AN INTRODUCTION TO THE LOEWNER DIFFERENTIAL EQUATION

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Abstract. In this paper, I set out to define the compact **H**-hulls and their mapping out functions. Several results are then presented: characterization of half plane capacity using Brownian motion, and a universal estimate over all compact **H**-hulls with the same capacities. I then give a proof on how compact **H**-hulls generated by a simple curve in the upper half plane satisfy the Loewner Differential Equation. Finally, I discuss about the Schramm-Loewner Evolution, a one parameter family of solutions to the Loewner Differential Equation with Brownian motion as the driving function.

Contents

1. Introduction

Loewner differential equations was discovered by Charles Loewner in 1923 to study slit mappings. Starting with a simple curve in the upper half plane **H**, we can find a unique conformal mapping that sends $\mathbf{H}\backslash\gamma(0,t]$ to **H**, and in some sense "normalized" at infinity. Loewner discovered that the chain of conformal mappings generated a simple curve as it grows satisfies a specific differential equation, called the Loewner differential equation. Conversely, when we have the Loewner differential equation, with a "driving function", we can recover the curve, or more generally a sequence of what's called the compact **H**-hulls. In the late 1990s, Oded Schramm discovered that if the driving function is Brownian motion, the curve we recover - Schramm-Loewner Evolution - is a universal scaling limit of a number of discrete physical models. The incredible thing is that when the variance of the Brownian motion varies, the nature of these physical models changes drastically. This paper will give a proof on why functions generated by the curve satisfy LDE, and briefly overview SLE.

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2. Compact H-Hulls and Mapping-Out Functions

Definition 2.1. A subset K of the upper half plane $\mathbb{H}$ is called a **compact H-hull** if K is closed and bounded and $\mathbf{H}\backslash K$ is a simply connected domain.

Theorem 2.2. *Given a compact $\boldsymbol{H}$-hull K, there exists a unique conformal transformation $g = g_K : \boldsymbol{H}\backslash K \to \boldsymbol{H}$ such that*

$$\lim_{z\to\infty}[g(z) - z] = 0$$

The expansion of g at infinity is

$$g(z) = z + \frac{a}{z} + \sum_{j=2}^{\infty} b_j z^{-j}, \quad b_j \in \mathbb{R}$$

Proof. By the Riemann Mapping Theorem, we know that there exists a conformal map g from $\mathbf{H}\backslash K$ to $\mathbf{H}$. Since K is compact, we can assume it is contained in a ball of radius r. Consider the set $A = \{z : |z| > r\}$. Since g is conformal, it can be extended continuously to the real axis, thus by Shwarz Reflection Principle, $g(z) = \overline{g(\bar{z})}$ defines an analytic extension of g from $\mathbf{H}\cap A$ to A. If $f(z) = 1/g(1/z)$, then f is holomorphic in $\{z : |z| < r\}$ and $f(0) = 0$. Then by Taylor expansion,

$$f(z) = \sum_{n=1}^{\infty} a_n z^n$$

thus g has an expansion about ∞ of the form

$$g(z) = b_{-1}z + b_0 + \sum_{n=1}^{\infty} b_n z^{-n}$$

Since g sends $\mathbb{R}$ to $\mathbb{R}$, all coefficients are real. Since g sends points in the $\mathbf{H}$ to points in $\mathbf{H}$, $b_{-1} > 0$. Then $g_K(z) = \frac{g(z)-b_0}{b_{-1}}$ gives the desired map. □

The constant a is of particular importance and is called the half plane capacity of K and denoted $hcap(K)$. The reason that a is called the half plane capacity is because it satisfies the following inequality that's characteristic of "capacities".

If A_1 and A_2 are compact **H**-hulls, then

$$\text{hcap}(A_1) + \text{hcap}(A_2) \geq \text{hcap}(A_1 \cup A_2) + \text{hcap}(A_1 \cap A_2)$$

Example 2.3. If A is a vertical line segment $[x, x+i\delta] = \{x + yi : 0 \leq y \leq \delta\}$ for some $x \in \mathbf{R}$ and $\delta > 0$, then

$$g_A(z) = x + ((z-x)^2 + \delta^2)^{1/2} = z + \frac{\delta^2}{2z} + \dots$$

So $\text{hcap}([x, x+i\delta]) = \frac{\delta^2}{2}$.

