Finite topological spaces

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Spaces

Hausdorff's original definition:

A topological space $(X, \mathscr{T})$ is a set X and
a set $\mathscr{T}$ of "open" subsets of X such that

(i) $\emptyset$ and X are in $\mathscr{T}$,

(ii) Any union of sets in $\mathscr{T}$ is in $\mathscr{T}$, and

(iii) Any finite intersection of sets in $\mathscr{T}$ is in $\mathscr{T}$.

$f\colon X \longrightarrow Y$ is continuous if $f^{-1}(U)$ is open in X when U is open in Y.

Alexandroff spaces

Alexandroff space:
ANY intersection of open sets is open.

Any finite space is an Alexandroff space.

T_0-space: topology distinguishes points:
$x \in U$ iff $y \in U \ \forall\, U \in \mathscr{T}$ implies $x = y$.

Kolmogorov quotient $K(A)$:
Quotient space: $x \sim y$ if $x \in U$ iff $y \in U \ \forall\, U \in \mathscr{T}$

Proposition (McCord)

$q\colon A \longrightarrow K(A)$ is a homotopy equivalence.

A-Space $\equiv$ Alexandroff T_0-space; F-Space $\equiv$ finite T_0-space

Closed points and bases

T_1-space: Points are closed
"Reasonable" spaces are T_1; A-spaces are not.

Lemma

Alexandroff T_1 spaces are discrete, but any finite T_0-space has at least one closed point.

Space = A-space for now. Fix X. Define

$$U_x \equiv \cap\{U | x \in U\}.$$

Lemma

$\{U_x\}$ is the unique minimal basis for the topology on X.

A-spaces and posets

Define a partial order $\leq$ on X by

$$x \leq y \;\; iff \;\; x \in U_y; \;\; that\ is, \;\; U_x \subset U_y.$$

Clearly transitive and reflexive, and $T_0 \Longrightarrow$ antisymmetric.

Conversely, for a poset X, define $U_x = \{x' | x' \leq x\}$.

This specifies a basis for an A-space topology on the set X.
It is the unique minimal basis for the topology.

$f\colon X \longrightarrow Y$ is continuous $\Longleftrightarrow$ f preserves order.

Theorem

The category $\mathscr{P}$ of posets is isomorphic to the category $\mathscr{A}$ of A-spaces.

Counting finite spaces

Now restrict to F-spaces.

Lemma

$f\colon X \longrightarrow X$ is a homeomorphism
iff f is either one-to-one or onto.

We can describe n-point topologies by a certain restricted kind of $n \times n$-matrix and enumerate them.

Combinatorics problem: Count the isomorphism classes of posets with n points; equivalently count the homeomorphism classes of spaces with n points. This is hard.

Count for $1 \leq 4$

$n = 1$ 2 spaces, 1 of them T_0 (discrete and trivial topologies)

$n = 2$ 3 spaces, 2 of them T_0

$n = 3$ 9 spaces, 5 of them T_0

$n = 4$ 33 spaces, 16 of them T_0

Let $X = \{a, b, c, d\}$. We list minimal bases $\{U_x\}$.

Y or N indicates T_0 or not; if not, there are
fewer than four distinct sets in the basis.

Spaces with 4 points

1	a, b, c, d	Y
2	a, b, c, X	Y
3	a, b, c, (a,b,d)	Y
4	a, b, c, (a,d)	Y
5	a, b, X	N
6	a, b, (a,b,c), X	Y
7	a, b, (a,c,d)	N
8	a, b, (a,b,c), (a,b,d)	Y
9	a, b, (a,c), X	Y
10	a, b, (a,c), (a,c,d)	Y
11	a, b, (a,c), (a,b,d)	Y
12	a, b, (c,d)	N
13	a, b, (a,c), (a,d)	Y
14	a, b, (a,c), (b,d)	Y
15	a	N
16	a, (a,b)	N
17	a, (a,b), (a,b,c), X	Y
18	a, (b,c), X	N
19	a, (a,b), (a,c,d)	N
20	a, (a,b), (a,b,c), (a,b,d)	Y
21	a, (b,c), (b,c,d)	N
22	a, (a,b), (a,c), (a,b,c), X	Y
23	a, (a,b), (a,c), (a,b,d)	Y
24	a, (a,b), (c,d)	N
25	a, (a,b), (a,c), (a,d)	Y
26	a, (a,b,c), X	N
27	a, (b,c,d)	N
28	(a,b), X	N
29	(a,b), (c,d)	N
30	(a,b), (a,b,c), X	N
31	(a,b), (a,b,c), (a,b,d)	N
32	(a,b,c), X	N
33	X	N

Homotopies and homotopy equivalence

$f, g \colon X \longrightarrow Y$: $\quad f \leq g$ if $f(x) \leq g(x) \;\forall\; x \in X$.

