Perfectoid spaces

Matthew Morrow (CNRS & Jussieu Institute of Mathematics, Paris)
matthew.morrow@imj-prg.fr

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Algebraic, arithmetic geometry

Field $(F, 0, 1, \times, +)$ $\rightsquigarrow$ Polynomials $f(X_1, \ldots, X_d) \in F[X_1, \ldots, X_d]$
E.g., $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\ldots$ and their solutions($\leftrightarrow$ "Algebraic varieties")
E.g., Elliptic curve $Y^2 = X^3 + aX^2 + bX + c$,
Fermat equation $X^n + Y^n = Z^n$.

Different flavour depending on F:

- Analytic: $\mathbb{C}$, $\mathbb{R}$ – analysis, topology, smoothness.
- Arithmetic: $\mathbb{Q}$,$\mathbb{Q}_p$,$\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$– number theory, *extra symmetry*.

Extra symmetry

Example:

- Elliptic curve: $Y^2 = X^3 + X^2 - 1$.
- Field: $\mathbb{Q}(i) = \{a + bi : a, b \in \mathbb{Q}\} \subseteq \mathbb{C}$ where $i = \sqrt{-1}$.
- Solutions: $\{(x, y) \in \mathbb{Q}(i)^2 : y^2 = x^3 + x^2 - 1\} \ni (-1, i)$.
- Symmetry: set of solutions closed under complex conjugation

$$(x, y) \mapsto (\overline{x}, \overline{y}).$$

(In general, under symmetries of roots = Galois group)

Key principles:

1. These groups know everything (Galois theory).
2. Groups are hard; study their actions.
3. They act on solutions! (and other invariants)

Problem: $\mathbb{Q}$ is still really hard(subtle interaction between prime numbers).
Solution: replace $\mathbb{Q}$ by $\mathbb{Q}_p$ to focus on a single prime number p.

p-adic numbers $\mathbb{Q}_p$

p a prime number $\rightsquigarrow$ *the p-adic absolute value* $|\cdot|_p$ on $\mathbb{Q}$:

$$\left|\frac{a}{b}p^n\right|_p := p^{-n} \qquad \text{where } a, b \in \mathbb{Z} \text{ not divisible by } p.$$

Absolute value: $|0| = 0$, $|1| = 1$, $|xy| = |x||y|$, $|x+y| \leq |x| + |y|$

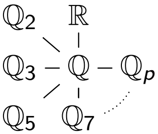
$\mathbb{Q}_p :=$ topological completion of $\mathbb{Q}$ (still a field)

p-adic expansion: $\sum_{n \gg -\infty} a_n p^n$ where $a_n \in \{0, \ldots, p-1\}$,

$$2/5 = 1 + 3 + 2 \cdot 3^2 + 1 \cdot 3^3 + 0 \cdot 3^4 + 2 \cdot 3^5 + \cdots$$

Theorem (Ostrowski 1916)

The only absolute values on $\mathbb{Q}$ are the usual absolute value and the p-adic absolute values; $\mathbb{Q}_p$ is the unique completion where $p^n \to 0$ as $n \to \infty$.



The perfectoid philosophy

Arithmetic geometry over $\mathbb{Q}_p$:
- Polynomials over $\mathbb{Q}_p$
- Algebraic varieties over $\mathbb{Q}_p$,their solutions,symmetry groups, ...

characteristic 0

$p \neq 0$

perfect- $\subseteq$ oids

Arithmetic geometry over $\mathbb{F}_p$:
- Polynomials over $\mathbb{F}_p$
- Algebraic varieties over $\mathbb{F}_p$, their solutions, symmetry groups, ...

characteristic p

$p = 0$

Why? Working over $\mathbb{F}_p$ is easier!

- Any ring A in which $p\,1_A = 0$ has extra symmetry called *Frobenius*:

$$(ab)^p = a^p b^p, \qquad (a+b)^p = a^p + \sum_{i=1}^{p-1} \binom{p}{i} a^i b^{p-i} + b^p = a^p + b^p$$

- Many results of arithmetic geometry have only been proved so far in characteristic p.

