TYPES OF HOMOTOPY IN SIMPLICIAL COMPLEXES AND FINITE SPACES

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Abstract. In this expository paper, we summarize some important results from homotopy theory of finite spaces and simplicial complexes, which establishes a hierarchy of types of homotopy equivalences in these two categories. We investigate the relationship between the types of homotopy equivalences of finite spaces and simplicial complexes, and ask whether one can define a new class of homotopy equivalence of simplicial complexes that lies between homotopy equivalence and simple homotopy equivalence.

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1. Introduction

Historically, R.E. Stong and M.C. McCord were two of the first mathematicians to study the homotopy theory of finite spaces and its connection with polyhedra. Stong took a combinatorial approach and introduced the notion of *linear* and *co-linear* points, which were later termed *upbeat* and *downbeat* points by Peter May. Stong showed that the removal of beat points generates all the homotopy equivalences between finite spaces [13]. By contrast, McCord's work established the connection between weak equivalences of Alexandroff spaces and weak equivalences of simplicial complexes: given a T_0-space X, one can build its *order complex* $\mathcal{K}(X)$ which consists of the non-empty chains in X. Conversely, one can associate to every simplicial complex K an Alexandroff T_0-space $\mathcal{K}(K)$ whose elements are given

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by the simplices of K. McCord [8] also proved the existence of weak homotopy equivalences $X \to \mathcal{K}(X)$ and $K \to \mathcal{X}(K)$.

Since then, it has been long recognized that one can use the homotopy theory of finite spaces to study the homotopy theory of simplicial complexes. For example, if two finite spaces X and Y are homotopy equivalent, then so are their corresponding order complexes $\mathcal{K}(X)$ and $\mathcal{K}(Y)$. In fact, T. Osaki [11] showed that $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ are actually *simple* homotopy equivalent, which is a more refined notion of homotopy equivalence in the world of simplicial complexes.

These observations led us to investigate the following question in this paper: *Can we further refine the notion of homotopy equivalence in the world of simplicial complexes to obtain some formal class of homotopy equivalence between simple homotopy equivalence and general homotopy equivalence by using the homotopy theory of finite spaces?*

In order to provide the necessary tools for exploring this question, we organize this paper as follows. Section 2 is a review of some basics concepts of finite space and simplicial complexes. We summarize McCord's work on the connection between simplicial complexes and finite spaces and introduce the fundamental notions of homotopy theory in both settings.

In Section 3, we focus on finite spaces and investigate how one-point reductions of finite spaces affect their order complexes. We show that removing beat points from a finite space X does not affect the homotopy type of either X or $\mathcal{K}(X)$. We then review the notions of a *weak point* and *simple homotopy equivalence* between finite spaces as introduced by Barmak and Minian [1]. We finish this section by introducing the more general definition of a γ-point, also due to Barmak and Minian. We will see that γ-point reduction also preserves the weak homotopy type of the original finite space.

We conclude Section 3 with a partial answer to our question: to leave the world of simple homotopy equivalence of simplicial complexes, one must look beyond one-point reductions of finite spaces. This answer will follow from Theorem 3.22: Let X be an F-space, and let $x \in X$. Suppose that the inclusion $i : X\backslash\{x\} \to X$ is a weak homotopy equivalence. Then the induced simplicial map $\mathcal{K}(X\backslash\{x\}) \to \mathcal{K}(X)$ is a simple homotopy equivalence.

This leads us to Section 4, where we propose some potential methods of multiple-point reductions, starting with removing two points at a time. The key idea is that we want to exclude cases where the removal of two points simply boils down to an iteration of one-point reductions. To do so, we consider pairs of points linked by some special relationship that is not preserved by removing both points in isolation. There are two possible routes to explore, the first being the generalization of the concept of a beat point to a *beat pair* while the second being the notion of a *homologically admissible matching*, which is a relaxation of an *admissible matching* on posets as introduced in [10]. In some limited cases, we show that removing a pair of points (x, y) that belongs to a homologically admissible matching induces a homotopy equivalence in the corresponding simplicial complexes.

2. Preliminaries

This section is a quick review of some preliminary definitions and results that underlie the subsequent discussion on finite spaces, simplicial complexes, and partially ordered sets. We construct pairwise correspondences between these spaces and introduce homotopy theory on simplicial complexes and finite spaces.

2.1. A first taste of category theory.

Although we will mainly be working with finite spaces and simplicial complexes, it sometimes provides some structural advantage to be able to consider these objects from a category theoretical perspective. Therefore, we first introduce some very basic terminology from category theory.

Definition 2.1. A *category* $\mathscr{C}$ consists of three things:

(1) A collection $\mathrm{Ob}(\mathscr{C})$ of objects.
(2) A collection Map (X, Y) of morphisms for each pair $X, Y \in \mathrm{Ob}(\mathcal{C})$, including an "identity" morphism $\mathrm{id}_X \in \mathrm{Mor}(X, X)$ for each X.
(3) A "composition of morphisms" function $\circ : \mathrm{Map}(X, Y) \times \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$ for each triple $X, Y, Z \in \mathrm{Ob}(\mathscr{C})$ such that $f \circ \mathrm{id} = f, \mathrm{id} \circ f = f$, and $(f \circ g) \circ h = f \circ (g \circ h)$.

The definition of a category generalizes the notion of a set. Given a category $\mathscr{C}$, we say that it is *small* if both $\mathrm{Ob}(\mathscr{C})$ and Map (X, Y) are sets for all X and Y in $\mathscr{C}$. In fact, category theory provides a general framework to unify different areas of mathematics. Yet just as any other mathematical object, categories are mathematical objects on their own right. Hence one might wonder: what are the morphisms between categories? This leads us to the following notion of a *functor*.

Definition 2.2. A (covariant) *functor* F from a category $\mathscr{C}$ to a category $\mathscr{D}$ assigns to each object C in $\mathscr{C}$ an object $F(C)$ in $\mathscr{D}$ and to each morphism $f : C \to D$ in $\mathscr{C}$ a morphism $F(f) : F(C) \to F(D)$ in $\mathscr{D}$ such that

$$F(\mathrm{id}_C) = \mathrm{id}_{F(C)}, \;\; F(f \circ g) = F(f) \circ F(g).$$

Examples of categories include the category of sets and functions, the category of topological spaces and continuous functions, the category of groups and homomorphisms, and so on.

Definition 2.3. Two categories $\mathscr{C}$, $\mathscr{D}$ are *isomorphic* if there exist functors $F : \mathscr{C} \to \mathscr{D}, G : \mathscr{D} \to \mathscr{C}$ such that

$$F \circ G = \mathrm{id}_{\mathscr{D}}, G \circ F = \mathrm{id}_{\mathscr{C}}.$$

Just as how we may wish to speak of one function as being continuously deformed into another (an idea which will be made precise when we introduce homotopy theory), we want to describe when one functor can be naturally transformed into another.

Definition 2.4. A natural transformation $\alpha : F \to G$ between functors $F, G : \mathscr{C} \to \mathscr{D}$ is a map of functors. It assigns a morphism $\alpha_A : F(A) \to G(A)$ to each object A of $\mathscr{C}$ such that the following diagram commutes for each morphism $f : A \to B$ of $\mathscr{C}$:

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ {\scriptstyle \alpha_A}\downarrow & & \downarrow{\scriptstyle \alpha_B} \\ G(A) & \xrightarrow[G(f)]{} & G(B) \end{array}$$

As the name suggests, natural transformations are "natural" in the sense that they do not depend on arbitrary choices.

2.2. **Finite spaces and posets.** Given a finite set X, one can think about it both as a space endowed with a particular topology and a set with a partial order. Indeed, these two perspectives are naturally related. To see this, we first summarize some relevant definitions on topological spaces.

First, we recall that a *topology* on a set X is a collection of subsets of X, denoted by $\mathcal{U}$, which contains as its elements the empty set $\varnothing$ and the entire set X, is closed under arbitrary union, and is closed under finite intersection. The elements of $\mathcal{U}$ are the open sets in X under this specific topology. The following definition modifies the third property above by requiring that $\mathcal{U}$ is also closed under arbitrary intersection.

Definitions 2.5. Given a topological space $(X,\mathcal{U})$, we say that X is an *Alexandroff space* if $\mathcal{U}$ is closed under arbitrary intersections.