Proposition 2.4. *If K is compact $\boldsymbol{H}$-hull, $r > 0$ and $x \in \mathbb{R}$, then rK and $K + x$ are compact $\boldsymbol{H}$-hulls with mapping out functions*

$$g_{rK}(z) = rg_K(z/r), \quad g_{K+x}(z) = g_K(z-x) + x$$

respectively.

Proof. Since rK is a dilation and $K+x$ is a horizontal translation, it's easy to see that their complement is simply connected in the upper half plane. Expanding the left side gives us

$$g_K(z-x)+x = z + \frac{a}{z-x} + O\left(|z|^{-2}\right)$$

$$rg_K(z/r) = r\left(\frac{z}{r} + r\frac{a}{z} + O\left(|z|^{-2}\right)\right) = z + r^2\frac{a}{z} + O\left(|z|^{-2}\right)$$

Since they satisfy the definition of a mapping out function, the uniqueness of mapping out functions tells us they are indeed g_{K+x} and g_{rK} respectively. □

The above proposition means that we can often times reduce general compact **H**-hulls into the simple case where $K \subset \mathbb{D}$. We also have the following corollary that describes capacities under transformations.

Corollary 2.5. *(Dilation and Scaling Invariance) It follows that*

$$\operatorname{hcap}(rK) = r^2 \operatorname{hcap}(K) \quad \operatorname{hcap}(K+x) = \operatorname{hcap}(K)$$

Definition 2.6. The radius of a compact **H**-hull K $\operatorname{rad}(K) = \inf\{r > 0 : K \subseteq r\mathbb{D} + x \text{ for some } x \in \mathbb{R}\}$

We now proceed to give an estimate for the mapping out function, which we will use to prove the Loewner Differential Equation. To do that, we will need to characterize the half plane capacity using Brownian Motion. We will first give a proposition that links harmonic (holomorphic) functions with martingales, so that we can apply optional sampling theorem later on.

Proposition 2.7. *Suppose f is a harmonic function in $\mathbb{C}$. Let B_t be a complex Brownian Motion. Then $Y_t = f(B_t)$ is a continuous local martingale.*

Proof. Ito's formula gives us

$$f(B_t) - f(B_0) = \sum_{j=1}^{d} \int_0^t f_j(s, B_s)\, dB_s^j$$

Since there is no ds in the integral, we know that Y_t is local martingale. □

Theorem 2.8. *(Alternative Definition) Let K be a compact **H**-hull, with its mapping-out function $g_K : \boldsymbol{H}\backslash K \to \boldsymbol{H}$. If B is a complex Brownian Motion starting at $z_0 \in \mathbf{H}\backslash K$, and τ denotes the stopping time of B leaving $\boldsymbol{H}\backslash K$, then*

$$\operatorname{Im}(z_0) = \operatorname{Im}(g_K(z_0)) + \mathbf{E}^{z_0}\left[\operatorname{Im}(B_\tau)\right] \tag{2.9}$$

and

$$\operatorname{hcap}(K) = \lim_{y\to\infty} y\boldsymbol{E}^{iy}\left[\operatorname{Im}(B_\tau)\right] \tag{2.10}$$

Proof. We consider the function $\phi(z) = \operatorname{Im}(z - g_K(z))$. By definition of g_K, $z - g_K(z)$ is bounded. Since g_K is a holomorphic function, it follows from Cauchy-Riemann Equations that the real and imaginary parts of $z - g_K(z)$ are harmonic. Therefore, by proposition 2.7 and optional sampling theorem, we have

$$\mathbf{E}[\phi(B_0)] = \mathbf{E}[\phi(B_\tau)]$$

Thus,

$$\phi(z) = \mathbf{E}[\operatorname{Im}(B_\tau)] - \mathbf{E}[\operatorname{Im}(g_K(B_\tau))]$$