Proposition

Let X and Y be finite. Then $f \leq g$ implies $f \simeq g$.

Proposition

(i) *If there is a $y \in X$ such that X is the smallest open (or closed) subset containing y, then X is contractible.*

(ii) *If X has a unique maximum or minimal point, then X is contractible.*

(iii) *Each U_x is contractible.*

Beat points

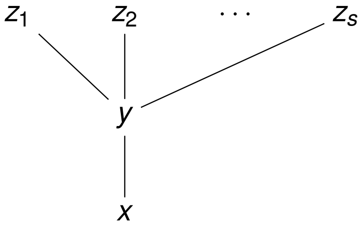
Definition

Let X be finite.

(a) $x \in X$ is *upbeat* if there is a $y > x$ such that $z > x$ implies $z \geq y$. Note that y is unique if X is T_0.

(b) $x \in X$ is *downbeat* if there is a $y < x$ such that $z < x$ implies $z \leq y$.

Upbeat:



Downbeat: upside down.

Minimal spaces and cores

X is minimal if it has no upbeat or downbeat points.

A subspace Y of X is a core if Y is minimal and is a deformation retract of X.

Theorem (Stong)

(i) *Any finite space X has a core.*

(ii) *If X is minimal and $f \simeq id\colon X \longrightarrow X$, then $f = id$.*

(iii) *Minimal homotopy equivalent finite spaces are homeomorphic.*

REU results of Alex Fix and Stephen Patrias

Can now count homotopy types with n points.

Hasse diagram $Gr(X)$ of a poset X:
Directed graph with vertices $x \in X$ and an edge
$x \to y$ if $y < x$ but there is no other z with $x \leq z \leq y$.

Translate minimality of X to a property of $Gr(X)$.
Count the number of graphs with that property.

Find a fast enumeration algorithm. Run it.

Get the number of homotopy types with n points.

Compare with the number of homeomorphism types.

Fix-Patrias theorem

n	$\simeq$	$\cong$
1	1	1
2	2	2
3	3	5
4	5	16
5	9	63
6	20	318
7	56	2,045
8	216	16,999
9	1,170	183,231
10	9,099	2,567,284
11	101,191	46,749,427
12	1,594,293	1,104,891,746

Exploit known results from combinatorics.
Astonishing conclusion:

Theorem

(Fix and Patrias) The number of homotopy types of F-spaces is asymptotically equivalent to the number of homeomorphism types of F-spaces.

Weak homotopy equivalences

A map $f\colon X \longrightarrow Y$ is a weak equivalence if

$$f_*\colon \pi_n(X,x) \longrightarrow \pi_n(Y, f(x))$$

is a bijection for all $n \geq 0$ and all $x \in X$.
This is the right notion of equivalence!!!

Theorem (Whitehead)

If X and Y are CW complexes (reasonable spaces), a weak equivalence $f\colon X \longrightarrow Y$ is a homotopy equivalence.

Headed towards the right homotopy theory of finite spaces

From A-spaces to simplicial complexes

Category $\mathscr{A}$ of A-spaces ($\cong$ posets);
Category $\mathscr{B}$ of classical simplicial complexes.

Theorem (McCord)

There is a functor $\mathscr{K} : \mathscr{A} \longrightarrow \mathscr{B}$ and a natural weak equivalence

$$\psi\colon |\mathscr{K}(X)| \longrightarrow X.$$

The n-simplices of $\mathscr{K}(X)$ are

$$\{x_0, \cdots, x_n | x_0 < \cdots < x_n\},$$

and $\psi(u) = x_0$ if u is an interior point of the simplex spanned by $\{x_0, \cdots, x_n\}$.