Since a finite topological space is just a topological space with only finitely many points, we have the following lemma.

Lemma 2.6. *Any finite space is Alexandroff.*

Given a set X with a topology $\mathcal{U}$, we can put different separation axioms on $(X,\mathcal{U})$. This gives rise to the following definitions.

Definitions 2.7. Let $(X,\mathcal{U})$ be a topological space.

(1) X is a T_0-space if $\mathcal{U}$ distinguishes points. That is, for any two points $x, y \in X$, there is an open neighborhood of one that does not contain the other.
(2) X is a T_1-space if each point of X is a closed set.
(3) X is a T_2-space, or *Hausdorff* space, if any two points of X have disjoint open neighborhoods.

There is a hierarchy among these types of spaces: $T_2 \implies T_1 \implies T_0$. Since an Alexandroff T_1-space is necessarily discrete, we will focus on Alexandroff T_0-spaces for the purpose of this paper. To lighten the notation, we will denote Alexandroff T_0-spaces by A-spaces and finite T_0-spaces by F-spaces. The next definition of a *basis* captures the idea of a "natural" or canonical set of open sets that generates the topology on a space.

Definition 2.8. A *basis* for the topology on X is a set $\mathcal{B}$ of subsets of X such that

(1) For each $x \in X$, there is at least one $B \in \mathcal{B}$ such that $x \in B$;
(2) If $x \in B_1 \cap B_2$ where $B_1, B_2 \in \mathcal{B}$, then there exists some $B \in \mathcal{B}$ such that $x \in B \subset B_1 \cap B_2$.

The topology $\mathcal{U}$ *generated* by the basis $\mathcal{B}$ is precisely the set of subsets U that are unions of sets in $\mathcal{B}$. In particular, we can define the *minimal basis* for an Alexandroff space X as follows: for each $x \in X$, let U_x be the intersection of all open sets containing x. Then the set $B = \{U_x \mid x \in X\}$ forms a basis for the space X, which we call the *minimal basis* because any other basis B' for X must contain the minimal basis. Indeed, let B' be a basis, and suppose that U_x is not an element in B'. Then it must be the union of some elements in B', one of which necessarily contains x. Hence this element must be U_x. This shows that the minimal basis is contained in B'.

We will now show that A-spaces are equivalent to *posets* by relating Alexandroff spaces to the combinatorial notions of preorder and partial order.

Definition 2.9. A *preorder* on a set X is a reflexive and transitive relation, which we denote by $\leq$. A preorder is a *partial order* if it is antisymmetric, i.e., $x \leq y$ and $y \leq x$ imply that $x = y$. A set X with a partial order $(X, \leq)$ is called a *poset*.

Let us first go from posets to A-spaces. Given a preorder $\leq$ on a set X, we can define a topology on X by taking as our basis elements

$$U_x = \{y \in X \mid y \leq x\}.$$

for each $x \in X$. Then U_x is the smallest open set containing x in this topology. If we take U to be the intersection of open sets $\{U_i\}_{i \in I}$ where I is some arbitrary index set, then for any $x \in U$, $U_x \subset U_i$ for each i, so U must be the union of the open sets $\{U_x\}_{x \in U}$. This shows that X is an Alexandroff space under this particular topology. If the preorder is in fact a partial order, then X is an A-space since $x \leq y$ and $y \leq x$ if and only if $U_x = U_y$.

For the other direction, note that from the definition of a minimal basis, one can define a relation $\leq$ on an Alexandroff space X by stipulating that $x \leq y$ whenever $x \in U_y$ or, equivalently, $U_x \subset U_y$. It is clear that $\leq$ is reflexive and transitive, so $(X, \leq)$ is a set with a preorder. Now, if X is an A-space, then $x \leq y$ and $y \leq x$ imply that $U_x = U_y$, which means that every open set containing either x or y must also contain the other. In other words, the T_0 separation axiom gives us the antisymmetric property required for a poset. This proves the following proposition.

Proposition 2.10. *For a set X, the Alexandroff topologies on X are in bijective correspondence with the preorders on X. The space $(X, \mathcal{U})$ is T_0 if and only if the relation $\leq$ associated to $\mathcal{U}$ is a partial order.*

From the observation above, it is reasonable to predict that the morphisms in A-spaces exhibit equivalent behavior to their counterparts in posets. We verify that this is indeed the case.

Proposition 2.11. *Let X, Y be A-spaces. Then a map $f : X \to Y$ is continuous if and only if f preserves order on the corresponding posets.*

Proof. First, suppose that $f : X \to Y$ is continuous. Let $x, y \in X$ such that $x \leq y$, that is, $x \in U_y$. We want to show that $f(x) \leq f(y)$. Since $U_{f(y)}$ is open, so is its preimage $V_y = f^{-1}(U_{f(y)})$ under f. Since $y \in V_y$, it follows from the definition of U_y that $U_y \subset V_y$. Now $x \in U_y$ implies that $f(x) \in f(U_y) \subset f(V_y) = U_{f(y)}$. Hence $f(x) \leq f(y)$.

Conversely, suppose that f preserves order. Since $B = \{U_y \mid y \in Y\}$ forms a basis of the topology on Y, it suffices to show that $f^{-1}(U_y)$ is open for every $y \in Y$.

Let $y \in Y$, and fix some $a \in f^{-1}(U_y)$ such that $a \neq f^{-1}(y)$. Then $f(a) \in U_y$, so $f(a) \leq y$. To simplify notation, let $x = f(a), y = f(b)$ for $a, b \in X$. Since f preserves order, we have $a \leq b$, so $a \in U_b$. The definition of U_b now implies that $U_b \subset f^{-1}(U_y)$. Hence every $a \in f^{-1}(U_y)$ belongs to some open set that is itself contained in $f^{-1}(U_y)$, so $f^{-1}(U_y)$ is open. This completes the proof. □

In the language of category theory, this proposition says that the category $\mathscr{P}$ of posets is isomorphic to the category $\mathscr{A}$ of A-spaces. The fact that A-spaces and posets encode the same data means that we may choose to work in either or both of them. When we wish to think from a combinatorial perspective, it is often useful to represent posets by their *Hasse diagrams*, which helps us visualize an A-space. Given a finite A-space X, its associated Hasse diagram, which we denote by $\mathcal{H}(X)$, is a digraph whose vertices are the elements of X and edges are the ordered pairs (x, y) such that $x \leq y$ and there exists no $z \in X$ with $x \leq z \leq y$.

Example 2.12. Consider the space $X = \{a, b, c, d\}$ whose proper open sets are $\{b, d\}, \{c, d\}$ and $\{d\}$. Its Hasse diagram looks like the following:

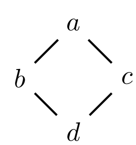

FIGURE 1. $\mathcal{H}(X)$

The edges should be read as downward directed arrows. Sometimes we may wish to consider the same space X associated with the reversed preorder, which we will denote by X^{op}. It is useful to note that the open sets in X are precisely the closed sets in X^{op}, and that the Hasse diagrams of X and X^{op} are flipped.

The fact that we can represent each A-space by its Hasse diagram indicates the presence of some combinatorial structure on A-spaces. Observing that similar combinatorial structures also arise in other topological objects, such as cell complexes, we expect to see some connections between A-spaces and cell complexes.

2.3. **Simplicial and CW complexes.** We now introduce the players of the other side of our story: *simplicial complexes* and their generalization, *CW complexes.* In algebraic topology, simplicial complexes provide a general class of spaces that is sufficient for most purposes. As cellular spaces, CW complexes are more general than simplicial complexes, but they are equivalent to the eyes of a homotopy theorist: for any CW complex, there exists a simplicial complex with exactly the same homotopy groups.

Definitions 2.13. An (abstract) *simplicial complex* K consists of a set V_K of vertices and a set S_K of finite nonempty subsets of V_K, which we call *simplices.* One requires that every vertex is an element of some simplex, and any subset of a simplex is a simplex. Such subsets are called the *faces* of a given simplex.

The *dimension* of a simplex K is defined to be $\dim K = \text{card}(V_K) - 1$, and the dimension of a complex is the maximal dimension of its simplices.