Since g_K takes $\partial\mathbf{H}\backslash K$ and $\mathbb{R}$ to $\mathbb{R}$, $\mathrm{Im}(g_K(B_\tau)) = 0$. Therefore, rearranging the equation and we obtain

$$\mathrm{Im}\,(z_0) = \mathrm{Im}\,(g_K\,(z_0)) + \mathbf{E}^{z_0}\,[\mathrm{Im}\,(B_\tau)]$$

Taking $z = iy$ in (2.9), we have

$$\begin{aligned} y\mathbf{E}^{iy}[\mathrm{Im}(B_\tau)] &= y(\mathrm{Im}(iy) - \mathrm{Im}(g_K(iy)) \\ &= y(y - (y + \frac{hcap(K)}{y} + \sum_{j=2}^{\infty} b_j y^{-j})) \\ &= hcap(K) + \sum_{j=1}^{\infty} b_{j+1} y^{-j} \xrightarrow{y\to\infty} hcap(K) \end{aligned}$$

□

Among other things that follow from theorem 2.8, we see that $\mathrm{Im}(g_K(z)) \leq \mathrm{Im}(z)$ for all $z \in \mathbf{H}\backslash K$, and the half plane capacity is strictly positive if K is nonempty.

Proposition 2.11. *(Monotonicity of half plane capacity) Let K, K' be two compact* ***H****-hulls and $K \subseteq K'$. Then $g_{K'} = g_{g_K(K'\backslash K)} \circ g_K$ and* $\mathrm{hcap}(K') = \mathrm{hcap}(g_K(K'\backslash K)) + \mathrm{hcap}(K)$*.*

Proof. Since K' is a compact **H**-hull, $\mathbf{H}\backslash K'$ is simply connected. Thus, the complement of $g_K(K'\backslash K)$ is simply connected as g_K is conformal. Therefore, $g_K(K'\backslash K)$ is a compact **H**-hull. The uniqueness of the mapping out function gives us the identity. The monotonicity of half plane capacity follows by expanding the mapping out functions. □

Note that if we have a sequence of increasing compact **H**-hulls $\{K_t\}$ indexed by t, $\frac{dhcap}{dt}$ gives an idea of how quick the hulls are growing.

We end this section of with a theorem that gives some kind of uniformity over all compact hulls with the same capacity. The definition of the mapping out function g_K gives the following equation:

$$\left| g_K(z) - z - \frac{\mathrm{hcap}(K)}{z} \right| < \frac{c}{|z|^2}, \quad \text{for all } |z| > R$$

It turns out that c doesn't depend on K, but only on $\mathrm{hcap}(K)$.

Theorem 2.12. *There exists a constant $c < \infty$ such that for all compact* ***H****-hulls K and $|z| \geq 2\,\mathrm{rad}(K)$ we have*

$$\left| g_K(z) - z - \frac{\mathrm{hcap}(K)}{z} \right| \leq c\frac{\mathrm{rad}(K)\,\mathrm{hcap}(K)}{|z|^2}$$

Proof. By scaling and translation invariance, we may assume that $\mathrm{rad}(K) = 1$, and $K \subset \mathbb{D}$. Let $h(z) = z - g_K(z) + \mathrm{hcap}(K)/z$ and let

$$v(z) = \mathrm{Im}[h(z)] = \mathrm{Im}[z - g_K(z)] + \mathrm{hcap}(K)\,\mathrm{Im}[\frac{1}{z}] = \mathrm{Im}[z - g_K(z)] - \frac{\mathrm{Im}[z]}{|z|^2}\,\mathrm{hcap}(K)$$

