k} \phi_i^*(c_a))\phi_j^*(c_b) = (\phi_i^* \otimes \phi_j^*)(\sum_{a+b=k} c_a \otimes c_b).$$

Since $\phi_i^* \otimes \phi_j^*$ is a monomorphism, this proves that $p_{ij}^*(c_k) = \sum_{a+b=k} c_a \otimes c_b$.

The uniqueness of the c_i is proven by induction on n. There is nothing to show if $n = 1$. Assume the uniqueness of the c_i in $H^*(BU(m))$ for $m < n$. Then, for $i < n$, the c_i in $H^*(BU(n))$ are uniquely determined since $i_n^*(c_i)$ is prescribed and i_n^* is an isomorphism in degrees less than $2n$. For $i = n$, $p_{1,n-1}^*(c_n)$ is prescribed and $p_{1,n-1}^*$ is a monomorphism.

For the statements about $BSU(n)$ in Theorem 1.1, we will see in Theorem 1.6 below that $H^*(BSU(n))$ is a polynomial ring on generators of degree $2i$, $2 \le i \le n$. Certainly $j_n^*(c_1) = 0$. Either by naturality from the proof of Theorem 1.6 or by the Serre spectral sequence of the fibration $BSU(n) \longrightarrow BU(n) \longrightarrow BU(1)$, we see that $j_n^* : H^*(BU(n)) \longrightarrow H^*(BSU(n))$ is an epimorphism, and it follows that its restriction to $P\{c_2, \ldots, c_n\}$ is an isomorphism. The commutative diagrams

$$\begin{array}{ccc} BSU(n-1) & \xrightarrow{j_{n-1}} & BU(n-1) \\ \downarrow{\scriptstyle i_{n-1}} & & \downarrow{\scriptstyle i_n} \\ BSU(n) & \xrightarrow[j_n]{} & BU(n) \end{array} \qquad \begin{array}{ccc} BSU(i) \times BSU(l) & \xrightarrow{j_i \times j_l} & BU(i) \times BU(l) \\ \downarrow{\scriptstyle p_{il}} & & \downarrow{\scriptstyle p_{il}} \\ BSU(i+l) & \xrightarrow[j_{i+l}]{} & BU(i+l) \end{array}$$

show that the c_i in $H^*(BSU(n))$ behave the same way under i_n^* and the p_{ij}^* as do the c_i in $H^*(BU(n))$. This completes the proof of Theorem 1.1. □

To prove Theorem 1.5, we need the following algebraic result.

Theorem 1.7. *Let Γ and Λ be graded polynomial algebras on n generators of positive degree over a field R. Let $f : \Lambda \to \Gamma$ be an R-algebra homomorphism. If Γ is finitely generated as a Λ- module (via f), then Γ is a free Λ-module. In particular, f is a monomorphism.*

Proof. See [**8**, Lemma 4.8] □

Proof of Theorem 1.5. In order to apply Theorem 1.7, we show that H^*BT^n is a finitely generated $H^*BU(n)$ module via ϕ_n^* (with integral coefficients). First, note that $H^*(U(n)/T^n)$ is finitely generated over $\mathbb{Z}$. This follows since $U(n)/T^n$ is a compact manifold, as it's the continuous image of the compact manifold $U(n)$. Now consider the Serre spectral sequence of $U(n)/T^n \to BT^n \to BU(n)$. Since $E_2^{*,0} = H^*BU(n)$, each E_r is an $H^*BU(n)$-module. Furthermore, if $x \in H^*BU(n)$ then $d_r(xy) = xd_r(y)$, since $d_r(x) = 0$ and x is an even degree class. Thus each d_r is an $H^*BU(n)$-module homomorphism and therefore, $\{E_r, d_r\}$ is a spectral sequence of $H^*BU(n)$-modules. Now ϕ_n^* is the composite

$$H^*BU(n) = E_2^{*,0} \to H^*BT^n$$

and hence E_∞ has the module structure induced by ϕ_n^*. Since $E_2 = H^*BU(n) \otimes H^*(U(n)/T^n)$ is finitely generated as an $H^*BU(n)$-module and $H^*BU(n)$ is Noetherian (by the Hilbert basis theorem), the subquotient E_∞ of E_2 is finitely generated. Hence H^*BT^n is finitely generated as a $H^*BU(n)$-module. By Theorem 1.7, $\phi_n^* : H^*(BU(n); F) \to H^*(BT^n; F)$ is a monomorphism for any field F and hence also for $F = \mathbb{Z}$. (In fact, since $H^*BU(n)$ is torsion free, it suffices to consider $F = \mathbb{Q}$). By inspection, S and $H^*BU(n)$ have the same dimension in each degree. It follows that $\phi_n^* : H^*(BU(n); F) \to S \otimes F$ is an isomorphism for every field F.

A map between finitely generated free abelian groups which induces an isomorphism upon tensoring with $\mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$ (for any prime p) is an isomorphism. Hence, $\phi_n^* : H^*BU(n) \to S$ is an isomorphism. □

We insert a further important property of Chern classes that follows from our way of defining them.

Corollary 1.8. *If $\bar{\xi}$ is the conjugate bundle of a complex n-plane bundle ξ, then $c_i(\bar{\xi}) = (-1)^i c_i(\xi)$.*

PROOF. Let $c : U(n) \longrightarrow U(n)$ be the homomorphism given by complex conjugation of matrix entries. Then $\bar{\xi}$ is classified by the composite of the classifying map of ξ and the induced map $c : BU(n) \longrightarrow BU(n)$. On diagonal matrices, c restricts to the product of n copies of the conjugation map $c : S^1 \longrightarrow S^1$, which has degree -1. Therefore, in $H^*(BT^n)$, $c^*(x_i) = -x_i$. This implies that $c^*(\sigma_i) = (-1)^i \sigma_i$, and the conclusion follows from the definition of the Chern classes. □

Corollary 1.9. *As algebras,*

$$H^*BU = P\{c_1, c_2, \ldots\}$$
$$\text{and } H^*BSU = P\{c_2, c_3, \ldots\}.$$

As coalgebras,

$$H_*BU(n) = \Gamma\{\gamma_1, \gamma_2, \ldots, \gamma_n\},$$
$$H_*BU = \Gamma\{\gamma_1, \gamma_2, \ldots\},$$
$$\text{and } H_*BSU = \Gamma\{\gamma_2, \gamma_3, \ldots\},$$

where γ_i is dual to c_i.

PROOF. It is immediate by duality that $H_*BU(n)$ and $H_*BSU(n)$ are as stated. The result for H_*BU and H_*BSU then follows by passage to limits and implies (i) by duality. □

In the Appendix, we will see that BU is an H-space whose product corresponds to Whitney sum of bundles. It will then follow from Theorem 1.1.(iii) that the coproduct ψ on H^*BU is

$$\psi(c_n) = \sum_{i+j=n} c_i \otimes c_j.$$

We can now use the suspension homomorphisms $\sigma^* : H^nBG \to H^{n-1}G$ and $\sigma_* : H_{n-1}G \to H_nBG$ to obtain convenient generators for the homology of cohomology of $U(n)$ and $SU(n)$.

Corollary 1.10. *As Hopf algebras,*

$$H^*U(n) = E\{x_1, x_3, \ldots, x_{2n-1}\},$$
$$H^*SU(n) = E\{x_3, x_5, \ldots, x_{2n-1}\},$$
$$H^*U = E\{x_1, x_3, \ldots\},$$
$$\text{and } H^*SU = E\{x_3, x_5, \ldots\}$$

where $x_{2i-1} = \sigma^*(c_i)$ in each case. Dually,

$$\begin{aligned} H_*U(n) &= E\{a_1, a_3, \ldots, a_{2n-1}\}, \\ H_*SU(n) &= E\{a_3, a_5, \ldots, a_{2n-1}\}, \\ H_*U &= E\{a_1, a_3, \ldots\}, \\ \text{and } H_*SU &= E\{a_3, a_5, \ldots\} \end{aligned}$$

as Hopf algebras, where $\sigma_*(a_{2i-1}) = \gamma_i$ and a_i is dual to x_i in each case. Moreover, the homomorphisms induced in homology and cohomology by the natural inclusions

$$\begin{array}{ccccc} SU(n-1) & \longrightarrow & SU(n) & \longrightarrow & SU \\ \downarrow & & \downarrow & & \downarrow \\ U(n-1) & \longrightarrow & U(n) & \longrightarrow & U \end{array}$$

send each generator to the generator of the same name if it is present and to zero otherwise.

PROOF. It is immediate from [See where it is] that the cohomology algebras and homology coalgebras are as specified. The consistency of these generators under the canonical inclusions follows directly from the same property for the Chern classes and the naturality of suspensions. The Hopf algebra structures now follow trivially as in Corollary **??**. □

There are several other ways in which the Chern classes arise. For a discussion of some of them and their interrelationships, see [**3**, p.363 ff]. One of the ways is as follows. Since $U(n)/U(n-1) \cong S^{2n-1}$, we have the bundle $S^{2n-1} \to BU(n-1) \to BU(n)$. Its Serre sequence has $E_2 \cong H^*BU(n) \otimes H^*S^{2n-1}$, from which we obtain the Gysin sequence (see the proof of Theorem 3.1)

$$\cdots \longrightarrow H^{p-1}BU(n-1) \xrightarrow{\alpha^*=0} H^{p-2n}BU(n) \xrightarrow{\phi^*=d_{2n}} H^pBU(n) \longrightarrow H^pBU(n-1) \longrightarrow \cdots$$

by splicing together the exact sequences

$$0 \longrightarrow E_\infty^{p,0} \longrightarrow H^pBU(n-1) \longrightarrow E_\infty^{p+1-2n,2n-1} \longrightarrow 0$$

and

$$\begin{array}{ccccccccc} 0 \longrightarrow E_\infty^{p-2n,2n-1} & \longrightarrow & E_2^{p-2n,2n-1} & \xrightarrow{d_{2n}} & E_2^{p,0} & \longrightarrow & E_\infty^{p,0} \longrightarrow 0 \\ & & \| & & \| & & \\ & & H^{p-2n}BU(n) & & H^pBU(n) & & \end{array}$$

(see Chapter 4 for details). We have $\alpha^* = 0$ because i_n^* is epimorphic. Since c_n generates $\ker(i_n^*)$, $\phi^*(1) = d_{2n}(\iota_{2n-1})$ must be $\pm c_n$. It can be shown that $\phi^*(1) = c_n$. Hence, we could define c_n to be $\phi^*(1)$ and, using the fact that i_n^* is an isomorphsim in degrees less than $2n$, inductively define c_i to be $(i_n^*)^{-1}(c_i)$ for $i < n$. Note that $\phi^*(1)$ can also be described as the transgression of ι_{2n-1}. We will not pursue this approach further.

2. The symplectic classes and $H^*(BSp(n))$

In this section we show show that $H^*BSp(n)$ has polynomial generators analogous to the Chern classes in $H^*BU(n)$. Let

$$i_n : Sp(n-1) \to Sp(n),\ \mu_n : Sp(n) \to SU(2n),$$
$$\nu_n : U(n) \to Sp(n),\ \text{and } p_{ij} : Sp(i) \times Sp(j) \to Sp(i+j)$$

be the canonical maps and let $\phi_n : T^n \to Sp(n)$ be the maximal torus consisting of diagonal matrices with complex entries. Let $i_n, \mu_n, \nu_n, p_{ij}$, and ϕ_n also denoted the induced map on classifying spaces. Recall that $\mu_1 : Sp(1) \to SU(2)$ is an isomorphism of topological groups.

Theorem 2.1. *There exist unique classes $k_i \in H^{4i}BSp(n)$, called the* symplectic classes*, which satisfy the following axioms.*

(i) *$k_1 \in H^4BSp(1)$ is $-c_2$ under the identification $\mu_1 : Sp(1) \xrightarrow{\cong} SU(2)$.*
(ii) *$i_n^*(k_i) = k_i$.*
(iii) *$p_{ij}^*(k_l) = \sum_{a+b=l} k_a \otimes k_b$*
(iv) *$k_0 = 1$ and $k_i = 0$ for $i > n$*

Furthermore, $H^*BSp(n) = P\{k_1, \ldots, k_n\}$ and

(v) *$\mu_n^*(c_i) = 0$ if i is odd, and $\mu_n^*(c_{2i}) = (-1)^i k_i$*
(vi) *$\nu_n^*(k_i) = \sum_{a+b=2i} (-1)^{a+i} c_a c_b$*

Proof. We proceed as in the case of $BU(n)$. Consider the maximal torus $\phi_n : T^n \to Sp(n)$. By Theorem 1.6, $H^*BSp(n) = P\{z_1, \ldots, z_n\}$ with $|z_i| = 4i$, and $H^*BT^n = P\{y_1, \ldots, y_n\}$ with $|y_i| = 2$. Since $H^*(Sp(n)/T^n)$ is finitely generated, Theorem 1.7 implies $\phi_n^* : H^*BSp(n) \to H^*BT^n$ is a monomorphism. As in the complex case, T^n is invariant under permutations of coordinates. Hence the image of ϕ_n^* is contained in the symmetric polynomials in H^*BT^n. Define $c : Sp(n) \to Sp(n)$ to be conjugation by the matrix

$$\begin{pmatrix} j & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix}$$

Clearly $c(T^n) = T^n$. In fact, $c|_{T^n}$ is complex conjugation of the first factor since $-j\alpha j = \bar{\alpha}$, for $\alpha \in \mathbb{C}$. Hence $Bc^* : H^*BT^n \to H^*BT^n$ is given by $Bc^*(y_1) = -y_1$, and $Bc^*(y_i) = y_i$ for $i > 1$. We have shown that $Bc^* : H^*BSp(n) \to H^*BSp(n)$ is the identity. Since we can do this for each coordinate it follows that the image of ϕ_n^* in H^*BT^n is contained in the subgroup of polynomials in $y_1, \ldots, y_n$ invariant under permutations and sign reversals, $y_i \mapsto -y_i$. This subring is generated by the symmetric polynomials in the y_i^2, $\sigma_i(y_1^2, \ldots, y_n^2)$. Thus

$$\phi_n^* : H^*BSp(n) \longrightarrow P\{\sigma_i(y_1^2, \ldots, y_n^2)\} \subset H^*BT^n.$$

The generators of $H^*BSp(n)$ and of $P\{\sigma_i(y_1^2, \ldots, y_n^2)\}$ lie in the same dimensions. Hence we conclude as in Theorem 1.1 that ϕ_n^* is an isomorphism between $H^*BSp(n)$ and $P\{\sigma_i(y_1^2, \ldots y_n^2)\}$. We therefore define

$$k_i = (\phi_n^*)^{-1}(\sigma_i(y_1^2, \ldots, y_n^2)).$$

Clearly $H^*BSp(n) = P\{k_1, \dots, k_n\}$. Properties (ii)-(iv) and uniqueness follow exactly as in Theorem 1.1. To prove (i), consider the following commutative diagrams ($\mathbb{H}$ has $\mathbb{C}$-basis $\{i, j\}$ as a right $\mathbb{C}$-module)

$$\begin{CD}
T^1 @>{\alpha}>> @. T^2 \\
@V{\phi_1}VV @. @VV{\phi_2}V \\
Sp(1) @>{\mu_1}>> SU(2) @>{j_2}>> U(2)
\end{CD}$$

and the induced diagram on classifying spaces in cohomology

$$\begin{CD}
H^*BT^1 @<{\alpha^*}<< @. H^*BT^2 \\
@A{\phi_1^*}AA @. @AA{\phi_2^*}A \\
H^*BSp(1) @<{\mu_1^*}<< H^*BSU(2) @<{j_2^*}<< H^*BU(2)
\end{CD}$$

where $\alpha(\lambda) = (\lambda, \bar{\lambda})$ so that $\alpha^*(y_1) = y_1$, $\alpha^*(y_2) = -y_1$. Clearly $\alpha^*\phi_2^*(c_2) = \alpha^*(\sigma_2(y_1, y_2)) = \alpha^*(y_1y_2) = -y_1^2$. By commutativity, $\phi_1^*\mu_1^*(c_2) = \phi_1^*\mu_1^*j_2^*(c_2) = -y_1^2$. Since ϕ_1^* is a monomorphism, $\mu_1^*(c_2) = -k_1$, proving (i). Since (i) is the $n = 1$ case of (v), we prove (v) by induction on n. First, it is clear that $\mu_n^*(c_i) = 0$ for odd i since $H^*BSp(n)$ is zero in degrees not divisible by 4. Now consider the commutative diagram

$$\begin{CD}
Sp(1) \times Sp(n-1) @>{\mu_1 \times \mu_{n-1}}>> SU(2) \times SU(2n-2) \\
@V{p_{1,n-1}}VV @VV{p_{2,2n-2}}V \\
Sp(n) @>{\mu_n}>> SU(2n).
\end{CD}$$

Since $p^*_{1,n-1}$ is a monomorphism it is sufficient to show that $p^*_{1,n-1}\mu_n^*(c_{2i}) = p^*_{1,n-1}((-1)^i k_i)$. This is immediate:

$$\begin{aligned}
(\mu_1^* \otimes \mu_{n-1}^*)(p^*_{2,2n-2}(c_{21})) &= (\mu_1^* \otimes \mu_{n-1}^*)(1 \otimes c_{2i} + c_2 \otimes c_{2i-2}) \\
&= (-1)^i(1 \otimes k_i + k_1 \otimes k_{i-1}), \text{ by inductive hypothesis} \\
&= p^*_{1,n_1}((-1)^i k_i),
\end{aligned}$$

proving (v).

We also prove (vi) by induction on n. Note that $v_1 = \phi_1 : U(1) = T^1 \to Sp(1)$ and hence $v_1^*(k_1) = c_1^2$ as required. For $n > 1$, consider the commutative diagram

$$\begin{CD}
U(1) \times U(n-1) @>{\nu_1 \times \nu_{n-1}}>> Sp(1) \times Sp(n-1) \\
@V{p_{1,n-1}}VV @VV{p_{1,n-1}}V \\
U(n) @>{\nu_n}>> Sp(n).
\end{CD}$$

As above, it is sufficient to show that $p^*_{1,n-1}\nu_n^*(k_i) = p^*_{1,n-1}\left(\sum_{a+b=2i}(-1)^{a+i}c_ac_b\right)$. We simply calculate:

$$\begin{aligned}
p^*_{1,n-1}\nu_n^*(k_i) &= (\nu_1^* \otimes \nu_{n-1}^*)p^*_{1,n-1}(k_i) = (\nu_1^* \otimes \nu_{n-1}^*)(1 \otimes k_i + k_1 \otimes k_{i-1}) \\
&= 1 \otimes \sum_{a+b=2i}(-1)^{a+i}c_ac_b + c_1^2 \otimes \sum_{a+b=2i-2}(-1)^{a+i-1}c_ac_b
\end{aligned}$$

by inductive hypothesis, while

$$p_{1,n-1}^*\left(\sum_{a+b=2i}(-1)^{a+i}c_ac_b\right)$$
$$= \sum_{a+b=2i}(-1)^{a+i}(1\otimes c_ac_b + c_1\otimes c_{a-1}c_b + c_1\otimes c_ac_{b-1} + c_1^2\otimes c_{a-1}c_{b-1})$$
$$= \sum_{a+b=2i}(-1)^{a+i}(1\otimes c_ac_b + c_1^2\otimes c_{a-1}c_{b-1})$$

since the middle terms cancel. This proves (vi), completing the proof. □

If ξ is a $U(n)$ bundle, write $\xi_{\mathbb{H}}$ for the $Sp(n)$ bundle given by the inclusion $U(n)\to Sp(n)$. If ξ is an $Sp(n)$ bundle, write $\xi^{\mathbb{C}}$ for the $SU(2n)$ bundle given by $Sp(n)\to SU(2n)$. If ξ is an $Sp(n)$ bundle over A, we define the *total symplectic class* $k(\xi)$ of ξ to be $k_0(\xi)+k_1(\xi)+k_2(\xi)+\cdots\in H^{**}A$. The following corollary is an immediate consequence of Theorem 2.1.

Corollary 2.2. *All characteristic classes of $Sp(n)$ bundles can be expressed as polynomials in the symplectic classes. If $\xi : E\to A$ is an $Sp(n)$ bundle, the symplectic classes $k_i(\xi)\in H^{4i}(A)$ satisfy:*

(i) if ξ is an $Sp(1)\cong SU(2)$ bundle then $k_1(\xi) = -c_2(\xi^{\mathbb{C}})$.
(ii) $k_i(\xi) = k_i(\xi\oplus 1)$, where 1 denotes the trivial $Sp(1)$ bundle over A.
(iii) $k(\xi\oplus\eta) = k(\xi)k(\eta)$, for any Sp bundle η.
(iv) $k_0(\xi) = 1$ and $k_i(\xi) = 0$ for $i > n$.
(v) $c_i(\xi^{\mathbb{C}}) = 0$ if i is odd and $c_{2i}(\xi^{\mathbb{C}}) = (-1)^ik_i(\xi)$.
(vi) if ξ is a $U(n)$ bundle then

$$k_i(\xi_{\mathbb{H}}) = \sum_{a+b=2i}(-1)^{a+i}c_a(\xi)c_b(\xi)$$

We are now able to compute the cohomology of the homogeneous spaces $U(2n)/Sp(n)$, $SU(2n)/Sp(n)$, and $Sp(n)/U(n)$.

Corollary 2.3. *As algebras,*

$$H^*U(2n)/Sp(n) = E\{x_{4i-3}\mid 1\le i\le n\}$$
$$\text{and } H^*SU(2n)/Sp(n) = E\{x_{4i-3}\mid 2\le i\le n\}$$

with $|x_{4i-3}| = 4i-3$. The induced homomorphisms $H^*U(2n)/Sp(n)\to H^*U(2n)$ and $H^*SU(2n)/Sp(n)\to H^*SU(2n)$ take x_{4i-3} to x_{4i-3}.

Proof. Consider the Eilenberg-Moore spectral sequence of $U(2n)/Sp(n)\to BSp(n)\to BU(2n)$. We must calculate $E_2 = \mathrm{Tor}_{H^*BU(2n)}(\mathbb{Z}, H^*BSp(n))$. Let A be $H^*BU(2n) = P\{c_1,\ldots,c_{2n}\}$ and let B be $H^*BSp(n) = P\{k_1,\ldots,k_n\}$. Of course B is an A-module via $\mu_n^* : A\to B$. Let X be the Koszul resolution of $\mathbb{Z}$ over A; $X = E_A\{y_1,y_2,\ldots,y_{2n}\}$, with $|y_i| = (-1,2i)$ and differential $d : X\to X$ given by $d(y_i) = c_i$. Now $X\otimes_A B = E_B\{y_1,\ldots,y_{2n}\}$ with differential $d(y_i) = \mu_n^*(c_i)$. Hence we can split $X\otimes_A B$ as $X_1\otimes_A X_2$, where X_1 is $E_B\{y_1,y_3,\ldots,y_{2n-1}\}$ with zero differential and X_2 is $E_B\{y_2,y_4,\ldots,y_{2n}\}$ with differential $d(y_{2i}) = (-1)^ik_i$. Clearly X_2 is the Koszul resolution of $\mathbb{Z}$ over B. It follows that $H(X_1) = X_1$ and

$H(X_2) = \mathbb{Z}$. Since X_1 is a free B-module, the Kunneth theorem implies that

$$\begin{aligned} E_2 \cong H(X_1 \otimes X_2) &= H(X_1) \otimes H(X_2) \\ &= E_B\{y_{2i-1} \mid 1 \le i \le n\} \otimes \mathbb{Z} \\ &= E\{y_{2i-1} \mid 1 \le i \le n\}. \end{aligned}$$

Since E_2 lies in the second quadrant, every differential must send each generator to zero. Hence $E_2 = E_\infty$ and $H^*U(2n)/Sp(n) = E\{y_{2i-1}\} = E\{x_{4i-3} \mid 1 \le i \le n\}$, where we have renamed the generators so that the subscript will reflect the degree. The second statement is immediate from the map of fibrations

$$\begin{array}{ccccc} U(2n) & \longrightarrow & EU(2n) & \longrightarrow & BU(2n) \\ \downarrow & & \downarrow & & \| \\ U(2n)/Sp(n) & \longrightarrow & BSp(n) & \longrightarrow & BU(2n) \end{array}$$

since we have chosen as generators for $H^*U(2n)$ the suspensions of the Chern classes (Corollary **??**). The computation of $H^*SU(n)/Sp(n)$ is completely analogous. □

Corollary 2.4. *As an algebra, $H^*(Sp(n)/U(n); R) = P_R\{c_1, \dots c_n\}/I$ for any coefficient ring R, where $I = \operatorname{im}(\nu_n^*)$ is generated by*

$$\left\{ \sum_{a+b=2i} (-1)^{a+i} c_a c_b \mid 1 \le i \le n \right\}.$$

The induced homomorphism $H^*(BU(n); R) \to H^*(Sp(n)/U(n); R)$ is the obvious epimorphism.

Proof. First take R to be a field and let all cohomology have coefficients in R. In the Eilenberg-Moore spectral sequence of $Sp(n)/U(n) \to BU(n) \to BSp(n)$, $E_2 = \operatorname{Tor}_{H^*BSp(n)}(R, H^*BU(n))$, where $H^*BU(n)$ is an $H^*BSp(n)$ module via $\nu_n^* : H^*BSp(n) \to H^*BU(n)$. By Theorem 1.7, $H^*BU(n)$ is free over $H^*BSp(n)$. Therefore, $E_2 = R \otimes_{H^*BSp(n)} H^*BU(n)$ and hence $H^*Sp(n)/U(n) = E_\infty = E_2$, proving the first statement when R is a field. Since $H^*(Sp(n)/U(n); \mathbb{Z}/p\mathbb{Z})$ is concentrated in even degrees, the Bockstein spectral sequence collapses ($E_1 = E_\infty$) for each prime p. It follows that $H^*(Sp(n)/U(n); \mathbb{Z})$ is torsion-free and hence $H^*(Sp(n)/U(n); R) = H^*(Sp(n)/U(n); \mathbb{Z}) \otimes R$ for any ring R by the universal coefficient theorem. Now take $R = \mathbb{Z}$. The natural homomorphism

$$\mathbb{Z} \otimes_{H^*BSp(n)} H^*BU(n) = P\{c_i\} \longrightarrow H^*Sp(n)/U(n)$$

is an isomorphism when tensored with any field, hence must be an isomorphism. This proves the first statement when $R = \mathbb{Z}$. The general result follows by universal coefficients since $H^*Sp(n)/U(n)$ is torsion-free. The second statement is immediate since the induced homomorphism $H^*BU(n) \to H^*Sp(n)/U(n)$ is the composite

$$H^*BU(n) \to \mathbb{Z} \otimes_{H^*BSp(n)} H^*BU(n) \to H^*Sp(n)/U(n). \qquad \square$$

The next two corollaries are analogous to Corollaries 1.9 and 1.10 and are proved in exactly the same manner.

Corollary 2.5. *As an algebra,*

$$H^*BSp = P\{k_1, k_2, \dots\}.$$

As coalgebras,

$$H_*BSp(n) = \Gamma\{\kappa_1, \kappa_2, \ldots, \kappa_n\}$$
$$\text{and } H_*BSp = \Gamma\{\kappa_1, \kappa_2, \ldots\},$$

where κ_i *is dual to* k_i.

We will show in the Appendix that BSp is an H-space with product corresponding to the Whitney sum of bundles. As for BU the induced coproduct on H^*BSp will be given by $\psi(k_n) = \sum_{i+j=n} k_i \otimes k_j$.

Corollary 2.6. *As Hopf algebras,*

$$H^*Sp(n) = E\{x_3, x_7, \ldots, x_{4n-1}\} \text{ and } H^*Sp = E\{x_3, x_7, \ldots\}$$

where $x_{4i-1} = \sigma^*(k_i)$ *in each case. Dually, as Hopf algebras,*

$$H_*Sp(n) = E\{a_3, a_7, \ldots, a_{4n-1}\} \text{ and } H_*Sp = E\{a_3, a_7, \ldots\}$$

where $\sigma_*(x_{4i-1}) = \kappa_i$ *and* a_i *is dual to* x_i *in each case.*

Moreover, the homomorphisms induced in homology and cohomology by the inclusions $Sp(n-1) \to Sp(n) \to Sp$ *send each generator to the generators of the same name if is present and to zero otherwise.*

Corollary 2.7. *As algebras,*

$$H^*U/Sp = E\{x_1, x_5, x_9, \ldots\},$$
$$H^*SU/Sp = E\{x_5, x_9, \ldots\},$$
$$\text{and } H^*Sp/U = P\{c_1, c_2, \ldots\}/I$$

where $I = \operatorname{im} \nu^*$ *is generated by*

$$\left\{ \sum_{a+b=2i} (-1)^{a+i} c_a c_b \mid i \geq 1 \right\}.$$

Proof. This is immediate by passage to limits since the maps induced in cohomology by $U(n)/Sp(n) \to U(n+1)/Sp(n+1)$, $SU(n)/Sp(n) \to SU(n+1)/Sp(n+1)$, and $Sp(n)/U(n) \to Sp(n+1)/U(n+1)$ are the obvious epimorphisms. □

We can now compute the homomorphisms induced in cohomology by each of the maps in the bundles

$$U \xrightarrow{\nu} Sp \xrightarrow{q} U$$
$$Sp/U \xrightarrow{i} BU \xrightarrow{\nu} BSp$$
$$Sp \xrightarrow{\mu} U \xrightarrow{p} U/Sp$$
$$U/Sp \xrightarrow{j} BSp \xrightarrow{\mu} BU.$$

We consider only the infinite case since the results for the finite case then follow trivially.

Corollary 2.8. *With the above notation,*

(i) $\nu^*(x_{4i-1}) = (-1)^i 2x_{4i-1}$, *and* $q^* = 0$
(ii) $i^*(c_i) \equiv c_i \mod I$, *where* I *is the ideal defined in Corollary 2.4, and* $\nu^*(k_i) = \sum_{a+b=2i} (-1)^{a+i} c_a c_b$

(iii) $\mu^*(x_{4i-3}) = 0, \mu^*(x_{4i-1}) = (-1)^i x_{4i-1}$, *and* $p^*(x_{4i-3}) = x_{4i-3}$
(iv) $j^* = 0$, $\mu^*(c_{2i-1}) = 0$, *and* $\mu^*(c_{2i}) = (-1)^i k_i$.