Given a simplicial complex K, a subset $L \subset K$ is called a *subcomplex* of K if the vertices and simplices of L are subsets of the vertices and simplices of K. Moreover, we say that L is a *full subcomplex* of K if every simplex in K whose vertices are in V_L is also a simplex of L.

Definition 2.14. A map $g : K \to L$ of abstract simplicial complexes is a function $g : V_K \to V_L$ that takes simplices to simplices.

It is very important to distinguish simplicial complexes from *ordered* simplicial complexes, since orderings are essential to understanding the relationship between simplicial complexes and finite spaces.

Definition 2.15. An *ordered* simplicial complex K is a simplicial complex equipped with a partial order on its vertices such that the partial order restricts to a total order on the set of vertices of each simplex in K. A map of ordered simplicial complexes is a map of simplicial complexes that preserves the order on the corresponding poset of vertices.

The above definition of a simplicial complex is very abstract and combinatorial. Often times, we would like to visualize a given simplicial complex by "drawing" it out. The following definitions enable us to embed a simplicial complex in the familiar Euclidean space.

Definition 2.16. Let $\{v_1, v_2 \ldots, v_n\}$ be a set of points in $\mathbb{R}^n$. This set is said to be *geometrically independent* if the set of vectors $v_2 - v_1, \ldots, v_n - v_1$ are linearly independent.

Definition 2.17. Let $\{v_1, v_2 \ldots, v_n\}$ be a geometrically independent set in $\mathbb{R}^n$. Then the n-simplex σ spanned by $v_1, v_2 \ldots, v_n$ is defined as

$$\sigma = \left\{ x = \sum_{i=1}^{n} \alpha_i v_i \right\} \subset \mathbb{R}^n$$

where $\sum_{i=1}^n \alpha_i = 1, \alpha_i \geq 0$ for all $1 \leq i \leq n$. Any simplex spanned by a subset of $\{v_1, v_2 \ldots, v_n\}$ is called a *face* of σ.

A *(geometric) simplicial complex* $K \subset \mathbb{R}^n$ is a collection of simplices in $\mathbb{R}^n$ such that

(1) Every face of a simplex of K is in K;
(2) The intersection of any two simplices of K is a face of each of them.

The *geometric realization* $|K|$ of a simplicial complex K is the union of the simplices of K, each regarded as a subspace of $\mathbb{R}^n$, with the topology whose closed sets are the sets that intersect each simplex in a closed subset. In other words, $U \subset |K|$ is open if and only if $U \cap \bar{\sigma}$ is open in $\bar{\sigma}$ for each $\sigma \in S_K$. If K is finite, this is the same as the topology of $|K|$ as a subspace of $\mathbb{R}^n$, but this is not true in general.

Remark 2.18. Given a map $g : K \to L$ of simplicial complexes, g determines the continuous map $|g| : |K| \to |L|$ that sends $\sum_{i=1}^n \alpha_i v_i$ to $\sum_{i=1}^n \alpha_i g(v_i)$. Although we do not require g to be one–to–one on vertices, $|g|$ is nevertheless well-defined and continuous. Note that if g is a bijection on vertices and simplices, then $|g|$ is a homeomorphism.

Now that we have defined simplicial complexes and the morphisms between them, we will introduce some vocabulary in the world of simplicial complexes. We first start with the *subdivision* of a simplicial complex, which will later be used to construct the correspondence between finite spaces and simplicial complexes.

Definitions 2.19. Let $S = [v_0, \ldots, v_n]$ be an n-simplex in $|K|$. Then elements in S can be expressed as linear combinations of the form $\Sigma_i \alpha_i v_i$ with $\sum_{i=1}^n \alpha_i$ and $\alpha_i \geq 0$ for each i. The *barycenter*, or "center of gravity," of S is the point $b = \sum_{i=1}^n \alpha_i v_i$ where all the barycentric coefficients are equal, i.e., $\alpha_i = 1/(n+1)$ for all i.

Inductively, we define the *barycentric subdivision* of S to be the decomposition of $[v_0, \ldots, v_n]$ into n-simplices $[b, u_0, \ldots, u_{n-1}]$ such that $[u_0, \ldots, u_{n-1}]$ is the barycentric subdivision of a face of S. Starting with $n = 0$, the barycentric subdivision of $[v_0]$ is just $[v_0]$ itself; the case for dimension 1 and 2 is shown below, from where one can induct upwards. We will denote the barycentric subdivision of S by S'.

The barycenters of K are exactly the vertices of K'. Moreover, the simplices $S' \in K'$ are sets spanned by the geometrically independent sets $\{b_{S_0}, \ldots, b_{S_n}\}$ where each S_i is a proper face of S_{i-1}.

Barycentric subdivision is a special case of the more general notion of a subdivision. Given a simplicial complex K, its *subdivision* is a simplicial complex L such that each simplex of L is contained in a simplex of K and each simplex of K is the union of finitely many simplices of L. Note that if L is a subdivision of K, then their geometric realizations are the same as topological spaces. In particular, $|K| = |K'|$.

Another useful construction that we have on simplicial complexes is the *simplicial join* of two simplicial complexes, which will be used later when we develop simple homotopy theory.

Definition 2.20. Let K, L be simplicial complexes. The *simplicial join* $K * L$ is defined to be

$$K * L = K \cup L \cup \{\sigma \cup \tau \mid \sigma \in K, \tau \in L\}.$$

In particular, given a vertex $a \notin K$, the *simplicial cone* aK is the join of a with K.

Note that taking the simplicial join is different from taking the union of K and L. In taking a simplicial join, we are not just collecting the discrete simplices in K and L but rather building a new simplicial complex from the "base" provided by K and L.

For a concrete example, let L be the 2-simplex as defined in Definition 2.17. If K consists of a single point a, then $K * L = aL$ is a genuine cone over the 2-simplex. If K is the discrete space with two points, then $K * L$ is the (unreduced) suspension of L. If we think of K and L as geometric objects in the Euclidean space, then $\dim K * L = \dim K + \dim L + 1$. In general, if K is an abstract simplicial complex, we define the cone aK by adding a new vertex a and choosing the simplices of aK to be all subsets of any union of a with a simplex in K.

So far, we have introduced all the concepts in the world of simplicial complexes that are necessary for the purposes of this paper. We will finish this section with *CW complexes*, which are more general classes of spaces built up from basic building blocks called cells. If one compares the definition of CW complexes and simplicial complexes, one notices that all simplicial complexes are CW complexes.

First, let us fix some notations. We denote by D^n the n-disk,

$$D^n = \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n : ||x||^2 \leq 1\},$$

and by S^{n-1} its boundary, the $(n-1)$-unit sphere,

$$S^{n-1} = \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n : ||x||^2 = 1\}.$$

When $n = 0$, we take the boundary of D^0 to be the empty set. Note that we will also denote by e^n the n-cell as another symbol for D^n. By definition, every CW complex admits a filtration, where each filtration step is obtained from the previous one by attaching cells. We now make precise this process of cell attachment.

Given a space X and a continuous map $f : \partial e^n \to X$, one can construct a new space $X \cup_f e^n$ defined by the pushout diagram,

$$\begin{array}{ccc} \partial e^n & \xrightarrow{f} & X \\ \downarrow{\scriptstyle i} & & \downarrow \\ e^n & \longrightarrow & X \cup_f e^n \end{array}$$

More concretely, the space $X \cup_f e^n$ is obtained from the disjoint union $X \sqcup e^n$, where we identify $i(y) \in e^n$ with $f(y) \in X$ for all $y \in \partial e^n$. The resulting set is equipped with the quotient topology. The space $X \cup_f e^n$ is said to be obtained from X by attaching an n-cell. The map $f : \partial e^n \to X$ is called the *attaching map*, and $e^n \to X \cup_f e^n$ is called the *characteristic map* of the cell e^n.

Definition 2.21. Let X be a topological space. A *CW decomposition* of X is a sequence of subspaces,

$$X^0 \subseteq X^1 \subseteq X^2 \subseteq \cdots \subseteq X^n \subseteq \cdots, \quad n \in \mathbb{N}$$

where

1. The space X^0 is a discrete set of singleton points;
2. The space X^n is obtained from X^{n-1} by attaching a (possibly) infinite number of n-cells $\{e^n_\sigma\}_{\sigma \in J_n}$ via attaching maps $f_\sigma : \partial e^n_\sigma \to X^{n-1}$.
3. We have $X = \bigcup X^n$ with the weak topology.