Let σ be the first time a standard Brownian motion starting at z hits $\partial D \cup \mathbb{R}$ and let $p(z, i\theta), 0 < \theta < \pi$ denote the density of B_σ on the upper half circle. Then by (2.9) and the strong markov property of Brownian motion, we have

$$\mathrm{Im}(z - g_K(z)) = \mathbf{E}^z[\mathrm{Im}(B_\tau)] = \int_0^\pi \mathbf{E}^{e^{i\theta}}[\mathrm{Im}(B_\tau)]p(z, i\theta)d\theta \tag{2.13}$$

In order to estimate the $p(z, i\theta)$, we consider the infinite half strip $D = \{x + iy : x > 0, 0, y < \pi\}$ and the map $z \mapsto e^z$ which maps the strip to $\mathbf{H} \backslash \mathbb{D}$ and the left edge to the upper half circle. Let $\mathrm{hm}_D(z,\ iq)$ denote the harmonic measure measure of D with base point z on the left edge. Then using separation of variables, we can write this function explicitly:

$$\mathrm{hm}_D(x + iy, iq) = \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-nx} \sin(nq) \sin(ny)$$

Therefore, there exists a constant c such that for all $x \geq 1 and 0 < y < \pi$,

$$|\,\mathrm{hm}_D(x + iy, iq) - \frac{2}{\pi} e^{-x} \sin(q) \sin(y)| \leq ce^{-2x} \sin(q) \sin(y)$$

We can write this as

$$\mathrm{hm}_D(x + iy, iq) = \frac{2}{\pi} e^{-x} \sin(q) \sin(y)(1 + \mathrm{O}(e^{-x}))$$

Under the map that sends this strip to $\mathbf{H} \backslash \mathbb{D}$, we have that

$$p(z, i\theta) = \mathrm{hm}(z, e^{iq}) = \frac{2}{\pi} \frac{\mathrm{Im}(z)}{|z|^2} \sin(\theta)(1 + O(|z|^{-1})) \tag{2.14}$$

Therefore,

$$\mathrm{Im}(z - g_K(z)) = \frac{2}{\pi} \frac{\mathrm{Im}(z)}{|z|^2} (1 + O(|z|^{-1}) \int_0^{\pi} \mathbf{E}^{e^{i\theta}} [\mathrm{Im}(B_\tau)] \sin(\theta) d\theta$$

On the other hand, combining (2.10), (2.13) and (2.14), we can express half plane capacity as follows,

$$\begin{aligned}
\mathrm{hcap}(K) &= \lim_{y \to \infty} y \int_0^{\pi} \mathbf{E}^{iy}[\mathrm{Im}(B_\tau)] p(iy, i\theta) d\theta \\
&= \lim_{y \to \infty} y \int_0^{\pi} \mathbf{E}^{iy}[\mathrm{Im}(B_\tau)] \frac{2}{\pi} \frac{1}{y} \sin(\theta)(1 + O(y^{-1})) d\theta \\
&= \frac{2}{\pi} \int_0^{\pi} \mathbf{E}^{iy}[\mathrm{Im}(B_\tau)] \sin(\theta) d\theta
\end{aligned}$$

Thus, $v(z) \leq c\,\mathrm{hcap}(K) \frac{\mathrm{Im}(z)}{|z|^2}$ when $|z| \geq 2$. We use the fact that if D is a domain, $z \in D$ and $u : D \to \mathbb{R}$ is harmonic, then there exists a constact c such that

$$|\partial_x^k \partial_y^{1-k} v(z)| \leq \frac{c||v||_\infty}{\mathrm{dist}(z, \partial D)}$$

We refer the readers to [1], exercise 2.21 for a proof of this fact. Applying this fact to v, we get that

$$|\partial_x v(z)|, |\partial_y v(z)| \leq \frac{c\,\mathrm{hcap}(K)}{|z|^3}$$

By the Cauchy-Riemann Equation, the same bound holds for $|h'(z)|$. Since $h(iy) \to 0$ as $y \to \infty$, for $y \geq 2$,

$$|h(iy)| = \left| \int_1^{\infty} h'(ity) y dt \right| \leq \int_1^{\infty} |h'(ity)| y dt \leq \frac{c\,\mathrm{hcap}(K)}{y^2}$$