Proof. First, note that the composites $q\nu$, νi, and μj are null homotopic. We have already determined $i^*(c_i), \nu^*(k_i), p^*(x_{4i-3})$, and $\mu^*(c_i)$ in Theorem 2.1 and Corollaries 2.3 and 2.4. By naturality of suspension,

$$\nu^*(x_{4i-3}) = \nu^*\sigma^*(k_i) = \sigma^*\nu^*(k_i) = \sum_{a+b=2i} (-1)^{a+i}\sigma^*(c_a c_b).$$

Since σ^* annihilates products, $\nu^*(x_{4i-3}) = (-1)^i 2\sigma^*(c_{2i}) = (-1)^i 2x_{4i-1}$. Thus, $\nu^* : H^*Sp \to H^*U$ is a monomorphism. For ν^*q^* to be zero, we must have $q^* = 0$. Now, $\mu^*(x_{4i-3}) = \mu^*p^*(x_{4i-3}) = 0$, while $\mu^*(X_{4i-1}) = \mu^*\sigma^*(c_{2i}) = \sigma^*((-1)^i k_i) = (-1)^i x_{4i-1}$. Finally, $\mu^* : H^*BU \to B^*BSp$ is an epimorphism and thus $j^* = 0$. □

3. The Stiefel-Whitney Classes and $H^*(BO(n);\mathbb{F}_2)$

In this section, all cohomology will have $\mathbb{F}_2$ coefficients. Let

$$i_n : BO(n-1) \to BO(n),\ i_n : BSO(n-1) \to BSO(n),$$
$$j_n : BSO(n) \to BO(n),\ \mu_n : BU(n) \to BSO(2n),$$
$$\nu_n : BO(n) \to BU(n), \text{ and } p_{ij} : BO(i) \times BO(j) \to BO(i+j)$$

be the canonical maps. Recall that $O(1) = \mathbb{Z}/2\mathbb{Z}$ and hence, $BO(1) = K(\mathbb{Z}/2\mathbb{Z}, 1) \cong \mathbb{R}\mathrm{P}^\infty$. There is an obvious inclusion of $(\mathbb{Z}/2\mathbb{Z})^n$ into $O(1)^n$ as diagonal matrices with entries ± 1 on the diagonal. This subgroup will play the role of the maximal torus played in the unitary and symplectic groups. Let $\phi_n : BO(1)^n \to BO(n)$ be the map induced by the inclusion $O(1)^n \subset O(n)$. Denote by $c_i \in H^{2i}(BU(n);\mathbb{F}_2)$, the mod 2 reduction of $c_i \in H^{2i}(BU(n);\mathbb{Z})$.

Theorem 3.1. *There exist unique classes $w_i \in H^iBO(n)$, called the* Stiefel-Whitney classes, *which satisfy the following axioms.*

(i) $w_1 \in H^1BO(1) = H^1\mathbb{R}\mathrm{P}^\infty$ *is the unique non-zero element*
(ii) $i_n^*(w_i) = w_i$
(iii) $p_{ij}^*(w_k) = \sum_{a+b=k} w_a \otimes w_b$
(iv) $w_0 = 1$ *and* $w_i = 0$ *for* $i > n$

Moreover, $H^*BO(n) = P\{w_1, \dots, w_n\}$ *and, if we let* $w_i = j_n^*(w_i) \in H^*BSO(n)$, *then* $H^*BSO(n) = P\{w_2, \dots, w_n\}$ *and (with* $w_1 = 0$*) properties (ii) - (iv) remain valid. Furthermore,*

(v) $\mu_n^*(w_{2i-1}) = 0$ *and* $\mu_n^*(w_{2i}) = c_i$ *where* $\mu_n : U(n) \to O(2n)$ *or* $\mu_n : U(n) \to SO(2n)$
(vi) $\nu_n^*(c_i) = w_i^2$ *where* $\nu_n : O(n) \to U(n)$ *or* $\nu_n : SO(n) \to SU(n)$.

Proof. We will show that $H^*BO(n)$ is a polynomial algebra $P\{y_1, \dots, y_n\}$ in several steps. Then we will be able to use the techniques used in the unitary and symplectic case to explicitly define the Stiefel-Whitney classes.

We know that $H^*SO(n) = E\{x_1, \dots, x_{n-1}\}$ as a coalgebra from Chapter 2. If the x_i are universally transgressive then Theorem **??** implies that $H^*BSO(n) = P\{y_2, \dots, y_n\}$ with $\sigma^* y_i = x_{i-1}$. To show that the x_i are universally transgressive, consider the map of fibrations

$$\begin{array}{ccccc} SO(n) & \longrightarrow & ESO(n) & \longrightarrow & BSO(n) \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle F} & & \| \\ SO(n)/SO(q) & \longrightarrow & BSO(q) & \longrightarrow & BSO(n). \end{array}$$

Recall that $H^*SO(n)/SO(q) = E\{x_q, \ldots, x_{n-1}\}$ and $f_1^*(x_i) = x_i$. In the Serre sequence of the bottom row, $d_r x_q = 0$ for dimensional reasons where $r < q+1$. Hence, in the Serre sequence of the top row, $d_r x_q = d_r E_r(f)(x_q) = E_r(f)(d_r x_q) = 0$ for $r < q+1$. Now x_q must be transgressive, $d_{q+1}(x_q) \neq 0$, since $H^*ESO(n)$ is trivial. Hence $H^*BSO(n) = P\{y_2, \ldots, y_n\}$.

The *Gysin sequence* will enable us to finish showing that $H^*BO(n) = P\{y_1, \ldots, y_n\}$.

Lemma 3.2. *Let $S^n \to E \xrightarrow{p} B$ be a fibration with trivial local coefficients and let $H^* = H^*(-;R)$ for any commutative ring R. Then there is an exact sequence (the* Gysin sequence*)*

$$\cdots \longrightarrow H^iE \longrightarrow H^{i-n}B \xrightarrow{\cdot\gamma} H^{i+1}B \xrightarrow{p^*} H^{i+1}E \longrightarrow \cdots$$

where $\cdot\gamma$ denotes multiplication by an element $\gamma \in H^{n+1}B$.

Proof. Consider the Serre spectral sequence of p. We have

$$E_2^{r,s} = \begin{cases} H^rB & s = 0 \\ H^rB \otimes \langle\iota\rangle & s = n \\ 0 & \text{otherwise.} \end{cases}$$

Hence d_{n+1} is the only nontrivial differential. Let $\gamma = d_{n+1}(\iota)$ in $E_2^{n+1,0} = H^{n+1}B$. We have exact sequences

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \ker(d_{n+1}) & \longrightarrow & E_2^{i,n} & \xrightarrow{d_{n+1}} & E_2^{i+n+1,0} & \longrightarrow & \mathrm{Cok}(d_{n+1}) & \longrightarrow & 0 \\ & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\ 0 & \longrightarrow & E_\infty^{i,0} & \longrightarrow & H^iB & \xrightarrow{\cdot\gamma} & H^{n+i+1}B & \longrightarrow & E_\infty^{i+n+1,0} & \longrightarrow & 0 \end{array}$$

and $0 \to E_\infty^{i,n} \to H^iE \to E_\infty^{i-n,n} \to 0$. Since $p^* : H^iB \to H^iE$ factors as $H^iB = E_2^{i,0} \to E_\infty^{i,0} \subset H^iE$, splicing these exact sequences together yields the result. $\square$

Consider the Gysin sequence of $S^n \to BSO(n) \to BSO(n+1)$. We have

$$H^0BSO(n+1) \xrightarrow{d_{n+1}(\iota)} H^{n+1}BSO(n+1) \xrightarrow{i_{n+1}^*} H^{n+1}BSO(n).$$

Since $H^*BSO(n+1) = P\{y_2, \ldots, y_{n+1}\}$, i_{n+1}^* has nonzero kernel in degree $n+1$. Hence $d_{n+1}(\iota) \neq 0$ in the Serre sequence of $S^n \to BSO(n) \to BSO(n+1)$. Now consider the Serre sequence of $S^n \to BO(n) \to BO(n+1)$ (note that the local coefficients are trivial because the only automorphism of $H^nS^n = \mathbb{F}_2$ is the identity). From the diagram

$$\begin{array}{ccccc} S^n & \longrightarrow & BSO(n) & \xrightarrow{i_n} & BSO(n+1) \\ \| & & \downarrow{\scriptstyle j_n} & & \downarrow{\scriptstyle j_{n+1}} \\ S^n & \longrightarrow & BO(n) & \xrightarrow{i_n} & BO(n+1) \end{array}$$

we observe that $0 \neq d_{n+1}(\iota) \in H^{n+1}BO(n+1)$ since $j_{n+1}^* d_{n+1}(\iota) = d_{n+1}(\iota) \neq 0$ in $H^{n+1}BSO(n+1)$.

We now have all the preliminary results needed to show $H^*BO(n) = P\{y_1, \ldots, y_n\}$. Recall that $BO(1) \cong \mathbb{R}\mathrm{P}^\infty$ and hence $H^*BO(1) = P\{y_1\}$. Assume for induction that $H^*BO(n) = P\{y_1, \ldots, y_n\}$. Let $y_{n+1} = d_{n+1}(\iota) \in H^{n+1}BO(n+1)$. Since $y_{n+1} \neq 0$, exactness of the Gysin sequence implies that $i_{n+1}^* H^iBO(n+1) \to H^iBO(n)$ is an isomorphism for $i \leq n$ and an epimorphism for all i. This implies that multiplication by y_{n+1} is a monomorphism $H^iBO(n+1) \to H^{i+n+1}BO(n+1)$ for all i. It follows that $H^*BO(n+1) = P\{y_1, \ldots, y_{n+1}\}$, completing the induction.

Now all of Theorem 3.1 except (v) and (vi) goes through exactly as in the unitary case with two minor changes. First, replace the maximal torus $T^n \subset U(n)$ by $O(1)^n \subset O(n)$. Second, replace the proof that $j_n^* : H^*BU(n) \to H^*SU(n)$ is an epimorphism by the following. Clearly, $j_1^* : H^*BO(1) \to H^*BSO(1)$ is an epimorphism. Assume for induction that j_n^* is an epimorphism and consider the diagram

$$\begin{array}{ccccc} S^n & \longrightarrow & BSO(n) & \xrightarrow{i_n} & BSO(n+1) \\ \| & & \downarrow{\scriptstyle j_n} & & \downarrow{\scriptstyle j_{n+1}} \\ S^n & \longrightarrow & BO(n) & \xrightarrow{i_n} & BO(n+1). \end{array}$$

Since i_n^* is an isomorphism in degrees less than $n+1$, $y_i \in \mathrm{im}(j_{n+1}^*)$ for $i \leq n$. Now in the Serre spectral sequences of the two bundles $d_{n+1}(\iota_n) = y_{n+1}$. Hence $j_{n+1}(y_{n+1}) = j_{n+1}(d_{n+1}(\iota_n)) = d_{n+1}(\iota_n) = y_{n+1}$. Hence j_{n+1}^* is an epimorphism.

We prove (v) by induction on n. When $n = 1$, $\mu_1 : U(1) \to SO(2)$ is an isomorphism and hence $\mu_1^*(w_2) = c_1$. Since $H^*BU(n)$ lies in even degrees, $\mu_n^*(w_{2i-1}) = 0$. Consider the diagram

$$\begin{array}{ccc} S^{2n-1} & = & S^{2n-1} \\ \downarrow & & \downarrow \\ BU(n-1) & \xrightarrow{i_{2n-1}\mu_{n-1}} & BSO(2n-1) \\ \downarrow{\scriptstyle i_n} & & \downarrow{\scriptstyle i_{2n}} \\ BU(n) & \xrightarrow{\mu_n} & BSO(2n). \end{array}$$

If $i < n$ then $\mu_n^*(w_{2i}) = c_i$ by inductive hypothesis, since i_n^* is an isomorphism in degree $2i$. It is clear that ι_{2n-1} must transgress to c_n and w_{2n} in $H^*BU(n)$ and $H^*BSO(2n)$. By naturality of transgression, $\mu_n^*(w_{2n}) = c_n$.

To prove (vi), consider the diagram

$$\begin{array}{ccc} BO(1)^n & \xrightarrow{\nu_1^n} & BU(1)^n \\ {\scriptstyle \phi_n}\downarrow & & \downarrow{\scriptstyle \phi_n} \\ BO(n) & \xrightarrow{\nu_n} & BU(n) \end{array}$$

From the Serre spectral sequence of $U(1) \cong U(1)/O(1) \to BO(1) \xrightarrow{\nu_1} BU(1)$ it is clear that ν_1^* is a monomorphism. Hence $\nu_1^*(c_1) = w_1^2$. If we write $H^*BO(1)^n$ as $P\{x_1, \ldots, x_n\}$ then $\phi_n^*\nu_n^*(c_i) = (\nu_1^n)^*\phi_n^*(c_i) = \sigma_i(x_1^2, \ldots, x_n^2)$. With $\mathbb{F}_2$ coefficients,

$\sigma_i(x_1^2,\ldots,x_n^2) = (\sigma_i(x_1,\ldots,x_n))^2$ and hence $\nu_n^*(c_i) = w_i^2$. From the diagram

$$\begin{array}{ccc} BSU(n) & \xrightarrow{j_n} & BU(n) \\ \uparrow{\scriptstyle \nu_n} & & \uparrow{\scriptstyle \nu_n} \\ BSO(n) & \xrightarrow{j_n} & BO(n) \end{array}$$

it is clear that $\nu_n^*(c_i) = w_i^2$ for the special groups also. □

If ξ is an $O(n)$ (resp. $SO(n)$) bundle, let $\xi_{\mathbb{C}}$ denote its complexification, given by $\nu_n : O(n) \to U(n)$. If ξ is a $U(n)$ bundle, let $\xi^{\mathbb{R}}$ denote the underlying real bundle, given by $\mu_n : U(n) \to SO(2n)$. If ξ is an $O(n)$ bundle over A, define the *total Stiefel-Whitney class* $w(\xi)$ to be

$$w_0(\xi) + w_1(\xi) + \cdots \in H^{**}(A).$$

Of course we have the following corollary.

Corollary 3.3. *All $\mathbb{F}_2$ characteristic classes of $O(n)$ and $SO(n)$ bundles can be expressed as polynomials in the Stiefel-Whitney classes. Let ξ be an $O(n)$ bundle over A. The Stiefel-Whitney classes $w_i(\xi) \in H^i(A;\mathbb{F}_2)$ satisfy:*

(i) $w_1(\gamma) \in H^1BO(1)$ *is the unique nonzero element if* γ *is the nontrivial* $O(1)$ *bundle over* $BO(1)$.
(ii) $w_i(\xi) = w_i(\xi \oplus 1)$, *where* 1 *denotes the trivial line bundle over* A.
(iii) $w(\xi \oplus \eta) = w(\xi)w(\eta)$ *for any* $O(m)$ *bundle* η *over* A.
(iv) $w_0(\xi) = 1$, $w_i(\xi) = 0$ *if* $i > n$, *and* $w_1(\xi) = 0$ *if and only if* ξ *is an* $SO(n)$ *bundle.*
(v) *if* ξ *is a* $U(n)$ *bundle, then* $w_{2i-1}(\xi^{\mathbb{R}}) = 0$ *and* $w_{2i}(\xi^{\mathbb{R}}) = c_i(\xi)$.
(vi) $c_i(\xi_{\mathbb{C}}) = w_i(\xi)^2$.

Note that in (v) and (vi) the Chern classes referred to are the mod 2 reductions of the ordinary Chern classes. The last statement in (iv) follows from the fact that

$$BSO(n) \xrightarrow{j_n} BO(n) \xrightarrow{B\det} BO(1)$$

is a fibration.

Corollary 3.4. *With $\mathbb{F}_2$ coefficients, we have the following.*

(i) As algebras,

$$H^*BO = P\{w_1, w_2, \ldots\},$$
$$\text{and } H^*BSO = P\{w_2, w_3, \ldots\}.$$

(ii) As coalgebras,

$$H_*BO(n) = \Gamma\{\omega_1, \omega_2, \ldots, \omega_n\},$$
$$H_*BSO(n) = \Gamma\{\omega_2, \omega_3, \ldots, \omega_n\},$$
$$H_*BO = \Gamma\{\omega_1, \omega_2, \ldots\},$$
$$\text{and } H_*BSO = \Gamma\{\omega_2, \omega_3, \ldots\}$$

where ω_i is dual to w_i. The homomorphisms induced in homology and cohomology by the natural inclusions send each generator to the generator of the same name if present and to zero otherwise.

PROOF. $H_*BO(n)$ and $H_*BSO(n)$ are as stated by duality. The consistency of the generators follows from the consistency of the Stiefel-Whitney classes in $H^*BO(n)$ and $H^*BSO(n)$ under the inclusion maps. The statements about H^*BO, H^*BSO, H_*BO, and H_*BSO now follow by passage to limits. □

We will show in the Appendix that BO and BSO are H-spaces with multiplication corresponding to Whitney sum. It will then follow from Theorem 3.1.(ii) that the coproduct on H^*BO and H^*BSO is given by $\psi(w_n) = \sum w_i \otimes w_{n-i}$.

In the Appendix, we will give an alternative description of H_*BO which is conceptually simpler.

Corollary 3.5. *As Hopf algebras,*

$$H^*SO(n) = \Delta\{x_1, \ldots, x_{n-1}\},$$
$$\text{and } H^*SO = \Delta\{x_1, x_2, \ldots\}$$

where $x_{i-1} = \sigma^* w_i$. *Dually,*

$$H_*SO(n) = E\{a_1, \ldots, a_{n-1}\},$$
$$\text{and } H_*SO = E\{a_1, a_2, \ldots\}$$

as algebras, where $\sigma_* a_{i-1} = \omega_i$ *and* a_i *is dual to* x_i.

PROOF. Recall that the notation $\Delta\{x_1, \ldots, x_{n-1}\}$ means that $\{x_1, \ldots x_{n-1}\}$ is a simple system of primitive generators. In the Eilenberg-Moore spectral sequence of the universal bundle $SO(n) \to ESO(n) \to BSO(n)$, we have

$$\mathrm{Tor}_{P\{w_2,\ldots,w_n\}}(\mathbb{F}_2, \mathbb{F}_2) = E\{x_1, \ldots, x_{n-1}\}.$$

Since the Stiefel-Whitney classes suspend, $E_2 = E_\infty$. However, since we have $\mathbb{F}_2$ coefficients, we can only conclude that $\{x_1, \ldots, x_{n-1}\}$ is a simple system of generators. The x_i are shown to be primitive as in Corollary **??**. The computation of H^*SO is identical. The statements about $H_*SO(n)$ and H_*SO follow by duality. □

We shall complete the determination of $H^*SO(n)$ as an algebra in the next section.

Proposition 3.6. *As an algebra,*

$$H^*SO(2n)/U(n) = \Delta\{x_2, x_4, \ldots, x_{2n-2}\} \subset \Delta\{x_1, \ldots, x_{2n-1}\} = H^*SO(2n).$$

The indicated inclusion is the homomorphism induced in cohomology by the natural map $SO(2n) \to SO(2n)/U(n)$.

PROOF. Consider the Eilenberg-Moore spectral sequence of

$$SO(2n)/U(n) \longrightarrow BU(n) \xrightarrow{\mu_n} BSO(2n).$$

The E_2 term is the homology of $X = E_{H^*BU(n)}\{a_2, a_3, \ldots, a_{2n}\}$ with respect to the differential $d(a_{2i-1}) = 0$ and $d(a_{2i}) = c_i$ where a_i has degree $(-1, i)$. Now X splits as the tensor product of $E_{H^*BU(n)}\{a_3, a_5, \ldots, a_{2n-1}\}$ with zero differential and $E_{H^*BU(n)}\{a_2, a_4, \ldots, a_{2n}\}$ with differential $d(a_{2i}) = c_i$. We recognize the second factor as the Koszul resolution of $H^*BU(n)$. Hence, the homology of X is $E_{H^*BU(n)}\{a_3, a_5, \ldots, a_{2n-1}\} \otimes_{H^*BU(n)} \mathbb{F}_2 = E\{a_3, a_5, \ldots, a_{2n-1}\}$. Since everything lies in even degrees, $E_2 = E_\infty$ and thus $H^*SO(2n)/U(n) = \Delta\{a_3, a_5, \ldots, a_{2n-1}\} =$

$\Delta\{x_2, x_4, \ldots, x_{2n-2}\}$, where we have renamed the elements so their subscripts reflect their degrees. The second statement follows by naturality of hte Eilenberg-Moore spectral sequence from the diagram

$$\begin{array}{ccccc} SO(2n) & \longrightarrow & ESO(2n) & \longrightarrow & BSO(2n) \\ \downarrow & & \downarrow & & \| \\ SO(2n)/U(n) & \longrightarrow & BU(n) & \longrightarrow & BSO(2n). \end{array}$$

Proposition 3.7. *As algebras,*

$$H^*U(n)/O(n) = E\{w_1, \ldots, w_n\} = H^*BO(n)/(w_i^2),$$
$$\textit{and } H^*SU(n)/SO(n) = E\{w_2, \ldots, w_n\} = H^*BSO(n)/(w_i^2).$$

The maps induced in cohomology by the natural maps $U(n)/O(n) \to BO(n)$ and $SU(n)/SO(n) \to BSO(n)$ are the obvious epimorhpisms.

PROOF. Consider the Eilenberg-Moore spectral sequence of

$$U(n)/O(n) \longrightarrow BO(n) \xrightarrow{\nu_n} BU(n).$$

By Theorem 1.7, $H^*BO(n)$ is a free $H^*BU(n)$ module and hence

$$\begin{aligned} H^*U(n)/O(n) = E_\infty = E_2 &= \mathbb{F}_2 \otimes_{H^*BU(n)} H^*BO(n) \\ &= H^*BO(n)/(w_i^2) \\ &= E\{w_1, \ldots, w_n\}. \end{aligned}$$

Since the homomorphism $H^*BO(n) \to H^*O(n)/U(n)$ is the composite

$$H^*BO(n) \to E_2^{0,*} \to E_\infty^{0,*} \to H^*U(n)/O(n),$$

it is clearly the evident epimorphism. The homogeneous space $SU(n)/SO(n)$ is handled similarly. □

Note that since $E_\infty^{p,q} = 0$ for $p \neq 0$, we can conclude that $H^*U(n)/O(n)$ is actually an exterior algebra rather than just an algebra with a simple system of generators. There is then no extension problem in determining $H^*U(n)/O(n)$ from E_∞. Alternatively, we can note that the algebra homomorphism $H^*BO(n) \to H^*U(n)/O(n)$ determines the algebra structure of $H^*U(n)/O(n)$.

By passage to limits, we obtain the following corollary.

Corollary 3.8. *(insert)*

(i) *As an algebra,*

$$H^*SO/U = \Delta\{x_2, x_4, \ldots\} \subset \Delta\{x_1, x_2, x_3, \ldots\} = H^*SO,$$

where the homomorphism induced by the natural map $SO \to SO/U$ is the indicated inclusion.

(ii) *As algebras,*

$$H^*U/O = E\{w_1, w_2, \ldots\} = H^*BO/(w_i^2),$$
$$\textit{and } H^*SU/SO = E\{w_2, w_3, \ldots\} = H^*BSO/(w_i^2)$$

where the maps induced in cohomology by the natural maps $U/O \to BO$ and $SU/SO \to BSO$ are the evident epimorphisms.

Consider the fibrations

$$SO \xrightarrow{\nu} SU \xrightarrow{q} SU/SO \qquad\qquad SU/SO \xrightarrow{i} BSO \xrightarrow{\nu} BSU$$

$$U \xrightarrow{\mu} SO \xrightarrow{p} SO/U \qquad\qquad SO/U \xrightarrow{j} BU \xrightarrow{\mu} BSO.$$

We know that $i^*(w_i) = w_i$ and $\nu^*(c_i) = w_i^2$. By naturality of suspension, $\nu^*(x_{2i-1}) = \nu^*\sigma^*(c_i) = \sigma^*(w_i^2) = 0$. It can be shown that $q^* = 0$ also, by showing that $SU \to SU/SO \to BSO$ is equivalent to a fibration and noting that i^* is an epimorphism. We have shown that $\mu^*(w_{2i-1}) = 0, \mu(w_{2i}) = c_i$. Since μ^* is an epimorphism, $j^* = 0$. By naturality of suspension, $\mu^*(x_{2i}) = \sigma^*(0) = 0$ and $\mu^*(x_{2i-1}) = \sigma^*(c_i) = x_{2i-1}$. Finally, we know that $p^*(x_{2i}) = x_{2i}$.

4. Steenrod Operations, the Wu formula, and BSpin

The Steenrod operations will enable us to determine the algebra structure of $H^*(SO(n); \mathbb{F}_2)$ and $H^*(SO(2n)/U(n); \mathbb{F}_2)$. We will also use them in calculating $H^*(B\mathrm{Spin}; \mathbb{F}_2)$. They provide important relations between Stiefel-Whitney classes by means of the Wu formula.

Let p be a prime. The Steenrod operations are natural transformations

$$\mathcal{P}^i : H^*(X; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p) \quad (i \geq 0)$$
$$\beta : H^*(X; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p)$$

such that

(i) $\mathcal{P}^i$ has degree $2i(p-1)$ if $p > 2$, degree i if $p = 2$
(ii) $\mathcal{P}^0 = 1$
(iii) $\mathcal{P}^i(x) = 0$ if $p > 2$ and $|x| < 2i$ or if $p = 2$ and $|x| < i$
(iv) $\mathcal{P}^i(x) = x^p$ if $p > 2$ and $|x| = 2i$ or if $p = 2$ and $|x| = i$
(v) $\mathcal{P}^i(xy) = \sum \mathcal{P}^j(x)\mathcal{P}^{i-j}(y)$.

When $p = 2$, $\mathcal{P}^i$ is denoted Sq^i. The following lemma will be extremely useful.

Lemma 4.1. *Let $|x| = 1$. If $p > 2$, then $\mathcal{P}^i(x) = 0$ for $i > 0$. If $p = 2$, then $\mathcal{P}^i(x^k) = (i, k-i)x^{i+k}$. Let $|y| = 2$. If $p > 2$, then $\mathcal{P}^i(y^k) = (i, k-i)y^{k+i(p-1)}$. If $p = 2$ and $\beta(y) = 0$, then $\mathcal{P}^{2i}(y^k) = (i, k-i)y^{k+i}$ and $\mathcal{P}^{2i+1}(y^k) = 0$.*

Note that for $i, j \geq 0, (i, j)$ denotes the binomial coefficient $\frac{(i+j)!}{i!j!}$ where $0! = 1$ as usual. As a convenience, we define (i, j) to be 0 if $i < 0$ or $j < 0$.

This lemma determines all Steenrod operations in BT^n and $BO(1)^n$, and hence in all classifying spaces of classifying Lie groups since either $H^*BG \to H^*BT^n$ or, when $p = 2$, $H^*BG \to H^*BO(1)^n$ is a monomorphism for some n. By suspension, the Steenrod operations in the classical Lie groups are also determined.

The following lemma simplifies the calculation of binomial coefficients mod p.

Lemma 4.2. *Let $i = \sum a_k p^k$ and $j = \sum b_k p^k$, be the p-adic expansion of i and j. Then $(i, j) \equiv \prod_k (a_k, b_k) \mod p$ and $(i, j) \equiv 0 \mod p$ if and only if $a_k + b_k \geq p$ for some k.*

Proof. Look at the coefficient of $x^i y^j$ in $(x + y)^{i+j} \mod p$. □

We can now derive the *Wu formula.* In the rest of this chapter, all cohomology will have $\mathbb{F}_2$ coefficients.

Theorem 4.3. *The Steenrod operations in $H^*BO(n)$ are given by*

$$\begin{aligned}\mathrm{Sq}^i w_j &= \sum_{t=0}^{i} (t, j-i-1) w_{i-t} w_{j+t} \\ &= w_i w_j + (1, j-i-1) w_{j-1} w_{j+1} + \cdots + (i, j-i-1) w_{i+j}\end{aligned}$$

for $0 \le i < j$.

Remark 4.4. The proof we are about to give applies with minor modifications to show that

$$\mathrm{Sq}^{2i} c_j = \sum_{t=0}^{i} (t, j-i-1) c_{i-t} c_{j+t} \text{ in } H^*(BU(n); \mathbb{F}_2) \text{ if } i < j$$

and

$$\mathrm{Sq}^{4i} k_j = \sum_{t=o}^{i} (t, j-i-1) k_{i-t} k_{j+t} \text{ in } H^*(BSp(n); \mathbb{F}_2) \text{ if } i < j.$$

Clearly $\mathrm{Sq}^i c_j = 0$ for odd i and $\mathrm{Sq}^i k_j = 0$ for $i \equiv 0 \mod 4$.

Proof. Since $\mathrm{Sq}^0 w_1 = w_1$ is the only relevant operation in $H^*BO(1)$, the theorem is true when $n = 1$. Consider $p = p_{1,n} : BO(1) \times BO(n) \to BO(n+1)$ and assume the theorem holds for $BO(n)$. Recall that p^* is a monomorphism and $p^*(w_j) = 1 \otimes w_j + w_1 \otimes w_{j-1}$. Now,

$$p^* \mathrm{Sq}^i w_j = \mathrm{Sq}^i p^* w_j = 1 \otimes \mathrm{Sq}^i w_j + w_1 \otimes \mathrm{Sq}^i w_{j-1} + w_1^2 \otimes \mathrm{Sq}^{i-1} w_{j-1}$$

since $\mathrm{Sq}^1 w_1 = w_1^2$. On the other hand,

$$\begin{aligned}&p^* \sum_{t=0}^{i} (t, j-i-1) w_{i-t} w_{j+t} \\ &= \sum_{t=0}^{i} (t, j-i-1)(1 \otimes w_{i-t} + w_1 \otimes w_{i-t-1})(1 \otimes w_{j+t} + w_1 \otimes w_{j+t-1}) \\ &= \sum_{t=0}^{i} (t, j-i-1)(1 \otimes w_{i-t} w_{j+t} + w_1 \otimes (w_{i-t-1} w_{j+t} + w_{i-t} w_{j+t-1}) + w_1^2 \otimes w_{i-t-1} w_{j+t-1}) \\ &= 1 \otimes \mathrm{Sq}^i w_j + w_1 \otimes \sum_{t=0}^{i} (t, j-i-1)(w_{i-t-1} w_{j+t} + w_{i-t} w_{j+t-1}) + w_1^2 \otimes \mathrm{Sq}^{i-1} w_{j-1}.\end{aligned}$$

Hence the formula will be proved if we show that

$$\mathrm{Sq}^i w_{j-1} = \sum_{t=0}^{i} (t, j-i-1)(w_{i-t-1} w_{j+t} + w_{i-t} w_{j+t-1}).$$

This sum is

$$\begin{aligned}&\sum_{t=0}^{i-1} (t, j-i-1) w_{i-t-1} w_{j+t} + \sum_{t=0}^{i} (t, j-i-1) w_{i-t} w_{j+t-1} \\ &= \sum_{t=1}^{i} (t-1, j-i-1) w_{i-t} w_{j+t-1} + \sum_{t=0}^{i} (t, j-i-1) w_{i-t} w_{j+t-1}.\end{aligned}$$

If $i = j-1$, this is clearly w_{j-1}^2 since all other terms cancel.