A (finite) CW complex is a space X equipped with a (finite) CW decomposition.

2.4. **Finite spaces and simplicial complexes.** We now have the necessary machinery to explain how to go back and forth between A-spaces and simplicial complexes. For the purpose of this paper, we will restrict our attention to F-spaces and finite simplicial complexes.

First, let X be an F-space. McCord developed a method of associating to X an abstract simplicial complex $\mathcal{K}(X)$ whose simplicies are exactly the finite nonempty chains in X. A *chain* in X consists of a set of elements $x_1, x_2, \ldots, x_n \in X$ ordered by the partial order given by the poset associated to X (see Proposition 2.10). We call $\mathcal{K}(X)$ the *order complex* of X. Since all the elements in a finite nonempty chain are comparable, the partial order on X is in fact a total order on each chain. Therefore, the partial order on X gives rise to an ordering on $\mathcal{K}(X)$.

If $f : X \to Y$ is an order-preserving map, then f induces a simplicial map $\mathcal{K}(f)$ between $\mathcal{K}(X)$ and $\mathcal{K}(Y)$. In fact, the operator $\mathcal{K}$ is a functor which makes the following diagram commute:

$$\begin{CD} |\mathcal{K}(X)| @>{|\mathcal{K}(f)|}>> |\mathcal{K}(Y)| \\ @VVV @VVV \\ X @>>{f}> Y \end{CD}$$

One can also go the other way around and assign to each finite geometric simplicial complex K a corresponding F-space via another functor $\mathcal{X}$. Let K' be the barycentric subdivision of K. The elements of $\mathcal{X}(K)$ are the barycenters b_σ of simplices $\sigma \in K$, i.e., the vertices of K'. We define a partial order on $\mathcal{X}(K)$ by stipulating that $b_\sigma \leq b_\tau$ if and only if $\sigma \subset \tau$ in K. If we think of $S \in K$ as a subcomplex of K, then $\mathcal{X}(S)$ is the open subspace $U_{b_S} \subset \mathcal{X}(K)$.

To define how $\mathcal{X}$ acts on maps between simplicial complexes, let $\varphi : K \to L$ be a simplicial map. For any simplex S in K, define $\mathcal{X}(\varphi) : \mathcal{X}(K) \to \mathcal{X}(L)$ by $\mathcal{X}(\varphi)(b_S) = b_{\varphi(S)}$. Since $\mathcal{X}(\varphi)$ preserves order, it is a continuous map. Moreover, note that $\mathcal{K}\mathcal{X}(K) = K'$ and $\mathcal{K}\mathcal{X}(\varphi) = \varphi'$. This gives us the following commutative diagram:

$$\begin{CD} |K| @>{|\varphi|}>> |L| \\ @VVV @VVV \\ \mathcal{X}(K) @>>{\mathcal{X}(\varphi)}> \mathcal{X}(L) \end{CD}$$

2.5. **Homotopy theory.** Having explained how one can go back and forth between F-spaces and finite simplicial complexes, we will now define and compare types of homotopy equivalences in these two worlds. This will serve as the motivation for our study of finite space reductions in Sections 3 and 4.

We begin with homotopy theory for general topological spaces and then look at finite spaces and CW complexes separately. When we say that two spaces are homotopy equivalent, the idea is that one can be obtained from the other via continuous deformations.

Definition 2.22. Let X, Y be topological spaces. A map $f : X \to Y$ is a *homotopy equivalence* if there is a map $g : Y \to X$ such that $f \circ g \simeq \mathbf{1}_Y$ and $g \circ f \simeq \mathbf{1}_X$. If such f and g exist, we say that X and Y are *homotopy equivalent* or that they have the same *homotopy type*, and write $X \simeq Y$.

It is not difficult to see that the notion of homotopy is an example of natural transformations between categories, where the objects are topological spaces and functors are continuous maps. Note that homotopy equivalence is an equivalence relation. We also have the weaker notion of a *weak homotopy equivalence* between topological spaces.

Definitions 2.23. Let X be a topological space. The nth *homotopy group* $\pi_n(X, x_0)$ is the set of homotopy classes of based maps $f : S_n \to X$, where $x_0 \in X$ is an arbitrary fixed point.

A map $f : X \to Y$ is a *weak homotopy equivalence* if it induces isomorphisms $\pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ for all $n \geq 0$ and any basepoint $x_0 \in X$.

If two spaces X and Y are homotopy equivalent, then they are weakly homotopy equivalent. This is illustrated by the commutative diagram below.

$$\begin{array}{ccc} \pi_n(X,x) & \xrightarrow{(gf)_*} & \pi_n(X, gf(x)) \\ {\scriptstyle f_*}\downarrow & \searrow \nearrow & \downarrow{\scriptstyle f_*} \\ \pi_n(Y, f(x)) & \xrightarrow[(fg)_*]{} & \pi_n(Y, fgf(x)) \end{array}$$

Suppose that X and Y are homotopy equivalent, and let $f : X \to Y$, $g : Y \to X$ be homotopy equivalences. Then $f \circ g \simeq \mathbf{1}_Y$, $g \circ f \simeq \mathbf{1}_X$. Since π_n is invariant under homotopy, $(fg)_*$ and $(gf)_*$ are isomorphisms by the "2-of-6 property" of homotopical categories (see [12]). Hence X and Y are weakly homotopy equivalent.

The notions of homotopy equivalence and weak homotopy equivalence coincide in the setting of CW complexes, according to a theorem by Whitehead. Before stating the theorem, we first define the notion of a *deformation retract*, which appears in the second part of the theorem.

Definition 2.24. Let Y be a space. A subspace $X \subset Y$ is a *deformation retract* if there is a homotopy $h : Y \times I \longrightarrow Y$ such that $h(y, 0) = y, h(x, t) = x$, and $h(y, 1) \in X$ for all $x \in X, y \in Y$, and $t \in I$. In this case, we call the homotopy h a *deformation* of Y onto X.

Theorem 2.25 (Whitehead)**.** *Let X, Y be connected CW complexes. If a map $f : X \to Y$ is a weak homotopy equivalence, then it is a homotopy equivalence. Moreover, if f is the inclusion of a subcomplex $X \hookrightarrow Y$, then X is a deformation retract of Y.*

A proof of this theorem can be found in Chapter 4 of [4], where the key ideas are the homotopy extension and lifting property. As a corollary, this theorem holds for spaces that are homotopy equivalent to CW complexes (it is a fact that every space is weakly homotopy equivalent to a CW complex in the sense of being *approximated* by a CW complex; see Section 5.5 of [7] for a proof).

Nevertheless, the Whitehead theorem fails patently for finite spaces: there are plenty of counterexamples of weak homotopy equivalences between finite spaces that are far from being homotopy equivalences. As an example, consider the "quasi-circle" given in Chapter 4 of [4], which is a noncontractible space whose homotopy groups are all trivial. Another example is the "wallet" (discussed below), which shows that the Whitehead Theorem may fail for a finite space even if it is homotopically trivial. This crucial difference between finite spaces and CW complexes will play an important role in our subsequent discussion.

In the rest of this section, we will consider the relationship among different kinds of homotopy equivalences in simplicial complexes and finite spaces. A key definition here is the notion of *simple homotopy equivalence* due to Whitehead.

Although simple homotopy equivalence is defined for both simplicial complexes and finite spaces, we will postpone the definition for finite spaces until the next section, where we will introduce the idea of a *weak point* that lies at the heart of this definition. For now, we will prove a convenient lemma which we will frequently make use of later when studying one-point reductions of finite spaces.

In Section 2.1, we defined continuous functions on a finite space and showed that they correspond to order-preserving maps on posets. If $f, g : X \to Y$ are continuous and $f \leq g$, then f is homotopic to g. This implies the following lemma.

