FROBENIUS ALGEBRA STRUCTURE IN HOPF ALGEBRAS AND COHOMOLOGY RING WITH POINCARÉ DUALITY

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Abstract. This paper aims to present the Frobenius algebra structures in finite-dimensional Hopf algebras and cohomology rings with Poincaré duality. We first introduce Frobenius algebras and their two equivalent definitions. Then, we give a concise construction of FA structure within an arbitrary finite-dimensional Hopf algebra using non-zero integrals. Finally, we show that a cohomology ring with Poincaré duality is a Frobenius algebra with a non-degenerate bilinear pairing induced by cap product.

Contents

1. Introduction

A Frobenius algebra (FA) is a vector space that is both an algebra and coalgebra in a compatible way. Structurally similar to Hopf algebras, it is shown that every finite-dimensional Hopf algebra admits a FA structure [8]. In this paper, we will present a concise version of this proof, focusing on the construction of non-degenerate bilinear pairings.

Another structure that is closely related to Frobenius algebras is cohomology ring with Poincaré duality. Using cap product, we will show there exists a natural FA structure in cohomology rings where Poincaré duality holds.

To understand these structural similarities, we need to define Frobenius algebras and some compatibility conditions.

2. Preliminaries

Definition 2.1. An *algebra* is a vector space A over a field $\mathbf{k}$, equipped with a linear multiplication map $\mu : A \otimes A \to A$ and unit map $\eta : \mathbf{k} \to A$ such that

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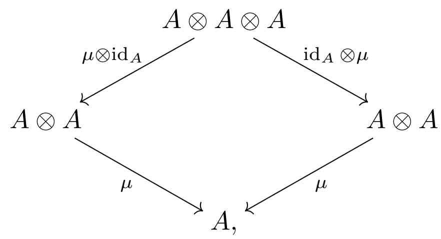
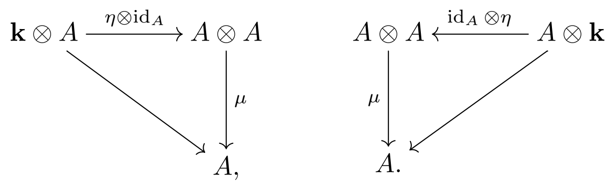
the multiplication is associative and unital, i.e. such that the following diagrams commute:





Example 2.2. For an integer n, the space of all n-by-n matrices over a field $\mathbf{k}$ (e.g. the real numbers) is an algebra. The multiplication $\mu : M^n \otimes M^n \to M^n$ is given by the ordinary matrix multiplication, and the unit map $\eta : \mathbf{k} \to M^n$ by $1_{\mathbf{k}} \mapsto I_{n \times n}$, where $I_{n \times n}$ is the identity matrix.

Definition 2.3. A *coalgebra* is a vector space A over a field $\mathbf{k}$ with a linear comultiplication map $\delta : A \to A \otimes A$ and counit map $\epsilon : A \to \mathbf{k}$ such that the comultiplication is coassociative and counital, i.e. satisfying the dual of the algebra diagrams.

Example 2.4. An interesting example of coalgebra is the *trigonometric coalgebra.* Let $T_{\mathbf{k}}$ be a 2-dimensional $\mathbf{k}$-vector space with basis $\{c, s\}$. Define the comultiplication $\delta : T \to T \otimes T$ and counit $\epsilon : T \to \mathbf{k}$ by

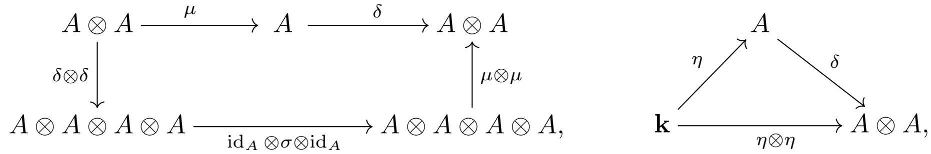
$$\delta(s) = s \otimes c + c \otimes s, \qquad \epsilon(s) = 0,$$
$$\delta(c) = c \otimes c - s \otimes s, \qquad \epsilon(c) = 1,$$

and check that coassociativity and counit condition hold. When $\mathbf{k} = \mathbb{R}$, the comultiplication map represents the following trigonometric identities:

$$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y),$$
$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y).$$

Therefore, $T_{\mathbb{R}}$ represents the 2-dimensional subspace generated by cosine and sine in the space of real functions [2].