For $i < j-1$,

$$(t, j-i-2) = (t, j-i-1) - (t-1, j-i-1) \equiv (t, j-i-1) + (t-1, j-i-1) \mod 2.$$

Our sum is therefore

$$\sum_{t=0}^{i} (t, j-i-2) w_{i-t} w_{j+t-1} = \mathrm{Sq}^i w_{j-1}$$

as desired. □

Clearly the same formula (with $w_1 = 0$) holds in $H^*BSO(n)$. We can use it to compute the cohomology algebras of $SO(n)$ and $SO(2n)/U(n)$.

Corollary 4.5. *In $H^*SO(n)$,*

$$\mathrm{Sq}^i x_j = (i, j-i)\, x_{j+i} \quad \textit{for } i \le j,$$

where $x_k = 0$ if $k > n-1$.

Proof. Since $x_j = \sigma^*(w_{j+1})$, we have

$$\begin{aligned} \mathrm{Sq}^i x_j &= \sigma^* \mathrm{Sq}^i w_{j+1} \\ &= \sigma^* \sum_{t=0}^{i} (t, j-i) w_{i-t} w_{j+1+t} \\ &= (i, j-i) x_{i+j}, \end{aligned}$$

where the last equality holding because σ^* annihilates decomposable elements (see the Appendix). □

Corollary 4.6. *In $H^*SO(n)$, $x_j^2 = x_{2j}$ if $2j < n$ and $x_j^2 = 0$ if $2j \ge n$. Hence, if s_i is minimal so that $2^{s_i}(2i-1) \ge n$, then $H^*SO(n) = P\{x_{2i-1} \mid 1 \le i \le [n/2]\}/(x_{2i-1}^{2^{s_i}})$ as a Hopf algebra and $H^*SO(2n)/U(n)$ is the subalgebra generated by the elements $x_{2i-1}^2 = x_{4i-2}$. Thus $H^*SO = P\{x_{2i-1} \mid i \ge 1\}$ and $H^*SO/U = P\{x_{2i-1}^2 = x_{4i-2} \mid i \ge 1\} \subset H^*SO$.*

Proof. Corollary 4.5 implies $x_j^2 = \mathrm{Sq}^j(x_j) = x_{2j}$, where $x_{2j} = 0$ if $2j \ge n$. The rest follows immediately. □

In order to determine the 2-torsion in $H^*(SO(n); \mathbb{Z})$ and related spaces we will need to know the Bockstein. Recall that $\beta = \mathrm{Sq}^1$.

Corollary 4.7. *In $H^*SO(n)$, $\beta(x_{2i-1}) = 0$ and $\beta(x_{2j-1}) = x_{2j}$. In $H^*SO(2n)/U(n)$, $\beta = 0$. In $H^*BO(n), \beta(w_j) = w_1 w_j + (j-1) w_{j+1}$ and in $H^*BSO(n)$, $\beta(w_j) = (j+1) w_{j+1}$.*

The E_2 term of the mod 2 Bockstein spectral sequence of X is the homology of $H^*(X; \mathbb{Z}/2\mathbb{Z})$ with respect to the differential β. We will denote it by $H^\beta X$. In all the cases we are interested in, the torsion has order 2. That is, $E_2 = E_\infty$.

[BOB check formatting and wording of this prop please]

Proposition 4.8. *We compute the following algebras.*

(i) *As algebras*

$$H^\beta SO(2n) = E\{x_{2n-1}, x_{2i-1}x_{2i} + x_{4i-1} \mid 1 \le i \le n-1\}$$

$$\textit{and } H^\beta SO(2n+1) = E\{x_{2i-1}x_{2i} + x_{4i-1} \mid 1 \le i \le n-1\}.$$

(ii) As an algebra

$$H^\beta BO(n) = P\{w_{2i}^2 \mid 1 \leq \lfloor n/2 \rfloor\}.$$

(iii) As algebras

$$H^\beta BSO(2n) = P\{w_{2n}, w_{2i}^2 \mid 1 \leq i \leq n\}$$
$$\text{and } H^\beta BSO(2n+1) = P\{w_{2i}^2 \mid 1 \leq i \leq n-1\}.$$

(iv) As algebras

$$H^\beta U(2n)/O(2n) = E\{w_1, w_{2i}w_{2i+1} \mid 1 \leq i \leq n-1\}$$
$$\text{and } H^\beta U(2n+1)/O(2n+1) = E\{w_1, w_{2i}w_{2i+1} \mid 1 \leq i \leq n\}.$$

(v) As algebras

$$H^\beta SU(2n)/SO(2n) = E\{w_{2n}, w_{2i}w_{2i+1} \mid 1 \leq i \leq n-1\}$$
$$\text{and } H^\beta SU(2n+1)/SO(2n+1) = E\{w_{2i}w_{2i+1} \mid 1 \leq i \leq n\}.$$

(vi) As an algebra

$$H^\beta SO(2n)/U(n) = H^* SO(2n)/U(n),$$

thus $H^*(SO(2n)/U(n);\mathbb{Z})$ has no 2-torsion.

PROOF. We prove each case separately.

(i) $\{x_{2i-1}, x_{2i}, x_{2i-1}x_{2i} + x_{4i-1}\}$ is a basis for a subcomplex of $H^*SO(n)$ and $H^*SO(n)$ is the tensor product of these subcomplexes. The result follows immediately.

(ii) $H^*BO(2n)$ is the tensor product of subcomplexes $P\{w_{2i}, w_{2i+1} + w_1 w_{2i}\}$, $1 \leq i < n$, and $P\{w_1, w_{2n}\}$. $H^*BO(2n+1)$ is the tensor product of $P\{w_{2i}, w_{2i+1} + w_1 w_{2i}\}$, $i \leq i \leq n$ and $P\{w_1\}$. Now $H^\beta P\{w_{2i}, w_{2i+1} + w_1 w_{2i}\} = P\{w_{2i}^2\}$, $H^\beta P\{w_1, w_{2n}\} = P\{w_{2n}^2\}$ and $H^\beta P\{w_1\} = 0$. The result follows.

(iii) The relevant subcomplexes are $P\{w_{2i}w_{2i+1}\}$ and, for $H^*BSO(2n)$, $P\{w_{2n}\}$. The result follows as in (ii).

(iv) As complexes,

$$H^*U(2n)/O(2n) = E\{w_1, w_{2n}\} \otimes \left(\bigotimes_{i=1}^{n-1} E\{w_{2i}, w_{2i+1} + w_1 w_{2i}\} \right)$$

and

$$H^*U(2n+1)/O(2n+1) = E\{w_1\} \otimes \left(\bigotimes_{i=1}^{n} E\{w_{2i}, w_{2i+1} + w_1 w_{2i}\} \right).$$

(v) As complexes,

$$H^*SU(2n)/SO(2n) = \left(\bigotimes_{i=1}^{n-1} E\{w_{2i}, w_{2i+1}\} \right) \bigotimes E\{w_{2n}\}$$

and

$$H^*SU(2n+1)/SO(2n+1) = \bigotimes_{i=1}^{n} E\{w_{2i}, w_{2i+1}\}.$$

(vi) We have shown this in Corollary 4.7.

□

Corollary 4.9. *In all of the above cases $E_2 = E_\infty$ and hence all torsion is of order 2.*

Proof. In case (i), this follows from our calculation of $H^*(SO(n); \mathbb{F}_2)$. In cases (ii) and (iii), E_2 is concentrated in even degrees and hence $E_2 = E_\infty$. Cases (iv) and (v) follow from our calculation of the relevant cohomology algebras away from 2 in Chapter 6. That the only torsion is 2-torsion in cases (ii)-(vi) will also follow from Chapter 6. □

Recall that $\mathrm{Spin}(n)$ is defined to be the universal covering group of $SO(n)$ ($n \geq 3$). We therefore have a short exact sequence $\mathbb{Z}/2\mathbb{Z} \to \mathrm{Spin}(n) \to SO(n)$ (and in the limit, $\mathbb{Z}/2\mathbb{Z} \to \mathrm{Spin} \to SO$) with $\mathbb{Z}/2\mathbb{Z}$ normal in $\mathrm{Spin}(n)$ (or Spin). In the next chapter we will see that $H^*(\mathrm{Spin}(n); \Lambda) = H^*(B\mathrm{Spin}(n); \Lambda) = H^*(BSO(n); \Lambda)$ if 2 is invertible in Λ. In this chapter we will calculate $H^*(\mathrm{Spin}; \mathbb{F}_2)$ and $H^*(B\mathrm{Spin}; \mathbb{F}_2)$. The cohomology of $\mathrm{Spin}(n)$ and $B\mathrm{Spin}(n)$ with $\mathbb{F}_2$ coefficients is known but is more difficult to compute.

Before we can calculate $H^*B\mathrm{Spin}$ we need to define an alternative set of polynomial generators for H^*BSO. Let $u_i = w_i$ if $i \neq 2^i + 1$ and let $u_2 = w_2$, $u_{2^{j^{i+1}}+1} = \mathrm{Sq}^{2^j} u_{2^j+1} = \mathrm{Sq}^{2^j}\mathrm{Sq}^{2^{j-1}} \cdots \mathrm{Sq}^2\mathrm{Sq}^1 u_2$. By the Wu formula, we have

$$\begin{aligned} u_2 &= w_2 \\ u_3 &= w_3 \\ u_5 &= w_5 + w_2 w_3 \\ &\vdots \end{aligned}$$

It is easy to see by induction that in general

$$u_i = w_i + \text{ decomposables.}$$

From this it follows that $H^*BSO = P\{u_i \mid i \geq 2\}$. The importance of these classes lies in the fact that the clases u_{2^i+1} are tied to w_2 by Steenrod operations. Hence if $f : X \to BSO$ satisfies $f^*(w_2) = f^*(u_2)$, then it must also satisfy $f^*(u_{2^i+1}) = 0$ for all i.

Since we have a short exact sequence $\mathbb{Z}/2\mathbb{Z} \to \mathrm{Spin} \to SO$, the induced sequence $B\mathbb{Z}/2\mathbb{Z} = K(\mathbb{Z}/2\mathbb{Z}, 1) \to B\mathrm{Spin} \to BSO$ is a fibration. The E_2 term of the Serre spectral sequence is $E_2 = H^*BSO \otimes H^*B\mathbb{Z}/2\mathbb{Z} = P\{u_i \mid i \geq 2\} \otimes P\{\iota\}$, where $\iota \in H^1B\mathbb{Z}/2\mathbb{Z}$ is the fundamental class. (Recall that $B\mathbb{Z}/2\mathbb{Z} = K(\mathbb{Z}/2\mathbb{Z}, 1) = \mathbb{R}\mathrm{P}^\infty$, so $H^*B\mathbb{Z}/2\mathbb{Z} = P\{\iota\} = P\{w_1\}$.) Since Spin is simply connected, $B\mathrm{Spin}$ is 2-connected and hence $H^1B\mathrm{Spin} = H^2B\mathrm{Spin} = 0$. It follows that ι (and $w_2 = u_2$) do not survive to E_∞. The only differential which can kill them is d_2. Hence, $d_2(\iota) = w_2 = u_2$. In terms of suspension and transgression, $\tau(\iota) = u_2$ and $\sigma^*(u_2) = \iota$. Now σ^* and its inverse τ, commute with Steenrod squares (see the Appendix). Hence

$$\begin{aligned} \tau(\iota^{2^i}) &= \tau(\mathrm{Sq}^{2^{i-1}}\mathrm{Sq}^{2^{i-2}} \cdots \mathrm{Sq}^2\mathrm{Sq}^1\iota) \\ &= \mathrm{Sq}^{2^{i-1}} \cdots \mathrm{Sq}^1 u_2 \\ &= u_{2^i+1}. \end{aligned}$$

Since $\iota^{2^i} \in E_2^{0,2^i}$, $\tau(\iota^{2^i}) = d_{2^i+1}(\iota^{2^i})$. This is all we need to compute E_∞.

Theorem 4.10. *$H^*B\text{Spin} = P\{u_j \mid j \neq 2^i+1\}$ and $H^*\text{Spin} = P\{x_{2i-1} \mid i \geq 2\}$. The suspension homomorphism is $\sigma^*(u_j) = x_{j-1}$ where x_{2i} is defined as x_i^2. The maps induced in cohomology by the natural maps $\text{Spin} \to SO$ and $B\text{Spin} \to BSO$ are the evident epimorphisms.*

PROOF. In the Serre spectral sequence of $B\mathbb{Z}/2\mathbb{Z} \to B\text{Spin} \to BSO$ we know that $d_{2^i+1}(\iota^{2^i}) = u_{2^i+1}$. Loosely speaking, this means that in E_{2^i+1} term, u_{2^i+1} and all odd powers of ι^{2^i} are killed, and in the other terms of the spectral sequences nothing happens. Thus, in the E_∞ term we have only $P\{u_j \mid j \neq 2^i+1\}$. We prove this by showing inductively that all terms following $E_{2^{i-1}+1}$ up to and including E_{2^i+1} are $P\{\iota^{2^i}\} \otimes P\{u_j \mid j \neq 2, 3, 5, \ldots, 2^{i-1}+1\}$ and that the differentials are zero except for d_{2^i+1} which we computed above. This is certainly true for $i = 0$. Assuming the hypothesis holds up to E_{2^i+1}, we have

$$\begin{aligned} d_{2^i+1}(\iota^{n2^i}) &= n\iota^{(n-1)2^i} u_{2^i+1} x \\ &= \begin{cases} 0 & n \text{ even} \\ \iota^{(n-1)2^i} u_{2^i+1} x & n \text{ odd} \end{cases} \end{aligned}$$

Hence, $\ker(d_{2^i+1}) = P\{\iota^{2^{i+1}}\} \otimes P\{u_j \mid j \neq 2, \ldots, 2^{i-1}+1\}$ while $\operatorname{im}(d_{2^i+1})$ is the ideal generated by u_{2^i+1}. Thus, E_{2^i+2} is as claimed. For dimensional reasons, all differentials must be zero until the term $E_{2^{i+1}+1}$. This completes the induction.

From the Eilenberg-Moore spectral sequence of $\text{Spin} \to E\text{Spin} \to B\text{Spin}$, we have $E_\infty = E\{x_i \mid i \neq 2^j\}$ and hence $H^*\text{Spin} = \Delta\{x_i \mid i \neq 2^j\}$ with $\sigma^* u_i = x_{i-1}$. By naturality of suspension it follows that the homomorphism $H^*SO \to H^*\text{Spin}$ sends $x_i \to x_i$. Therefore $x_{2i} = x_i^2$ in $H^*\text{Spin}$ and hence $H^*\text{Spin} = P\{x_{2i-1} \mid i \geq 2\}$. $\square$

Remark 4.11. Note that it would be equivalent to say that $H^*B\text{Spin} = P\{w_j \mid j \neq 2^i+1\} = P\{w_4, w_6, w_7, w_8, w_{10}, \ldots\}$. Theorem 4.10 says that a Spin bundle (that is, an SO-bundle whose classifying map can be lifted to Spin) must have u_2, u_3, u_5, u_9, etc all zero. Of course, if $u_2(\xi) = 0$ then $u_{2^i+1}(\xi) = \text{Sq}^{2^{i-1}} \cdots \text{Sq}^1 u_2(\xi) = 0$ also. It is possible to show by a deeper analysis of classifying spaces that $B\text{Spin}$ is the homotopy theoretic fiber of $u_2 = w_2 : BSO \to K(\mathbb{Z}/2\mathbb{Z}, 2)$. This gives us the reverse implication: if $w_2(\xi) = 0$, then ξ is a Spin-bundle. These remarks apply equally well to $\text{Spin}(n)$ and $SO(n)$.

We now prove the following lemma which was used above.
[BOB I think we should merge this with Lem 5.2?]

Lemma 4.12. *If $i = \sum a_k p^k$ and $j = \sum b_k p^k$, then $(i, k) = \prod (a_k, b_k)$.*

PROOF. Consider the polynomial algebra $P_{\mathbb{Z}/p\mathbb{Z}}\{x, y\}$ over $\mathbb{Z}/p\mathbb{Z}$ in x and y. The coefficient of $x^i y^j$ in $(x+y)^{i+j}$ can be expressed in two ways by the following sequence of identities:

$$\begin{aligned}
\sum_{s+t=i+j} (s,t)x^s y^t &= (x+y)^{i+j} \\
&= \prod_k (x+y)^{(a_k+b_k)p^k} \\
&= \prod_k (x^{p^k} + y^{p^k})^{a_k+b_k} \\
&= \prod_k \sum_{s_k+t_k=a_k+b_k} (s_k,t_k) x^{s_k p^k} y^{t_k p^k} \\
&= \sum_{s_k+t_k=a_k+b_k} \left(\prod_k (s_k,t_k) \right) x^{\sum s_k p^k} y^{\sum t_k p^k}, \quad \text{for each } k.
\end{aligned}$$

Comparing coefficients we that $(i,j) = \prod (a_k,b_k) \mod p$. □

5. Euler and Pontrjagin classes in rings containing 1/2

In this section, *all cohomology will have coefficients in a ring* Λ *containing 1/2*. The important examples are $\mathbb{F}_p$ and $\mathbb{Z}_{(p)}$ for $p > 2$, $\mathbb{Q}$, and the universal example $\mathbb{Z}[1/2]$. We will define the canonical polynomial generators, the Euler class, and the Pontrjagin classes, for $H^*(BO(n);\Lambda)$ and $H^*(BSO(n);\Lambda)$. In the following chapters, we will see that they are the Λ-reductions of integral Euler and Pontrjagin classes. However, since the presence of 2-torsion in the integral cohomology obscures their formal properties, they are most naturally studied away from the prime 2. Let i_n, j_n, p_{ij}, μ_n, and ν_n be as in Chapter 4.

Theorem 5.1. *There exist unique classes* $P_i \in H^{4i}BO(n)$ *and* $P_i \in H^{4i}BSO(n)$, *called the* Pontrjagin classes, *and a unique class* $\chi \in H^n BSO(n)$, *called the* Euler class, *such that*

(i) $P_1 = j_2^*(P_1) = c_1^2 \in H^4BU(1) = H^4BSO(2)$ *and* $\chi = c_1 \in H^2BU(1) = H^2BSO(2)$
(ii) $i_n^*(P_i) = P_i, i_n^*(\chi) = 0,$ *and* $j_n^*(P_i) = P_i$
(iii) $p_{ij}^*(P_k) = \sum_{a+b=k} P_a \otimes P_b$ *and* $p_{ij}^*(\chi) = \chi \otimes \chi$
(iv) $P_0 = 1, P_i = 0$ *if* $i > [n/2]$, *while* $\chi = 0$ *if* n *is odd and* $\chi^2 = P_k$ *if* $n = 2k$.

Moreover, $H^*BO(2n) = H^*BO(2n+1) = H^*BSO(2n+1) = P\{P_1,\dots,P_n\}$ *and* $H^*BSO(2n) = P\{P_1,\dots,P_{n-1},\chi\}$. *In addition,*

(v) $\nu_n^*(c_{2i+1}) = 0$ *and* $\nu_n^*(c_{2i}) = (-1)^i P_i$ *where* $\nu_n : BO(n) \to BU(n)$, $\nu_n : BSO(n) \to BU(n)$, *or* $\nu_n : BSO(n) \to BSU(n)$.
(vi) $\mu_n^*(P_i) = \sum_{a+b=2i} (-1)^{a+i} c_a c_b$ *where* $\mu_n : BU(n) \to BO(2n)$ *or* $\mu_n : BU(n) \to BSO(2n)$ *and* $\mu_n^*(\chi) = c_n$ *for* $\mu_n : BU(n) \to BSO(2n)$.

Proof. We will prove the result for $BSO(n)$ first and use it to obtain the result for $BO(n)$. Let $\phi_{2n+\epsilon} : T^n \to SO(2n+\epsilon)$, $\epsilon = 0$ or 1, be the maximal torus consisting of matrices of the form

$$\begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_n \end{pmatrix} \text{ or } \begin{pmatrix} A_1 & & & & 0 \\ & A_2 & & & \\ & & \ddots & & \\ & & & A_n & \\ 0 & & & & 1 \end{pmatrix}$$

in $SO(2n)$ or $SO(2n+1)$, where each $A_i \in SO(2) = T^1$. In Corollary 2.9, we saw that $H^*SO(2n) = E\{x_3, x_7, \ldots, x_{4n-5}, y_{2n-1}\}$ and $H^*SO(2n+1) = E\{x_3, x_7, \ldots, x_{4n-1}\}$. Hence $H^*BSO(2n) = P\{z_4, z_8, \ldots, z_{4n-4}, z_{2n}\}$ and $H^*BSO(2n+1) = P\{z_4, z_8, \ldots, z_{4n}\}$. Since $H^*BSO(2n+\epsilon)$ is a polynomial algebra on n generators, it follows as before that $\phi^*_{2n+\epsilon}$ is a monomorphism. We can permute the factors of T^n by conjugation by an element of $SO(2n+\epsilon)$. Hence, $\operatorname{im}\phi^*_{2n+\epsilon}$ is contained in the symmetric polynomials. If $\epsilon = 1$, the matrix

$$\overline{A} = \begin{pmatrix} I & & & & & & & 0 \\ & \ddots & & & & & & \\ & & I & & & & & \\ & & & A & & & & \\ & & & & I & & & \\ & & & & & \ddots & & \\ & & & & & & I & \\ 0 & & & & & & & -1 \end{pmatrix} \qquad \text{where } A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is in $SO(2n+1)$. Conjugation by $\overline{A}$ has the effect of complex conjugation of a factor of T^n, which induces negation of the corresponding polynomial generator in H^*BT^n. Hence, $\operatorname{im}\phi^*_{2n+1}$ is contained in the subalgebra generated by the $\sigma_i(y_1^2, \ldots, y_n^2) \in H^*BT^n$. Since $H^*BSO(2n+1) = P\{z_4, z_8, \ldots, z_{4n}\}$, it follows that $\phi^*_{2n+1} : H^*BSO(2n+1) \longrightarrow P\{\sigma_i(y_1^2, \ldots, y_n^2) \mid i = 1, \ldots, n\}$ is an isomorphism. Hence, we define P_i to be $(\phi^*_{2n+1})^{-1}(\sigma_i(y_1^2, \ldots, y_n^2))$ and χ to be 0. When $\epsilon = 0$, the matrix corresponding to complex conjugation by one factor of T^n is not in $SO(2n)$. However, the matrices corresponding to complex conjugation of an even number of factors of T^n are in $SO(2n)$. Hence, $\operatorname{im}(\phi^*_{2n})$ is contained in the subalgebra invariant under permutations and an even number of sign changes. This has as generators $\sigma_i(y_1^2, \ldots, y_n^2)$, $1 \le i \le n-1$, and $\sigma_n(y_1, \ldots, y_n)$. Since the generators of $H^*BSO(2n)$ lie in the same degrees, ϕ^*_{2n} is an isomorphism onto this subalgebra. Accordingly, we define $P_i = (\phi^*_{2n})^{-1}(\sigma_i(y_1^2, \ldots, y_n^2))$ and $\chi = (\phi^*_{2n})^{-1}(\sigma_n(y_1, \ldots, y_n))$.

Since $O(n)/SO(n) = \mathbb{Z}/2\mathbb{Z}$, we have a 2-fold covering (principal $\mathbb{Z}/2\mathbb{Z}$- bundle) $\mathbb{Z}/2\mathbb{Z} \to BSO(n) \to BO(n)$. We now proceed to calculate $H^*BO(n)$ from $H^*BSO(n)$ using this covering. We may as well take any covering $p : E \to B$ whose group of covering transformations, π, is finite. As usual we have $p_* : C_*E \to C_*B$, where C_* denotes the singular chain complex functor. However, since p is a covering we also have a map $p_! : C_*B \to C_*E$ where $p_!(c)$ is the sum of all the lifts of c (of course, one has to do this for simplices and extend by linearity). It is easy to see that $p_!$ commutes with the boundary operator. The composite $p_*p_! : C_*B \to C_*B$ is multiplication by $|\pi|$, the order of π and the composite $p_!p_*$ takes x to $\sum_{g\in\pi} gx$. Letting $p_!$ also denote the map induced in cohomology we have the relations $p_!p^* = |\pi| : H^*B \to H^*B$ and $p^*p_! = \sum_{g\in\pi} g : H^*E \to H^*E$. The homomorphism $p_! : H^*E \to H^*B$ is called the *transfer* homomorphism.

Lemma 5.2. *If $|\pi|$ is invertible in Λ then $p^* : H^*(B;\Lambda) \to H^*(E;\Lambda)^\pi$ is an isomorphism (where $H^*(E;\Lambda)^\pi$ denotes the elements fixed by π).*

Proof. Multiplication by $|\pi|$ is monic so p^* is monic. If $x \in (H^*E)^\pi$ then $p^*p_!(\frac{1}{\pi}x) = x$ so p^* maps onto $(H^*E)^\pi$. Now $|\pi|p^*(x) = p^*p_!p^*(x) = \sum gp^*(x)$. Therefore $p^*(x) = \frac{1}{\pi}\sum_{g\in\pi} gp^*(x)$, which is obviously invariant under π. That is, $p^*(H^*B) \subset (H^*E)^\pi$. □

By Lemma 5.2, we conclude that $H^*BO(n) = H^*BSO(n)^{\mathbb{Z}/2\mathbb{Z}}$. It remains only to compute the action of $\mathbb{Z}/2\mathbb{Z} = O(n)/SO(n)$ on $H^*BSO(n)$, to which we can apply Lemma 6.6. To this end, let G be a topological group, H by a normal subgroup such that G/H is discrete. For $g \in G$, define $\phi_g : H \to$ by $\phi_g(h) = g^{-1}hg$.

The nontrivial element of $O(2n+\epsilon)/SO(2n+\epsilon)$ is represented by the matrix

$$A = \begin{pmatrix} -1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix}$$

among others. This representative normalizes $T^n \subset SO(2n+\epsilon)$ and hence induces an action on BT^n. Conjugation of T^n by A induces complex conjugation of the first factor of T^n. Hence, the action induced on H^*BT^n sends y_1 to $-y_1$ and fixes the other generators. It follows from the definition of the Euler and Pontrjagin classes that $B\phi_A$ sends χ to $-\chi$ and fixes each P_i. Therefore,

$$H^*BO(2n) = H^*BO(2n+1) = P\{P_1, \ldots, P_n\} \subset H^*BSO(2n+\epsilon),$$

where we define $P_i \in H^*BO(n)$ to be $(j_n^*)^{-1}(P_i)$.

We can now finish the proof of Theorem 5.1. Statement (i) is immediate from the definition of P_1 and χ since $U(1) \cong SO(2) \cong T^1$. In statement (ii), $j_n^*(P_i) = P_i$ is definition while $i_n^*(P_i) = P_i$ and $i_n^*(\chi) = 0$ follows by an easy diagram chase using the fact that the map $H^*BT^n \to H^*BT^{n-1}$, induced by inclusion of the first $n-1$ factors $T^{n-1} \to T^n$, is the quotient map sending y_n to 0. In order to prove (iii), we will use (v). Since $P_k = \nu_{i+j}^*((-1)^k c_{2k})$, we have

$$\begin{aligned} p_{ij}^*(P_k) &= p_{ij}^*\nu_{i+j}^*((-1)^k c_{2k}) \\ &= (\nu_i^* \otimes \nu_j^*)p_{ij}^*((-1)^k c_{2k}) \\ &= (-1)^k \sum_{a+b=k} (-1)^a P_a \otimes (-1)^b P_b \\ &= \sum_{a+b=k} P_a \otimes P_b. \end{aligned}$$

To show that $p_{ij}^*(\chi) = \chi \otimes \chi$, first note that if exactly one of i and j is odd, then equality is trivial since both sides are zero. If $i = 2n+1$, $j = 2m+1$ then

$$p_{ij}^*(\chi^2) = p_{ij}^*(P_{n+m+1})$$
$$= \sum_{a+b=n+m+1} P_a \otimes P_b.$$

But $P_a = 0$ for $a > n$ and $P_b = 0$ for $b > m$. Thus $p_{ij}^*(\chi^2) = 0$. Now $\chi \otimes \chi = 0$ since $\chi = 0$ in $H^*BSO(2n+1)$. The proof when i and j are both even is immediate from the commutative diagram

$$\begin{array}{ccc}
T^n \times T^m & = & T^{n+m} \\
\downarrow{\scriptstyle \phi_{2n} \times \phi_{2m}} & & \downarrow{\scriptstyle \phi_{2n+2m}} \\
SO(2n) \times SO(2m) & \xrightarrow{p_{2n,2m}} & SO(2n+2m),
\end{array}$$

using the fact that $\sigma_{n+m}(y_1, \ldots, y_{n+m}) = \sigma_n(y_1, \ldots, y_n)\sigma_m(y_{n+1}, \ldots, y_{n+m})$. Statement (iv) is immediate from the definitions. Now the uniqueness of the Euler and Pontrjagin classes follows inductively from (i) and (iii) as for the Chern classes.