Lemma 2.26. *Any finite space with a maximum or a minimum is contractible, i.e., it is homotopy equivalent to a point.*

Proof. Let X be a finite space, with $x \in X$ a maximum. Let g be the constant map that equals x and f the identity map $\mathbf{1}_X$. Then $f \leq g$, so $f \simeq g$, which means that the entire space X and the singleton set $\{x\}$ have the same homotopy type. Hence X is contractible. The proof where X has a minimum is analogous. □

We will leave the world of finite spaces for now and define simple homotopy equivalences of simplicial complexes.

Definition 2.27. Let K be a finite simplicial complex and $L \subset K$ be a subcomplex. We say that K collapses to L via an *elementary simplicial collapse* and write $K \searrow^e L$ if there exists a simplex $S \in K$ and a vertex $a \in K$ that is not contained in S such that

$$K = L \cup aS \text{ and } L \cap aS = a\partial S.$$

In other words, K collapses to L via an elementary simplicial collapse if there are only two simplices $S, S' \in K$ disjoint from L such that S is a free face of S', i.e., S' is the only simplex disjoint from L that contains S as a face.

Definition 2.28. We say that K *(simplicially) collapses* to L or L *(simplicially) expands* to K if L can be obtained from K via a sequence of elementary collapses. We denote this by $K \searrow L$ or $L \nearrow K$. Two complexes K and L have the same *simple homotopy type* if there exists a sequence of simplicial complexes $K = K_1, K_2, \ldots, K_n = L$ such that $K_i \searrow K_{i+1}$ or $K_i \nearrow K_{i+1}$ for all $1 \leq i \leq n$.

For a concrete example, consider the sequence of elementary collapses below, which can be found in [1].

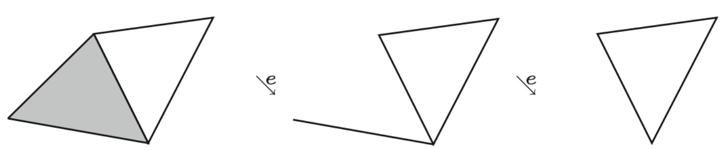

FIGURE 2. Elementary simplicial collapses

We say that a simplicial complex K is *collapsible* if it collapses to one of its vertices. For example, any simplicial cone aK is collapsible. The key observation here is that simple homotopy equivalence is a special case of homotopy equivalence, as we show below.

Proposition 2.29. *If two simplicial complexes are simple homotopy equivalent, then they are homotopy equivalent.*

Proof. Let K, L be two simplicial complexes. Without loss of generality, let $L \subset K$ be a subcomplex and suppose that K collapses to L via an elementary simplicial collapse. Then there exists some simplex $S \in K$ and a vertex $a \in K, a \notin S$ such that $K = L \cup aS$ and $L \cap aS = a\partial S$. Note that the inclusion $i : L \cap aS \hookrightarrow aS$ is a homotopy equivalence. Applying the gluing theorem (Theorem A.2.5 in [1]) to the diagram below,

$$\begin{array}{ccc} L \cap aS & \xrightarrow{i} & aS \\ {\scriptstyle \cap}\downarrow & & \downarrow \\ L & \xrightarrow[I]{} & K \end{array}$$

we see that the inclusion $I : L \hookrightarrow K$ is also a homotopy equivalence. $\square$

This yields the following proper containment of types of homotopies between simplicial complexes, where $\mathcal{S}$ denote the set of simple homotopy equivalences. A theorem by Whitehead shows that a homotopy equivalence between simplicial complexes is a simple homotopy equivalence precisely when the Whitehead torsion τ vanishes (see [9] for details).

$$\mathcal{S} \subset \{\text{Homotopy equivalence}\} = \{\text{Weak Homotopy Equivalence}\}$$

In finite spaces, as we will show in the next section, a different relation holds:

$$\{\text{Homotopy equivalence}\} \subset \mathcal{S} \subset \{\text{Weak equivalence}\}$$

In both cases, the containment is proper. A natural question to ask is that, does there exist some kind of homotopy equivalence between simple homotopy and homotopy equivalence of CW complexes? In other words, can we define a new class of homotopy equivalences that will "fill in" the first chain of set containment? The close correspondence between finite spaces and simplicial complexes suggests that we may find an answer by examining the hierarchy of homotopy equivalences of finite spaces. This will be the main topic of the next section.

3. One-point reduction of finite spaces

In this section, we study three types of one-point reductions of finite spaces, namely the removal of beat points, weak points, and γ-points, and consider what kind of homotopy equivalences they induce on the corresponding order complexes.

3.1. **Beat points.** Stong coined the definition of linear and colinear points in 1966 to describe the homotopy types of finite spaces. Following the convention of Peter May, we will call linear and colinear points upbeat and downbeat points in this paper. Stong showed that the removal and inclusion of beat points generate all homotopy equivalences between finite spaces. That is, two finite spaces are homotopy equivalent if and only if one can be obtained from another by successively removing or adding beat points.

Definition 3.1. Let X be a finite space.

(1) A point $x \in X$ is *upbeat* if there is a $y > x$ such that $z > x$ implies $z \geq y$;
(2) A point $x \in X$ is *downbeat* if there is a $y < x$ such that $z < x$ implies $z \leq y$;
(3) A point $x \in X$ is a *beat point* if it is either an upbeat or a downbeat point.

Equivalently, x is an upbeat point if $\hat{F}_x$ (defined in 3.1) has a minimum and a downbeat point if $\hat{U}_x$ has a maximum, where $\hat{U}_x = U_x \backslash \{x\} = \{y \in X \mid y \leq x\} \backslash \{x\}$

and $\hat{F}_x = F_x \backslash \{x\} = \{y \in X \mid x \leq y\} \backslash \{x\}$. Note that F_x can also be thought of as the closure of $\{x\}$ under the T_0 topology.

In the Hasse diagram of X, a point is an upbeat point if it has only one edge connected to it from above, and a downbeat point if it has only one connected edge from below. Note that an upbeat point in X is a downbeat point in X^{op}, and vice versa. If a T_0-space has no beat points, then we call it a *minimal finite space*.

Definitions 3.2. Given a finite space X, we say that a subspace $Y \subset X$ is a *core* of X if it is a minimal finite space and a deformation retract of X.

Using the notation from Definition 3.1, if we think in terms of the Hasse diagram of X and consider an upbeat point $x \in X$, then, intuitively, erasing the edge between x and y should not change the homotopy type of X. The next theorem says that this is indeed the case.

Theorem 3.3. *If x is a beat point, then $i : X \backslash \{x\} \hookrightarrow X$ is a deformation retract. In particular, $X \backslash \{x\}$ is homotopy equivalent to X.*

Proof. Without loss of generality, let x be an upbeat point. Then there exists some $y > x$ such that $z > x$ implies $z \geq y$. Define $f : X \longrightarrow X \backslash \{x\}$ to be the map where $f(z) = z$ if $z \neq x$ and $f(x) = y$. We will use this map to construct the desired deformation. To do so, we first prove that f is a homotopy equivalence. Note that $f \geq \mathbf{1}_X$.

To see that f preserves order, let $u, v \in X$ and without loss of generality suppose that $u \leq v$. Our goal is to show that $f(u) \leq f(v)$. This is obvious if $u = v = x$ or neither of u, v equals x. Otherwise, suppose that $u = x < v$, which implies that $f(u) = y, f(v) = v \geq y$. If $u < x = v$, then $f(u) = u < x < y = f(v)$. This shows that f is order-preserving and therefore continuous. Hence there exists a homotopy h between f and $\mathbf{1}_X$ such that $h(x, t) = f(x)$ for all t and all $x \in X$, which gives us the desired deformation.

Note that the proof where x is a downbeat point is identical by using the fact that upbeat points in X are exactly the downbeat points in X^{op}. □

As a corollary, every F-space X admits a core, since one can start with X and proceed inductively, defining X_i from X_{i-1} by deleting a beat point. This process must terminate after finitely many steps when there are no more upbeat or downbeat points left, thereby yielding the core of X as desired.

Further, Stong proved in [13] that every homotopy equivalence between minimal finite spaces is in fact a homeomorphism. This implies that the core of any finite space X is unique up to isomorphism, since if C, C' are two distinct cores of X, then they must be homeomorphic as a result of the trivial homotopy of X to itself. In other words, if we think of finite spaces as classified into homotopy types, each type has a representative which is "minimal" in the sense that it is the smallest finite space that is homotopy equivalent to every space in the homotopy class. This unique-up-to-homeomorphism representative must be a minimal finite space, since it is homotopy equivalent to its core.