From simplicial complexes to A-spaces

Let $Sd\,K$ be the barycentric subdivision of a simplicial complex K; let b_σ be the barycenter of a simplex σ.

Theorem

There is a functor $\mathscr{X} : \mathscr{B} \longrightarrow \mathscr{P} \cong \mathscr{A}$ *and a natural weak equivalence*

$$\phi\colon |K| \longrightarrow \mathscr{X}(K).$$

The points of $\mathscr{X}(K)$ *are the barycenters* b_σ *of simplices of* K*, and* $b_\sigma < b_\tau$ *if* $\sigma \subset \tau$*. Moreover,* $\mathscr{K}(\mathscr{X}(K)) = Sd\,K$ *and*

$$\phi_K = \psi_{\mathscr{X}(K)}\colon |K| \cong |Sd\,K| \longrightarrow \mathscr{X}(K).$$

(It is natural to also write $\mathscr{X} = Sd$.)

Comparison of maps

Problem: not many maps between finite spaces!
Solution: subdivision: $Sd\,X \equiv \mathscr{X}(\mathscr{K}(X))$. Iterate.

Theorem

There is a natural weak equivalence $\xi\colon Sd\,X \longrightarrow X$.

Theorem (Classical)

Let $f\colon |K| \longrightarrow |L|$ be continuous, where K and L are simplicial complexes, K finite. For some large n, there is a simplicial map $g\colon K^{(n)} \longrightarrow L$ such that $f \simeq |g|$.

Theorem (Consequence)

Let $f\colon |\mathscr{K}(X)| \longrightarrow |\mathscr{K}(Y)|$ be continuous, where X and Y are A-spaces, X finite. For some large n there is a continuous map $g\colon X^{(n)} \longrightarrow Y$ such that $f \simeq |\mathscr{K}(g)|$.

Non-Hausdorff cones and suspensions

Definition

let X be a space.

1. *Define the non-Hausdorff cone $\mathbb{C}X$ by adjoining a new point $+$ and letting the proper open subsets of $\mathbb{C}X$ be the non-empty open subsets of X.*
2. *Define the non-Hausdorff suspension $\mathbb{S}X$ by adjoining two points $+$ and $-$ such that $\mathbb{S}X$ is the union under X of two copies of $\mathbb{C}X$.*

Let SX be the (unreduced) suspension

$$X \times \{-1\} \backslash X \times [-1, 1] / X \times \{1\}.$$

Finite spheres

Definition

Define a natural map

$$\gamma = \gamma_X \colon SX \longrightarrow \mathbb{S}X$$

by $\gamma(x,t) = x$ *if* $-1 < t < 1$, $\gamma(1) = +$ *and* $\gamma(-1) = -$.

Theorem

γ *is a weak equivalence.*

Corollary

$\mathbb{S}^n S^0$ *is a finite space with* $2n+2$ *points weakly equivalent to* S^n.

Characterization of finite spheres

The height $h(X)$ of a poset X is the maximal length h of a chain $x_1 < \cdots < x_h$ in X.

$$h(X) = dim\,|\mathscr{K}(X)| + 1.$$

Barmak and Minian:

Proposition

Let $X \neq *$ be a minimal finite space. Then X has at least $2h(X)$ points. It has exactly $2h(X)$ points iff it is homeomorphic to $\mathbb{S}^{h(X)-1}S^0$.

Corollary

If $|\mathscr{K}(X)|$ is homotopy equivalent to a sphere S^n, then X has at least $2n+2$ points. If it has exactly $2n+2$ points it is homeomorphic to $\mathbb{S}^n S^0$.

Remark

If X has 6 elements, then $h(X)$ is 2 or 3. There is a 6 point space that is weak equivalent to S^1 but is not homotopy equivalent to $\mathbb{S}S^0$.

Really finite H-spaces

An *H-space* X is a space with a product $X \times X \longrightarrow X$ and a 2-sided unit element e up to homotopy: $x \to ex$ and $x \to xe$ are homotopic to the identity map of X. Let X be a finite H-space. Really finite.

Theorem (Stong)

If X is minimal, these maps are homeomorphisms and e is both a maximal and minimal point of X. Therefore $\{e\}$ is a component of X.