To prove (v), first note that we need only consider $\nu_{2n} : BSO(2n) \to BU(2n)$ since j_{2n}^*, j_{2n+1}^*, and $i_{2n}^* : H^*BSO(2n+1) \to H^*BSO(2n)$ are all monomorphisms. We wish to compare the maximal tori of $SO(2n)$ and $U(2n)$ under the map ν_{2n}. However, this is not possible using the standard maximal tori. Hence, we use a different maximal torus for $U(2n)$. This presents no problems because all maximal tori induce the same homomorphism in cohomology. To see this we need only recall that any two maximal tori are conjugate and that conjugation

$$\begin{array}{ccc}
 & & G \\
 & \nearrow{\scriptstyle \phi_1} & \downarrow{\scriptstyle g(\ )g^{-1}} \\
T & \xrightarrow{\phi_2} & G
\end{array}
\qquad
\begin{array}{ccc}
H^*BG & & \\
\downarrow{\scriptstyle 1} & \searrow{\scriptstyle B\phi_2^*} & \\
H^*BG & \xrightarrow{B\phi_1^*} & H^*BT
\end{array}$$

by $g \in G$ induces the identity map $H^*BG \to H^*BG$. The maximal torus we need is obtained from the standard one by conjugating with

$$\tilde{A} = \begin{pmatrix} A & & & 0 \\ & A & & \\ & & \ddots & \\ 0 & & & A \end{pmatrix}$$

where $A = 1/\sqrt{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \in U(2)$. We can express $\nu_2\phi_2 : T^1 \to U(2)$ by

$$\nu_2\phi_2(\alpha + i\beta) = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

and $\overline{\phi_2} = A\phi_2 A^{-1} : T^2 \to U(2)$ by

$$\overline{\phi_2}(\lambda_1, \lambda_2) = 1/2\begin{pmatrix} \lambda_1 + \lambda_2 & i(\lambda_2 - \lambda_1) \\ -i(\lambda_2 - \lambda_1) & \lambda_1 + \lambda_2 \end{pmatrix}.$$

We have a commutative diagram

$$\begin{array}{ccc}
T^n & \xrightarrow{\gamma^n} & (T^2)^n \\
\downarrow{\scriptstyle \phi_2^n} & & \downarrow{\scriptstyle \overline{\phi_2^n}} \\
SO(2)^n & \xrightarrow{\nu_2^n} & U(2)^n \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle q} \\
SO(2n) & \xrightarrow{\nu_{2n}} & U(2n)
\end{array}$$

where $\gamma(c) = (c, \bar{c})$. Let $H^*BT^n = P\{x_1, \ldots, x_n\}$ and $H^*BT^{2n} = P\{x_1, y_1, \ldots, x_n, y_n\}$ so that $(\gamma^n)^*(x_i) = x_i$ and $(\gamma^n)^*(y_i) = -x_i$. Then

$$\begin{aligned}
\phi_{2n}^* \nu_{2n}^*(c_i) &= (\gamma^n)^* \overline{\phi_{2n}^*}(c_i) \\
&= (\gamma^n)^*(\sigma_i(x_1, y_1, \ldots, x_n, y_n)) \\
&= \sigma_i(x_1, -x_1, \ldots, x_n, -x_n).
\end{aligned}$$

Using the relation

$$\sigma_i(x_1, -x_1, \ldots, x_n, -x_n) = \sigma_i(x_2, -x_2, \ldots, x_n, -x_n) - x_1^2 \sigma_{i-2}(x_2, -x_2, \ldots, x_n, -x_n)$$

it is easy to see that this is 0 if i is odd and is $(-1)^k \sigma_k(x_1^2, \ldots, x_n^2)$ if $i = 2k$. This proves (v).

Again for (vi) we need only consider $\mu_n : U(n) \to SO(2n)$. We have a commutative diagram

$$\begin{array}{ccccc}
 & & Sp(n) & & \\
 & \overset{\phi_n}{\nearrow} & & \overset{\nu_n}{\nwarrow} & \\
T^n & & \xrightarrow{\qquad \phi_n \qquad} & & U(n) \\
 & \underset{\phi_{2n}}{\searrow} & & \underset{\mu_n}{\swarrow} & \\
 & & SO(2n) & &
\end{array}$$

Now $\mu_n^* P_i$ is the unique element which is mapped to $\phi_{2n}^*(P_i) = \sigma_i(y_1^2, \ldots, y_n^2)$. Hence $\mu_n^*(P_i) = \nu_n^*(k_i) = \sum_{a+b=2i} (-1)^{a+i} c_a c_b$. Finally $\phi_{2n}^*(\chi) = \sigma_n(y_1, \ldots, y_n) = \phi_n^*(c_n)$ and therefore $\mu_n^*(\chi) = c_n$. This completes the proof of Theorem 5.1. □

If ξ is an $O(n)$ bundle, let $\xi_{\mathbb{C}}$ denote the $U(n)$ bundle induced by $\nu_n : O(n) \to U(n)$. If ξ is a $U(n)$ bundle, let $\xi^{\mathbb{R}}$ denote the $SO(2n)$ bundle induced by $\mu_n : U(n) \to SO(2n)$. Define the *total Pontrjagin class* $P(\xi)$ of an $O(n)$ bundle ξ to be $1 + P_1(\xi) + P_2(\xi) + \cdots$. Recall that Λ is a ring in which 2 is invertible.

Corollary 5.3. *All Λ-characteristic classes of $O(n)$ bundles can be expressed as polynomials in the Pontrjagin classes. All Λ-characteristic classes of $SO(n)$ bundles can be expressed as polynomials in the Pontrjagin classes and the Euler class. These classes satisfy:*

(i) if ξ is an $SO(2) = U(1)$ bundle, then $P_1(\xi) = (c_1(\xi_{\mathbb{C}}))^2$ and $\chi(\xi) = c_1(\xi_{\mathbb{C}})$. If ξ' denotes ξ regarded as an $O(2)$ bundle then $P_i(\xi') = P_i(\xi)$.

(ii) $P_i(\xi \oplus 1) = P_i(\xi)$, $\chi(\xi \oplus 1) = 0$, and $P_i(\xi') = P_i(\xi)$ if ξ is an $SO(n)$ bundle, ξ' is ξ regarded as an $O(n)$ bundle and 1 denotes the trivial $SO(1)$ or $O(1)$ bundle.

(iii) $P(\xi \oplus \eta) = P(\xi)P(\eta)$ *if* ξ *and* η *are* $O(n)$ *and* $O(m)$ *bundles (or* $SO(n)$ *and* $SO(m)$ *bundles).* $\chi(\xi \oplus \eta) = \chi(\xi)\chi(\eta)$ *if* ξ *and* η *are* $SO(n)$ *and* $SO(m)$ *bundles.*

(iv) $P_0(\xi) = 1$ *and* $P_i(\xi) = 0$ *if* $i > [n/2]$ *and* ξ *is an* $SO(n)$ *or* $O(n)$ *bundle.* $\chi(\xi) = 0$ *if* ξ *is an* $SO(2n+1)$ *bundle and* $\chi(\xi)^2 = P_n(\xi)$ *if* ξ *is an* $SO(2n)$ *bundle.*

(v) $c_{2i+1}(\xi_{\mathbb{C}}) = 0$ *and* $c_{2i}(\xi_{\mathbb{C}}) = (-1)^i P_i(\xi)$ *if* ξ *is an* $O(n)$ *bundle or* $SO(n)$ *bundle.*

(vi) $P_i(\xi^{\mathbb{R}}) = \sum_{a+b=2i}(-1)^{a+i}c_a(\xi)c_b(\xi)$ *and* $\chi(\xi^{\mathbb{R}}) = c_n(\xi)$ *if* ξ *is a* $U(n)$ *bundle.*

Remark 5.4. By (v) we could have defined $P_i(\xi)$ as $(-1)^i c_{2i}(\xi_{\mathbb{C}})$. This is often done. Many properties of the P_i follow directly from the analogous properties of the c_i.

The formula for $P_i(\xi^{\mathbb{R}})$ in (vi) can be written

$$\begin{aligned} 1 - P_1(\xi^{\mathbb{R}}) + P_2(\xi^{\mathbb{R}}) - \cdots &= (1 + c_1(\xi) + c_2(\xi) + \cdots)(1 - c_1(\xi) + c_2(\xi) - \cdots) \\ &= c(\xi)c(\bar{\xi}), \end{aligned}$$

since this product has non zero components only in degrees $4i$.

From (iii) and (iv) it follows that if $\chi(\xi) \neq 0$ (in Λ-cohomology) then ξ cannot split as the Whitney sum of two odd dimensional bundles.

Since $c_{2i+1}(\xi_{\mathbb{C}}) = 0$ in Λ-cohomology by (v), it follows that $c_{2i+1}(\xi_{\mathbb{C}})$ has order a power of two in integral cohomology (take $\Lambda = \mathbb{Z}[1/2]$ to see this). We will improve this is result in Chapter 8.

Corollary 5.5. *With coefficients in which* 2^{-1} *exists, we have the following.*

(i) As Hopf algebras,

$$H^*SO(2n) = E\{x_3, x_7, \ldots, x_{4n-5}\} \otimes E\{y_{2n-1}\},$$
$$H^*SO(2n+1) = E\{x_3, x_7, \ldots, x_{4n-1}\},$$

and $H^*SO = E\{x_{4i-1} \mid i \geq 1\}$ *where* $x_{4i-1} = \sigma^*(P_i)$ *and* $y_{2n-1} = \sigma^*(\chi)$.

(ii) As algebras

$$H^*BSO = H^*BO = P\{P_i \mid i \geq 1\}.$$

(iii) As algebras

$$H^*U(n)/O(n) = E\{x_{4i-3} \mid 1 \leq i \leq \lfloor \frac{n+1}{2} \rfloor\},$$
$$H^*SU(n)/SO(n) = E\{x_{4i-3} \mid 2 \leq i \leq n/2\} \otimes P\{\chi\}/(\chi^2) \ (n \text{ even}),$$
and $H^*SU(n)/SO(n) = E\{x_{4i-3} \mid 2 \leq i \leq n/2\}$ $(n$ *odd*$)$.

The natural maps $U(n) \to U(n)/O(n)$ *and* $SU(n) \to SU(n)/SO(n)$ *send* x_{4i-3} *to itself and the natural map* $SU(n)/SO(n) \to BSO(n)$ *sends* χ *to itself.*

(iv) As algebras,

$$H^*U/O = E\{x_{4i-3} \mid i \geq 1\},$$
$$\text{and } H^*SU/SO = E\{x_{4i-3} \mid i \geq 2\}.$$

Again the natural maps $U \to U/O$ *and* $SU \to SU/Sp$ *send* x_{4i-3} *to itself.*

(v) As an algebra,

$$H^*SO(2n)/U(n) = P\{c_1, \ldots, c_n\}/I$$

where I is the ideal generated by c_n and

$$\left\{ \sum_{a+b=2i} (-1)^{a+i} c_a c_b \mid 1 \leq i \leq n-1 \right\}.$$

The natural map $SO(2n)/U(n) \to BU(n)$ sends c_i to $c_i \mod I$.

(vi) As an algebra

$$H^*SO/U = P\{c_i \mid i \geq 1\}/I$$

where I is the ideal generated by

$$\left\{ \sum_{a+b=2i} (-1)^{a+i} c_a c_b \mid 1 \leq i \leq n-1 \right\}$$

and the natural map $SO/U \to BU$ sends c_i to $c_i \mod I$.

Further, none of the above spaces has p-torsion for odd primes p.

PROOF. Results (i) and (ii) follow in exactly the same manner as the analogous results in previous chapters. Once we have shown (iii) and (v), (iv) and (vi) will also follow in standard fashion. Having shown (i) through (vi), it will follow that none of these spaces have p-torsion for odd primes p since we may take $\Lambda = \mathbb{Z}[1/2]$ and p-torsion in integral cohomology would have p-torsion in Λ-cohomology. Recall that in Corollary 2.4 we have shown that $P\{c_i\}/I$ is torsion-free.

To prove (iii), consider the Eilenberg-Moore spectral sequence of $U(n)/O(n) \longrightarrow BO(n) \longrightarrow BU(n)$. The E_2 term is the homology of $E_{P\{c_i\}}\{a_i\} \otimes H^*BO(n)$ where $1 \leq i \leq n$, $d(a_i) = \nu_n^*(c_i)$, and $|a_i| = (-1, 2i)$. Hence, $E_2 = E_\infty$ and the result follows. Now in the Eilenberg-Moore spectral sequence of $SU(n)/SO(n) \longrightarrow BSO(n) \longrightarrow BSU(n)$ for odd n, the E_2 term is the same except that x_1 is not present. Hence the result follows as above. If n is even, however, we have $E_2 = E\{x_{4i-3} \mid 2 \leq i \leq n/2\} \otimes P\{\chi\}/(\chi^2)$ since $\chi^2 = P_{n/2}$ is in the image of ν_n^* while χ is not. Clearly $E_2 = E_\infty$ since the generators lie in $E_2^{0,*}$ and $E_2^{-1,*}$. Now E_∞ is not free so our previous results do not allow us to conclude that $H^* = H^*SU(n)/SO(n) \cong E_\infty$. However, E_∞ is close enough to being free. Let $F_0 \subset F_{-1} \subset F_{-2} \subset \cdots$ be the filtration of H^* which gives $E_\infty : E_\infty^{p,q} = F_pH^{p+q}/F_{p+1}H^{p+q}$. Since $E\{x_{4i-3} \mid 2 \leq i \leq n/2\}$ is free there is a homomorphism $E\{x_{4i-3}\} \to H^*$ which projects to the obvious homomorphism $E\{x_{4i-3}\} \to E_\infty$. This and the inclusion $P\{\chi\}/(\chi^2) = F_0 \subset H^*$ define a homomorphism $f : E\{x_{4i-3}\} \otimes P\{\chi\}/(\chi^2) \to H^*$ which is filtration preserving if the domain is given the obvious filtration. Clearly, f induces an isomorphism of associated graded algebras, hence must be an isomorphism. Therefore, $H^*SU(n)/SO(n)$ is as claimed. In general this shows that 'deviation from freeness' when confined to the 'bottom' filtration does not prevent the conclusion $A \cong E^0A$.

To prove (v) consider the Eilenberg-Moore spectral sequence of $SO(2n)/U(n) \longrightarrow BU(n) \longrightarrow BSO(2n)$, where the last map is μ_n. As in Corollary 2.4 we conclude that $H^*BU(n)/\operatorname{im}(\mu_n^*) \longrightarrow H^*SO(2n)/U(n)$ is an isomorphism. It is clear that $\operatorname{im}(\mu_n^*) = I$ is as stated. □

Note that once we have shown that BO and BSO are H-spaces whose product is given by the Whitney sum map it will follow from Theorem 5.1 that the coproduct on $H^*BO = H^*BSO$ is given by $\psi(P_n) = \sum_{i+j=n} P_i \otimes P_j$.

We now calculate $H^*B\mathrm{Spin}(n)$ and $H^*\mathrm{Spin}(n)$ (with Λ coefficients, $1/2 \in \Lambda$). We have the fibration

$$B\mathbb{Z}/2\mathbb{Z} \longrightarrow B\mathrm{Spin}(n) \longrightarrow BSO(n).$$

Since $O(1) = \mathbb{Z}/2\mathbb{Z}$ we know from Theorem 5.1 that $H^*B\mathbb{Z}/2\mathbb{Z} = H^*BO(1) = H^0BO(1) = \Lambda$. Hence in the Serre sequence of the above fibration $E_2 = E_2^{*,0} = H^*BSO(n)$. It follows immediately that $E_2 = E_\infty = H^*B\mathrm{Spin}(n)$. This establishes the first part of the next result.

Proposition 5.6. *The natural maps* $B\mathrm{Spin}(n) \to BSO(n)$ *and* $\mathrm{Spin}(n) \to SO(n)$*, and their limits* $B\mathrm{Spin} \to BSO$ *and* $\mathrm{Spin} \to SO$ *induce isomorphisms in* Λ*-cohomology.*

PROOF. We have just shown that $B\mathrm{Spin}(n) \to BSO(n)$ is a Λ-cohomology isomorphism. By the map of fibrations

$$\begin{array}{ccccc} \mathrm{Spin}(n) & \longrightarrow & E\mathrm{Spin}(n) & \longrightarrow & B\mathrm{Spin}(n) \\ \downarrow & & \downarrow & & \downarrow \\ SO(n) & \longrightarrow & ESO(n) & \longrightarrow & BSO(n) \end{array}$$

and the naturality of the Eilenberg-Moore spectral sequence, it follows that $\mathrm{Spin}(n) \to SO(n)$ is also an isomorphism in Λ-cohomology. The statement about the limit spaces is an obvious consequence. □

6. Integral Euler, Pontrjagin and Stiefel-Whitney classes

Let H^* denote integral cohomology and let Λ be a ring in which 2 is invertible. As before, we shall use the following names for the indicated natural maps:

$$j_n : BSO(n) \to BO(n) \qquad i_n : BSO(n-1) \to BSO(n) \qquad i_n : BO(n-1) \to BO(n)$$
$$p_{ij} : BSO(i) \times BSO(j) \to BSO(i+j) \qquad p_{ij} : BO(i) \times BO(j) \to BO(i+j)$$
$$\mu_n : BU(n) \to BSO(2n) \qquad \nu_n : BSO(n) \to BSU(n) \qquad \nu_n : BO(n) \to BU(n).$$

By our calculation of the mod 2 Bockstein spectral sequence of $BO(n)$ and $BSO(n)$ (Proposition 4.8), we know that all torsion in $H^*BO(n)$ and $H^*BSO(n)$ has order 2 and that

$$\begin{aligned} \frac{H^*BO(2n)}{\text{torsion}} \otimes \mathbb{Z}/2\mathbb{Z} &\cong \frac{H^*BO(2n+1)}{\text{torsion}} \otimes \mathbb{Z}/2\mathbb{Z} \\ &\cong \frac{H^*BSO(2n+1)}{\text{torsion}} \otimes \mathbb{Z}/2\mathbb{Z} \\ &\cong P\{w_2^2, w_4^2, \ldots, w_{2n}^2\} \end{aligned}$$

and

$$\frac{H^*BSO(2n)}{\text{torsion}} \otimes \mathbb{Z}/2\mathbb{Z} \cong P\{w_2^2, w_4^2, \ldots, w_{2(n-1)}^2, w_{2n}\}.$$

This tells us that there are classes of infinite order which reduce mod 2 to the classes of w_{2i}^2 (and for $BSO(2n)$, w_{2n}) and which generate the torsion free part modulo torsion and elements divisible by 2. We will give a more precise description shortly.

The following trivial fact will be used repeatedly.

Lemma 6.1. *Let B be a space such that all torsion elements of H^*B have order 2. Let K_1 denote the kernel of $H^*B \to H^*(B;\mathbb{F}_2)$ and let K_2 be the kernel of $H^*B \to H^*(B;\mathbb{Z}[1/2])$. Then $K_1 \cap K_2 = 0$. Hence, an element of H^*B is completely determined by its images in $H^*(B;\mathbb{F}_2)$ and $H^*(B;\mathbb{Z}[1/2])$*

Proof. K_1 is the set of elements divisible by 2 and K_1 is the set of torsion elements. Since all torsion has order 2, no torsion element is divisible by 2. □

Definition 6.2. The i$^{\text{th}}$ *Pontrjagin class* P_i in $H^*BO(n)$ or $H^*BSO(n)$ is defined to be $(-1)^i\nu_n^*(c_{2i})$.

Note that $\nu_n^*(c_i)$ reduces mod 2 to w_i^2 by Theorem 3.1.(vi). Since w_{2i}^2 survives to E_∞ of the Bockstein spectral sequence, $\nu_n^*(c_{2i})$ has infinite order. Since $w_{2i+1}^2 = \beta(w_{2i}, w_{2i+1})$, $\nu_n^*(c_{2i+1})$ has order 2.

Theorem 6.3. *The integral Pontrjagin classes have the following properties:*

(i) The mod 2 reduction of P_i is w_{2i}^2. The Λ-reduction of P_i is P_i. The integral classes P_i are characterized by this condition.
(ii) $i_n^*(P_i) = P_i$ and $j_n^*(P_i) = P_i$.
(iii) $p_{ij}^*(P_k) \equiv \sum_{a+b=k} P_a \otimes P_b$ modulo 2-torsion.
(iv) $\nu_n^*(c_{2i}) = (-1)^i P_i$
(v) $\mu_n^*(P_i) = \sum_{a+b=2i}(-1)^{a+i}c_ac_b$
(vi) $H^*BO(2n+\epsilon)$ and $H^*BSO(2n+\epsilon)$, $\epsilon = 0$ or 1, contain $P\{P_1,\dots,P_n\}$ as a subalgebra.

Proof. We treat each case separately.

(i) $\nu_n^*(c_{2i}) = w_{2i}^2$ mod 2 by Theorem 3.1.(vi) and $\nu_n^*(c_{2i}) = (-1)^iP_i$ in Λ-coefficients by Theorem 5.1.(v). Hence the first two statements follow. The third statement is immediate from Lemma 6.1.
(ii) This follows immediately from the corresponding statement about the Chern class c_{2i}.
(iii) This is easy:

$$\begin{aligned} p_{ij}^*(P_k) &= p_{ij}^*((-1)^k\nu_n^*(c_{2k})) \\ &= (-1)^k(\nu_i^*\otimes\nu_j^*)p_{ij}^*(c_{2k}) \\ &= (-1)^k(\nu_i^*\otimes\nu_j^*)\sum_{a+b=2k} c_a\otimes c_b \\ &= \sum_{a+b=k} P_a\otimes P_b + \sum_{a+b=k}\nu_i^*(c_{2a+1})\otimes\nu_j^*(c_{2b-1}). \end{aligned}$$

We can drop the sign $(-1)^k$ in the second sum because the elements involved have order 2. Part (iii) is now immediate since the second sum has order 2.
(iv) This is the definition.
(v) This holds in $\mathbb{Z}[1/2]$ coefficients. Since the homomorphism $H^*BU(n) \to H^*(BU(n);\mathbb{Z}[1/2])$ is monic, it must also hold in integral coefficients.

(vi) Since there are no relations among the P_i in Λ-cohomology, part (i) implies that there are no relations among the P_i in integral cohomology. $\square$

Let the total Pontrjagin class of an $O(n)$ bundle ξ be $P(\xi) = 1 + P_1(\xi) + P_2(\xi) + \cdots$. If ξ is an $O(n)$-bundle, let $\xi_{\mathbb{C}}$ be its complexification. If ξ is a $U(n)$-bundle, let $\xi^{\mathbb{R}}$ be the underlying $SO(2n)$ bundle. We have the following interpretation of Theorem 6.3 in terms of characteristic classes.

Corollary 6.4. *The Pontrjagin classes satisfy:*

(i) the mod 2 reduction of $P_i(\xi)$ is $(w_{2i}(\xi))^2$. The Λ-reduction of $P_i(\xi)$ is $P_i(\xi)$.
(ii) $P_i(\xi \oplus 1) = P_i(\xi)$ if 1 is the trivial line bundle. If ξ is an $SO(n)$ bundle and ξ' is ξ regarded as an $O(n)$ bundle, then $P_i(\xi) = P_i(\xi')$.
(iii) $P(\xi \oplus \eta) \equiv P(\xi)P(\eta)$ modulo 2-torsion
(iv) $c_{2i}(\xi_{\mathbb{C}}) = (-1)^i P_i(\xi)$
(v) $P_i(\xi^{\mathbb{R}}) = \sum_{a+b=2i} c_a(\xi)c_b(\xi)$, or, equivalently, $1 - P_1(\xi^{\mathbb{R}}) + P_2(\xi^{\mathbb{R}}) - \cdots = c(\xi)c(\bar{\xi})$.

Note the following useful fact. If we work mod 2, then $\nu_n^*(c_{2i+1}) = w_{2i+1}^2 = \beta(w_{2i}w_{2i+1})$. Therefore, $\nu_n^*(c_{2i+1})$ has order 2 and so does $c_{2i+1}(\xi_{\mathbb{C}})$.

The Euler class $\chi \in H^n BSO(n)$ was defined in Definition 3.2. Using Theorem 3.3, we choose the orientation of the universal $SO(2n)$ bundle which makes χ reduce to the Euler class in $H^{2n}(BSO(2n); \Lambda)$ defined in Chapter 6. In odd dimensions, $\chi = -\chi$ so it does not matter how we orient the universal $SO(2n+1)$ bundle.

Theorem 6.5. *The Euler class $\chi \in H^n BSO(n)$ satisfies:*

(i) The mod 2 reduction of χ is w_n. The Λ reduction of χ is χ (which is zero in $H^*BSO(2n+1)$). The class χ is characterized by these two conditions. In $H^*BSO(2n)$, χ has infinite order, while $2\chi = 0$ in $H^*BSO(2n+1)$.
(ii) $i_n^*(\chi) = 0$
(iii) $p_{ij}^*(\chi) = \chi \otimes \chi$
(iv) $\nu_n^*(c_n) = (-1)^{\lfloor n/2 \rfloor} \chi^2$
(v) $\mu_n^*(\chi) = c_n$ ($\mu_n : BU(n) \to BSO(2n)$)
(vi) $\chi^2 = P_n$ in $H^*BSO(2n)$. $H^*BSO(2n)$ contains $P\{P_1, \ldots, P_{n-1}, \chi\}$ as a subalgebra. $H^*BSO(2n+1)$ contains $P\{P_1, \ldots, P_n\} \otimes P_{\mathbb{Z}/2\mathbb{Z}}\{\chi\}$ as a subalgebra.
(vii) The canonical automorphism $BSO(n) \to BSO(n)$ obtained from the bundle $O(n)/SO(n) = \mathbb{Z}/2\mathbb{Z} \to BSO(n) \to BO(n)$ sends χ to $-\chi$.

Proof. We treat each case separately.

(i) For the first two statements, see Theorem 3.3 and the remarks following Definition 3.2. The third statement is implied by Lemma 6.1. Since $\chi \in H^*(BSO(2n); \mathbb{Z}[1/2])$ is nonzero, $\chi \in H^*BSO(2n)$ has infinite order. Since $\chi \in H^*(BSO(2n+1); \mathbb{Z}[1/2])$ is zero, $\chi \in H^*BSO(2n+1)$ is torsion and hence of order 2.
(ii) Mod 2, $i_n^*(w_n) = 0$ and with Λ coefficients $i_n^*(\chi) = 0$. The result follows by Lemma 6.1.
(iii) Mod 2, $p_{ij}^*(w_{i+j}) = w_i \otimes w_j$ by Theorem 3.1.(iii). With Λ coefficients $p_{ij}^*(\chi) = \chi \otimes \chi$ by Theorem 5.1.(iii). By Lemma 6.1.(i), this proves the result.

(iv) Mod 2, $\nu_n^*(c_n) = w_n^2 = -w_n^2$. With Λ coefficients $\nu_{2n}^*(c_{2n}) = (-1)^n P_n = (-1)^n \chi^2$ while $\nu_{2n+1}^*(c_{2n+1}) = 0 = \chi$. By Lemma 6.1 and part (i) the result follows.

(v) This is true with $\mathbb{Z}[1/2]$ coefficients. Since $H^*BU(n) \to H^*(BU(n);\mathbb{Z}[1/2])$ is a monomorphism, it also holds integrally.

(vi) With Λ coefficients, $\chi^2 = P_n$ in $H^*BSO(2n)$. Mod 2, χ is w_{2n} and P_n is w_{2n}^2 in $H^*BSO(2n)$. Lemma 6.1 completes the proof that $\chi^2 = P_n$ in $H^*BSO(2n)$. Since there are no relations among $P_1, \dots, P_{n-1}$ and χ in $H^*(BSO(2n);\Lambda)$, there can be none in $H^*BSO(2n)$. In $H^*BSO(2n+1)$ we can write any relation between $P_1, \dots, P_n$ and χ in the form $\chi A + B = 0$ where B does not involve χ. Since χ reduces to 0 in Λ coefficients and there are no relations between the P_i in $H^*(BSO(2n+1);\Lambda)$, B must be trivial and hence our relation has the form $\chi A = 0$. Mod 2, the P_i and χ reduce to $w_2^2, \dots, w_{2n}^2$ and w_{2n+1} among which there are no relations. Hence A must be divisible by 2: $\chi A = 2\chi A'$. But $2\chi = 0$ and thus the relation is trivial.

(vii) In Chapter V, we showed that $\mathbb{Z}/2\mathbb{Z}$ acts by reversing orientations. Since $\chi = (\gamma^*)^{-1}(p^* u_\gamma)$ where (γ, u_γ) is the universal $SO(n)$ bundle

$$BSO(n) \xleftarrow{\gamma} D_\gamma \xrightarrow{p} TSO(n)$$

the result follows immediately. $\square$

Corollary 6.6. *The Euler class of an $SO(n)$-bundle satisfies:*

(i) The mod 2 reduction of $\chi(\xi)$ is $w_n(\xi)$. The Λ-reduction of $\chi(\xi)$ is $\chi(\xi)$. If ξ is odd dimensional then $2\chi(\xi) = 0$.
(ii) $\chi(\xi \oplus 1) = 0$ where 1 denotes the trivial line bundle.
(iii) $\chi(\xi \oplus \eta) = \chi(\xi)\chi(\eta)$.
(iv) If ξ is an $SO(n)$ bundle then $c_n(\xi_{\mathbb{C}}) = (-1)^{\lfloor n/2 \rfloor}\chi(\xi)^2$.
(v) If ξ is a $U(n)$ bundle then $\chi(\xi^{\mathbb{R}}) = c_n(\xi)$.
(vi) If ξ is an $SO(2n)$ bundle then $\chi(\xi)^2 = P_n(\xi)$.
(vii) If $\tilde{\xi}$ denotes ξ with the opposite orientation then $\chi(\tilde{\xi}) = -\chi(\xi)$.

The Euler class and the Pontrjagin classes are the most important integral characteristic classes. They account for all of the torsion-free classes. We now consider a family of torsion classes. In $H^*(BSO(n);\mathbb{F}_2)$ we have $\beta(w_{2i}) = w_{2i+1}$. Hence, there exists a (necessarily unique) class of order 2 in $H^*BSO(n)$ which reduces mod 2 to w_{2i+1}.

Definition 6.7. The $(2i+1)^{\text{st}}$ *integral Stiefel-Whitney class* $w_{2i+1} \in H^{2i+1}BSO(n)$ is the unique class of order 2 which reduces mod 2 to w_{2i+1}.

Note that in $H^*BSO(2n+1)$, w_{2n+1} is the same as χ.