Similarly, by integrating along the circle with radius $r \geq 2$, we have $|h(re^{i\theta})| \geq |h(ir)| + \frac{c\,\mathrm{hcap}(K)}{y^2}$. Therefore,

$$\left| g_K(z) - z - \frac{\mathrm{hcap}(K)}{z} \right| \leq c \frac{\mathrm{rad}(K)\,\mathrm{hcap}(K)}{|z|^2}$$

□

3. Loewner Differential Equation

Suppose we have a simple a simple curve $\gamma : [0,\infty) \to \mathbf{H} \cup \mathbb{R}$ with $\gamma(0) = 0$ and $\gamma(0,\infty) \subset \mathbf{H}$. Then, for each t, $\gamma(0,t]$ forms a compact $\mathbf{H}$-Hull as $D_t = \mathbf{H}\backslash\gamma(0,t]$ is simply connected. Let $g_t(z) = g_{\gamma(0,t])}$ be the corresponding mapping out function of D_t. In this section, we will show that $\{g_t\}$ satisfies what's called the Loewner Differential Equation. Note that we haven't had any randomness in the equation yet.

Theorem 3.1. *(Half plane Loewner Differential Equation)*
Suppose γ is a simple curve re-parameterized so that $hcap(\gamma(0,t]) = 2t$ (We will justify why we can do so later on). Then for every $z \in \boldsymbol{H}$, the map $t \mapsto g_t(z)$ satisfies

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad 0 \le t < T_z$$

where $U_t = g_t(\gamma(t))$, $T_z = \inf\{s : \gamma(s) = z\}$. Moreover, the map $t \mapsto U_t$ is continuous.

Before proving this theorem, we need to make sure that $U_t = g_t(\gamma(t))$ is indeed well-defined. We will use Wolff's Lemma here, which roughly says that small connected sets are mapped to small connected sets under conformal transformation. We refer the readers to section 8.3 in [6] for a treatment using extremal distance and prime ends.

Lemma 3.2. *(Wolff's Lemma) Let H be an open set in $\mathbb{C}$ and $h : H \to D(0,R)$ be a conformal map. If $c \in \mathbb{C}$ and $C(r) = H \cap \{|z-c| = r\}$, then there exists a decreasing sequence $r_n \to 0$ such that $l(h(C(r_n))) \to 0$ as $n \to \infty$.*

Proof. By Shwarz's inequality, we have

$$\begin{aligned} l(h(C(r)))^2 &= \left(\int_{C(r)} |h'(z)|\,|dz|\right)^2 \le \int_{C(r)} |dz| \int_{C(r)} |h'(z)|^2\,|dz| \\ &\le 2\pi r \int_{c+re^{it}\in H} \left|h'\left(c+re^{it}\right)\right|^2 r\,dt \end{aligned}$$

Therefore,

$$\int_0^\infty l(h(C(r)))^2 \frac{dr}{r} \le 2\pi \iint_H |h'(z)|^2\,dxdy = 2\pi\,\mathrm{area}\,h(H)$$

Since h maps H to a finite disk, the area of h(H) is finite. Thus, $l(h(C(r)))$ converges to 0 as $r \to 0$. □

Proposition 3.3. *$g_t : \boldsymbol{H}\backslash\gamma(0,t] \to \boldsymbol{H}$ can be extended continuously to the boundary point $\gamma(t)$, thus the limit $U_t = \lim_{z\to\gamma(t)} g_t(z)$ exists.*

Proof. Since γ is simple, it is locally connected, for any $\epsilon > 0$, there exists $\delta < \epsilon$ such that ball $B(\gamma(t),\delta) \cap \gamma$ is connected. Suppose, for the sake of contradiction, that γ crosses the ball $B(\gamma(t),\delta)$ more than once before hitting $\gamma(t)$, then it must enter the ball and then leave the ball as it is injective. However, this would mean that the path $B(\gamma(t),\delta)\cap\gamma$ is disconnected. Therefore, γ enters the ball only once, and $B(\gamma(t),\delta)\backslash\gamma$ is path-connected. One can thus find arbitrarily small δ such that any curve that approaches the tip $\gamma(t)$ needs to pass through the $B(\gamma(t),\delta)$. By lemma 3.2, we can then find a sequence $\delta_n \to 0$ such that $l(g_t(C(\gamma(t), \delta_n)\backslash\gamma))) \to 0$ as n increases. Since $g_t(C(\gamma(t), \delta_n)\backslash\gamma))$ must contain the image of all curves hitting $\gamma(t)$ and it is simply connected, $g_t(C(\gamma(t), \delta_n)\backslash\gamma))$ must indeed approach the same point, which is $\gamma(t)$ □

The lemma below shows that it suffices to prove Theorem 3.1 for right derivatives.

Lemma 3.4. *Suppose $f : [0, \infty) \to \mathbb{R}^n$ is a function whose right derivative*

$$f'_+(t) = \lim_{h \downarrow 0} \frac{f(t+h) - f(t)}{h}$$

exists for all t and the map $t \mapsto f'_+(t)$ is continuous. Then f is C^1 with $f'(t) = f'_+(t)$.

Proof. We may assume without loss of generality that $f(0) = 0$ and $f'_+(t) = 0$ for all t since otherwise we consider the function

$$g(t) = f(t) - f(0) + \int_0^t f'_+(x)dx$$

Fix $\epsilon > 0$. Let $p = \inf\{t : |f(t)| > \epsilon t\}$. Since $f'_+(0) = 0$, $p > 0$. Suppose, for the sake of contradiction, that $p < \infty$. Since f is C^1, $f(p) = \epsilon p$. Because $f'_+(p) = 0$, there exists $\delta > 0$ such that $f(p + x) < \epsilon p + \epsilon x = \epsilon(p + x)$ for all $0 < x < \delta$. However, this contradicts the definition of p being the infimum of $\{t : |f(t)| > \epsilon t\}$. Thus $p = \infty$ and f is constant thus differentiable. □

Now, we prove the main component of theorem 3.1 by computing the right derivative of g_t as t changes.

Proof. Fix $t > 0$. We look at the small curve $\gamma(t, t + \epsilon)$ and consider the image A under g_t. The curve starts at U_t. By proposition 2.11, we have $g_{t+\epsilon} = g_A \circ g_t$. Let $w = g_t(z)$ and we apply theorem 2.12 to g_A (translated by U_t):

$$\left| g_A(w) - w - \frac{\text{hcap}(A)}{w - U_t} \right| \leq c \frac{\text{rad}\,(A)\,\text{hcap}(A)}{|w - U_t|^2}$$

for all $|w - U_t| \geq 2\,\text{rad}(A)$. Therefore,

$$\left| g_{t+\epsilon}(z) - g_t(z) - \frac{\text{hcap}(A)}{g_t(z) - U_t} \right| \leq c \frac{\text{rad}\,(A)\,\text{hcap}(A)}{|g_t(z) - U_t|^2}$$

for all $|g_t(z) - U_t| \geq 2\,\text{rad}(A)$. Now, we divide both sides by ϵ and take the limit as $\epsilon \to 0$. Rearranging the equation, we get

$$\lim_{\epsilon \to 0} \frac{g_{t+\epsilon}(z) - g_t(z)}{\epsilon} = \lim_{\epsilon \to 0} \frac{\text{hcap}(A)/\epsilon}{g_t(z) - U_t} + \lim_{\epsilon \to 0} c \frac{\text{rad}\,(A) \cdot \text{hcap}(A)/\epsilon}{|g_t(z) - U_t|^2}$$

However, since we have reparameterized γ so that $\text{hcap}(\gamma((0, t])) = 2t$, $\lim_{\epsilon \to 0} \frac{\text{hcap}(A)}{\epsilon} = 2$. Also by Wolff's lemma, $rad(A) \to 0$ as $\epsilon \to 0$. Thus, we have

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad 0 \leq t < T_z$$

□

Definition 3.5. The maps $t \to U_t$ are called the **Loewner transformation**.