Theorem (Stong)

X is an H-space with unit e iff e is a deformation retract of its component in X, hence X is an H-space iff a component of X is contractible. If X is a connected H-space, X is contractible.

Example (Hardie, Vermeulen, Witbooi)

Let $\mathbb{T} = \mathbb{S}S^0$, $\mathbb{T}' = Sd\mathbb{T}$. There is product $\mathbb{T}' \times \mathbb{T}' \longrightarrow \mathbb{T}$ that realizes the product on S^1 after realization.

Finite groups and finite spaces

X, Y F-spaces and G-spaces, G a finite group.

Theorem (Stong)

X has an equivariant core, namely a sub G-space that is a core and a G-deformation retract of X.

Corollary

Let X be contractible. Then X is G-contractible and has a point fixed by every self-homeomorphism.

Corollary

If $f\colon X \longrightarrow Y$ is a G-map and a homotopy equivalence, then it is a G-homotopy equivalence.

Towards Quillen's conjecture

Let G be a finite group and p a prime.

Definition

$\mathscr{S}_p(G)$ is the poset of non-trivial p-subgroups of G, ordered by inclusion. Observe that G acts on $\mathscr{S}_p(G)$ by conjugation and that $P \in \mathscr{S}_p(G)$ is normal if and only if P is a G-fixed point.

A p-torus is an elementary Abelian p-group. Let $r_p(G)$ be the rank of a maximal p-torus in G.

Definition

$\mathscr{A}_p(G) \subset \mathscr{S}_p(G)$ is the sub-poset of p-tori.

Proposition

If G is a p-group, $\mathscr{A}_p(G)$ and $\mathscr{S}_p(G)$ are contractible.

Note: genuinely contractible, not just weakly.

A key diagram

$$\begin{array}{ccc} |\mathscr{K}\mathscr{A}_p(G)| & \xrightarrow{|\mathscr{K}(i)|} & |\mathscr{K}\mathscr{S}_p(G)| \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi} \\ \mathscr{A}_p(G) & \xrightarrow[i]{} & \mathscr{S}_p(G) \end{array}$$

The vertical maps ψ are weak equivalences.

Proposition

$i\colon \mathscr{A}_p(G) \longrightarrow \mathscr{S}_p(G)$ is a weak equivalence. Therefore $|\mathscr{K}(i)|$ is a weak equivalence and hence a homotopy equivalence.

Example

If $G = \Sigma_5$, $\mathscr{A}_2(G)$ and $\mathscr{S}_2(G)$ are not homotopy equivalent.

Quillen's conjecture

Theorem

If $\mathscr{S}_p(G)$ or $\mathscr{A}_p(G)$ is contractible, then G has a non-trivial normal p-subgroup. Conversely, if G has a non-trivial normal p-subgroup, then $\mathscr{S}_p(G)$ is contractible, hence $\mathscr{A}_p(G)$ is weakly contractible.

Conjecture (Quillen)

If $\mathscr{A}_p(G)$ is weakly contractible, then G contains a non-trivial normal p-subgroup.

(Hypothesis holds iff $|\mathscr{K}\mathscr{A}_p(G)|$ is contractible.)

Status of the conjecture

Easy: True if $r_P(G) \leq 2$.

Quillen: True if G is solvable.

Aschbacker and Smith: True if $p > 5$ and G has no component of the form $U_n(q)$ with $q \equiv -1 \pmod{p}$ and q odd.

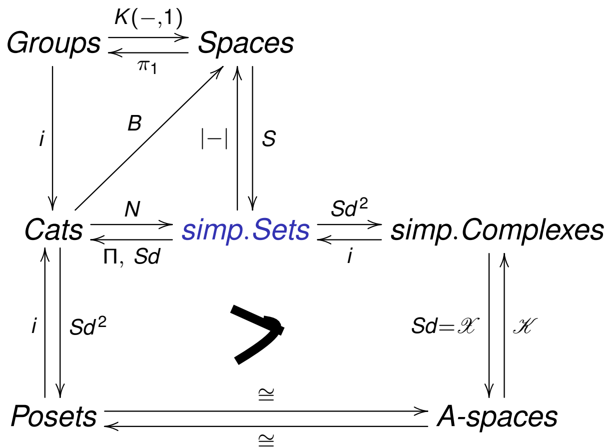
(Component of G: normal subgroup that is simple modulo its center).