Some properties of the w_{2i+1} are given in the next proposition.

Theorem 6.8. *The integral Stiefel-Whitney classes $w_{2i+1} \in H^*BSO(n)$ satisfy:*

(i) w_{2i+1} reduces mod 2 to w_{2i+1} and has order 2. The class w_{2i+1} is characterized by these properties.
(ii) $i_n^*(w_{2i+1}) = w_{2i+1}$.
(iii) $\mu_n^*(w_{2i+1}) = 0$.
(iv) $\nu_n^*(c_{2i+1}) = w_{2i+1}^2$.

(v) *$H^*BSO(2n+\epsilon)$, $\epsilon = 0$ or 1, contains $P_{\mathbb{Z}/2\mathbb{Z}}\{w_3, w_5, \dots, w_{2n-1}\}$ as a subalgebra. In $H^*BSO(2n+1)$, $w_{2n+1} = \chi$.*

Note that in (v) we have omitted w_{2n+1} in the case $\epsilon = 1$ because w_{2n+1} is χ.

PROOF. We treat each case separately.

(i) To say that w_{2i+1} has order 2 is equivalent to saying that it reduces to 0 in Λ coefficients. Hence this follows by Lemma 6.1.
(ii) This is true in $\mathbb{F}_2$ and Λ coefficients. Apply Lemma 6.1.
(iii) $H^*BU(n)$ is torsion free so this is forced on us.
(iv) Same as (ii).
(v) Mod 2, there are no nontrivial relations among the w_{2i+1}. Since these classes have order 2, this is sufficient to imply (as in Theorem 6.5.(vi) that there are no nontrivial relations integrally. □

Corollary 6.9. *The integral Stiefel-Whitney classes of a bundle satisfy:*

(i) *$w_{2i+1}(\xi)$ has order 2 and reduces mod 2 to $w_{2i+1}(\xi)$.*
(ii) *$w_{2i+1}(\xi \oplus 1) = w_{2i+1}(\xi)$.*
(iii) *$w_{2i+1}(\xi^{\mathbb{R}}) = 0$ for a $U(n)$ bundle ξ.*
(iv) *$c_{2i+1}(\xi_{\mathbb{C}}) = w_{2i+1}(\xi)^2$.*
(v) *If ξ is an $BSO(2n+1)$ bundle, then $w_{2n+1}(\xi) = \chi(\xi)$.*

Conspicuously absent is any result on integral Stiefel-Whitney classes of Whitney sums. This is because $w_{2i+1}(\xi \oplus \eta)$ is difficult to describe.

The single most important fact about the integral Stiefel-Whitney classes is that the ordinary (mod 2) Stiefel-Whitney classes w_{2i+1} of an oriented bundle are reductions of integral classes which satisfy (i)-(v) above. For example, if B is a space such that H^*B has no 2-torsion then, for any $SO(n)$-bundle ξ over B, $w_{2i+1}(\xi) = 0$.

In summary, we have examined the following subalgebras of $H^*BO(n)$ and $H^*BSO(n)$:

$$P\{P_1, \dots, P_{n-1}, \xi\} \otimes P_{\mathbb{Z}/2\mathbb{Z}}\{w_3, w_5, \dots, w_{2n-1}\} \subset H^*BSO(2n)$$
$$P\{P_1, \dots, P_n\} \otimes P_{\mathbb{Z}/2\mathbb{Z}}\{w_3, w_5, \dots, w_{2n+1} = \chi\} \subset H^*BSO(2n+1)$$
$$P\{P_1, \dots, P_n\} \subset H^*BO(2n) \text{ and } H^*BO(2n+1).$$

There are, of course, other classes in the integral cohomology of these spaces. For example, $\nu_n^*(c_{2i+1}) =$ "w_{2i+1}^2 " in $H^*BO(n)$ ("w_{2i+1}" because theres no w_{2i+1} here). These form another polynomial subalgebra $P_{\mathbb{F}_2}\{w_1^2, w_3^2, \dots, w_{2n-1}^2\}$ in $H^*BO(2n-1)$ and $H^*BO(2n)$. Also, from the Bockstein spectral sequence it is apparent that $w_1 w_{2i} + w_{2i+1} \in H^*(BO(n); \mathbb{F}_2)$ is the reduction of a unique integral class of order 2. However, these classes are of no great importance in practice.

CHAPTER V

The Thom Isomorphism

WHERE IS Lemma Lem:6.3 ?

We define Thom spaces and orientations, then prove the Thom Isomorphism Theorem. After a discussion of orientations and orientability, we show that the Stiefel-Whitney and Euler classes of a bundle can be defined in terms intrinsic to the bundle, i.e., without reference to a classifying map for the bundle. We then define the homology tangent bundle of a topological manifold, define orientations in this context, and give a version of the Thom Isomorphism Theorem for topological manifolds. We then prove the topological invariance of the Stiefel-Whitney and Euler classes of a manifold by showing that the tangent bundle of a smooth manifold is equivalent to its homology tangent bundle.

1. Thom spaces, orientations, and the Thom isomorphism

Let H^* denote cohomology with coefficients in a ring Λ (commutative with unit). Given a principal $O(n)$-bundle $\xi : E \longrightarrow B$ we have the associated vector bundle

$$\mathbb{R}^n \longrightarrow E' = E \times_{O(n)} \mathbb{R}^n \longrightarrow B$$

and the associated disk and sphere bundles

$$D^n \longrightarrow D_\xi = E \times_{O(n)} D^n \longrightarrow B$$

and

$$S^{n-1} \longrightarrow S_\xi = E \times_{O(n)} S^{n-1} \longrightarrow B.$$

If we let E'_0 denote the complement of the zero cross-section in E' then we have an obvious map of relative fibrations

$$\begin{array}{ccccc} (\mathbb{R}^n, \mathbb{R}^n - 0) & \longrightarrow & (E', E'_0) & \longrightarrow & B \\ \uparrow & & \uparrow & & \| \\ (D^n, S^{n-1}) & \longrightarrow & (D_\xi, S_\xi) & \longrightarrow & B \end{array}$$

which is easily seen to be a homotopy equivalence.

Definition 1.1. The *Thom space* of ξ is the quotient space $T\xi = D\xi/S\xi$.

A commonly used alternative notation is $M\xi$. The Thom space of the universal G-bundle is often denoted TG or MG, e.g., $TU(n), MU(n), TU, MU$. These universal Thom spaces are the representing spaces for bordism and cobordism just as BU, BO, etc. are representing spaces for complex or real K-theory.

Observe that $T(\xi)$ is homotopy equivalent to the mapping cone of $S_\xi \to B$. Clearly the Thom space is a functorial construction: a map of fibrations $f : (D_\xi, S_\xi) \to (D_{\xi'}, S_{\xi'})$ induces a map $\overline{f} : T(\xi) \to T(\xi')$. Fiber homotopic maps

F_1 and F_2 induce homotopic maps $\overline{F}_1$ and $\overline{F}_2$ (this follows directly from the fact that $T(\xi)$ may be obtained as the mapping cone of $S_\xi \to B$). It follows that, up to homotopy, $T(\xi)$ is an invariant of the fiber homotopy type of $(D_\xi, S_\xi) \to B$ or $S_\xi \to B$.

Frequently, we are given an n-plane bundle with group $GL(n, \mathbb{R})$ rather than $O(n)$. However, since $GL(n, \mathbb{R})/O(n)$ is contractible [**18**], the natural map $BO(n) \to BGL(n, \mathbb{R})$ is a homotopy equivalence and hence there is no significant distinction between $O(n)$ bundles $GL(n, \mathbb{R})$ bundles. This is the homotopy theoretic proof that every manifold has a Riemannian metric.

Note also that the above discussion applies to $U(n)$ bundles and $Sp(n)$ bundles. Recall $\tilde{H}^*T(\xi) \cong H^*(D_\xi, S_\xi)$.

Definition 1.2. A Λ-*orientation* (an *orientation* if $\Lambda = \mathbb{Z}$) of ξ is a class

$$u_\xi \in H^n T\xi = H^n(D\xi, S\xi)$$

which restricts to a generator of

$$H^n(D\xi_b, S\xi_b) = H^n(D^n, S^{n-1}) = \Lambda$$

under the inclusion $\xi^{-1}(b) = (D\xi_b, S\xi_b) \subset (D_\xi, S_\xi)$ for every $b \in B$.

If we have a bundle map

$$\begin{array}{ccc} (D_{\xi'}, S_{\xi'}) & \xrightarrow{\tilde{f}} & (D_\xi, S_\xi) \\ \downarrow{\scriptstyle \xi'} & & \downarrow{\scriptstyle \xi} \\ B' & \xrightarrow{f} & B \end{array}$$

and a Λ-orientation $u \in H^n(D_\xi, S_\xi)$ of ξ, then $\tilde{f}^*(u) \in H^n(D_{\xi'}, S_{\xi'})$ is a Λ-orientation of ξ'. In fact, this holds whenever we have a map of fibrations $\tilde{f} : S_{\xi'} \to S_\xi$ whose degree on each fiber is a unit of Λ.

If $\alpha : \Lambda_1 \to \Lambda_2$ is a ring homomorphism (preserving unit) and if

$$u \in H^n(D_\xi, S_\xi; \Lambda_1)$$

is a Λ_1-orientation of ξ then

$$\alpha_*(u) \in H^n(D_\xi, S_\xi; \Lambda_2)$$

is a Λ_2-orientation of ξ. Hence, an orientation (that is a $\mathbb{Z}$-orientation) provides us with a Λ-orientation for every coefficient ring Λ.

The following theorem is fundamental.

Theorem 1.3. *The following are equivalent:*

(1) ξ has a Λ-orientation $u_\xi \in H^n(D_\xi, S_\xi)$.
(2) there is a class $\overline{u} \in H^n(D_\xi, S_\xi)$ which restricts to a generator of $H^n(D\xi_b, S\xi_b)$ for some b in each path component of B.
(3) the local coefficient system $\mathcal{H}^*(D\xi_b, S\xi_b)$ of ξ is trivial.

If (1)-(3) hold then the homomorphism $\Phi : H^k B \to H^{n+k}(D_\xi, S_\xi)$ defined by $\Phi(\alpha) = \xi^*(\alpha) \cup u_\xi$ is an isomorphism where $\xi : D_\xi \to B$ and

$$\cup : H^* D_\xi \otimes H^*(D_\xi, S_\xi) \longrightarrow H^*(D_\xi, S_\xi).$$

The map Φ is natural with respect to orientation preserving bundle maps.

Proof. Obviously (1) implies (2). For (2) implies (3), consider the Serre spectral sequence of $(D_\xi, S_\xi) \to B$. We have $E_2 = H^*(B; \mathcal{H}^*(D^n, S^{n-1}))$. Since $\mathcal{H}^i(D^n, S^{n-1}) = 0$ if $i \neq n$, $E_2 = E_\infty$. Clearly we may assume B is connected. Then $E_2^{0,*} = H^0(B; \mathcal{H}^*(D^n, S^{n-1})) = (H^*(D^n, S^{n-1}))^{\pi_1 B}$, the elements of $H^*(D^n, S^{n-1})$ fixed by $\pi_1 B$.

Let $i_b : (D^n, S^{n-1}) \to (D_\xi, S_\xi)$ be the inclusion of the fiber over $b \in B$. Since i_b^* factors as

$$H^n(D_\xi, S_\xi) = E_\infty^{0,n} = E_2^{0,n} = (H^n(D^n, S^{n-1}))^{\pi_1 B} \subset H^n(D^n, S^{n-1})$$

and i_b^* is onto by assumption, all elements of $H^n(D^n, S^{n-1})$ are fixed by $\pi_1 B$ and thus the local coefficient system is trivial. Thus (2) implies (3).

Assume (3) holds. Then $E_2 = H^*B \otimes \iota_n$ where $\iota_n \in E_2^{0,n} = H^n(D^n, S^{n-1})$ is a generator. Obviously, $H^*(D_\xi, S_\xi) = E_\infty = E_2$. Let u_ξ correspond to $1 \otimes \iota_n$ under these identifications. Then u_ξ restricts to a generator of each fiber by the naturality of the Serre spectral sequence applied to the diagram

$$\begin{array}{ccc} (D^n, S^{n-1}) & \xrightarrow{i_b} & (D_\xi, S_\xi) \\ \downarrow & & \downarrow \\ \{b\} & \longrightarrow & B. \end{array}$$

Hence (3) implies (1).

Now assume (1)-(3) hold. There is a pairing of spectral sequences $E_r(D_\xi) \otimes E_r(D_\xi, S_\xi) \to E_r(D_\xi, S_\xi)$ which yields the cap product on E_∞. Now $E_r^{p,q}(D_\xi) = 0$ if $q \neq 0$ and $E_r^{*,0}(D_\xi) = H^*B$. On E_2 the pairing is just left multiplication by H^*B and Φ corresponds to the isomorphism sending x to $x \otimes \iota_n$. Since this is an isomorphism, Φ is also.

The statement that Φ is natural is expressed by the equality $\tilde{f}^*\xi^*(x) \cup \overline{f}^* u_\xi = \xi'^* f^*(x) \cup u'_\xi$ when $\overline{f}^* u_\xi = u'_\xi$ in the situation

$$\begin{array}{ccc} T_{\xi'} & \xrightarrow{\overline{f}} & T_\xi \\ \uparrow & & \uparrow \\ D_{\xi'} & \xrightarrow{\tilde{f}} & D_\xi \\ \downarrow{\scriptstyle \xi'} & & \downarrow{\scriptstyle \xi} \\ B' & \xrightarrow{f} & B \end{array}$$

□

Clearly we may also think of Φ as an isomorphism $H^k B \to H^{n+k}(T(\xi))$.

By condition (3) we see that any bundle has a $\mathbb{Z}/2\mathbb{Z}$-orientation since the only automorphism of $\mathbb{Z}/2\mathbb{Z}$ is the identity and that an $SO(n)$ bundle has an orientation (and hence a Λ-orientation for any Λ) since $BSO(n)$ is simply connected.

A word about the connection between this notion of orientation and orientations of vector spaces is in order. An orientation of $\mathbb{R}^n$ is an equivalence class of ordered bases, where two bases are equivalent if the linear transformation sending one to the other has positive determinant. Taking $\mathbb{Z}$ coefficients, this is the same as a choice of generator for $H^n(\mathbb{R}^n, \mathbb{R}^n - 0) = \mathbb{Z}$. The correspondence is given by taking a simplex linearly embedded in $\mathbb{R}^n$ representing the chosen generator of $H^n(\mathbb{R}^n, \mathbb{R}^n - 0)$ and letting the i^{th} basis vector of the corresponding basis be the vector connecting vertex $i-1$ and vertex i. Since $(D^n, S^{n-1}) \subset (\mathbb{R}^n, \mathbb{R}^n - 0)$ is a homotopy equivalence, it is equivalent to supply a generator of $H^n(D^n, S^{n-1})$. Letting $i_b : (D^n, S^{n-1}) \to (D_\xi, S_\xi)$ be the inclusion of the fiber over $b \in B$ we see that an orientation $u \in H^n(D_\xi, S_\xi)$ provides us with orientations $i_b^*(u)$ of each fiber. These orientations are locally compatible in the sense that for each $b \in B$, there is a neighborhood V of b and a class $\pi \in H^n(\xi^{-1}(V), \xi^{-1}(V) \otimes S_\xi)$ such that π restricts to the chosen orientation of each fiber lying over a point of V. Conversely, given orientations of the fibers which are locally compatible in this sense we can recover an orientation $u \in H^n(D_\xi, S_\xi)$ inducing them by Mayer-Vietoris sequences and a limit argument [**17**, p. 212]. If one considers orientations with respect to a generalized cohomology theory, orientations as in Definition 1.2 are no longer equivalent to locally compatible families of orientations (see [**1**, pp. 253-255]). Definition 1.2 is the one which generalizes.

Observe that any two Λ-orientations of a bundle differ by a unit of Λ. Hence $\mathbb{Z}/2\mathbb{Z}$-orientations are unique, while $\mathbb{Z}$-orientations occur in pairs $\{u, -u\}$.

Definition 1.4. A bundle is Λ-*orientable* if it has a Λ-orientation. A Λ-*oriented bundle* is a pair (ξ, u_ξ) consisting of a bundle ξ and a particular Λ-orientation $u_\xi \in H^n T(\xi)$.

2. Orientations and the classifying spaces $BO(n)$ and $BSO(n)$

These ideas can be expressed most clearly in terms of classifying maps and characteristic classes. We need the following fact.

Theorem 2.1. *Let $\Lambda = \mathbb{Z}$. Then conditions (1)-(3) of Theorem 1.3 are equivalent to either of the following:*

(1') the classifying map ξ can be lifted to $BSO(n)$
(2') $w_1(\xi) = 0$.

Proof. The equivalence of (1') and (2') was demonstrated in Corollary (iv). As noted before (1') implies (3) since if $w_1(\xi) \neq 0$ then ξ is not orientable. Let $\xi : E \to B$ have classifying map $f : B \to BO(n)$. Since $w_1(\xi) \neq 0$, $f^* : H^1(BO(n); \mathbb{Z}/2\mathbb{Z}) \to H^1(B; \mathbb{Z}/2\mathbb{Z})$ is nonzero. By universal coefficients and the Hurewicz theorem, $f_* : \pi_1 B \to \pi_1 BO(n) = \mathbb{Z}/2\mathbb{Z}$ is nonzero. Pick $\alpha : S^1 \to B$ such that $f_*(\alpha) \neq 0$. Then $\alpha^*\xi$ is the nontrivial $O(n)$ bundle over S^1. Now if ξ were orientable, $\alpha^*\xi$ would be also. Since there is only one nontrivial $O(n)$ bundle over S^1, it remains only to exhibit a nonorientable $O(n)$ bundle over S^1. Since the transformation

$$\begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

in $O(n)$ induces the nontrivial automorphism of $H^n(D^n, S^{n-1})$, this is easy. □

Remark 2.2. The following remarks are immediate.

(1) Condition (1') is equivalent to saying that the group of the bundle can be reduced to $SO(n)$.

(2) If we are given an n-plane bundle $\xi : E \to B$ with group $GL(n, \mathbb{R})$, let E_0 be the complement of the 0-cross section $B \to E$. Then $(E, E_0) \to B$ is a bundle pair with fiber $(\mathbb{R}^n, \mathbb{R}^n - 0)$. Although we cannot define a Thom space from the pair (E, E_0) as we did with (D_ξ, S_ξ), Theorem 1.3 holds with the pair (D_ξ, S_ξ) replaced by (E, E_0) and Theorem 2.1 holds with $SO(n)$ replaced by $SL(n, \mathbb{R})$.

Recall the fibration sequence

$$\mathbb{Z}/2\mathbb{Z} \longrightarrow BSO(n) \longrightarrow BO(n) \longrightarrow B\mathbb{Z}/2\mathbb{Z}.$$

For any space X, we have an exact sequence

$$[X, \mathbb{Z}/2\mathbb{Z}] \xrightarrow{i} [X, BSO(n)] \xrightarrow{j} [X, BO(n)] \xrightarrow{k} [X, B\mathbb{Z}/2\mathbb{Z}]. \tag{2.3}$$

The elements of $[X, BSO(n)]$ correspond to *oriented* bundles since an orientation of the universal $SO(n)$ bundle induces an orientation on each $SO(n)$-bundle which is preserved by $SO(n)$ bundle maps. The image of $[X, BSO(n)]$ in $[X, BO(n)]$ consists of *orientable* bundles. The various lifts of a map $f : X \to BO(n)$ to $BSO(n)$ correspond to the various choices of orientation. $\mathbb{Z}/2\mathbb{Z}$ acts on $[X, BSO(n)]$ by reversing orientations (sending (ξ, u_ξ) to $(\xi, -u_\xi)$. To prove the last statement, recall the situation of Lemma **??**. Let $\xi : ESO(n) \to BSO(n)$ be the universal $SO(n)$ bundle, let $\psi : BSO(n) \to BSO(n)$ give the action of $\mathbb{Z}/2\mathbb{Z} = O(n)/SO(n)$ on $BSO(n)$ and let $\xi' : E' \to BSO$ be classified by ψ. Then we have an $SO(n)$ bundle map

$$\begin{array}{ccc} E' & \xrightarrow{\xi'} & E \\ \downarrow & & \downarrow \xi \\ BSO(n) & \xrightarrow{\psi} & BSO(n). \end{array}$$

We want to show that ξ' is the same $O(n)$ bundle as ξ but with the opposite orientation. This is accomplished by exhibiting an $O(n)$ bundle map

$$\begin{array}{ccc} E' \times_{SO(n)} S^{n-1} & \longrightarrow & E \times_{SO(n)} S^{n-1} \\ \downarrow & & \downarrow \\ BSO(n) & \xrightarrow{1} & BSO(n) \end{array}$$

covering the identity and noting that it has degree -1 on the fibers. Details are similar to Lemma **??**.

We can identify $[X, \mathbb{Z}/2\mathbb{Z}]$ with the set of orientations of the trivial $O(n)$ bundle over X in an obvious way. Then sequence (2.3) can be interpreted as

$$\left\{\begin{array}{c}\text{orientations}\\ \text{of the}\\ \text{trivial bundle}\end{array}\right\} \xrightarrow{i} \left\{\begin{array}{c}\text{oriented}\\ \text{n-bundles}\end{array}\right\} \xrightarrow{j} \{\text{n-bundles}\} \xrightarrow{k} \left\{\begin{array}{c}\text{two-fold}\\ \text{covers}\end{array}\right\}$$

which may be split into

$$\left\{\begin{array}{c}\text{orientations}\\ \text{of the}\\ \text{trivial bundle}\end{array}\right\} \xrightarrow{i} \left\{\begin{array}{c}\text{oriented}\\ \text{n-bundles}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{orientable}\\ \text{n-bundles}\end{array}\right\}$$

$$\left\{\begin{array}{c}\text{orientable}\\ \text{n-bundles}\end{array}\right\} \longrightarrow \left\{\text{ n-bundles }\right\} \xrightarrow{k} \left\{\begin{array}{c}\text{two-fold}\\ \text{covers}\end{array}\right\}$$

where $\mathrm{im}(j) \subset [X, BO(n)]$ is identified with the set of orientable bundles. The map k takes an $O(n)$ bundle $E \longrightarrow B$ to the double cover

$$E \times_{O(n)} O(n)/SO(n) = E/SO(n) \longrightarrow B.$$

By exactness of (2.3), we observe the amusing fact that, for connected base space B, the (principal) $O(n)$-bundle $E \to B$ is orientable if and only if E is disconnected (since $SO(n)$ is connected, E is disconnected if and only if $E/SO(n)$ is disconnected).

3. The Stiefel-Whitney classes and Euler class revisited

A remarkable feature of the Thom isomorphism is that it enables us to define Stiefel-Whitney classes and an Euler class in terms intrinsic to the bundle.

Theorem 3.1.

$$w_i = \Phi^{-1}\mathrm{Sq}^i\Phi(1) \in H^i(BO(n); \mathbb{Z}/2\mathbb{Z})$$

Note: Clearly this implies that $w_i(\xi) = \Phi^{-1}\mathrm{Sq}^i u_\xi$ for any $O(n)$-bundle ξ.

Proof. We shall prove this by explicitly analyzing the Thom space $TO(n)$. Let $\gamma : EO(n) \to BO(n)$ be the universal $O(n)$ bundle. Then

$$S_\gamma = E \times_{O(n)} S^{n-1} = E \times_{O(n)} O(n)/O(n-1) = E/O(n-1) = BO(n-1)$$

and there is an equivalence

$$\begin{array}{ccc} S_\gamma & = & BO(n-1) \\ \downarrow & & \downarrow{\scriptstyle i} \\ BO(n) & = & BO(n) \end{array}$$

of fibrations. From the diagram

$$\begin{array}{ccc} S_\gamma & \longrightarrow & B \\ \downarrow & & \| \\ D_\gamma & \xrightarrow{\gamma} & B \end{array}$$

in which $\gamma : D_\gamma \to B$ is an equivalence, we see that the inclusion $S_\gamma \to D_\gamma$ is equivalent to $BO(n-1) \xrightarrow{i} BO(n)$. Since i^* is an epimorphism in $\mathbb{F}_2$-cohomology, the long exact sequence becomes

$$0 \longrightarrow H^*TO(n) \xrightarrow{(\gamma^*)^{-1}p^*} H^*BO(n) \xrightarrow{i^*} H^*BO(n-1) \longrightarrow 0,$$

where $p : D_\gamma \to TO(n)$ is the natural map and H^* denotes $H^*(-; \mathbb{F}_2)$. Hence, p^* is a monomorphism which identifies $H^*TO(n)$ with the ideal in $H^*BO(n)$ generated by w_n. It follows that $p^*u_\gamma = \gamma^*(w_n)$. We want to show $\Phi(w_i) = \mathrm{Sq}^i u_\gamma$. Since p^* is a monomorphism, it is enough to show $p^*\Phi(w_i) = p^*\mathrm{Sq}^i u_\gamma$. But

$$\begin{aligned} p^*\Phi(w_i) &= p^*(\gamma^*(w_i)u_\gamma) \\ &= \gamma^*(w_i)\gamma^*(w_n) \end{aligned}$$

and

$$\begin{aligned} p^*\mathrm{Sq}^i u_\gamma &= \gamma^*\mathrm{Sq}^i w_n \\ &= \gamma^*(w_i w_n) \end{aligned}$$

so we are done. □

Definition 3.2. The *Euler class* χ of an oriented bundle (ξ, u_ξ) is defined to be $\Phi^{-1}(u_\xi^2) \in H^*B$.

Note that we could have defined χ as $(\xi^*)^{-1}p^*u_\xi$,

$$\begin{array}{ccc} H^*D_\xi & \xleftarrow{p^*} & H^*T_\xi \\ \cong \big\downarrow (\xi^*)^{-1} & & \\ H^*B & & \end{array}$$

since $\Phi((\xi^*)^{-1}p^*u_\xi) = p^*(u_\xi)u_\xi = u_\xi^2$.

If ξ is an n-bundle with n odd then

$$u_\xi^2 = (-1)^{n^2}u_\xi^2 = -u_\xi^2.$$

Thus $2u_\xi^2 = 0$ and hence $2\chi(\xi) = 0$. With $\mathbb{F}_2$ coefficients, $u_\xi^2 = \mathrm{Sq}^n u_\xi$. Hence the mod 2 reduction of χ is w_n.

The next thing we must do is check that this definition agrees with the definition given in Chapter 6 when $1/2 \in \Lambda$. In the following theorem, let χ denote the Euler class defined in Chapter 6. Assume Λ contains $1/2$.

Theorem 3.3. *The Λ-reduction of $\Phi^{-1}(u_\xi^2)$ is $\chi \in H^n(BSO(n);\Lambda)$ for an appropriately chosen orientation u_γ of the universal bundle $\gamma : ESO(n) \to BSO(n)$.*

Proof. Let H^* denote $H^*(-;\Lambda)$. As in Theorem 3.1, $S_\gamma \to D_\gamma$ is equivalent to $i : BSO(n-1) \to BSO(n)$ so we have the long exact sequence

$$\cdots \longrightarrow H^{i-1}BSO(n-1) \longrightarrow H^iTSO(n) \xrightarrow{p^*} H^iBSO(n) \xrightarrow{i^*} H^iBSO(n-1) \longrightarrow \cdots$$

The even and odd cases differ:

$$\begin{aligned} H^*BSO(2k) &= P\{P_1, \ldots, P_{k-1}, \chi\} \\ H^*BSO(2k+1) &= P\{P_1, \ldots, P_k\} \end{aligned}$$

When $n = 2k+1$, $2u_\gamma^2 = 0$. Since 2 is invertible in Λ, $u_\gamma^2 = 0$. Thus $\Phi^{-1}(u_\gamma^2) = 0$. This is as desired, since $\chi = 0$ in $H^*BSO(2k+1)$.

When $n = 2k$ we have the exact sequence

$$0 = H^{2k-1}BSO(n-1) \longrightarrow H^{2k}TSO(n) \xrightarrow{p^*} H^{2k}BSO(n) \xrightarrow{i^*} H^{2k}BSO(n-1).$$

The kernel of i^* is generated by χ so $p^*u_\gamma = \pm\chi$. Choose u_γ so that $p^*u_\gamma = \chi$. By the remark following Definition 3.2, this implies $\Phi(x) = u_\gamma^2$. □

Note that this determines a choice of canonical orientation for $SO(2k)$-bundles. Further properties of the integral Euler class χ will be developed in the next chapter.

4. The Thom isomorphism in generalized cohomology theories

The Thom isomorphism is true in a more general setting. To show this, we first express the Thom isomorphism in a more geometric form, following [**10**]. There are three ingredients:

- a ring spectrum R,
- an R-oriented n-plane bundle $\xi : E \to B$, and
- the Thom diagonal $\Delta_T : T\xi \longrightarrow T\xi \wedge B_+$.

The *Thom diagonal* expresses $T\xi$ as a comodule over B_+ as follows. The map $S\xi \longrightarrow B$ from the sphere bundle to B is homotopy equivalent to the inclusion $S\xi \longrightarrow D\xi$ of the sphere bundle into the disk bundle. The Thom diagonal is then given by the middle row in the following diagram. The vertical maps are the collapse maps $p : D\xi_+ \longrightarrow D\xi_+/S\xi_+ = D\xi/S\xi = T\xi$. It is simple to verify that $(p \wedge 1)\Delta$ factors through p, inducing Δ_T, and that composing with $1 \wedge p$ gives the diagonal map for $D\xi/S\xi$.

$$\begin{array}{ccc} D\xi_+ & \xrightarrow{\Delta} & D\xi_+ \wedge D\xi_+ \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p\wedge 1} \\ D\xi/S\xi & \xrightarrow{\Delta_T} & (D\xi/S\xi) \wedge D\xi_+ \\ \| & & \downarrow{\scriptstyle 1\wedge p} \\ D\xi/S\xi & \xrightarrow{\Delta} & (D\xi/S\xi) \wedge (D\xi/S\xi) \end{array}$$

Replacing $D\xi$ by B and $D\xi/S\xi$ by $T\xi$, this takes the more pleasing form:

$$\begin{array}{ccc} B_+ & \xrightarrow{\Delta} & B_+ \wedge B_+ \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p\wedge 1} \\ T\xi & \xrightarrow{\Delta_T} & T\xi \wedge B_+ \\ \| & & \downarrow{\scriptstyle 1\wedge p} \\ T\xi & \xrightarrow{\Delta} & T\xi \wedge T\xi \end{array}$$

An *R-orientation* of ξ is a cohomology class $u \in R^nT\xi$ which restricts to a generator of $R^*T\epsilon_{n,b} = R^*S^n$ over each point b of the base. Here $\epsilon_{n,b}$ is the (necessarily trivial) n-plane bundle over $b \in B$. See [**1**] and [**13**].

If R is a ring spectrum with multiplication $\mu_R : R \wedge R \longrightarrow R$, consider the composition

$$R \wedge T(\xi) \xrightarrow{1\wedge\Delta_T} R \wedge T(\xi) \wedge B_+ \xrightarrow{1\wedge u\wedge 1} R \wedge \Sigma^n R \wedge B_+ \xrightarrow{\mu_R\wedge 1} R \wedge \Sigma^n B_+.$$

By construction, applying $\pi_*(-)$ to this composite yields the homological Thom isomorphism

$$R_*T\xi \xrightarrow{\cong} R_*\Sigma^n B_+,$$

and this implies that the composite induces an equivalence $R \wedge T\xi \xrightarrow{\simeq} R \wedge \Sigma^n B_+$. Since this is an equivalence of R-modules, we also get an equivalence of function spectra

$$F(T\xi, R) \simeq F_R(R \wedge T\xi, R) \xleftarrow{\simeq} F_R(R \wedge \Sigma^n B_+, R) \simeq F(\Sigma^n B_+, R).$$

Here F_R denotes the function spectrum of R-module maps. Taking homotopy of this yields the cohomological version of the Thom isomorphism,

$$R^*T\xi \overset{\cong}{\longleftarrow} R^*\Sigma^n B_+.$$

Observe that the Thom diagonal (together with the Künneth map) induces the cup product

$$\begin{array}{ccc} R^*T\xi \otimes R^*B_+ & \xrightarrow{\ \cup\ } & R^*T\xi \\ \Big\downarrow{\scriptstyle 1\otimes\xi^*} & & \Big\| \\ R^*D\xi/S\xi \otimes R^*D\xi_+ & \xrightarrow{\ \cup\ } & R^*D\xi/S\xi \end{array}$$

Using this product, we may express the Thom isomorphism $\Phi : R^kB_+ \longrightarrow R^{k+n}T\xi$ as $\Phi(x) = \xi^*(x)u_\xi$. (WARNING: ORDER ERROR. NOT TRIVIAL TO FIX)

The method just used to define Stiefel-Whitney classes and the Euler class can now be used in other situations. Let $\psi : R^* \to R^*$ be a cohomology operation of degree k (i.e., $\psi : R^i \to R^{i+k}$). Assume given a bundle $\xi : E \to B$ with an R-orientation $u_\xi \in R^nT_\xi$ with associated Thom isomorphism $\Phi : R^iB_+ \to R^{i+n}T_\xi$ given by $\Phi(x) = \xi^*(x)u_\xi$.

The class $c_\psi(\xi) = \Phi^{-1}\psi(u_\xi) \in R^kB_+$ is a characteristic class of the bundle ξ, characterized by the equation

$$\xi^*(c_\psi(\xi))u_\xi = \psi(u_\xi).$$

The naturality of ψ and Φ implies the naturality of c_ψ. Here are our two examples:

(i) $R^* = H^*(-;\mathbb{F}_2)$, $\psi = \mathrm{Sq}^i$ and our orientation is the unique $\mathbb{F}_2$-orientation of an $O(n)$-bundle (so all bundle maps are orientation preserving). Then $c_\psi = w_i$, the i^{th} Stiefel-Whitney class.

(ii) $R^* = H^*(-;\mathbb{F}_2)$, $\psi : H^n \to H^{2n}$ by $\psi(x) = x^2$ and our orientation is the ordinary orientation of a vector bundle. Then $c_\psi = \chi$, the Euler class. Of course this example works equally well in the general situation: if $u_\xi \in R^nT_\xi$ is an R-orientation of ξ define $\chi_R(\xi)$ to be $\Phi^{-1}(u_\xi^2)$. The remark following Definition 3.2 applies here also to show that $\chi_R(\xi) = (\xi^*)^{-1}p^*(u_\xi)$, $p : D_\xi \to T_\xi$ the natural map.

We have noted that the Thom space is an invariant of the fiber homotopy type of a bundle. Theorems 3.1 and 3.3 therefore show that the Stiefel-Whitney (and Euler) classes of a bundle depend only on the (oriented) fiber homotopy type of the bundle. This is of particular importance in the study of manifolds.

5. The homology tangent bundle of topological manifolds

Let M be a smooth manifold. We define characteristic classes of M by use of the tangent bundle τ of M. Thus we can speak of the Stiefel-Whitney classes of M ($w_i(M) = w_i(\tau)$), the Euler class of M if M is oriented ($\chi(M) = \chi(\tau)$), etc. Now it turns out that any *topological* manifold has a "homology tangent bundle" for which we can define a Thom isomorphism in the presence of an orientation. Thus we can define Stiefel-Whitney (and Euler) classes for (oriented) manifolds using the formulas of Theorems 3.1 and 3.3. Further, if M has a smooth structure then the tangent bundle and the homology tangent bundle are equivalent in an appropriate sense. It follows that the new definition extends the previous one and that *the Stiefel-Whitney and Euler classes of a manifold M depend only upon the topological type of M*. We proceed to the detail.

Let M be a topological manifold without boundary (add an open collar if M has a boundary). Let $\Delta \subset M \times M$ be the diagonal $\{(m, m)\}$. Let

$$\begin{array}{c}(M \times M, M \times M - \Delta) \\ \big\downarrow \pi \\ M\end{array}$$

be projection on the first factor. Then π has fibers

$$\pi^{-1}(m) = (m \times M, m \times (M - m)) \cong (M, M - m).$$

If V is a coordinate neighborhood of m then $\{V, V - m\}$ is an excisive couple. Hence $H^*(M, M - m) \cong H^*(V, V - m) \cong H^*(\mathbb{R}^n, \mathbb{R}^n - 0)$. Further, π is locally trivial [Spanier, p.293]. If $m, m' \in M$ are in the same component of M then $M - m \cong M - m'$. Thus, if M is connected, π is a fiber bundle pair with fiber $(M, M - m)$ for any $m \in M$.

Definition 5.1. The map π is called the *homology tangent bundle* of M. A Λ-*orientation* of M is a class $u \in H^n(M \times M, M \times M - \Delta; \Lambda)$ such that $i_m^*(u)$ is a generator of $H^n(m \times M, m \times (M - m)) \cong H^n(\mathbb{R}^n, \mathbb{R}^n - 0)$ for the inclusion $i_m : (m \times M, m \times (M - m)) \subset (M \times M, M \times M - \Delta)$ of each fiber.

The proof of Theorem 1.3 goes through to give the following version of the Thom Isomorphism Theorem.

Theorem 5.2. *Let M be an n-manifold without boundary. The following conditions are equivalent:*

(1) M has a Λ-orientation $u \in H^n(M \times M, M \times M - \Delta; \Lambda)$
(2) there is a class $\overline{u} \in H^n(M \times M, M \times M - \Delta; \Lambda)$ such that $i_m^*(\overline{u})$ is a generator of $H^n(m \times M, m \times (M - m); \Lambda)$ for some m in each component of M.
(3) the local coefficient system $\mathcal{H}^n(M, M - m; \Lambda)$ is trivial.

If (1)-(3) hold then the homomorphism

$$\Phi : H^i(M; \Lambda) \longrightarrow H^{i+n}(M \times M, M \times M - \Delta; \Lambda)$$

defined by $\Phi(x) = \pi^*(x)u$ is an isomorphism, where $\pi^*(x) \in H^*(M \times M; \Lambda)$. Φ is natural with respect to orientation preserving maps.

Taking Theorem 3.1 and 3.3 as our cue we now define characteristic classes of M.

Definition 5.3. The i^{th} *Stiefel-Whitney class of* M, $w_i(M) \in H^i(M; \mathbb{F}_2)$ is $\Phi^{-1}\mathrm{Sq}^i u$ where u is the unique $\mathbb{F}_2$-orientation of M. If (M, u) is an oriented manifold, $u \in H^n(M \times M, M \times M - \Delta)$, the *Euler class* χ *of* (M, u) is $\Phi^{-1}(u^2) \in H^n(M; \mathbb{Z})$.

It is easy to check that if we choose the opposite orientation $-u$ of M then the Euler class changes sign. Also, if n is odd then $2u^2 = 0$ so $2\chi = 0$. Since $H^n(M; \mathbb{Z})$ is torsion free, this implies that $\chi = 0$.

It remains to prove that these definitions coincide with our earlier definitions when M has a smooth structure. Let M be a smooth n-manifold with tangent bundle $\tau : TM \to M$. Choose a Euclidean metric $\mu : TM \to \mathbb{R}^+$ for τ (a Riemannian metric on M). The metric provides us with an exponential map defined on a neighborhood of the zero cross section of TM as follows. For $v \in TM$, let $g : \mathbb{R} \to M$ be the unique geodesic with $g(0) = \tau(v)$ and $g'(0) = v$. Then if $\mu(v)$ is small enough so that $g(1)$ is defined, we let $\exp(v) = g(1)$. Recall that for each fiber $\tau^{-1}(x)$ there is a small disk $D_{x,\epsilon} = \{v \in \tau^{-1}(x) \mid \mu(v) \leq \epsilon\}$ such that $\exp|_{D_{x,\epsilon}}$ is a diffeomorphism. Let $\epsilon : M \to \mathbb{R}$ be a positive continuous function such that for each $\alpha \in M$, $\exp_{D_{\epsilon(x),x}}$ is a diffeomorphism. Let $D_\tau = \{v \in TM \mid \mu(v) \leq \epsilon(\tau(v))\}$ and let $S_\tau = \{v \in TM \mid \mu(v) = \epsilon(\tau(v))\}$. Then D_τ is the n-disk bundle and S_τ be the $(n-1)$-sphere bundle associated to τ. We define a map

$$\psi : (D_\tau, S_\tau) \longrightarrow (M \times M, M \times M - \Delta)$$

by

$$\psi(v) = (\tau(v), \exp(v)).$$

Since ψ maps fibers to fibers, it is a map of fibrations

$$\begin{array}{ccc}
(D^n, S^{n-1}) & \longrightarrow & (M, M - x) \\
\downarrow & & \downarrow \\
(D_\tau, S_\tau) & \xrightarrow{\psi} & (M \times M, M \times M - \Delta) \\
\downarrow \tau & & \downarrow \pi \\
M & = & M.
\end{array}$$

Now ψ restricted to a fiber is a homotopy equivalence so by the five lemma, ψ is a homotopy equivalence. Hence, orientations of the tangent bundle τ and the homology tangent bundle π correspond via ψ^* and we have a commutative diagram

$$\begin{array}{ccc}
 & H^i(M) & \\
\swarrow \Phi & & \searrow \Phi \\
H^{n+i}(M \times M, M \times M - \Delta) & \xrightarrow{\psi^*} & H^{n+i}(D_\tau, S_\tau),
\end{array}$$

if we choose orientations u and u_τ so that $u_\tau = \psi^*(u)$. Clearly this implies that the two definitions of Stiefel- Whitney and Euler classes coincide. Since the homology tangent bundle depends only upon the topological structure of M, it follows that the Stiefel-Whitney classes and the Euler class are topological invariants; they are independent of any possible smooth structures on M.

CHAPTER VI

Applications and Examples

We compute the characteristic classes of various bundles and apply them to the solution of typical problems.

The tangent bundle of a manifold M is denoted τ_m or τ.

The first space to consider is the sphere S^n. The usual imbedding $S^n \subset \mathbb{R}^{n+1}$ has trivial normal bundle. Hence $\tau_{S^n} \oplus \epsilon_1 = \epsilon_{n+1}$ where ϵ_i is the trivial bundle of dimension i. That is, τ_{S^n} is stably trivial.

Remark 0.1. $w(\tau_{S^n}) = 1$ and $P(\tau_{S^n}) = 1$.

Proof. Note that $w(\tau_{S^n}) = w(\tau_{S^n} \oplus \epsilon_1) = w(\epsilon_{n+1}) = 1$ and similarly for P. More generally, the Stiefel-Whitney and Pontrjagin classes of stably equivalent bundles are equal. $\square$

Since τ_{S^n} is orientable, $\chi(\tau_{S^n})$ is defined. For odd n, $\chi(\tau_{S^n})$ is clearly zero since H^*S^n has no elements of order 2. For even n, the following lemma will enable us to determine $\chi(S^n)$. It is proved by studying the pair $(M \times M, M \times M - \Delta)$ where Δ is the diagonal. Recall that the Euler characteristic of a manifold is defined by

$$\chi_M = \sum_{i \geq 0} (-1)^i \operatorname{rank} H^i(M).$$

Lemma 0.2. *If (M, u) is an oriented compact n-manifold with orientation class (fundamental class) $u \in H_n(M)$ then $\langle \chi(\tau_M), u \rangle = \chi_M$. If M is a compact n-manifold and $u \in H_n(M; \mathbb{Z}/2\mathbb{Z})$ is the unique $\mathbb{Z}/2\mathbb{Z}$-orientation class then $\langle w_n(\tau_M), u \rangle \equiv \chi_M$ mod 2.*

Proof. See [Spanier, p.348]. Recall that the mod 2 Euler class is w_n. $\square$

Now $\chi_{S^{2n}} = 2$ so $\chi(\tau_{S^{2n}}) = 2u^*$ where u^* is dual to the fundamental class $u \in H_{2n}(S^{2n})$. Thus χ distinguishes $\tau_{S^{2n}}$ from the trivial bundle. Note that χ is the only characteristic class we have defined which is not stably invariant. In fact $\chi(\tau \oplus \epsilon_1) = 0$. Of course we know that $\tau_{S^{2n}}$ is nontrivial for geometric reasons also. If $\tau_{S^{2n}}$ were trivial, it would have a nowhere zero cross section $\nu : S^{2n} \to \tau_{s^{2n}}$ which would allow us to define a homotopy between the identity map and the antipodal map by $h(x,t) = \cos(t)x + \sin(t)\nu(x)$. Since the antipodal map of S^{2n} has degree -1, this is impossible.

The next simplest spaces are the projective spaces $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$. To avoid duplication, let K denote $\mathbb{R}$ or $\mathbb{C}$ and let KP^n be the associated projective space. To handle τ_{KP^n} we must relate it to simpler bundles. Recall that $BO(1) = O(\infty)/(O(1) \times O(\infty)) = \mathbb{R}P^\infty$ and $BU(1) = U(\infty)/\left(U(1) \times U(\infty)\right) = \mathbb{C}P^\infty$. The universal $O(1)$ and $U(1)$ bundles are the Hopf bundles

$$\begin{array}{ccc} \dfrac{O(\infty)}{e \times O(\infty)} = S^\infty & \text{and} & \dfrac{U(\infty)}{e \times U(\infty)} = S^\infty \\ \Big\downarrow & & \Big\downarrow \\ \dfrac{O(\infty)}{O(1) \times O(\infty)} = \mathbb{R}P^\infty & & \dfrac{U(\infty)}{U(1) \times U(\infty)} = \mathbb{C}P^\infty. \end{array}$$

If we restrict to $KP^n \subset KP^\infty$ we obtain the Hopf bundles

$$\begin{array}{ccc} \dfrac{O(n+1)}{e \times O(n)} = S^n & \text{and} & \dfrac{U(n+1)}{e \times U(n)} = S^{2n-1} \\ \Big\downarrow & & \Big\downarrow \\ \dfrac{O(n+1)}{O(1) \times O(n)} = \mathbb{R}P^\infty & & \dfrac{U(n+1)}{U(1) \times U(n)} = \mathbb{C}P^\infty \end{array}$$

The associated line bundles

$$\begin{array}{ccc} S^n \times_{O(1)} \mathbb{R} & \text{and} & S^{2n-1} \times_{U(1)} \mathbb{C} \\ \Big\downarrow & & \Big\downarrow \\ \mathbb{R}P^n & & \mathbb{C}\mathrm{P}^n \end{array}$$

admit a description which is more convenient for our purposes. Let $E = \{([x], v) \in KP^n \times K^{n+1} \mid v \in [x]\}$ where we consider the point $[x] \in KP^n$ as a line in K^{n+1}. Let $\gamma_n : E \to KP^n$ be projection on the first factor. We then have obvious isomorphisms of bundles

$$\begin{array}{ccc} S^n \times_{O(1)} \mathbb{R} \xrightarrow{f} E & \text{and} & S^{2n-1} \times_{U(1)} \mathbb{C} \xrightarrow{f} E \\ \Big\downarrow \qquad\qquad \Big\downarrow \gamma_n & & \Big\downarrow \qquad\qquad \Big\downarrow \gamma_n \\ \mathbb{R}\mathrm{P}^n = \mathbb{R}\mathrm{P}^n & & \mathbb{C}\mathrm{P}^n = \mathbb{C}\mathrm{P}^n \end{array}$$

where $f([x, \lambda]) = ([x], x\lambda)$. By abuse we will also call γ_n the Hopf bundle over KP^n. Recall that $H^*\mathbb{C}P^\infty = H^*BU(1) = P\{c_1\}$ and $H^*(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z}) = H^*(BO(1); \mathbb{Z}/2\mathbb{Z}) = P\{w_1\}$. If $i : KP^n \subset KP^\infty$ is the standard inclusion then $H^*\mathbb{C}P^\infty = P\{x\}/(x^{n+1})$ where $x = i^*(c_1)$ and $H^*(\mathbb{R}\mathrm{P}^n; \mathbb{Z}/2\mathbb{Z}) = P\{x\}/(x^{n+1})$ where $x = i^*(w_1)$.

Lemma 0.3. *The total Chern class of* $\gamma_n : E \to \mathbb{C}\mathrm{P}^n$ *is* $1 + x$. *The total Stiefel-Whitney class of* $\gamma_n : E \to \mathbb{R}\mathrm{P}^n$ *is* $1+x$. *The total Pontrjagin class of* $\gamma_n : E \to \mathbb{R}\mathrm{P}^n$ *is* 1.

Proof. Since γ_n is the restriction of the universal bundle, these all follow immediately from the corresponding facts about the universal bundles over $BO(1)$ and $BU(1)$. □

Now γ_n is a subbundle of the trivial bundle $\epsilon_{n+1} : KP^n \times K^{n+1} \to KP^n$. Its orthogonal complement is denoted $\omega_n : \tilde{E} \to KP^n$. The total space $\tilde{E}$ may be described as $\{([x], v) \in KP^n \times K^{n+1} \mid v \text{ is orthogonal to } x\}$. By definition we have $\gamma_n \oplus \omega_n = \epsilon_{n+1}$. We can now give a convenient description of τ_{KP^n}.

Proposition 0.4. $\tau_{KP^n} \cong \mathrm{Hom}(\gamma_n, \omega_n)$.

Proof. Suppose $f \in \mathrm{Hom}(\gamma, \omega)$ is in the fiber over $[x] \in KP^n$. Then f is a K-linear map $\gamma_{[x]} \to \omega_{[x]}$ (the subscript $[x]$ means take the fiber over $[x]$). If $0 \neq y \in \gamma_{[x]} \subset K^{n+1}$ define a path in K^{n+1} by $\alpha(t) = y + tf(y)$. Letting $\pi : K^{n+1} - 0 \to KP^n$ be the projection, $\pi\alpha$ is a path in KP^n with $\pi\alpha(0) = [x]$. Then $(\pi\alpha)'(0)$ is independent of the choice of y. Thus we have a map $\mathrm{Hom}(\gamma, \omega) \to \tau_{KP^n}$. It is easy to verify that it is a bundle map and hence an isomorphism □

Corollary 0.5. $\tau_{KP^n} \oplus \epsilon_1 \cong \mathrm{Hom}(\gamma_n, \epsilon_{n+1}) = (n+1)\,\mathrm{Hom}(\gamma_n, \epsilon_1)$

Note that is ξ is a bundle and m an integer, then $m\xi$ denotes the sum $\xi \oplus \cdots \oplus \xi$ with m factors.

Proof. Since $\mathrm{Hom}(\gamma_n, \gamma_n) = \epsilon_1$ and $\gamma_n \oplus \omega_n = \epsilon_{n+1}$, we have

$$\begin{aligned}
\tau_{KP^n} \oplus \epsilon_1 &= \mathrm{Hom}(\gamma_n, \omega_n) \oplus \mathrm{Hom}(\gamma_n, \gamma_n)\\
&= \mathrm{Hom}(\gamma_n, \epsilon_{n_1})\\
&= \mathrm{Hom}(\gamma_n, (n+1)\epsilon_1)\\
&= (n+1)\,\mathrm{Hom}(\gamma_n, \epsilon_1)
\end{aligned}$$ □

Now, a quick glance at the effect on transition functions shows that the operation which sends a bundle ξ to its dual bundle $\mathrm{Hom}(\xi, \epsilon_1)$ corresponds to the homomorphism $O(n) \to O(n)$ or $U(n) \to U(n)$ which sends A to $(A^{-1})^t$. Note that the inverse enters in because $\mathrm{Hom}(-, K)$ is contravariant.

Lemma 0.6. *If ξ is an $O(n)$ bundle then $\mathrm{Hom}(\xi, \epsilon_1) = \xi$. If ξ is a $U(n)$ bundle then $\mathrm{Hom}(\xi, \epsilon_1) = \overline{\xi}$ where $\overline{\xi}$ denotes the conjugate bundle.*

Proof. In $O(n)$, $A^{-1} = A^t$ so $(A^{-1})^t = A$. In $U(n)$, $A^{-1} = \overline{A}^t$ so $(A^{-1})^t = \overline{A}$. □

Now we have a grasp on τ_{KP^n}. Briefly,

$$\begin{aligned}
\tau_{\mathbb{R}P^n} \oplus \epsilon_1 &= (n+1)\gamma_n \qquad (\gamma_n = \text{ Hopf } O(1) \text{ bundle on } \mathbb{R}P^n)\\
\tau_{\mathbb{C}P^n} \oplus \epsilon_1 &= (n+1)\overline{\gamma}_n \qquad (\gamma_n = \text{ Hopf } U(1) \text{ bundle on } \mathbb{C}P^n).
\end{aligned}$$

Proposition 0.7.

$$\begin{aligned}
c(\tau_{\mathbb{C}P^n}) &= (1-x)^{n+1} \in H^{**}\mathbb{C}P^n\\
w(\tau_{\mathbb{R}P^n}) &= (1+x)^{n+1} \in H^{**}(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})
\end{aligned}$$

Proof. By Corollary **??**, $c(\overline{\gamma}_n) = 1 - x$. Hence $c(\tau_{\mathbb{C}P^n}) = c(\tau_{\mathbb{C}P^n} \oplus \epsilon_1) = c((n=1)\overline{\gamma}_n) = (1-x)^{n+1}$. Similarly, $w(\tau_{\mathbb{R}P^n}) = w((n+1)\gamma_n) = (1+x)^{n+1}$ since $w(\gamma_n) = 1 + x$. □

We may also write Proposition 0.7 in the form

$$\begin{aligned}
c_i(\tau_{\mathbb{C}P^n}) &= \binom{n+1}{i}(-x)^i\\
w_i(\tau_{\mathbb{R}P^n}) &= \binom{n+1}{i} x^i.
\end{aligned}$$

Other characteristic classes of $\tau_{\mathbb{R}P^n}$ and $\tau_{\mathbb{C}P^n}$ can be computed from these. For example,

$$\chi((\tau_{\mathbb{C}P^n})^{\mathbb{R}}) = c_n(\tau_{\mathbb{C}P^n}) = \binom{n+1}{n}(-x)^n = (-1)^n(n+1)x^n.$$

For another example, let $y \in H^2(\mathbb{R}P^n;\mathbb{Z})$ be the unique class which reduces to x^2 in $H^2(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z})$. Thus $2y = 0$ and y^i is the unique class which reduces to x^{2i} mod 2. Now $P_i(\xi)$ reduces to $(w_{2i}(\xi))^2$ mod 2. Since $w_{2i}(\tau_{\mathbb{R}P^n}) = \binom{n+1}{2i}x^{2i}$, $P_i(\tau_{\mathbb{R}P^n})$ must be $\binom{n+1}{2i}y^{2i}$. This gives an example of a case in which $P(\xi\oplus\eta) \neq P(\xi)P(\eta)$. Recall that $P(\xi\oplus\eta)$ is congruent to $P(\xi)P(\eta)$ modulo elements of order 2. We have $P(\gamma_n) = 1$ (so $P(\gamma_n)^{n+1} = 1$), while $P((n+1)\gamma_n) = P(\tau_{\mathbb{R}P^n}) = 1 + \binom{n+1}{1}y^2 + \binom{n+1}{4}y^4 + \cdots$.

By Proposition 0.7 $w_1(\tau_{\mathbb{R}P^n}) = (n+1)x$, that is

$$w_1(\mathbb{R}P^n) = \begin{cases} 0 & n \text{ odd} \\ x & n \text{ even.} \end{cases}$$

Hence $\mathbb{R}P^{2n+1}$ is orientable and $\mathbb{R}P^{2n}$ is not. The Euler class $\chi(\tau_{\mathbb{R}P^{2n+1}})$ must be zero since the Euler characteristic of $\mathbb{R}P^{2n+1}$ is $1 + (-1)^{2n+1} = 0$.

We now turn to non-immersion and non-imbedding results. Let M^m and N^n be smooth manifolds and let $f : M \to N$ be an immersion. Then τ_M is a subbundle of $f^*\tau_N$. The normal bundles of f, ν_f, satisfies $\tau_M \oplus \nu_f = f^*\tau_N$. It may be obtained by choosing a Euclidean metric on τ_N (which then induces one on $f^*\tau_N$) and letting ν_f be the orthogonal complement to τ_M in $f^*\tau_N$. We may also take τ_f to be the quotient bundle $(f^*\tau_N)/\tau_M$. The important feature of ν_f is that

$$w(\tau_M)w(\nu_f) = f^*w(\tau_N).$$

The most important special case is when $N = \mathbb{R}^n$. Then τ_N is trivial so we have $\tau_M \oplus \nu_f = \epsilon_n$. If $g : M \to \mathbb{R}^k$ is also an immersion then $\tau_M \oplus \nu_g = \epsilon_k$. It follows that ν_f and ν_g are stably equivalent, for

$$\nu_f \oplus \epsilon_k = \nu_f \oplus \tau_M \oplus \nu_g = \epsilon_n \oplus \nu_g.$$

Hence we write ν_m for the normal bundle of any immersion $M \to \mathbb{R}^n$. Since $\tau_M \oplus \nu_M$ is trivial, we have

$$w(\tau_M)w(\nu_M) = 1.$$

Let us define $\overline{w}(\tau_M)$ to be $w(\nu_M) = w(\tau_M)^{-1}$. Then we have the inductive formulas

$$\overline{w}_0 = 1$$

$$\overline{w}_i = \sum_{j=1}^{i} w_j \overline{w}_{i-j}$$

where we have dropped a sign since we are working mod 2.

Theorem 0.8. *If M^m immerses in $\mathbb{R}^{m+k}$, then $\overline{w}_i(\tau_M) = 0$ for $i > k$.*

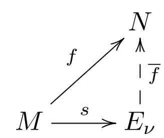
PROOF. $\overline{w}_i(\tau_M) = w_i(\nu_M)$. Since ν_M is a k dimensional bundle, $w_i(\nu_M) = 0$ for $i > k$. □

Corollary 0.9. *If M^m immerses in $\mathbb{R}^{m+1}$ then $w_i(\tau_M) = w_1(\tau_M)^i$.*

PROOF. By the above inductive formulas and the proposition, we have $\overline{w}_1 = w_1$ and, for $i > 1$, $w_i = w_{i_1}\overline{w}_1 = w_{i_1}w_1$. ☐

For imbeddingss, we can improve these results by using the *tubular neighborhood theorem*:

Theorem 0.10. *Let M be a manifold without boundary and let $f : M \to N$ be an imbedding with normal bundle ν. Let E_ν be the total space of ν and let $s : M \to E_\nu$ be the zero cross-section. Then we may extend f to an embedding $\overline{f} : E_\nu \to N$ such that $\overline{f}(E_\nu)$ is a neighborhood of $f(M)$*



PROOF. See [Liulvicus, *On Characteristic Classes*, 5.7] ☐

Note that if M were a manifold with boundary, the imbedding $\overline{f}$ would still exist by $\overline{f}(E_\nu)$ would no longer be a neighborhood of $f(M)$.

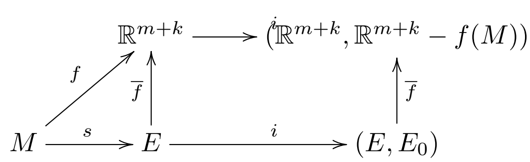
We apply this result in the following theorem.

Theorem 0.11. *Let M^m be a manifold without boundary which imbeds as a closed subset of $\mathbb{R}^{m+k}$ with normal bundle ν. Assume ν is orientable over the coefficient ring R. Then the Euler class $\chi(\nu)$ in R cohomology is 0. In particular, $\overline{w}_i(\tau_M) = w_i(\nu) = 0$ for $i \geq k$.*

PROOF. Let $f : M \to \mathbb{R}^{m+k}$ be the imbedding and let $s : M \to E$ be the zero cross-section of ν. Extend f to $\overline{f} : E \to \mathbb{R}^{m+k}$ by Theorem 0.10. Recall that if $u_\nu \in H^*(E, E_0; R)$ is an R-orientation of ν (where $E_0 = E - s(M)$ is the complement of the zero cross-section), then $\chi(\nu) = s^*i^*(u_\nu)$ where

$$H^*(E, E_0; R) \xrightarrow{i^*} H^*(E; R) \xrightarrow{s^*} H^*(M; R).$$

Consider the following diagram



Since $f(M)$ is closed and M has no boundary

$$\overline{f} : (E, E_0) \to (\mathbb{R}^{m+k}, \mathbb{R}^{m+k} - f(M))$$

is a cohomology isomorphism by excision. (Note that this would fail if either of these hypotheses were dropped). It follows that $u_\nu = \overline{f}^*(u)$ for some $u \in H^*(\mathbb{R}^{m+k}, \mathbb{R}^{m+k} - f(M); R)$. Then $\chi(\nu) = s^*i^*\overline{f}^*(u) = f^*i^*(u)$. But $i^*(u) = 0$ since $H^*(\mathbb{R}^{m+k}) = 0$. Hence $\chi(\nu) = 0$. The last statement follows since every bundle is $\mathbb{Z}/2\mathbb{Z}$-orientable and the mod 2 Euler class is the top Stiefel-Whitney class w_k. ☐

Note that the proof also shows that $\chi(\nu_f) = 0$ for an imbedding $f : M \to N$ (as a closed subset) if $H^k(N; R) = 0$ where k is the codimension of M in N.