Definition 2.5. Define the *twist map* $\sigma : A \otimes A \to A \otimes A$ as $a \otimes b \mapsto b \otimes a$. If B is both an algebra (B, μ, η) and a coalgebra (B, δ, ϵ) over field $\mathbf{k}$, then B is called a *bialgebra* if δ and ϵ are algebra homomorphisms (or equivalently μ and η are coalgebra homomorphisms), i.e. the following diagrams commute:



$$\begin{array}{ccc} & A & \\ {}^{\mu}\nearrow & & \searrow^{\epsilon} \\ A\otimes A & \xrightarrow[\epsilon\otimes\epsilon]{} & \mathbf{k}, \end{array} \qquad \begin{array}{ccc} & A & \\ {}^{\eta}\nearrow & & \searrow^{\epsilon} \\ \mathbf{k} & \xrightarrow[\mathrm{id}_A]{} & \mathbf{k}. \end{array}$$

3. Frobenius Algebras

There are many equivalent definitions of Frobenius algebras, and we will introduce two of them that best illustrate FA's structural similarities to Hopf algebras and Cohomology rings.

The first definition involves the *Frobenius relation*:

Definition 3.1. A *Frobenius Algebra* A is both a finite-dimensional algebra (A, μ, η) and coalgebra (A, δ, ϵ) over field $\mathbf{k}$, satisfying the *Frobenius relation*, i.e. such that the following diagram commutes:

$$\begin{array}{ccccc} & & A\otimes A & & \\ & {}^{\delta\otimes\mathrm{id}_A}\swarrow & & \searrow^{\mathrm{id}_A\otimes\delta} & \\ A\otimes A\otimes A & & & & A\otimes A\otimes A. \\ & {}_{\mathrm{id}_A\otimes\mu}\searrow & & \swarrow_{\mu\otimes\mathrm{id}_A} & \\ & & A\otimes A. & & \end{array}$$

Proposition 3.2. *The preceding Frobenius relation can be equivalently stated as the following communicative diagrams:*

$$\begin{array}{ccc} A\otimes A & \xrightarrow{\mu} & A \\ {\scriptstyle \delta\otimes\mathrm{id}_A}\downarrow & & \downarrow{\scriptstyle\delta} \\ A\otimes A\otimes A & \xrightarrow[\mathrm{id}_A\otimes\mu]{} & A\otimes A, \end{array} \qquad \begin{array}{ccc} A\otimes A & \xrightarrow{\mu} & A \\ {\scriptstyle \mathrm{id}_A\otimes\delta}\downarrow & & \downarrow{\scriptstyle\delta} \\ A\otimes A\otimes A & \xrightarrow[\mu\otimes\mathrm{id}_A]{} & A\otimes A. \end{array}$$

The proof is left as an exercise. (An elegant proof using string diagram can be found in [7], page 40.)

The second definition of FA relies on a non-degenerate pairing:

Definition 3.3. A pairing $\beta : A \otimes A \to \mathbf{k}$ is said to be *non-degenerate* if there exists a linear co-pairing $\lambda : \mathbf{k} \to A \otimes A$ such that they are categorically "self-dual":

$$A = \mathbf{k} \otimes A \xrightarrow{\lambda\otimes\mathrm{id}_A} (A \otimes A) \otimes A = A \otimes (A \otimes A) \xrightarrow{\mathrm{id}_A\otimes\beta} A \otimes \mathbf{k} = A.$$

Definition 3.4. A *Frobenius algebra* is a finite-dimensional $\mathbf{k}$-algebra (A, μ, η), equipped with an associative non-degenerate pairing $\beta : A \otimes A \to \mathbf{k}$, called a *Frobenius pairing*.