We also have the following useful corollary.

Corollary 3.4. *A finite space X is contractible if and only if one can remove beat points from X one at a time to obtain a space consisting of only one point.*

We now consider what kind of homotopy equivalence a beat point removal will induce on the order complex associated to the original finite space. As one would reasonably expect, removing a beat point from a finite space X does not change the simple homotopy type of $\mathcal{K}(X)$. This result was first proved by Osaki [11].

Theorem 3.5 (Osaki)**.** *If x is a beat point, then $\mathcal{K}(X)$ collapses to $\mathcal{K}(X\backslash\{x\})$.*

Since two finite spaces are homotopy equivalent if and only if one can be obtained from another by successively removing and adding beat points, this theorem generalizes readily to the following corollary.

Corollary 3.6. *If X and Y are homotopy equivalent, then $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ have the same simple homotopy type.*

There is one thing unsatisfactory, however, about this corollary: its converse is false. To see that, consider the following example given by Barmak and Minian [1].

Example 3.7. The finite space W (inspired by its resemblance to a wallet), which we draw below, has no beat points and is therefore non-contractible. Nevertheless, if one follows the definition of a order complex and draws out $\mathcal{K}(W)$, one sees that $\mathcal{K}(W)$ is contractible. In fact, Theorem 3.18 will imply that $\mathcal{K}(W)$ is simple homotopy equivalent to a point.

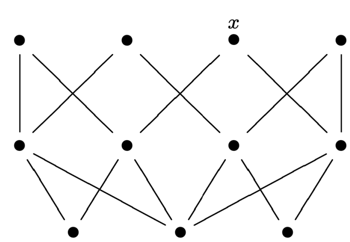

Figure 3. W

This example suggests that homotopy equivalence of finite spaces is a "stronger" relation than simple homotopy equivalence of simplicial complexes. To put it more precisely, the set of homotopy equivalences in simplicial complexes that are induced by removal of beat points from finite spaces, which we sometimes call *strong homotopy equivalence*, is a proper subset of the set of simple homotopy equivalences of simplicial complexes. Correspondingly, the removal of a beat point from a finite space is a "stronger" move than an elementary collapse in simplicial complexes.

This observation naturally gives rise to the following question: does there exist an "elementary move" in finite spaces that would precisely correspond to an elementary collapse in simplicial complexes? It is for precisely this reason that Barmak and Minian [1] introduced the notion of a weak point. In particular, we will show that the point $x \in W$ in the above example is a weak point, and that $\mathcal{K}(W)$ is homotopically trivial.

3.2. **Weak points.**

Definition 3.8. Let X be an F-space. We say that $x \in X$ is an *up weak point* if $\hat{F}_x$ is contractible and a *down weak point* or $\hat{U}_x$ is contractible. A point is a *weak point* if it is either an up weak point or a down weak point.

Note that a beat point is necessarily a weak point, since for any beat point x, either $\hat{U}_x$ has a maximum or $\hat{F}_x$ has a minimum, which makes at least one of these two sets contractible. To lighten the notation, we make the following definitions.

Definitions 3.9. Let X and Y be F-spaces. The *(non-Hausdorff) join* $X \oplus Y$ is defined as the disjoint union $X \sqcup Y$ such that the elements in X and Y are ordered by the given order, with the further condition that $x \leq y$ for all $x \in X$ and $y \in Y$.

Given an F-space X, the *link* of $x \in X$ is defined as $lk(x) = \hat{C}_x = \hat{U}_x \oplus \hat{F}_x$ ($\hat{U}_x$ and $\hat{F}_x$ are defined in Definition 3.1).

The join interacts nicely with the simplicial join via taking the order complex.

Remark 3.10. Let X, Y be F-spaces. Then $\mathcal{K}(X \oplus Y) = \mathcal{K}(X) * \mathcal{K}(Y)$.

The following lemma gives us an alternative way to characterize weak points.

Lemma 3.11. *Let X, Y be F-spaces. Then $X \oplus Y$ is contractible if and only if either X or Y is contractible.*

Proof. Without loss of generality, suppose that X is contractible. By Corollary 3.4, we can find a decreasing sequence of spaces

$$X = X_n \supset X_{n-1} \supset \dots X_1 = \{*\},$$

where we remove beat points from X one by one such that each X_i contains i points and $x_i \in X_i$ is a beat point. Note that x_i is also a beat point of $X_i \oplus Y$, so $X \oplus Y$ inductively deformation retracts to $\{*\} \oplus Y$, which has a minimum and is therefore contractible. The argument where Y is contractible is exactly analogous if one replaces minimum by maximum at the end.

Conversely, suppose that $X \oplus Y$ is contractible. Again by Corollary 3.4, there exists a decreasing sequence of spaces

$$X \oplus Y = (X \oplus Y)_n \supset (X \oplus Y)_{n-1} \supset \dots (X \oplus Y)_1 = \{*\},$$

where $(X \oplus Y)_i = \{z_1, z_2, \dots, z_i\}$ such that z_i is a beat point of $(X \oplus Y)_i$.

Fix some $2 \leq i \leq n$, and suppose that $z_i \in X_i$. Then z_i is a beat point of X_i unless it is a maximal point of X_i, Y_i has a minimum, and $X_i \backslash \{z_i\}$ has no maximum. Similarly, if $z_i \in Y_i$, then either z_i is a beat point of Y_i or X_i has a maximum and $Y_i \backslash \{z_i\}$ has no minimum. Thus for every i, at least one of the following statements is true: (1) either $X_{i-1} \hookrightarrow X_i$ or $Y_{i-1} \hookrightarrow Y_i$ is a deformation retract, and (2) one of X_i and Y_i is contractible. Hence X or Y is contractible, as desired. □

Proposition 3.12. *Let X be an F-space. Then $x \in X$ is a weak point if and only if $lk(x) = \hat{C}_x$ is contractible.*

As we showed in Theorem 3.3, if x is a beat point of X, then $X \backslash \{x\}$ is homotopy equivalent to X. This is no longer true if we replace beat points with weak points. Nevertheless, a weaker version of this result holds.

Proposition 3.13. *Let X be an F-space, and let $x \in X$ be a weak point. Then the inclusion $i : X \backslash \{x\} \hookrightarrow X$ is a weak homotopy equivalence.*

The proof of this proposition makes use of the notion of a *basis like open cover*, which we now define. This definition makes sense in general topological spaces, but we shall only be concerned with its application in F-spaces.

Definition 3.14. Let X be a topological space and $\mathcal{U}$ an open cover of X. We say that $\mathcal{U}$ is a *basis like open cover* if $\mathcal{U}$ is a basis for a topology on the underlying set of X (which is allowed to differ from the topology on X). In other words, for any $U_1, U_2 \in \mathcal{U}$ and $x \in U_1 \cap U_2$, there exists some $U_3 \in \mathcal{U}$ such that $x \in U_3 \subseteq U_1 \cap U_2$.

Note that for any F-space X, the minimal basis $\{U_x\}_{x\in X}$ is a basis like open cover. The following theorem due to McCord [8] is the key to the proof of the preceding proposition. Essentially, it says that a map is a weak homotopy equivalence if it is a weak homotopy equivalence on every element of the basis like open cover. That is, we can piece together the information on local behaviour of a map to determine its global properties.

Theorem 3.15 (McCord)**.** *Let X and Y be topological spaces and $f : X \to Y$ a continuous map. Suppose that there exists a basis like open cover $\mathcal{U}$ of Y such that for every $U \in \mathcal{U}$, the restriction*

$$f|_{f^{-1}(U)} : f^{-1}(U) \to U$$

is a weak homotopy equivalence. Then $f : X \to Y$ is a weak homotopy equivalence.

We now give the proof of Proposition 3.13.

Proof. Without loss of generality, suppose that x is a up weak point. Then $\hat{F}_x$ is contractible. Let $y \in X$. Then the set $i^{-1}(F_y) = F_y \backslash \{x\}$ has a minimum if $y \neq x$, and is contractible if $y = x$. Hence the restricted map

$$i\big|_{i^{-1}(F_y)} = i^{-1}(F^y) \to F_y,$$

is a weak homotopy equivalence, since the map $\pi_n(i^{-1}(F_y), y) \to \pi_n(F_y, y)$ is an isomorphism for all n. As remarked above, the minimal basis of X is a basis like open cover of X. Now applying Theorem 3.15 to the minimal basis of X shows that the restricted inclusion is a weak homotopy equivalence.