The perfectoid philosphy: $\mathbb{Q}_p$ itself

Theorem (Fontaine–Wintenberger 1979)

The fields $\mathbb{Q}_{p,\infty}$ *and the field of formal Laurent series*

$$\mathbb{F}_p((t)) := \{ \sum_{n \gg -\infty} a_n t^n : a_n \in \mathbb{F}_p \} \qquad \overset{?}{\leftrightarrow} \sum_{n \gg -\infty} a_n p^n \in \mathbb{Q}_p$$

$\big($*not* $+, \times$*:* $\mathbb{F}_5((t)) : 2 \cdot t + 4 \cdot t = 1 \cdot t, \quad \mathbb{Q}_5 : 2 \cdot 5 + 4 \cdot 5 = 1 \cdot 5 + 1 \cdot 5^2\big)$

are "arithmetically equivalent".

$$\underbrace{\mathbb{Q}_p \subseteq \mathbb{Q}_p(p^{1/p}) \subseteq \mathbb{Q}_p(p^{1/p^2}) \subseteq \cdots \subseteq \mathbb{Q}_p(p^{1/p^\infty})}_{\text{Add all } p\text{-power roots of } p} =: \mathbb{Q}_{p,\infty}$$

- Same absolute Galois group,i.e., *one variable* polynomials with coefficients in $\mathbb{Q}_{p,\infty}$ and $\mathbb{F}_p((t))$ have same symmetries.
- Theory of *diamonds*: $\mathbb{Q}_p \approx \mathbb{F}_p((t)) +$ symmetries of above tower.

Beyond $\mathbb{Q}_p$

By extracting many p-power roots we constructed $\mathbb{Q}_{p,\infty}$, which is "arithmetically equivalent" to a field of characteristic p, but only for one variable polynomials.

Idea: axiomatise the essential properties of $\mathbb{Q}_{p,\infty}$ to go further.

Definition

A $\mathbb{Q}_p$-algebra R is called perfectoid if "it contains many p^{th} roots and p-adic approximation arguments work".

Example: $f(X_1, \ldots, X_n) \in \mathbb{Q}[X_1, \ldots, X_n] \rightsquigarrow \mathbb{Q}_{p,\infty}[X_1^{1/p^\infty}, \ldots, X_n^{1/p^\infty}]/(f)$.

Theorem (The tilting correspondence, Scholze 2012)

Let R be a perfectoid algebra. Then there exists a tilt $R^\flat$, such that R and $R^\flat$ are "arithmetically equivalent".

Example: the tilt of $\mathbb{Q}_{p,\infty}$ is $\mathbb{F}_p((t))$.

Definition of the tilt

Theorem (The tilting correspondence, Scholze 2012)

Let R be a perfectoid algebra. Then there exists a tilt $R^\flat$, such that R and $R^\flat$ are "arithmetically equivalent".

Definition

Let R be a perfectoid algebra. Its *tilt* is the set of p-power compatible sequences :

$$R^\flat := \{(a_0, a_1, a_2, \dots) : a_i \in R,\ a_{i+1}^p = a_i \text{ for all } i \geq 0\}$$

- $(a_0, a_1, \dots) \times (b_0, b_1, \dots) := (a_0 b_0, a_1 b_1, \dots)$.
- $(a_0, a_1, \dots) + (b_0, b_1, \dots) := (\lim_{i\to\infty}(a_i + b_i)^{p^i}, \dots)$.

These operations make $R^\flat$ into a ring of characteristic p.

Summary

Variety V over $\mathbb{Q}_p$ i.e. polynomials $f(X_1, \ldots, X_d)$ $\rightsquigarrow$ perfectoid algebras R $\mathbb{Q}_{p,\infty}[X_1^{1/p^\infty}, \ldots, X_n^{1/p^\infty}]/(f)$ $\overset{\text{Tilt}}{\Longrightarrow}$ V is controlled by the $R^\flat$

"V has been embedded into characteristic p"

To make this precise, one...

1. defines *perfectoid spaces* X by building geometry from perfectoid algebras;
2. extends tilting to perfectoid spaces: $X \mapsto X^\flat$
3. introduces "diamonds" as "quotients of perfectoid spaces"
4. Final conclusion: $V \approx$ "$X^\flat$ mod symmetries"

Thank you!