Example 3.5. Recall the space of matrices M^n from Example 2.2. It is a Frobenius algebra if the pairing is defined as the trace of the matrix product:

$$\beta(a, b) = \mathrm{Trace}(ab).$$

A great way to understand Frobenius algebras is to study how their two definitions translate into each other:

Theorem 3.6. *The Frobenius algebras defined by Frobenius relation in Definition 3.1 is equivalent to that defined by non-degenerate bilinear pairing as in 3.4.*

Proof. To see Definition 3.1 implies 3.4, let A be a finite-dimensional algebra (A, μ, η) and coalgebra (A, δ, ϵ) that satisfies the Frobenius relation. Define a bilinear pairing $\beta : A \otimes A \to \mathbf{k}$ as $\beta = \epsilon \circ \mu$, and co-pairing $\lambda : \mathbf{k} \to A \otimes A$ as $\lambda = \delta \circ \eta$. The associativity of β is inherited from algebra multiplication μ, so we just need to show its non-degeneracy, i.e. the map $(\mathrm{id}_A \otimes \beta) \circ (\lambda \otimes \mathrm{id}_A)$ is the identity. By our definition of λ and β, we can rewrite:

$$A = \mathbf{k} \otimes A \xrightarrow{\eta \otimes \mathrm{id}_A} A \otimes A \xrightarrow{\delta \otimes \mathrm{id}_A} A \otimes A \otimes A \xrightarrow{\mathrm{id}_A \otimes \mu} A \otimes A \xrightarrow{\mathrm{id}_A \otimes \epsilon} A \otimes \mathbf{k} = A,$$

which by the Frobenius relation, equals to

$$A = \mathbf{k} \otimes A \xrightarrow{\eta \otimes \mathrm{id}_A} A \otimes A \xrightarrow{\mu} A \xrightarrow{\delta} A \otimes A \xrightarrow{\mathrm{id}_A \otimes \epsilon} A \otimes \mathbf{k} = A.$$

This map is the identity map $A \xrightarrow{\mathrm{id}_A} A$ due to the unital and co-unital conditions

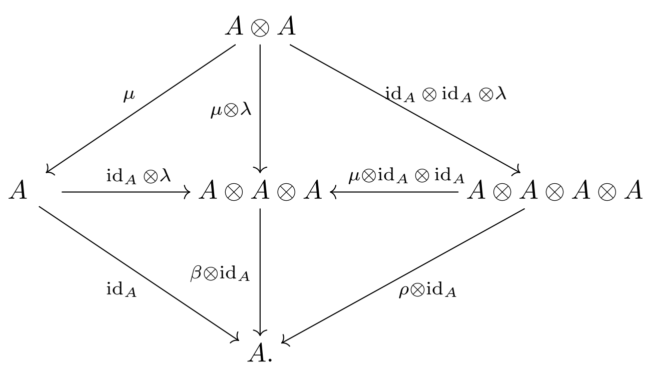
For the other direction, assume A to be a finite-dimensional algebra (A, μ, η) equipped with an associative non-degenerate pairing $\beta : A \otimes A \to \mathbf{k}$ and co-pairing $\lambda : \mathbf{k} \to A \otimes A$. Define comultiplications:

$$\delta, \delta' : A \to A \otimes A, \quad \delta = (\mu \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes \lambda), \quad \delta' = (\mathrm{id}_A \otimes \mu)(\lambda \otimes \mathrm{id}_A).$$

(Note that we denote $f \circ g$ as fg in the rest of this proof.) The key element of this proof is to show $\delta = \delta'$, and other results will soon follow. By associativity of β, we can define $\rho = (\mu \otimes \mathrm{id}_A) \circ \beta = (\mathrm{id}_A \otimes \mu) \circ \beta$. We will first show

$$(\rho \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes \mathrm{id}_A \otimes \lambda) = \mu = (\mathrm{id}_A \otimes \rho)(\lambda \otimes \mathrm{id}_A \otimes \mathrm{id}_A).$$

Consider the following diagram:



The lower right triangle is our definition of ρ, and the lower left triangle follows from β being non-degenerate. On the upper part, both triangles commute since the paths express the same compositions. Repeat symmetrically, we obtain

$$(\mathrm{id}_A \otimes \rho)(\lambda \otimes \mathrm{id}_A \otimes \mathrm{id}_A) = \mu.$$

Thus,

$$\begin{aligned}\delta = (\mu \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes \lambda) &= (\mathrm{id}_A \otimes \rho \otimes \mathrm{id}_A)(\lambda \otimes \mathrm{id}_A \otimes \mathrm{id}_A \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes \lambda) \\ &= (\mathrm{id}_A \otimes \rho \otimes \mathrm{id}_A)(\lambda \otimes \mathrm{id}_A \otimes \mathrm{id}_A \otimes \lambda) \\ &= (\mathrm{id}_A \otimes \mu)(\lambda \otimes \mathrm{id}_A) = \delta'.\end{aligned}$$

The rest of the conditions follow:

(1) The comultiplication is co-associative: this is shown with $\delta'(\mathrm{id}_A \otimes \delta) = \delta(\delta' \otimes \mathrm{id}_A)$.