The case where x is a down weak point follows immediately by applying the above argument to X^{op}, noting that $\mathcal{K}(X^{op}) = \mathcal{K}(X)$. □

To illustrate this proposition, let us return to Example 3.7, as promised at the end of the last section. To see that the point x is a weak point, we draw out the subspace $\hat{U}_x$ as follows.

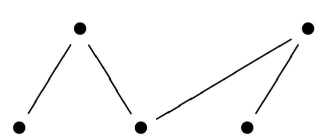

FIGURE 4. $\hat{U}_x$

Clearly, $\hat{U}_x$ is contractible, so x is a weak point. Hence Proposition 3.13 tells us that W is weak homotopy equivalent to $W \backslash \{x\}$, whose Hasse diagram looks like the following:

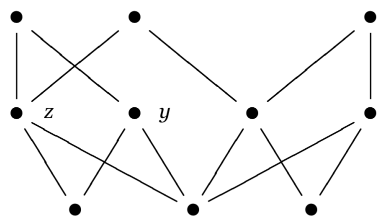

FIGURE 5. $W\backslash\{x\}$

$W\backslash\{x\}$ is contractible because we can remove beat points one by one (starting with the point y as labeled in the diagram, then proceed to z, and so on), eventually obtaining a space consisting of a single point. This motivates the following definition.

Definition 3.16. Let X be an F-space and $Y \subset X$ a subspace. We say that X collapses to Y by an *elementary collapse* (or that Y expands to X by an *elementary expansion*) if Y is obtained from X by removing a weak point. In this case, we denote $X \searrow^e Y$ or $Y \nearrow_e X$.

In general, given two F-spaces X and Y, we say that X collapses to Y (or Y expands to X) if there is a sequence of F-spaces $X = X_1, X_2, \ldots, X_n = Y$ such that for each $1 \leq i < n, X_i \searrow^e X_{i+1}$. In this case, we write $X \searrow Y$or $Y \nearrow X$. Two F-spaces X and Y are *simply equivalent* if one can be obtained from another via a sequence of elementary collapses and expansions.

Before stating the following corollary from Proposition 3.13, we make a quick note on convention: adopting the terminology of Barmak and Minian, we will say that two F-spaces as simply equivalent and two simplicial complexes as simple homtopy equivalent (or having the same simple homotopy type). The crux of Theorem 3.18 is that these two definitions are really describing the same relation for two kinds of objects.

Corollary 3.17. *Let X, Y be two simply equivalent F-spaces. Then they are weakly equivalent.*

The next theorem, which was proved by Barmak and Minian [1] as the main result of simple homotopy theory of finite spaces and simplicial complexes, essentially says that weak points do exactly what we want them to do. That is, removal of weak points is the F-space counterpart to an elementary simplicial collapse in simplicial complexes.

Theorem 3.18 (Barmak and Minian)**.**

1. *Let X and Y be F-spaces. Then X and Y are simply equivalent if and only if $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ have the same simple homotopy type. In particular, if $X \searrow Y$, then $\mathcal{K}(X) \searrow \mathcal{K}(Y)$.*
2. *Let K and L be finite simplicial complexes. Then K and L are simple homotopy equivalent if and only if $\mathcal{X}(K)$ and $\mathcal{X}(L)$ are simply equivalent. In particular, if $K \searrow L$, then $\mathcal{X}(K) \searrow \mathcal{X}(L)$.*

From a category theoretical perspective, the functors $\mathcal{K}$ and $\mathcal{X}$ give a one-to-one correspondence between finite spaces modulo simply equivalence types and finite simplicial complexes modulo simple homotopy types. This is illustrated by the following diagram, where $\mathcal{S}$ denotes the simply equivalence types in finite spaces and simple homotopy types in finite simplicial complexes.

$$\{F-\text{Spaces}\}/\mathcal{S} \underset{\mathcal{K}}{\overset{\mathcal{X}}{\rightleftarrows}} \{\text{Finite Simplicial Complexes}\}/\mathcal{S}$$

We say that an F-space is *collapsible* if it collapses to a point. Similarly, a simplicial complexes is said to be *collapsible* if it simplicially collapses to a single point. Since every beat point is a weak point, the set of contractible F-spaces is a proper subset of collapsible spaces. For example, the wallet W as constructed above is a collapsible space that is not contractible.

3.3. **γ-points.** Recall that our goal is to define a formal class of homotopy equivalences of simplicial complexes that are not simple homotopy equivalences. Having seen that removing weak points induces simple homotopy equivalences in simplicial complexes, we want to relax the condition even further. This motivates the definition of a γ-point.

Definition 3.19. Let X be an F-space. Then $x \in X$ is a γ-point if $\hat{C}_x$ is homotopically trivial. That is, $\pi_n(\hat{C}_x) = 0$ for all $n \geq 0$.

This definition gives us a new method of reduction of finite spaces.

Definition 3.20. We say that X γ-collapses to $X\backslash\{x\}$ by an *elementary γ-collapse* if $x \in X$ is a γ-point. More generally, an F-space X *γ-collapses* to a subspace $Y \subset X$ if there is a sequence of spaces

$$X = X_n \supset X_{n-1} \supset \cdots \supset X_k = Y \quad (n > k)$$

such that X_i γ-collapses to X_{i-1} via an elementary γ-collapse for all $k \leq i \leq n$. In this case, we write $X \nearrow^{\gamma} Y$. If X γ-collapses to a point, we say that X is *γ-collapsible.*

Note that every weak point is a γ-point, since a contractible space necessarily has all trivial homotopy groups. To see what kind of homotopy equivalence a γ-point reduction will induce on simplicial complexes, we first consider the relationship between $X\backslash\{x\}$ and X where $x \in X$ is a γ-point.

Proposition 3.21. *If $x \in X$ is a γ-point, then the inclusion $i : X\backslash\{x\} \to X$ is a weak homotopy equivalence.*

The proof for Proposition 3.13 does not apply directly because neither $\hat{F}_x$ nor $\hat{U}_x$ is necessarily contractible. Nevertheless, the following pushout diagram still holds:

$$\begin{array}{ccc} |\mathcal{K}(\hat{C}_x)| & \xrightarrow{\varphi} & |\mathcal{K}(C_x)| \\ \downarrow{\scriptstyle \psi} & & \downarrow \\ |\mathcal{K}(X\backslash\{x\})| & \longrightarrow & |\mathcal{K}(X)| \end{array}$$

Note that $\varphi : |\mathcal{K}(\hat{C}_x)| \to |\mathcal{K}(C_x)|$ is a homotopy equivalence, and that $\psi : |\mathcal{K}(\hat{C}_x)| \to |\mathcal{K}(X\backslash\{x\})|$ satisfies the homotopy extension property. Hence the map $|\mathcal{K}(X\backslash\{x\})| \to |\mathcal{K}(X)|$ is a homotopy equivalence. This implies that $i : X\backslash\{x\} \to X$ is a weak homotopy equivalence. The converse to this proposition, however, is true only when x is neither maximal nor minimal (Theorem 3.13 in [2]).

If $x \in X$ is a γ-point, one can show that the map $\mathcal{K}(X \backslash \{x\}) \to \mathcal{K}(X)$ is a simple homotopy equivalence (the proof uses the relativity principle of simple homotopy theory; see [3]). In fact, Barmak and Minian [2] proved the following more general result, which says that this is the case whenever we have a weak homotopy equivalence between finite spaces.

Theorem 3.22. *Let X be an F-space, and let $x \in X$. Suppose that the inclusion $i : X \backslash \{x\} \to X$ is a weak homotopy equivalence. Then the induced simplicial map $\mathcal{K}(X \backslash \{x\}) \to \mathcal{K}(X)$ is a simple homotopy equivalence.*

This theorem essentially shows that one-point reductions do not generate all weak homotopy types of finite spaces. In order to answer the question raised at the end of Section 2, we need to look beyond one-point reductions. Before proceeding to discuss how this might be done, we briefly discuss how some of the previous results can be generalized to a broader class of topological spaces.

Everything we have proved so far has taken place in the setting of finite spaces and finite simplicial complexes. In Section 2, we also introduced the more general construction of a CW complex. While we cannot directly take these results for granted in general CW complexes, we can consider them on subsets called *regular* and *h-regular* CW complexes.

Definition 3.23. Let K be a CW complex. We say that K is *regular* if, for each open cell e^n, the characteristic map $D^n \to e^n$ is a homeomorphism. Equivalently, the attaching map $S^{n-1} \to K$ is a homeomorphism onto its image ∂e^n.

For a regular CW complex K, the closure $\overline{e^n}$ of each cell is a subcomplex of K. There is also a more general notion of h-regular CW complex, where one only requires the attaching map of each cell to be a homotopy equivalence with its image and that the closed cells $\overline{e^n}$ are subcomplexes of K.

Theorem 3.18 fails even when we consider only regular CW complexes (see page 60 of [1] for a counterexample). Nevertheless, a weaker version of the second part of Theorem 3.18, as proved in the same book, shows that simplicial collapses of h-regular CW complexes do induce γ-collapses in the corresponding finite spaces.

4. Looking for a new type of homotopy equivalence

In this section, we discuss possible ways to define a class of homotopy equivalence of simplicial complexes that does not belong to simple homotopy equivalence. At the end of Section 2, we noted that a homotopy equivalence between simplicial complexes is a simple homotopy equivalence if both simplicial complexes have trivial Whitehead group. Given two finite simplicial complexes K and L, every homotopy equivalence $f : |K| \to |L|$ is a simple homotopy equivalence if and only if the Whitehead group $Wh(K) = Wh(\pi_1(K))$ is zero (see [1] and work by C.T.C. Wall).

Intuitively, the Whitehead group provides a straightforward way to answer our question: if we can calculate the Whitehead group of a simplicial complex, then we can fix some appropriate constant $c > 0$ and define a new class of homotopy equivalence to be the set of maps where the Whitehead group is less than or equal to c. The interested reader is referred to [9] for details. The key point here, however, is that the Whitehead group is hard to calculate, the Whitehead torsion even more so, and there are little results in the literature that we can use. This computational difficulty led us to consider an alternative strategy, namely to consider multiple-point reductions of finite spaces.

Our first attempt is to generalize the definition of a beat point to a beat pair.

Definition 4.1. Let X be an F-space, and let $x, y \in X$.

(1) The pair (x, y) is an *upbeat pair* if there exists $z > x, z > y$ such that for all $w \in X, w > x, w > y$ implies that $w \geq z$.
(2) The pair (x, y) is a *downbeat pair* if there exists $z < x, z < y$ such that for all $w \in X, w < x, w < y$ implies that $w \leq z$.
(3) The pair (x, y) is a *beat pair* if it is either an upbeat or a downbeat pair.

This definition, however, fails to exclude cases where removing beat pair reduces to an iteration of removing beat points. Consider the example below (taken from an illustration for beat points in [6]):

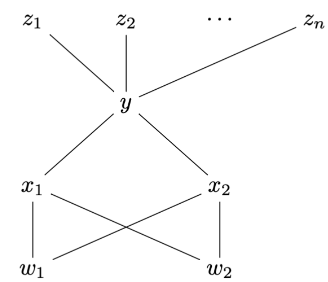

Figure 6. A beat pair

By the definition above, the pair (x_1, x_2) constitutes a beat pair, but x_1 and x_2 can also be regarded separately as beat points. Therefore, we want to look for a new notion of two-point reduction where we require a stronger relationship between the two points of interest so as to ensure that this new method does not boil down to one-point reductions. One possible way to do this is to relax the definition of an *admissible matching* on a poset.

Definition 4.2. Let X be a poset. An edge $(x, y) \in \mathcal{H}(X)$ is *admissible* if the subposet $\hat{U}_y - \{x\}$ is homotopically trivial. We say that a poset is *admissible* if all its edges are admissible.

A related notion from graph theory helps make the above definition more concrete. In graph theory, an admissible graph is a finite directed graph with no directed cycles. In other words, given an vertex x in a graph G, there is no loop (a sequence of edges with consistent direction) that starts at x and winds back to x.

We first consider what type of homotopy equivalence is induced on the order complex $\mathcal{K}(X)$ by removing an admissible edge from the poset X, with some additional assumptions.

Proposition 4.3. *Let X be a poset and $(x, y) \in X$ an admissible edge. Suppose that the inclusion $i : X\backslash\{x\} \hookrightarrow X$ is a weak homotopy equivalence. Then the inclusion $i : X\backslash(x, y) \hookrightarrow X$ is a homotopy equivalence.*

Proof. This proof is essentially applying Theorem 3.22 twice. First, we use Theorem 3.22 to show that $\mathcal{K}(X\backslash\{x\})$ is simple homotopy equivalent to $\mathcal{K}(X)$. It then suffices to consider the relationship between $\mathcal{K}(X\backslash\{x\})$ and $\mathcal{K}(X\backslash(x, y))$. Since

$\hat{U}_y^X \backslash \{x\} = \hat{U}_y^{X\backslash\{x\}}$ is homotopically trivial by assumption, so is the space $\hat{C}_y^{X\backslash\{x\}}$. Here the superscripts are written to clarify the ambient space. In fact, these three spaces are exactly the same if y is a maximal point. This means that y is a γ-point in $X\backslash\{x\}$, and so the inclusion $i : X\backslash(x,y) \hookrightarrow X\backslash\{x\}$ is a simple homotopy equivalence by Theorem 3.22. Hence $X\backslash(x,y), X\backslash\{x\}$, and X all have the same simple homotopy type. □

This proposition tells us that, in order to find a homotopy equivalence that is not a simple homotopy equivalence, we need to relax the restriction on the edge (x,y). This gives us the following notion of a *homologically admissible matching*, a name taken from [10].

Definition 4.4. Let X be a poset. An edge $(x,y) \in \mathcal{H}(X)$ is *homologically admissible* if the subposet $\hat{U}_y - \{x\}$ is acyclic, in the sense that its homology is the equivalent to the homology of the space consisting of only one point.

This is a generalization of Definition 4.2, since homology groups are commutative while homotopy groups are not in general, which means that space with trivial homotopy groups necessarily has all trivial homology groups, but the converse is false. This definition is largely motivated by the following proposition, which is proved in Chapter 6 of [1].

Proposition 4.5. *Let X be a poset, and let $x \in X$. Then the inclusion $i : X\backslash\{x\} \hookrightarrow X$ induces isomorphisms in all homology groups if and only if $\hat{C}_x$ is acyclic, i.e., homologically trivial.*

A homologically admissible matching, however, only indicates that $X\backslash(x,y)$, $X\backslash\{x\}$, and X have the same homology groups, but it alone does not guarantee that these three spaces are homotopy equivalent, which is what we want. To ensure that this is the case, we restrict our attention to a limited class of finite spaces called *simple spaces.*

Definition 4.6. Let X be a path connected space. We say that X is *simple* if $\pi_1(X)$ is abelian and acts trivially on its higher homotopy groups.

The following theorem, for which a proof may be found in [5] and goes back to Whitehead, shows that simple spaces provide a suitable setting in which to consider two-point reductions by removing a homologically admissible edge.

Theorem 4.7. *Let X, Y be simple spaces. Then any integral homology isomorphism $e : X \to Y$ is a weak homotopy equivalence.*

To the best of our knowledge, the question that we raised above is still open. Although this question originally arises in the setting of simplicial complexes, we have seen how one may try to find an answer by looking at other topological objects, such as finite spaces. To provide some motivation for continued investigation of this question, here is an incomplete list of some related areas of interest: simple homotopy theory, as introduced by Whitehead, underlies results like the s-cobordism theorem and the theory of surgery; a conjecture by Quillen, which concerns the poset of non-trivial elementary subgroups of a finite group, can be reinterpreted in the context of finite spaces; the combinatorial side of the theory, which mostly takes place in the setting of posets, gives rise to the relatively new subject of discrete Morse theory, which uses the tool of a Morse function to study properties of simplicial and CW complexes.

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