Corollary 0.12. *If M^m embeds in $\mathbb{R}^{n+1}$ as a closed subset then the normal bundle is trivial, $\nu = \epsilon_1$. Hence τ_M is stably trivial: $\tau_M \oplus \epsilon_1 = \epsilon_{n+1}$.*

PROOF. The normal bundle n is a line bundle with $w_1(\nu) = 0$. Since $BO(1) = K(\mathbb{Z}/2\mathbb{Z}, 1)$, w_1 classifies line bundles. Thus, ν is trivial. □

The requirement that M be embedded as a closed subset is necessary. For example, let M be the total space of the canonical line bundle over $\mathbb{R}P^1 = S^1$. Then M is just the open Moebius band which can be imbedded in $\mathbb{R}^3$ (though not as a closed subset). The following proposition will enable us to calculate $w(\tau_M)$.

Proposition 0.13. *Let $\xi : E \to B$ be a smooth vector bundle (E and B are smooth manifolds and ξ is a smooth map). Then*

$$T_\xi \cong \xi^*(\tau_B) \oplus \xi^*(\xi).$$

PROOF. The tangent bundle τ_ξ splits into the sum of the bundle of vectors tangent to the fiber and the bundle of vectors orthogonal to the fiber. These bundles are isomorphism to $\xi^*(\xi)$ and $\xi^*(\tau_B)$ respectively. □

Let $\gamma : M \to S^1$ be the canonical line bundle as above. Now τ_{S^1} is trivial so $\gamma^*(T_{S^1}) = \epsilon_1$. Also γ^* is a homotopy equivalence and $w_1(\gamma) \neq 0$ so $w_1(\gamma^*(\gamma)) \neq 0$. Hence $\overline{w}_1(\tau_M) = w_1(\tau_M) \neq 0$. It follows that M cannot be imedded as a closed subset of $\mathbb{R}^3$.

We can use the preceding results to obtain information on the immersion and imbedding of projective spaces in Euclidean space. We have shown that $w(\tau_{\mathbb{R}\mathrm{P}^n}) = (1 + x)^{n+1}$. Choose k so that $n < 2^k \leq 2n$. Then

$$(1 + x)^{2^k} = 1 + x^{2^k} = 1.$$

Thus $(1 + x)^{n+1}(1 + x)^{2^k-n-1} = 1$. We conclude that

$$\overline{w}(\tau_{\mathbb{R}\mathrm{P}^n}) = (1 + x)^{2^k-n-1} \qquad (n < 2^k \leq 2n).$$

Proposition 0.14. *If $\mathbb{R}\mathrm{P}^n$ immerses in $\mathbb{R}^M$ then $m \geq 2^k - 1$. If $\mathbb{R}\mathrm{P}^n$ embeds in $\mathbb{R}^m$ then $m \geq 2^k$. ($n < 2^k \leq 2n$).*

PROOF. Since $\mathbb{R}\mathrm{P}^n$ is compact any embedding has closed image. Now $\overline{w}_{2^k-n-1}(\tau_{\mathbb{R}\mathrm{P}^n}) \neq 0$ so if $\mathbb{R}\mathrm{P}^n$ embeds in $\mathbb{R}^m$ then $2^k - n - 1 < m - n$ so $m \geq 2^k$. If $\mathbb{R}\mathrm{P}^n$ immerses in $\mathbb{R}\mathrm{P}^m$, then $2^k - n - 1 < m - n + 1$ so $m \geq 2^k - 1$. □

Remark 0.15.
(i) By Whitney's embedding and immersion theorems $\mathbb{R}\mathrm{P}^n$ embeds in $\mathbb{R}^{2n}$ and immerses in $\mathbb{R}^{2n-1}$. When $n = 2^{k-1}$ the proposition shows this is the best possible.
(ii) At the other extreme, if $\mathbb{R}\mathrm{P}^n$ immerses in $\mathbb{R}^{n+1}$ then $n \geq s^k - 2$ so either $n = 2^k - 2$ or $n = 2^k - 1$. If $\mathbb{R}\mathrm{P}^n$ embeds in $\mathbb{R}^{n+1}$ then $n \geq 2^k - 1$ so $n = 2^k - 1$.

Recall that a manifold is said to be parallelizable if its tangent bundle is trivial. This is equivalent to requiring the existence of n cross-sections of the tangent bundle, linearly independent at each point, where n is the dimension of the manifold.

Lemma 0.16. *If $\mathbb{R}\mathrm{P}^n$ is parallelizable then $n = 2^k - 1$.*

PROOF. If $\tau_{\mathbb{R}\mathrm{P}^n}$ is trivial, then $w(\tau_{\mathbb{R}\mathrm{P}^n}) = 1$. Let $n + 1 = 2^k m$ with m odd. Then $w(\tau_{\mathbb{R}\mathrm{P}^n}) = (1 + x)^{2^k m} = (1 + x^{2^k})^m = 1 + x^{2^k} + \cdots \neq 1$ if $m > 1$. Hence $m = 1$ and $n = 2^k - 1$. □

In fact $\mathbb{R}P^1$, $\mathbb{R}P^3$, and $\mathbb{R}P^7$ are the only projective spaces which are parallelizable.

The question of the existence of division algebras over $\mathbb{R}$ is related to the parallelizability of S^{n-1} and $\mathbb{R}P^{n-1}$. A division algebra of dimension n over $\mathbb{R}$ gives a nonsingular bilinear pairing $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, It is this which is easiest to work with.

Proposition 0.17. *If there is a nonsingular bilinear pairing $\mu : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ then $\mathbb{R}P^{n-1}$ is parallelizable and hence $n = 2^k$ for some k.*

PROOF. Write ab for $\mu(a, b)$. Let $\{E_1, \dots, e_n\}$ be a basis of $\mathbb{R}^n$. For $a \in \mathbb{R}^n$ the elements $ae_1, ae_2, \dots, ae_n$ are linearly independent. Let $v_i(ae_1)$ be the projection of ae_{i+1} on the plane perpendicular to ae_1. Then for $x \in S^{n-1}$, $\{v_1(x), \dots, v_{n-1}(x)\}$ is a basis for the tangent space of S^{n-1} at x. Thus $v_1, \dots, v_{n-1}$ are linearly independent cross sections of $\tau_{S^{n-1}}$. Now $v_i(-x) = -v_i(x)$ so the v_i also define $n-1$ linearly independent cross sections of the tangent bundle of $\mathbb{R}P^{n-1}$. Thus S^{n-1} and $\mathbb{R}P^{n-1}$ are parallelizable. By Lemma 0.16 this implies $n - 1 = 2^k - 1$. □

For $n = 1, 2, 4$, and 8 such pairings exist; they are the real numbers, the complex numbers, the quaternions, and the Cayley numbers. It can be shown that these are the only possibilities. The existence of a structure of division algebra on $\mathbb{R}^n$ is closely related to the existence of elements of Hopf invariant one in $\pi_{2n-1}(S^n)$. For references and a proof that $n = 1, 2, 4$ and 8, see [Adams, Hopf Invariant One, Annals v.72].

Regarding cross sections of general bundles, we have the following result.

Proposition 0.18. *If the n-plane bundle ξ has k linearly independent cross sections then*

$$w_{n-k+1} = w_{n-k+2} = \cdots w_n = 0.$$

PROOF. If ξ has k linearly independent cross sections then they span a trivial k-dimensional subbundle of ξ. Thus we can split ξ as $\epsilon_k \oplus \gamma$ where γ is an $(n-k)$-plane bundle. Then $w_i(\xi) = w_i(\gamma)$ which is zero for $i > n - k$. □

Let us apply this to the projective space $\mathbb{R}\mathrm{P}^n$.

Corollary 0.19. *Let $n + 1 = 2^r m$ with m odd. Then $\tau_{\mathbb{R}\mathrm{P}^n}$ has at most $2^r - 1$ linearly independent cross sections.*

PROOF. $w(\tau_{\mathbb{R}\mathrm{P}^n}) = (1+x)^{n+1} = (1+x)^{2^r m} = (1+x^{2^r})^m = 1 + \cdots + \binom{m}{1} x^{2^r(m-1)}$. Since m is odd $\binom{m}{1} \neq 0$ and hence $w_{2^r(m-1)} \neq 0$. If k is the maximum number of linearly independent cross sections then $n - k \geq 2^r(m-1)$ so $k \leq 2^r - 1$. □

Obviously there are perfectly analogous results for complex bundles and Chern classes or for symplectic bundles and symplectic classes.

The Euler class will sometimes give more information about the non-existence of cross sections. Recall the following result.

Proposition 0.20. *If $2\chi(\xi) \neq 0$ then ξ has no odd-dimensional subbundles. In particular, ξ has no nowhere zero cross sections.*

PROOF. If $\xi = \nu \oplus \eta$, then $\chi(\xi) = \chi(\nu)\chi(\eta)$. If ν is odd-dimensional then $2\chi(\nu) = 0$. □

If $\xi = \tau_{S^{2n}}$ then $w(\xi) = 1$ so the Stiefel Whitney classes give us no information. However, $2\chi(\xi) \neq 0$ so we know there cannot be a nowhere zero cross section.

Characteristic classes has important applications in the theory of cobordism. We will only sketch these briefly.

Let M be a smooth, compact n-manifold and let $[m] \in H_n(M; \mathbb{Z}/2\mathbb{Z})$ be the $\mathbb{Z}.2\mathbb{Z}$-fundamental class of M. For any partition of n, $n = j_1 + j_2 + \cdots + j_k$, consider the class $w_{j_1} w_{j_2} \ddots w_{j_k}$ in $H^n(M; \mathbb{Z}.2\mathbb{Z})$ where $w_i = w_i(\tau_M)$. The Kronecker product

$$\langle w_{j_1} w_{j_2} \dots w_{j_k}, [m] \rangle$$

is an element of $\mathbb{Z}/2\mathbb{Z}$. The *Stiefel Whitney characteristic numbers* of M are these elements of $\mathbb{Z}/2\mathbb{Z}$ defined for each partition of n. Similarly, if M is a smooth, compact, orientable $4m$ manifold with fundamental class $[m] \in H_{4m}(M; \mathbb{Q})$ then we have *rational Pontrjagin characteristic numbers*

$$\langle P_{j_1} P_{j_2} \dots P_{j_k}, [m] \rangle \in \mathbb{Q}$$

for each partition $j_1 + \cdots + j_k = m$. In the same manner we obtain the *Chern numbers* of a complex manifold. Characteristic numbers are important because they are invariants of cobordism. If M and N are smooth, compact n-manifolds (without boundary) then we say that M and N are (unoriented) *cobordant* if there exists an $(n+1)$-manifold with boundary, W, such that $\partial W = M \cup N$. If M and N are oriented then we say that M and N are *oriented cobordant* if there exists an oriented $(n+1)$ manifold with boundary, W, such that $\partial W = M \cup (-N)$, where $-N$ means N with the opposite orientation. The application of characteristic numbers in cobordism then comes from the following facts.

M and N are unoriented cobordant if and only if they have the same Stiefel Whitney numbers. M and N oriented cobordant if and only if they have the same Stiefel Whitney numbers and the same rational Pontrjagin numbers. For an exposition of these results see [Liulevicos, On Char classes] or [Milnor, Char classes].

APPENDIX A

Bott periodicity

1. Definition of the Maps

Our goal here is to prove the following theorem:

Theorem 1.1. *There exist maps of H-spaces, which are all weak homotopy equivalences (hence homotopy equivalences):*

$$BU \xrightarrow{\phi_0} \Omega SU$$
$$BO \xrightarrow{\phi_1} \Omega SU/SO \qquad U/O \xrightarrow{\phi_2} \Omega Sp/U$$
$$Sp/U \xrightarrow{\phi_3} \Omega Sp \qquad BSp \xrightarrow{\phi_4} \Omega SU/Sp$$
$$U/Sp \xrightarrow{\phi_5} \Omega SO/U \qquad SO/U \xrightarrow{\phi_6} \Omega Spin$$

Corollary 1.2. *$\pi_i(U) = \pi_{i+2}(U)$ for $i \geq 0$, and $\pi_0 = 0$, $\pi_1(U) = \mathbb{Z}$.*

Corollary 1.3. *If G is any of O, Sp, U/Sp, Sp/U, O/U, or U/O, then $\pi_i(G) = \pi_{i+8}(G)$ for all $i \geq 0$; if G is BO or BSp, then $\pi_i(G) = \pi_{i+8}(G)$ for $i > 0$. For $i \geq 0$,*

$$\begin{aligned}\pi_i(O) = \pi_{i+1}(BO) &\overset{\phi_1}{=} \pi_{i+2}(U/O) \overset{\phi_2}{=} \pi_{i+3}(Sp/U) \overset{\phi_3}{=} \pi_{i+4}(Sp)\\ &= \pi_{i+5}(BSp) \overset{\phi_4}{=} \pi_{i+6}(U/Sp) \overset{\phi_5}{=} \pi_{i+7}(O/U) \overset{\phi_6}{=} \pi_{i+8}(O).\end{aligned}$$

Further, $\pi_0(O) = \pi_1(O) = \mathbb{Z}/2$, $\pi_3(O) = \pi_7(O) = \mathbb{Z}$, and $\pi_2(O) = \pi_4(O) = \pi_5(O) = \pi_6(O) = 0$.

Proof of Corollary 1.3. $\pi_0(O) = \pi_1(O) = \mathbb{Z}/2$ is clear; $\pi_2(O) = \pi_0(U/Sp) = 0$; $\pi_3(O) = \pi_1(U/Sp) = \mathbb{Z}$. $\pi_i(O) = \pi_{i-4}(Sp) = 0$ for $4 \leq i \leq 6$, and $\pi_7(O) = \pi_1(U/O) = \mathbb{Z}$, by $O \longrightarrow U \longrightarrow U/O$. $\square$

The method of proof is to produce maps ϕ_i which induce isomorphisms on integral homology. When there is no torsion, we will work directly with integral homology or cohomology. In the other cases, we shall prove that ϕ_i induces an isomorphism on mod p homology or cohomology for all primes p and shall then be able to conclude that ϕ_i induces an isomorphism on integral homology by means of the following lemma.

Lemma 1.4. *Let $f : X \longrightarrow Y$ be a continuous map between connected spaces having integral homology of finite type. If either f_* or f^* is an isomorphism for mod p coefficients and all primes p, then $f_* : H_*(X;\mathbb{Z}) \longrightarrow H_*(Y;\mathbb{Z})$ is also an isomorphism.*

Proof. Replacing Y by Mf, we may assume that f is an inclusion. By assumption, $H_*(Y,X;\mathbb{F}_p) = 0$ for all p, hence $H_*(Y,X;\mathbb{Z}) \otimes \mathbb{F}_p \subset H_*(Y,X;\mathbb{F}_p)$ is also zero and $H_*(Y,X;\mathbb{Z}) = 0$. The result follows. $\square$

We next show that the ϕ_i will induce isomorphisms on homotopy if they induce isomorphisms on integral homology. This conclusion follows from the fact that the ϕ_i are H-maps (the point being that not all of our spaces are simply connected).

Lemma 1.5. *Let $f : X \longrightarrow Y$ be a fibration and a map of connected H-spaces. Then each $\alpha \in f_*\pi_1(X, x_0)$ operates trivially on the homology of the fibre $F = f^{-1}(y_0)$, where x_0 and y_0 are the identities of X and Y.*

PROOF. Let $q : I \longrightarrow Y$ be a loop at y_0 such that $q = fr$ for a loop $r : I \longrightarrow X$ at x_0. For any $s : \Delta_n \longrightarrow F$, define $h(s) : I \times \Delta_n \longrightarrow X$ by $h(s)(t, u) = r(t)s(u)$. Then $h(s)$ satisfies the properties

(1) $h(s)(0, u) = x_0 s(u)$;
(2) $fh(s)(t, u) = q(t)y_0$;
(3) $h(s)(1 \times \delta_i) = h(\partial_i s)$, where $\delta_i : \Delta_{n-1} \longrightarrow \Delta_n$ is the inclusion of the i-th face;
(4) $h(s)(1, u) = x_0 s(u)$.

If we let S_nF denote the singular n-simplices of F, then the function $h_1 : S_nF \longrightarrow S_NF$, given by $h_1(s) = h(s)|_{1\times\Delta_n}$, induces the action of q on $H_n(F)$. Since h_1 is homotopic to the identity, this action is clearly trivial. □

Proposition 1.6. *Let $f : X \longrightarrow Y$ be a map of connected H-spaces which induces an isomorphism on integral homology. Then f is a homotopy equivalence.*

PROOF. Let $X' = \{(p, x) \mid x \in X, p : I \longrightarrow Y, p(1) = f(x)\}$, define $f' : X' \longrightarrow Y$ by $f'(p, x) = f(x)$, and define $(p, x)(p', x') = (pp', xx')$, where $(pp')(t) = p(t)p'(t)$. Then X is a deformation retract of X' and f' is a fibration and a map of H-spaces. Replacing X by X', we may assume that f is a fibration. Since X and Y are H-spaces, their fundamental groups are abelian, hence equal to their homology groups, hence equal to each other. By Lemma 1.5, $\pi_1(Y)$ operates trivially on $H_*(F)$, $F = f^{-1}(y_0)$. F is connected since $\pi_1(f)$ is an isomorphism and, in the integral Serre spectral sequence of f, $E^r = H_*(Y; H_*(F))$ and $E^2_{*,0} = H_*(Y)$. Since $f_* : H_*(X) \cong H_*(Y)$, $E^2_{*,0} = E^\infty_{*,0} = E^\infty_{*,*}$. By induction on q, $E^2_{p,q} = 0$ for all $q > 0$, hence $E^2_{0,q} = H_q(F) = 0$ for $q > 0$. In detail, if $E^2_{p,q} = 0$ for $q > 0$, then $E^2_{0,q+1} = 0$, since its elements are permanent cycles which cannot bound, and therefore $E^2_{p,q+1} = 0$ for all p, etc. Now $H_*(F) = H_0(F) = \mathbb{Z}$. Since F is an H-space, $\pi_1(F)$ is abelian, hence $\pi_1(F) = H_1(F) = 0$. By the Hurewicz theorem, $\pi_q(F) = 0$ for all q, and the result follows from the long exact homotopy sequence of the fibration f. □

With these technicalities out of the way, we proceed to the definition of the maps ϕ_i. In order to relate our various classical groups conveniently, we start with $Z = Z' \oplus Z''$, where Z' and Z'' are right inner product spaces of countable dimension over $\mathbb{H}$. Regarding Z as a direct sum of copies of $\mathbb{H}$, we may also define a structure of a left quaternionic inner product space on Z. The reader may check that this is independent of the implicit choice of basis, since left actions commute with right actions. Define

$$Y = \{z \in Z \mid zi = iz\}$$

and define

$$X = \{y \in Y \mid yj = jy\}.$$

The left inner product on Z restricts to a complex inner product on Y and to a real inner product on X. In fact, we have identifications of left inner product spaces:

$$\widetilde{Z} = Y_{\mathbb{H}} = \mathbb{H} \otimes_{\mathbb{C}} Y = X_{\mathbb{H}} = \mathbb{H} \otimes_{\mathbb{R}} X$$

and

$$Y = X_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} X.$$

where we write $\widetilde{Z}$ for Z regarded as a left inner product space.

Now, it is easy to check that

$$Sp(Z) \cap Sp(\widetilde{Z}) = \Big\{ T \in Sp(\widetilde{Z}) \mid T(X) \subset X \Big\}$$

using the fact that any $z \in Z$ can be written $z = \sum q_i x_i = \sum x_i q_i$ for elements $x_i \in X$ and $q_i \in \mathbb{H}$. We take the liberty of writing

$$O(X) = Sp(Z) \cap Sp(\tilde{Z})$$

since the two natural maps

$$\Big(X \xrightarrow{\alpha} X\Big) \longmapsto \mathbb{H} \otimes_{\mathbb{R}} X \xrightarrow{1 \otimes \alpha} \mathbb{H} \otimes_{\mathbb{R}} X$$

and

$$\Big(X \xrightarrow{\alpha} X\Big) \longmapsto X \otimes_{\mathbb{R}} \mathbb{H} \xrightarrow{\alpha \otimes 1} X \otimes_{\mathbb{R}} \mathbb{H}$$

are the same. Similarly

$$U(Y) = U(Z^{\mathbb{C}}) \cap Sp(\widetilde{Z}) = \{T \in Sp(\widetilde{Z}) \mid T(Y) \subset Y\}.$$

Thus, looking solely at right actions, we have the inclusions

$$Sp(Z') \times Sp(Z'') \subset Sp(Z) \subset U(Z^{\mathbb{C}}) \subset O(Z^{\mathbb{R}}), \tag{1.1}$$

and the intersection of this sequence with $Sp(\widetilde{Z})$ is

$$O(X') \times O(X'') \subset O(X) \subset U(Y) \subset Sp(\widetilde{Z}). \tag{1.2}$$

The last two inclusions can be viewed as resulting from tensoring up: given a map $\alpha : X \to X$ in $O(X)$, we get $\alpha_{\mathbb{C}} : X_{\mathbb{C}} \to X_{\mathbb{C}}$ in $U(Y)$, and similarly for $\beta : Y \to Y$ in $U(Y)$.

Take

$$\begin{aligned} BO(X) &= \frac{O(X)}{O(X') \times O(X'')} \\ BU(Y) &= \frac{U(Y)}{U(Y') \times U(Y'')}, \quad \text{and} \\ BSp(Z) &= \frac{Sp(Z)}{Sp(Z') \times Sp(Z'')}. \end{aligned}$$

Then the inclusion of (1.2) into (1.1) induces the further inclusions

$$BO(X) \subset BSp(Z), \quad U(Y)/O(X) \subset U(Z^{\mathbb{C}})/Sp(Z) \tag{1.3}$$

and

$$Sp(\widetilde{Z})/U(Y) \subset O(Z^{\mathbb{R}})/U(Z^{\mathbb{C}}), \quad Sp(\widetilde{Z}) \subset O(Z^{\mathbb{R}}). \tag{1.4}$$

Now, for $0 \leq \theta \leq \pi$, define $\alpha_i(\theta) : Z \longrightarrow Z$ by the formulas

$$\begin{aligned} \alpha_0(\theta)(z', z'') &= (z' e^{i\theta}, z'' e^{-i\theta}) \\ \alpha_1(\theta)(z', z'') &= \alpha_4(\theta)(z', z'') = (z' e^{i\theta/2}, z'' e^{-i\theta/2}) \\ \alpha_2(\theta)(z', z'') &= \alpha_5(\theta)(z', z'') = (z' e^{j\theta/2}, z'' e^{j\theta/2}) \\ \alpha_3(\theta)(z', z'') &= \alpha_6(\theta)(z', z'') = (z' e^{i\theta}, z'' e^{i\theta}) \end{aligned} \tag{1.5}$$

Using paths of length π, define $\widetilde{\phi}_i : O(Z^{\mathbb{R}}) \longrightarrow PSO(Z^{\mathbb{R}})$, $0 \leq i \leq 6$, by

$$\widetilde{\phi}_i(T)(\theta) = [T, \alpha_i(\theta)] = T\alpha_i(\theta)T^{-1}\alpha_i(\theta)^{-1}, 0 \leq \theta \leq \pi. \tag{1.6}$$

Since $\alpha_i(0) = \mathrm{id}$, the paths $\widetilde{\phi}_i$ all start at the identity of $SO(Z^{\mathbb{R}})$. If $i = 0, 3$, or 6, then $\alpha_i(\pi) = -\mathrm{id}$, and hence the paths $\widetilde{\phi}_i$ also end at the identity. Each $\widetilde{\phi}_i(T)(\theta)$ is a linear transformation of finite type, since T is, and is clearly a real isometry of determinant one. Thus, $\widetilde{\phi}_i(T)$ is a path in $SO(Z^{\mathbb{R}})$, and $\widetilde{\phi}_i : O(Z^{\mathbb{R}}) \longrightarrow PSO(Z^{\mathbb{R}})$ is continuous.

Proposition 1.7. *By restriction and passage to quotients, the maps $\widetilde{\phi}_i$ induce continuous functions, which are maps of H-spaces,*

$$\phi_0 : BU(Z) = \frac{U(Z)}{U(Z') \times U(Z'')} \longrightarrow \Omega SU(Z)$$

$$\phi_1 : BO(X) = \frac{O(X)}{O(X') \times O(X'')} \longrightarrow \Omega \frac{SU(Y)}{SO(X)}$$

$$\phi_2 : \frac{U(Y)}{O(X)} \longrightarrow \Omega \frac{Sp(\widetilde{Z})}{U(Y)}$$

$$\phi_3 : \frac{Sp(\widetilde{Z})}{U(Y)} \longrightarrow \Omega Sp(\widetilde{Z})$$

$$\phi_4 : BSp(Z) = \frac{Sp(Z)}{Sp(Z') \times Sp(Z'')} \longrightarrow \Omega \frac{SU(Z)}{Sp(Z)}$$

$$\phi_5 : \frac{U(Z)}{Sp(Z)} \longrightarrow \Omega \frac{SO(Z)}{U(Z)}$$

$$\phi_6 : \frac{SO(Z)}{U(Z)} \longrightarrow \Omega_0 SO(Z) \simeq \Omega Spin(Z),$$

where Ω_0 denotes the component of the loop space containing the trivial loop.

Proof. In each case, to prove that $\phi_i : G_i/H_i \longrightarrow \Omega G'_i/H'_i$ is well-defined, we must verify the following three points

(i) $\widetilde{\phi}_i(T)(\theta) \in G'_i$ if $T \in G_i$
(ii) $\widetilde{\phi}_i(T)(\theta) = 1$ if $T \in H_i$ (so that $[ST, \alpha_i(\theta)] = [S, \alpha_i(\theta)], S \in G_i$)
(iii) $\widetilde{\phi}_i(T)(\pi) \in H'_i$ for all $T \in G_i$.

Note that ϕ_1, ϕ_2, and ϕ_3 are obtained from ϕ_4, ϕ_5, and ϕ_6, respectively, by restricting to linear transformations which commute with left multiplication by quaternions. Therefore ϕ_1, ϕ_2, and ϕ_3 will be well-defined provided that ϕ_4, ϕ_5, and ϕ_6 are well-defined. Now, (i) is satisfied for ϕ_0 and ϕ_4 since, if $T \in U(Z)$, then $[T, \alpha_i(\theta)]$ is a composite of linear transformations of Z regarded as a complex vector space (since $\alpha_0(\theta)$ and $\alpha_4(\theta)$ involve only i and not j or k), and is easily seen to be isometric for the complex inner product. Condition (i) is trivially satisfied for ϕ_5 and ϕ_6.

Condition (ii) is satisfied for ϕ_0 and ϕ_4 since $\alpha_0(\theta)$ and $\alpha_4(\theta)$ are just scalar multiplications when restricted to Z' or to Z''. Condition (ii) is satisfied for ϕ_5 since $[T, \alpha_5(\theta)] = 1$ for $T \in Sp(Z)$. This follows from the fact that $\alpha_5(\theta)$ is right multiplication by a quaternionic scalar. Similarly, $[T, \alpha_6(\theta)] = 1$ for $T \in U(Z)$ since $\alpha_6(\theta)$ is right multiplication by a complex scalar, hence (ii) is satisfied for ϕ_6.

Condition (iii) is trivially satisfied for ϕ_0 and ϕ_6, since $\alpha_0(\pi) = 1 = \alpha_6(\pi)$. It remains to verify (iii) for ϕ_4 and ϕ_5. For ϕ_4, note that $\alpha_4(\pi)(z', z'') = (z'i, -z''i)$. Writing the components of $T \in Sp(Z)$ as T', T'', we must show that $[T, \alpha_4(\pi)] \in Sp(Z)$. Since $T^{-1} \in Sp(Z)$, it suffices to show that $\alpha_4(\pi)T\alpha_4(-\pi) \in Sp(Z)$. Computing,

$$\begin{aligned} \alpha_4(\pi)T\alpha_4(-\pi)(z', z'') &= \alpha_4(\pi)(T'(-z'i, z''i), T''(-z'i, z''i)) \\ &= (T'(-z'i, z''i)i, -T''(-z'i, z''i)i) \\ &= (T'(z', -z''), T''(-z', z'')), \end{aligned}$$

from which it is clear that $\alpha_4(\pi)T\alpha_4(-\pi)$ commutes with multiplication by j:

$$\alpha_4(\pi)T\alpha_4(-\pi)(z'j, z''j) = \alpha_4(\pi)T\alpha_4(-\pi)(z', z'')j.$$

Hence, $[T, \alpha_4(\pi)] \in Sp(Z)$. Finally, $\alpha_5(\pi)(z', z'') = (z'j, z''j)$. For $T \in U(Z)$ we must show that $[T, \alpha_5(\pi)] \in U(Z)$. We have the formula

$$\begin{aligned} \alpha_5(\pi)T\alpha_5(-\pi)(z', z'') &= -\alpha_5(\pi)(T'(z'j, z''j), T''(z'j, z''j)) \\ &= -(T'(z'j, z''j)j, T''(z'j, z''j)j). \end{aligned}$$

Since $T \in U(Z)$, it follows easily from this that

$$\alpha_5(\pi)T\alpha_5(-\pi)(z'i, z''i) = \alpha_5(\pi)T\alpha_5(-\pi)(z', z'')i.$$

This proves (iii) for ϕ_5 and completes the proof that the ϕ_i are well-defined.