To justify why we can reparameterize hcap so that $\text{hcap}(\gamma(0,t]) = 2t$, we use a lemma (Lemma 4.1 in [1]): There exists $c < \infty$ such that for every γ, if $s < t$,

$$\text{diam}\left(g_s\left[\gamma_t \backslash \gamma_s\right]\right) \le c\sqrt{\text{diam}\left(\gamma_t\right)}\sqrt{\text{diam}(\gamma[s,t])}$$

where $\gamma_t = \gamma(0,t]$. Therefore,

$$\text{hcap}(\gamma(t)) - \text{hcap}(\gamma(s)) \le c\,\text{diam}\left(\gamma_t\right)\text{diam}(\gamma[s,t])$$

In particular, hcap is continuous and we can reparameterize as we want.

However, Loewner Transformation does not necessarily come from a simple curve. In fact, If the family of compact **H**-hulls is increasing and satisfies the local growth property, in some sense "continuously increasing", then the mapping out functions generate Loewner transformation.

Definition 3.6. A family of compact **H**-hulls $(K_t)_{t\ge 0}$ is **increasing** if $K_s \subset K_t$ whenever $s < t$.

Definition 3.7. Let $K_{s,t} = g_{K_s}\left(K_t \backslash K_s\right)$. A family of compact **H**-hulls is said to satisfy the **local growth property** if

$$\lim_{h\downarrow 0} \text{rad}\left(K_{t,t+h}\right) = 0$$

Conversely, suppose one is given a continuous driving term $U_t : [0,+\infty) \to \mathbf{R}$. It follows from Picard existence and uniqueness theorem that for each $z \in \mathbf{H}$ there is a unique maximal time of existence $0 < T(z) \le +\infty$ such that the LDE with initial data $g_0(z)$ can be solved. It is highly non-trivial and we refer the readers to Chapter 4 of [1] for a detailed proof.

4. Schramm-Loewner Evolution

Definition 4.1. The Schramm-Loewner Evolution with parameter $\kappa \ge 0$, or SLE_κ, is the random collection of conformal maps g_t obtained from solving the Loewner Differential Equation

$$\dot{g}_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa}B_t}, \quad g_0(z) = 0 \quad (z \in \mathbf{H})$$

It was proven by Schramm that the Schramm-Loewner evolution is the only conformally invariant family of random curves that satisfy the domain Markov property, which we will state now.

Definition 4.2. Conformal Markov property:
Let $\mathbb{P}$ be the probability measure on simple curves γ with $\gamma(0+) = 0$. Suppose the beginning segment $\gamma_t = \gamma(0,t]$ is observed and let $g : \mathbf{H}\backslash\gamma_t \to \mathbf{H}$ be a conformal transformation with $g(\gamma(t)) = 0$, $g(\infty) = \infty$ (mapping out function). Then the conditional distribution of $g[\gamma[t,\infty))$ given γ_t is $\mathbb{P}$.

As mentioned, SLE is the scaling limit of a number of processes, one of which is the percolation exploration process.

Consider a hexagonal lattice in the upper half plane with mesh δ and for each site we flip a fair coin. If it is heads, we color the site black and if otherwise, we color it white. Furthermore, we apply boundary conditions to the lattice: the right half of the boundary (i.e. $\mathbb{R}_-$) is black and the left half of the boundary ($\mathbb{R}_+$) is white. This is a critical model, meaning that all the connected monochromatic components are almost surely finite, but the probability that two points are in the same component decays only polynomially in the distance.

Another way to construct model is that we have the same lattice as above and start at the origin, each step we flip a coin: if it is heads, the path turns right and if it is tails, the path turns left.

As such, if $\delta \to 0$, the interface between black and white sites converges in distribution to a random continuous curve γ in $\mathbf{H}$. This is proven by Stanislav Smirnov, and in fact the random continuous curve is SLE_6. Therefore, Loewner theory is essential in statistical physics, as it presents itself to the scaling limit of a number of well-studied models.

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