(2) Frobenius relation is satisfied: using associativity of μ, we can show

$$\begin{aligned}\delta'\mu &= (\mathrm{id}_A \otimes \mu)(\lambda \otimes \mathrm{id}_A)\mu \\ &= (\mathrm{id}_A \otimes \mu)(\mathrm{id}_A \otimes \mu \otimes \mathrm{id}_A)(\lambda \otimes \mathrm{id}_A \otimes \mathrm{id}_A) \\ &= (\mathrm{id}_A \otimes \mu)(\delta' \otimes \mathrm{id}_A).\end{aligned}$$

The other part of the Frobenius relation is shown similarly using δ.

(3) The co-unital maps commutes: Define counits

$$\epsilon, \epsilon' : A \to \mathbf{k}, \quad \epsilon = \beta(\eta \otimes \mathrm{id}_A), \quad \epsilon' = \beta(\mathrm{id}_A \otimes \eta).$$

We can show

$$\begin{aligned}(\epsilon \otimes \mathrm{id}_A)\delta' &= (\beta \otimes \mathrm{id}_A)(\eta \otimes \mathrm{id}_A \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes \mu)(\lambda \otimes \mathrm{id}_A) \\ &= \mu(\beta \otimes \mathrm{id}_A \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes \lambda \otimes \mathrm{id}_A)(\eta \otimes \mathrm{id}_A) \\ &= \mu(\eta \otimes \mathrm{id}_A) \qquad \text{(Non-degeneracy)} \\ &= \mathrm{id}_A . \qquad \text{(Unital map)}\end{aligned}$$

Similarly, $(\mathrm{id}_A \otimes \epsilon')\delta = \mathrm{id}_A$, so $\epsilon = \epsilon'$.

□

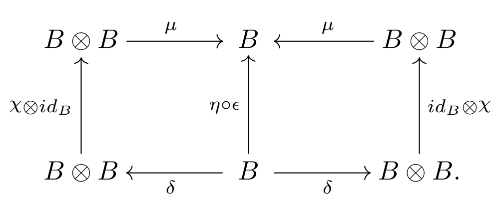
See another proof using 2-dimensional corbordism in [5].

4. Frobenius Algebra Structure in Hopf Algebras

In this section, we will explore the similarities and differences between Frobenius and Hopf algebras. The main theorem is that every finite-dimensional Hopf algebra admits a Frobenius algebra via integral construction. The result was first introduced as a consequence of Larson-Sweedler's theorem in 1969 [4]. We aim to adapt a concise version, focusing on the conclusions directly related to Frobenius algebras.

First, we introduce Hopf algebras:

Definition 4.1. A *Hopf Algebra* B is a bialgebra over field $\mathbf{k}$, equipped with a $\mathbf{k}$-linear map $\chi : B \to B$, called the *antipode*, such that the following diagram commutes:



Proposition 4.2. *Let H be a Hopf algebra. The antipode map $\chi : H \to H$ has the following properties:*

(1) $\chi(h_1 h_2) = \chi(h_2)\chi(h_1)$ *for all* $h_1, h_2 \in H$,

(2) $\epsilon(\chi(h)) = \epsilon(x)$ *for all* $h \in H$.

Example 4.3. (1) If G is a group (possibly infinite, e.g. $G = \mathbb{Z}$), then the group algebra $H = \mathbf{k}[G]$ is a Hopf algebra with canonical basis G. The coalgebra structure is given by

$$\delta(g) = g \otimes g,$$
$$\epsilon(g) = 1.$$

The antipode map is induced by the group inverse:

$$\chi(g) = g^{-1} \quad \forall g \in G$$

and extended linearly.

(2) An universal enveloping algebra $U(\mathfrak{g})$ of a Lie group $\mathfrak{g}$ is also a Hopf algebra. The coalgebra structure and antipode are induced by

$$\delta(x) = 1 \otimes x + x \otimes 1,$$
$$\epsilon(x) = 0,$$
$$\chi(x) = -x \quad \forall x \in \mathfrak{g}.$$

Note that as seen in the examples, Hopf algebras can be either finite- or infinite-dimensional. Yet, Frobenius algebras are limited to be finite-dimensional by definition. So in the rest of the paper, