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THE COFFEE CUP AND BELT TRICKS AND THEIR HOMOTOPY THEORETIC EXPLANATIONS

The coffee cup + belt move in a closed path in a certain space. The cup moves in $\mathbb{R}^3$, but we are more interested in its ORIENTATION, which is an <u>angle</u>, i.e. an element in $SO(2)$, the group of orthogonal 2×2 matrices with determinant 1. Each such matrix has the form

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

for angle θ.

Let $SO(3)$ denote the group of

Let $SO(3)$ denote the group of 3×3 orthogonal matrices with determinant 1. There is an inclusion homomorphism

$$SO(2) \longrightarrow SO(3)$$

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \longmapsto \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The coffee cup describes a closed path in $SO(2)$

$$t \longmapsto \begin{bmatrix} \cos 4\pi t & \sin 4\pi t \\ -\sin 4\pi t & \cos 4\pi t \end{bmatrix} \quad \text{for } 0 \le t \le 1$$

Claim This closed path in $SO(2)$ cannot be continuously deformed to the constant path

$\gamma(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}$ BUT

$$x \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{BUT}$$

its image in $SO(3)$ can be.

Both $SO(2)$ and $SO(3)$ can be topologized as follows.

$SO(2) \subset \mathbb{R}^4 =$ set of all 2×2 matrices

$SO(3) \subset \mathbb{R}^9 =$ set of all 3×3 matrices.

<u>Definition</u> Two continuous maps $f_0, f_1 : X \longrightarrow Y$ are HOMOTOPIC if $\exists$ continuous map $h : X \times [0,1] \longrightarrow Y$ such that

$$h(x,0) = f_0(x)$$

and $h(x,1) = f_1(x)$.

h is a HOMOTOPY between f_0 and f_1

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We can define $\forall s \in [0,1]$

$$f_s(x) = h(x,s) \in Y$$

These f_s form a family of continuous $X \longrightarrow Y$ parametrized by s.

Another interpretation:

Consider the space $\mathrm{Map}(X,Y)$ of all continuous map $X \longrightarrow Y$.

$$f_0, f_1 \in \mathrm{Map}(X,Y)$$

h defines a path in $\mathrm{Map}(X,Y)$ from f_0 to f_1.

<u>Variation</u> Suppose we have $x_0 \in X$ and $y_0 \in Y$ (BASE POINTS)

and $y_0 \in Y$ (BASE POINTS) and $f_0(x_0) = f_1(x_0) = y_0$. We can require that $h(x_0, s) = y_0$ for all $0 \le s \le 1$. Then h is a BASE POINT PRESERVING HOMOTOPY.

The claim is that the path $S^1 \to SO(2)$ described above is not homotopic to the constant map $S^1 \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, but

$$S^1 \to SO(2) \to SO(3)$$

IS HOMOTOPIC to

$$S^1 \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Each point in my body follows

some closed path in $O(3)$.

My feet did not move, so a path in my body from my right hand to my foot gives a path in the mapping space of closed paths $S^1 \to SO(3)$. We get the desired homotopy this way.

Consider set of homotopy classes of closed paths in (Y, y_0), i.e. homotopy classes of maps $(S^1, *) \to (Y, y_0)$, i.e. closed paths starting and

closed paths starting and
ending at y_0.

e.g. $(Y, y_0) = (SO(2), I)$ or $(SO(3), I)$

I = identity matrix.

Call this set $\pi_1(Y, y_0)$

<u>Theorem</u> This set has a natural group structure.

What is the binary operation?

Let $f, g : (S^1, *) \longrightarrow (Y, y_0)$
be closed paths

Let $f*g(t)$ for $0 \leq t \leq 1$
be

$$\begin{cases} f(2t) & 0 \leq t \leq \tfrac{1}{2} \\ g(2t-1) & \tfrac{1}{2} \leq t \leq 1 \end{cases}$$

Can check that if $f \simeq f'$
(f homotopic to f') and

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(f homotopic to f') and
$g \simeq g'$ then $(f*g) \simeq (f'*g')$

Can show that
$(f_1 * f_2) * f_3 \simeq f_1 * (f_2 * f_3)$
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The identity element is the constant y_0-valued path

Inverse of f is $\bar{f}$ defined by $\bar{f}(t) = f(1-t)$ for $0 \le t \le 1$.

This makes $\pi_1(Y, y_0)$ a group.

It is natural in that a map $(Y, y_0) \longrightarrow (Y', y_0')$ induces a group homomorphism

$\pi_1(Y, y_0) \to \pi_1(Y', y_0')$

$$\pi_1(Y, y_0) \longrightarrow \pi_1(Y', y_0').$$

Theorem

$\pi_1(SO(2), I) \cong \mathbb{Z}$ = integers

$\pi_1(SO(3), I) \cong \mathbb{Z}/2$ = integers mod 2

The inclusion map $SO(2) \to SO(3)$ induces a nontrivial homomorphism