Horrors: proof from the classification theorem!

Their 1993 article summarizes earlier results.

That is where the problem stands.

A BIGGER PICTURE



Simplicial sets

$\Delta \equiv$ standard simplicial category

Objects: ordinals $\mathbf{n} = \{0, 1, \cdots, n\}$, $n \geq 0$

Morphisms: nondecreasing (monotonic) functions

Simplicial set: Functor $K \colon \Delta^{op} \longrightarrow \mathit{Set}$

$s\mathscr{S}et$: category of simplicial sets.

$\Delta[n]$ is represented on Δ by $\mathbf{n}$,

$$\Delta[n](\mathbf{m}) = \Delta(\mathbf{m}, \mathbf{n})$$

Small Categories

$\mathscr{C}at$: Category of categories and functors

$\underline{\mathbf{n}}$: Poset $\mathbf{n}$ thought of as a category with a map $i \to j$ when $i \leq j$

$N \colon \mathscr{C}at \longrightarrow s\mathscr{S}et$

$$(N\mathscr{C})(\mathbf{n}) = \mathscr{C}at(\underline{\mathbf{n}}, \mathscr{C}); \text{ for example } N\underline{\mathbf{n}} = \Delta[n].$$

Full and faithful:

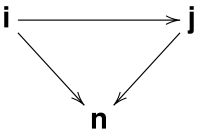
$$\mathscr{C}at(\mathscr{C}, \mathscr{D}) \cong s\mathscr{S}et(N\mathscr{C}, N\mathscr{D})$$

Subdivision of simplicial sets

$$Sd\Delta[n] \equiv \Delta[n]' \equiv N(Sd(\underline{\mathbf{n}})),$$

where

$$Sd(\underline{\mathbf{n}}) \equiv \underline{\mathbf{n}}' \equiv monos/\mathbf{n}.$$



Categorical tensor product: $SdK \equiv K \otimes_{\Delta} \Delta'$.

Lemma (Foygel)

$SdK \cong SdL$ does not imply $K \cong L$ but it does imply $K_n \cong L_n$ as sets, with corresponding simplices having corresponding faces.

Regular simplicial sets K

A nondegenerate $x \in K_n$ is regular if the subcomplex $[x]$ it generates is the pushout of the diagram

$$\Delta[n] \xleftarrow{\delta^n} \Delta[n-1] \xrightarrow{d_n x} [d_n x].$$

K is regular if all x are so.

Theorem

For any K, $Sd\, K$ is regular.

Theorem

If K is regular, then $|K|$ is a regular CW complex: $(e^n, \partial e^n) \cong (D^n, S^{n-1})$ for all closed n-cells e.

Theorem

If X is a regular CW complex, then X is triangulable; that is X is homeomorphic to some $|i(K)|$.

Properties A, B, C of simplicial sets (Foygel)

Let $x \in K_n$ be a nondegenerate simplex of K.

A For all x, all faces of x are nondegenerate.

B For all x, x has $n+1$ distinct vertices.

C Any $n+1$ distinct vertices are the vertices of at most one x.

Lemma

K has B iff for all x and all monos $\alpha, \beta \colon \mathbf{m} \longrightarrow \mathbf{n}$, $\alpha^* x = \beta^* x$ implies $\alpha = \beta$. (Distinct vertices imply distinct faces.)

Lemma

If K has B, then K has A.

There are no other general implications among A, B, C.

Properties A, B, C and subdivision (Foygel)

Lemma

K has A iff SdK has A.

Lemma

K has A iff SdK has B.

Lemma

K has B iff SdK has C.

Characterization of simplicial complexes

Lemma

K has A iff Sd^2K has C, and then Sd^2K also has B.

Lemma

K has B and C iff K is i of some simplicial complex.

Theorem

K has A iff Sd^2K is i of some simplicial complex.

Subdivision and horn-filling

Lemma

If SdK is a Kan complex, then K is discrete.

Lemma

If K does not have A, then SdK cannot be a quasicategory.

Theorem

If K has A, then $Sd\,K$ is N of some category.

Proof: Check the Segal maps criterion.

Properties A, B, and C on categories

Definition

A category $\mathscr{C}$ satisfies A, B, or C if $N\mathscr{C}$ satisfies A, B, or C.

Lemma

$\mathscr{C}$ has A iff for any $i\colon C \longrightarrow D$ and $r\colon D \longrightarrow C$ such that $r \circ i = id$, $C = D$ and $i = r = id$. (Retracts are identities.)

Lemma

$\mathscr{C}$ has B iff for any $i\colon C \longrightarrow D$ and $r\colon D \longrightarrow C$, $C = D$ and $i = r = id$.

Lemma

$\mathscr{C}$ has B and C iff $\mathscr{C}$ is a poset.

Subdivision of categories

Definition

Define a category $T\mathscr{C}$:
Objects: nondegenerate simplices of $N\mathscr{C}$. e.g.

$$\underline{C} = C_0 \longrightarrow C_1 \longrightarrow \cdots \longrightarrow C_q$$

$$\underline{D} = D_0 \longrightarrow D_1 \longrightarrow \cdots \longrightarrow D_r$$

Morphisms: maps $\underline{C} \longrightarrow \underline{D}$ are maps $\alpha\colon \mathbf{q} \longrightarrow \mathbf{r}$ in Δ such that $\alpha^*\underline{D} = \underline{C}$ (which implies that α is mono).

Define a quotient category $Sd\mathscr{C}$ of $T\mathscr{C}$ with the same objects:

$$\alpha \circ \beta_1 \sim \alpha \circ \beta_2 \colon \underline{C} \longrightarrow \underline{D}$$

if $\sigma \circ \beta_1 = \sigma \circ \beta_2$ for a surjection $\sigma\colon \mathbf{p} \longrightarrow \mathbf{q}$ such that $\alpha^*\underline{D} = \sigma^*\underline{C}$.
Here $\alpha\colon \mathbf{p} \longrightarrow \mathbf{r}$ and $\beta_i\colon \mathbf{q} \longrightarrow \mathbf{p}$, hence $\beta_i^*\alpha^*\underline{D} = \beta_i^*\sigma^*\underline{C} = \underline{C}$, $\;i = 1, 2$.

Properties A, B, and C and subdivision (Foygel)

Lemma

For any $\mathscr{C}$, $T\mathscr{C}$ has B. Therefore $Sd\mathscr{C}$ has B.

Lemma

$\mathscr{C}$ has B iff $Sd\mathscr{C}$ is a poset.

Theorem

For any $\mathscr{C}$, $Sd^2\mathscr{C}$ is a poset.

Compare with K has A iff Sd^2K is a simplicial complex.

Del Hoyo: Equivalence $\varepsilon\colon Sd\mathscr{C} \longrightarrow \mathscr{C}$.

(Relate to equivalence $\varepsilon\colon Sd\,K \longrightarrow K$?)

Fundamental category functor

Left adjoint Π to N (Gabriel–Zisman).

Objects of ΠK are the vertices.

Think of 1-simplices y as maps

$$d_1 y \longrightarrow d_0 y.$$

Form the free category they generate. Impose the relations

$$s_0 x = id_x \;\; for \;\; x \in K_0$$

$$d_1 z = d_0 z \circ d_2 z \;\; for \;\; z \in K_2.$$

The counit $\varepsilon\colon \Pi \circ N\mathscr{A} \longrightarrow \mathscr{A}$ is an isomorphism.

Commuting N with subdivision

ΠK depends only on the 2-skeleton of K. When $K = \partial\Delta[n]$ for $n > 2$, the unit $\eta\colon K \longrightarrow (N \circ \Pi)K$ is the inclusion $\partial\Delta[n] \longrightarrow \Delta[n]$. (Surprising)

Theorem

For any $\mathscr{C}$, $Sd\mathscr{C} \cong \Pi\, Sd\, N\mathscr{C}$ and $\varepsilon \cong \Pi\varepsilon\colon Sd\mathscr{C} \longrightarrow \Pi N\mathscr{C} \cong \mathscr{C}$.

Corollary

$\mathscr{C}$ has A if and only if $Sd\, N\mathscr{C} \cong N\, Sd\mathscr{C}$.

Remark

Even for posets P and Q, $SdP \cong SdQ$ does not imply $P \cong Q$.