We must prove that the ϕ_i are weak H-maps. We use the product induced from that on G'/H' on $\Omega G'/H'$ rather than the loop product. Let $\{e'_i \mid i \geq 1\}, \{e''_i \mid i \geq 1\}$ be orthonormal symplectic bases for Z' and Z''. Define $\mu : Z \longrightarrow Z$ by $\mu(e'_i) = e'_{2i-1}$ and $\mu(e''_i) = e''_{2i-1}$ and define $\nu : Z \longrightarrow Z$ by $\nu(e'_i) = e'_{2i}$ and $\nu(e''_i) = e''_{2i}$. We can use μ and ν to define maps $O(Z^{\mathbb{R}}) \longrightarrow O(Z^{\mathbb{R}})$ by

$$\mu(T)(e_{2i}) = e_{2i} \qquad \nu(T)(e_{2i}) = \nu(T(e_i))$$
$$\mu(T)(e_{2i-1}) = \mu(T(e_i)) \qquad \nu(T)(e_{2i-1}) = e_{2i-1}$$

so that the diagrams

$$\begin{array}{ccc} Z^{\mathbb{R}} & \xrightarrow{T} & Z^{\mathbb{R}} \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} \\ Z^{\mathbb{R}} & \xrightarrow{\mu(T)} & Z^{\mathbb{R}} \end{array} \qquad \begin{array}{ccc} Z^{\mathbb{R}} & \xrightarrow{T} & Z^{\mathbb{R}} \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \nu} \\ Z^{\mathbb{R}} & \xrightarrow{\nu(T)} & Z^{\mathbb{R}} \end{array}$$

will commute. The products on both the G/H and $\Omega(G'/H')$ are all induced from $\phi : O(Z) \times O(Z) \longrightarrow O(Z)$ defined by $\phi(S,T) = \mu(S) \cdot \nu(T)$ by restriction and projection. To see this, it suffices to show that $\widetilde{\phi}_i(ST)(\theta) = \widetilde{\phi}_i(S)(\theta) \cdot \widetilde{\phi}_i(T)(\theta)$ for all S, T, and θ

$$[\mu(S)\nu(T), \alpha_i(\theta)] = \mu[S, \alpha_i(\theta)] \cdot \nu[T, \alpha_i(\theta)].$$

Visibly, $\alpha_i(\theta) = \mu(\alpha_i(\theta))\nu(\alpha_i(\theta))$ and $\mu(S)\nu(T) = \nu(T)\mu(S)$ for any S and T. Using these facts, we obtain the desired relation by a simple expansion of the relevant commutators. □

Remark 1.8. This technique of separating even and odd coordinates is also used to show that the space of linear isometries $\mathcal{L}(V, \mathbb{R}^\infty) \simeq *$.

2. Commutative Diagrams in the ϕ_i

Observe that ϕ_0 is defined for Y and that ϕ_2, ϕ_3, ϕ_5, and ϕ_6 may be defined using Z' or Z'' instead of Z (since classifying spaces are not involved). We take Z'' to be a copy of Z', $Z = Z' \oplus Z'$, for convenience. We write μ for the inclusions of (1.1) and any maps induced from them and ν for the inclusions of (1.2) and any maps induced from them. Thus we have from (1.1) and (1.2) the commutative diagram

$$\begin{array}{ccccc} Sp(Z) & \xrightarrow{\mu} & U(Z) & \xrightarrow{\mu} & SO(Z) \\ \uparrow & & \uparrow & & \uparrow \\ O(X) & \xrightarrow{\nu} & U(Y) & \xrightarrow{\nu} & Sp(\widetilde{Z}) \end{array} \tag{2.1}$$

For notational convenience, we write $U(Z)$ and $SO(Z)$ rather than $U(Z^{\mathbb{C}})$ and $SO(Z^{\mathbb{R}})$. Let $U'(Z')$ denote those elements of $SO(Z')$ which commute with right multiplication by j, rather than i (i.e. use the inclusion $\mathbb{C} \subset \mathbb{H}$ via $i \mapsto j$ instead of the usual $i \mapsto i$), and $U'(Y') = U'(Z') \cap Sp(\widetilde{Z'})$. We must next define maps μ' and ν' so as to have diagrams

$$\begin{array}{ccccc} Sp(Z') & \xrightarrow{\mu} & U'(Z') & \xrightarrow{\nu'} & Sp(Z) & & \\ & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\mu} & & \\ & & SO(Z') & \xrightarrow{\nu'} & SU(Z) & \xrightarrow{\mu} & SO(Z) \end{array} \tag{2.2}$$

and

$$\begin{array}{ccccc} O(X') & \xrightarrow{\nu} & U'(Y') & \xrightarrow{\mu'} & SO(X) & & \\ & & \downarrow{\scriptstyle\nu} & & \downarrow{\scriptstyle\nu} & & \\ & & Sp(\widetilde{Z}') & \xrightarrow{\mu'} & SU(Y) & \xrightarrow{\nu} & Sp(\widetilde{Z}). \end{array} \tag{2.3}$$

Here (2.3) will be obtained from (2.2) by restriction to transformations which commute with left multiplication by quaternions. Thus to obtain (2.2) and (2.3), it suffices to define $\nu' : SO(Z') \longrightarrow SU(Z)$ such that $\nu'(U'(Z')) \subset Sp(Z)$. For $T : Z' \longrightarrow Z'$ in $SO(Z')$, define $\nu'(T) = \tau(T,T)\tau^{-1}$, where $\tau : Z \longrightarrow Z$ is given by

$$\tau(z', z'') = \frac{1}{\sqrt{2}}\,(z' + z''i,\ (z' - z''i)j)\,, \tag{2.4}$$

so that

$$\tau^{-1}(z', z'') = \frac{1}{\sqrt{2}}\,(z' - z''j,\ -(z' + z''j)i)\,.$$

Equivalently,

$$\begin{aligned} \nu'(T)(z', z'') &= \frac{1}{2}(T(z') - T(z''j) - T(z'i)i + T(z''k)i, \\ &\qquad T(z')j - T(z''j)j + T(z'i)k - T(z''k)k). \end{aligned} \tag{2.5}$$

Careful inspection of formula (2.5) reveals $\nu'(T)(z'i, z''i) = \nu'(T)(z', z'')i$; and if $T(z'j) = T(z')j$, then $\nu'(T)(z'j, z''j) = \nu'(T)(z', z'')j$. For $T \in SO(Z')$, $\nu'(T)$ is easily verified to be a complex isometry of determinant one and this gives diagrams (2.2) and (2.3). Note that if $T \in U(Z')$, so that $T(z'i) = T(z')i$, then (2.5) reduces to

$$\nu'(T)(z', z'') = (T(z'), -T(z''j)j). \tag{2.6}$$

The maps ν' and μ' are equivalent to the standard inclusions and induce the same maps on homology and cohomology. In fact, if Z is given a new quaternionic structure $\overline{Z}$ in such a manner that $\tau : \overline{Z} \longrightarrow Z$ is symplectic, then $\overline{Z} = Z' \otimes_{\mathbb{C}'} \mathbb{H}$ where $\mathbb{C}' = \{a + bj \,|\, a, b \in \mathbb{R}\}$ and $\overline{Z}^{\mathbb{C}} = Z' \otimes_{\mathbb{R}} \mathbb{C}$ ($\mathbb{C} = \{a + bi\}$), and ν' is just symplectification (on $U'(Z')$) or complexification (on $SO(Z')$) followed by conjugation by τ.

We shall also need the map $\lambda : U(Z) \longrightarrow U(Z)$ defined by

$$\lambda(T)(z) = [T, j_r](z) = -T(T^{-1}(zj)j), \tag{2.7}$$

where j_r denotes right multiplication by j.

Proposition 2.8. *The following diagrams are either commutative or homotopy commutative.*

$$\begin{array}{ccc}
\dfrac{U'(Y')}{O(X')} & \xrightarrow{\phi_2'} & \Omega\dfrac{Sp(\widetilde{Z}')}{U'(Y')} \\
\Big\downarrow{\scriptstyle \mu'} & \Delta_2 & \Big\downarrow{\scriptstyle \Omega(\mu')} \\
BO(X) & \xrightarrow{\phi_1} & \Omega\dfrac{SU(Y)}{SO(X)} \\
\Big\downarrow{\scriptstyle \nu} & \Delta_1 & \Big\downarrow{\scriptstyle \Omega(\lambda)} \\
BU(Y) & \xrightarrow{\phi_0} & \Omega SU(Y) \\
\Big\uparrow{\scriptstyle \mu'} & \Delta_3 & \Big\uparrow{\scriptstyle \Omega(\mu')} \\
\dfrac{Sp(\widetilde{Z}')}{U(Y')} & \xrightarrow{\phi_3} & \Omega Sp(\widetilde{Z}')
\end{array}
\qquad\qquad
\begin{array}{ccc}
\dfrac{U'(Z')}{Sp(Z')} & \xrightarrow{\phi_5'} & \Omega\dfrac{SO(Z')}{U'(Z')} \\
\Big\downarrow{\scriptstyle \nu'} & \Delta_5 & \Big\downarrow{\scriptstyle \Omega(\nu')} \\
BSp(Z) & \xrightarrow{\phi_4} & \Omega\dfrac{SU(Z)}{Sp(Z)} \\
\Big\downarrow{\scriptstyle \mu} & \Delta_4 & \Big\downarrow{\scriptstyle \Omega(\lambda)} \\
BU(Z) & \xrightarrow{\phi_0} & \Omega SU(Z) \\
\Big\uparrow{\scriptstyle \nu'} & \Delta_6 & \Big\uparrow{\scriptstyle \Omega(\nu')} \\
\dfrac{SO(Z')}{U(Z')} & \xrightarrow{\phi_6} & \Omega Spin(Z')
\end{array}$$

Proof. The left-hand diagrams are the restrictions of the right-hand diagrams to transformations which commute with left multiplication by quaternions, hence it suffices to prove that Δ_4, Δ_5, and Δ_6 either commute or homotopy commute.

(i) *The diagram* Δ_6 *is commutative:*
Proof: First, ν' is well-defined here in view of formula (2.6). By the definitions,

$$\begin{array}{rcl} \Omega(\nu')\phi_6(T)(\theta) & = & \nu'[T,\alpha_6(\theta)] \\ & = & \tau([T,\alpha_6(\theta)],[T,\alpha_6(\theta)])\tau^{-1} \\ & = & \tau[(T,T),(\alpha_6(\theta),\alpha_6(\theta))]\tau^{-1} \\ & = & [\tau(T,T)\tau^{-1},\tau(\alpha_6(\theta),\alpha_6(\theta))\tau^{-1}], \\ \phi_0\nu'(T)(\theta) & = & [\nu'(T),\alpha_0(\theta)] = [\tau(T,T)\tau^{-1},\alpha_0(\theta)]. \end{array}$$

$\alpha_6(\theta): Z' \longrightarrow Z'$ is right multiplication by $e^{i\theta}$, hence commutes with right multiplication by i. By (2.6), $\tau(\alpha_6(\theta),\alpha_6(\theta))\tau^{-1} = (\alpha_6(\theta), -j_r\alpha_6(\theta)j_r)$. Since $e^{i\theta}j = je^{-i\theta}$, $\tau(\alpha_6(\theta),\alpha_6(\theta))\tau^{-1} = (e_r^{i\theta}, e_r^{-i\theta}) = \alpha_0(\theta)$, as desired.
(ii) *The diagram* Δ_5 *is commutative:*
Proof: ϕ_5' is defined from $\alpha_5'(\theta) = e_r^{i\theta/2} : Z' \longrightarrow Z'$, since the roles of i and j must be reversed by the definition of $U'(Z')$. Here ν' is again well-defined in virtue of (2.6). Precisely, as in the previous proof,

$$\begin{array}{rcl} \tau(\alpha_5'(\theta),\alpha_5'(\theta))\tau^{-1} & = & (e_r^{i\theta/2}, -j_r e^{i\theta/2} j_r) \\ & = & (e_r^{i\theta/2}, e_r^{-i\theta/2}) = \alpha_4(\theta), \end{array}$$

and

$$\begin{array}{rcl} \Omega(\nu')\phi_5'(T)(\theta) & = & [\tau(T,T)\tau^{-1},\tau(\alpha_5'(\theta),\alpha_5'(\theta))\tau^{-1}] \\ & = & [\nu'(T),\alpha_4(\theta)] = \phi_4\nu'(T)(\theta). \end{array}$$

(iii) *The diagram* Δ_4 *is homotopy commutative:*
Proof: $\lambda(T) = [T, j_r] = 1$ if $T \in Sp(Z)$, hence $\Omega(\lambda)$ is well-defined. Clearly $\phi_0\mu(T)(\theta) = [T,\alpha_0(\theta)] = [T,\alpha_4(2\theta)]$, since $\alpha_0(\theta) = \alpha_4(2\theta)$. $j_r\alpha_4(\theta) = \alpha_4(-\theta)j_r$;

$$\begin{array}{rcl} \Omega(\lambda)\phi_4(T)(\theta) & = & [[T,\alpha_4(\theta)],j_r] \\ & = & T\alpha_4(\theta)T^{-1}\alpha_4(-\theta)j_4\alpha_4(\theta)T\alpha_4(-\theta)T^{-1}j_r^{-1} \\ & = & T\alpha_4(\theta)T^{-1}\alpha_4(-2\theta)T\alpha_4(\theta)T^{-1} \end{array}$$

(for $T \in BSp(Z)$, $[T, j_r] = 1$). If $h : BSp(Z) \times I \longrightarrow \Omega SU(Z)$ is defined by the formula $h(T,t)(\theta) = T\alpha_4(\theta + t\theta)T^{-1}\alpha_4(-2\theta)T\alpha_4(\theta - t\theta)T^{-1}$, then h is a homotopy from $\Omega(\lambda)\phi_4$ to $\phi_0\mu$. This completes the proof of the proposition. □

We shall need certain auxiliary diagrams to study some of the ϕ_i. Let $P(G,H)$ denote the paths of length π in G which start at the identity and end in H. Let $\pi : G \longrightarrow G/H$ and let $p : PG \longrightarrow G$ and $p : P(G,H) \longrightarrow H$ denote the endpoint projection maps. We have $\Omega(G/H) = P(G/H, H/H)$ and thus a commutative diagram

$$\begin{array}{ccccc} P(G,H) & \longrightarrow & P(G) & \xrightarrow{\pi p} & G/H \\ \downarrow{\scriptstyle P(\pi)} & & \downarrow{\scriptstyle P(\pi)} & & \| \\ \Omega(G/H) & \longrightarrow & P(G/H) & \xrightarrow{p} & G/H \end{array}$$

$P(\pi) : P(G,H) \longrightarrow \Omega(G/H)$ is a weak homotopy equivalence, which, by abuse, we treat as an identification. Note that, by construction, ϕ_1, ϕ_2, ϕ_4, and ϕ_5 all factor through the relevant $P(\pi)$.

Lemma 2.9. *The following are commutative diagrams of Serre fibrations:*

$$\begin{CD}
\Omega SO(Z')/U(Z') @. \\
@| @. \\
P(SO(Z'),U(Z')) @>{p}>> U(Z') \\
@V{\subset}VV @VV{i}V \\
PSO(Z') @>{\nu' p}>> EU(Z) \\
@V{\pi p}VV @VV{\pi}V \\
SO(Z')/U'(Z') @>{\nu}>> BU(Z)
\end{CD}
\qquad\qquad
\begin{CD}
\Omega SU'(Y')/SO(X') @. \\
@| @. \\
P(SU'(Y'),SO(X')) @>{p}>> SO(X') \\
@V{\subset}VV @VV{i}V \\
PSU'(Y') @>{\mu' p}>> ESO(X) \\
@V{\pi p}VV @VV{\pi}V \\
SU'(Y')/SO'(X') @>{\mu}>> BSO(X)
\end{CD}$$

Denote the left diagram by Γ_1 and the right by Γ_2.

Proof. $EU(Z) = U(Z)/e \times U(Z')$ and $ESO(X) = SO(X)/e \times SO(X')$, while $i(T) = T \times 1$. The maps μ' and ν' are well-defined here in view of formula (2.6) above. Diagrams Γ_1 and Γ_2 commute since $\pi\nu' = \nu'\pi$ and $\pi\mu' = \mu'\pi$. If $F \in P(SO(Z'), U(Z'))$, then $\nu' p(f) = \nu' f(\pi) = (f(\pi), j_r f(\pi) j_r^{-1})$ by (2.6), while $ip(f) = (f(\pi), 1)$. These are equal in $EU(Z)$ since the second coordinate has no effect in

$$\frac{U(Z') \times U(Z')}{e \times U(Z')} \subset \frac{U(Z' \oplus Z')}{e \times U(Z')} = EU(Z).$$

The upper square of Γ_2 commutes for the same reasons. □

Lemma 2.10. *With λ and λ' induced from (2.7), we have commutative diagrams*

$$\begin{array}{ccc}
U(Z')/Sp(Z') & \xrightarrow{\phi_5} & \Omega SO(Z')/U(Z') \\
\uparrow{\scriptstyle \pi} & \overset{\lambda}{\searrow} & \downarrow{\scriptstyle p} \\
U(Z') & \xrightarrow{\lambda'} & U(Z')
\end{array}
\qquad\qquad
\begin{array}{ccc}
U(Y')/O(X') & \xrightarrow{\phi_2} & \Omega Sp(\widetilde{Z}')/U(Y') \\
\uparrow{\scriptstyle \pi} & \overset{\lambda}{\searrow} & \downarrow{\scriptstyle p} \\
U(Y') & \xrightarrow{\lambda'} & U(Y')
\end{array}$$

Write $H_*(U;Z) = E\{x_{2i-1} | i \geq 1\}$, $x_{2i-1} = \sigma^*(c_i)^*$. Then $\lambda'_*(x_{2i-1}) = [1 - (-1)^i]x_{2i-1}$, $i \geq 1$.

Proof. The right-hand diagram is the restriction of the left-hand diagram to transformations which commute with left-multiplication by quaternions, hence it suffices to prove the result for the left-hand diagram. Note that $\lambda(T) = p\phi_5(T) = [T, \alpha_5(\pi)] = [T, j_r]$, hence $\lambda'(T) = \lambda\pi(T) = [T, j_r] = T j_r T^{-1} j_r^{-1} = \phi(1 \times \alpha\gamma)\Delta(T)$, where Δ is the diagonal, $\alpha(T) = j_r T j_r^{-1}$, $\gamma(T) = T^{-1}$, and ϕ is the group product. Since $\phi(\gamma \times 1)\Delta(T) = 1$, $0 = \phi_*(\gamma_* \otimes 1)\Delta_*(x_i) = x_i + \gamma_*(x_i)$ and $\gamma_*(x_i) = -x_i$. α is an automorphism, hence $\alpha_*(x_i) = (-1)^{\varepsilon_i} x_i$. In $H^*(BU;Z)$, $(-1)^{\varepsilon_i} c_i = (B\alpha)^*(c_i)$, since $\sigma^*(B\alpha)^* = \alpha^*\sigma^*$, and $\psi(B\alpha)^* = (B\alpha)^* \otimes (B\alpha)^*\psi$, hence

$$(-1)^{\varepsilon_i} \sum_{j+k=i} c_j \otimes c_k = \sum_{j+k=i} (-1)^{\varepsilon_j + \varepsilon_k} c_j \otimes c_k.$$

Thus $\varepsilon_j + \varepsilon_k = \varepsilon_i$, and $\varepsilon_i = i\varepsilon_1$. To compute ε_1, observe that x_1 is the image of the fundamental class of S^1 under $i : S^1 = U(1) \longrightarrow U$, $i(e^{i\theta}) = e_r^{i\theta}$ on the first coordinate of $U = U(Z')$ and $i(e^{i\theta})$ is the identity on the remaining coordinates. Since $j_r e_r^{i\theta} j_r = e_r^{-i\theta}$, it is clear that $\alpha_*(x_1) = -x_1$, $\varepsilon_1 = 1$. Now the result follows since $\lambda'_*(x_i) = \phi_*(1 \otimes \alpha_*\gamma_*)\Delta_*(x_i) = [1 - (-1)^i]x_i$, because x_i is primitive. □

3. Proof of the Periodicity Theorem

We prove in turn that each ϕ_{i*} or ϕ_i^* is an isomorphism either integrally or mod p for all primes p. This will suffice.

Lemma 3.1. $\phi_{0*} : H_*(BU;\mathbb{Z}) \longrightarrow H_*(\Omega SU;\mathbb{Z})$ *is an isomorphism.*

Proof. $H^*(BU) = P\{c_i\}$, hence $H_*(BU) = P\{\gamma_i\}$, where $< c_1^i, \gamma^i >= 1$. Thus $H_*(BU) = P(j_* \widetilde{H}_*(BU(1)))$, $j : BU(1) \longrightarrow BU$.

Consider $\phi_0 : BU(Y) \longrightarrow \Omega SU(Y)$[1], $Y = Y' \oplus Y''$, where Y' and Y'' have complex orthonormal bases ...? □

4. Proof of Complex Periodicity

Before delving into the proof of real Bott periodicity, we first work out the simpler complex case. The complex case will serve as a guide to working through the proof of real Bott periodicity. We start with a bit of intuition behind the proof.

Using the Bott map $\psi_0 : U(n+1) \to \Omega SU(n+1)$, we first show the map out of the quotient $\psi_0 : \mathbb{C}\mathrm{P}^n \to \Omega SU(n+1)$ is well-defined and further show how to obtain a map $\Psi_0 : BU \to \Omega SU$. The effort is spent on the finite stages, working by induction one cell at a time. Specifically, we show that the image of $H_*\mathbb{C}\mathrm{P}^n$ under ψ_{0*} generates $H_*\Omega SU(n+1)$. This is done via induction using the cofibration $\Sigma\mathbb{C}\mathrm{P}^{n-1} \to \Sigma\mathbb{C}\mathrm{P}^n \to S^{2n+1}$. By keeping track of the effect on the top cell, we obtain the desired result. Finally, in the limit we note that the ranks of $H_*(BU)$ and $H_*(\Omega SU)$ are the same, with the map $\Psi_{0*} : H_*BU \to H_*\Omega SU$ being onto the aforementioned generators. This will complete the proof.

It should be noted that, by abuse of notation, we use a map homotopic to ψ_0 below. The only difference is in the map α_0: before we split $\mathbb{C}^{2n} \cong \mathbb{C}^n \bigoplus \mathbb{C}^n$ with $\alpha_0(u = e^{i\theta})(z', z'') = (z'e^{i\theta}, z''e^{-i\theta})$. Now, we're working with $\mathbb{C}^{n+1} \cong \mathbb{C}^n \bigoplus \mathbb{C}^1$, with $\alpha_0(u)(z', z'') = (z', uz'')$. Even though I'm being sloppy here, these are essentially the same map.

Proposition 4.1 (Douady)**.** *The following diagram is commutative, with the canonical maps χ and $\tilde{\chi}$, and ε is a homeomorphism.*

$$\begin{array}{ccccc}
\Sigma\mathbb{C}\mathrm{P}^{n-1} & \longrightarrow & \Sigma\mathbb{C}\mathrm{P}^{n} & \xrightarrow{\tilde{\chi}} & \Sigma(S^{2n}) \\
\downarrow{\scriptstyle \overline{\phi}_0} & & \downarrow{\scriptstyle \overline{\phi}_0} & & \downarrow{\scriptstyle \varepsilon} \\
SU(n) & \longrightarrow & SU(n+1) & \xrightarrow{\chi} & \frac{SU(n+1)}{SU(n)} \cong S^{2n+1}
\end{array}$$

Proof. The map χ is given by acting on the last factor:

$$\chi(y) = y(e_{n+1}).$$

If $\dot{x} \in \mathbb{C}\mathrm{P}^n$, let x denote reflection in the hyperplane perpendicular to $\dot{x}$. The composite $\chi\overline{\phi}_0(u, \dot{x}) = x\alpha_0(u)x^{-1}\alpha_0(u)^{-1}(e_{n+1})$, for $(u, \dot{x}) \in \Sigma\mathbb{C}\mathrm{P}^n$. Now

$$\alpha_0(u)(e) = \begin{cases} e & e \in \mathbb{C}^n \\ u \cdot e & e \in \mathbb{C}^1 \end{cases}$$

Therefore, if $\dot{x} \in \mathbb{C}\mathrm{P}^n$ is a line in $\mathbb{C}^{n+1}$, we have that $x\alpha_0(u)x^{-1}$ scales the line x by u, and fixes the orthogonal complement to x in $\mathbb{C}^{n+1}$.

---

[1] $\phi_0(T)(\theta) = [T, \alpha_0(\theta)] = T\alpha_0(\theta)T^{-1}\alpha_0(\theta)^{-1}$, $0 \le \theta \le \pi$, $\alpha_0(\theta)(y', y'') = (e^{i\theta}y', e^{-i\theta}y'')$

Choose a unit vector $\hat{x} \in \dot{x}$. Decompose the last basis vector

$$e_{n+1} = \langle e_{n+1}, \hat{x} \rangle \hat{x} + e_{n+1} - \langle e_{n+1}, \hat{x} \rangle \hat{x}$$

into its components in $\dot{x}$ and in $\dot{x}^{\perp}$, respectively. Now plugging into the above formula, we find

$$\begin{aligned} \chi\phi_0(u, \dot{x}) &= x\alpha_0(u)x^{-1}(e_n) \\ &= u\langle e_n, \hat{x} \rangle \hat{x} + e_n - \langle e_n, \hat{x} \rangle \hat{x} \\ &= e_n + (u-1)\langle e_n, \hat{x} \rangle \hat{x}. \end{aligned}$$

Therefore, $\chi\phi_0(u, \dot{x}) = e_n$, so that $(u, \dot{x})$ maps to the basepoint in $SU(n+1)/SU(n) \cong S^{2n+1}$, precisely when $u = 1$ or $\hat{x} \perp e_n$, i.e. $\hat{x} \in \mathbb{C}\mathrm{P}^{n-1}$. Therefore, the map $\varepsilon : \Sigma(\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^{n-1}) \to SU(n+1)/SU(n)$ is well defined.

Since the domain of ε is compact and the codomain is connected, it suffices to show its image is open. By invariance of domain, it suffices to prove ε is injective. To see this, suppose $\varepsilon(u, \dot{x}) = e \neq e_n$. Then $0 \neq e - e_n = (u-1)\langle e, e_n \rangle \hat{x}$. Hence, $e - e_n$ and x determine the same line in $\mathbb{C}^{n+1}$ and hence the same point in $\mathbb{C}\mathrm{P}^n$. Re-writing this, we find

$$u = 1 + \frac{e - e_n}{\langle e, e_n \rangle \hat{x}}.$$

Therefore, ε is injective and hence, a homeomorphism. □

Proposition 4.2. *$H_*(\Omega SU(n+1))$ is generated by $\phi_{0*} : H_*\mathbb{C}\mathrm{P}^n \to H_*(\Omega SU(n+1))$.*

Proof. We proceed by induction on n. Suppose $\phi_{0*} : H_*\mathbb{C}\mathrm{P}^{n-1} \to H_*\Omega SU(n)$ generates. By [Thm 3.1.3], we know that $H_*\Omega SU(n+1) \cong \mathbb{Z}[x_2, x_4, \ldots, x_{2n-2}, x_{2n}]$, and by induction, the generators $x_2, \ldots, x_{2n-2}$ are in the image of ϕ_{0*}. Consider applying homology to the previous diagram

$$\begin{array}{ccccc} H_*\mathbb{C}\mathrm{P}^{n-1} & \longrightarrow & H_*\mathbb{C}\mathrm{P}^n & \xrightarrow{\tilde{\chi}_*} & H_*(S^{2n}) \\ \downarrow{\scriptstyle \phi_{0*}} & & \downarrow{\scriptstyle \phi_{0*}} & & \downarrow{\scriptstyle \varepsilon_*} \\ H_*\Omega SU(n) & \longrightarrow & H_*\Omega SU(n+1) & \xrightarrow{\chi} & H_*(\Omega S^{2n+1}) \end{array}$$

The Wang sequence of the fibration □

5. *BSp*

The proof that $\Phi_4 : BSp \longrightarrow \Omega(SU/Sp)$ is an equivalence is almost identical to the proof of cpomplex periodicity.

First, we observe that there is a fiber sequence

$$\frac{SU(2n)}{Sp(n)} \longrightarrow \frac{SU(2n+2)}{Sp(n+1)} \xrightarrow{\chi} S^{4n+1}$$

by applying Verdier's axiom to the first map, written as the composite of two maps.

Naturality of ϕ_4 gives a commutative square

$$\begin{array}{ccc} \dfrac{Sp(n+1)}{Sp(n)\times Sp(1)} & \xrightarrow{\phi_4} & \Omega\left(\dfrac{SU(2n+2)}{Sp(n+1)}\right) \\ \downarrow & & \downarrow \\ \dfrac{Sp(n+m)}{Sp(n)\times Sp(m)} & \xrightarrow{\phi_4} & \Omega\left(\dfrac{SU(2n+2m)}{Sp(n+m)}\right) \end{array}$$

Precisely, if we write $\mathbb{H}^{n+1} = \mathbb{H}^n \oplus \mathbb{H} \cong \mathbb{C}^{2n} \oplus \mathbb{C}^2$ and similarly for $\mathbb{H}^{n+m}$, then $\alpha_4(\theta) : \mathbb{H}^{n+m} \longrightarrow \mathbb{H}^{n+m}$ is given by

$$\alpha_4(\theta)(z', z'') = (zie^{i\theta/2}, z''e^{-i\theta/2}).$$

Since $\phi_4(x)(\theta) = [x, \alpha_4(\theta)]$, the square commutes.

As in the complex case, we then have the following simple geometric lemma.

Proposition 5.1. *The following diagram is commutative, with the map χ as above and the canonical map $\tilde{\chi}$. The map ε is a homeomorphism.*

$$\begin{array}{ccccc} \Sigma\mathbb{H}\mathrm{P}^{n-1} & \longrightarrow & \Sigma\mathbb{H}\mathrm{P}^{n} & \xrightarrow{\tilde{\chi}} & \Sigma(S^{4n}) \\ \downarrow{\scriptstyle \overline{\phi}_0} & & \downarrow{\scriptstyle \overline{\phi}_0} & & \downarrow{\scriptstyle \varepsilon} \\ \dfrac{SU(2n)}{Sp(n)} & \longrightarrow & \dfrac{SU(2n+2)}{Sp(n+1)} & \xrightarrow{\chi} & S^{4n+1} \end{array}$$

PROOF. χ acts by evaluating on the last factor, $\chi(y) = y(e_{2n+2},$

□

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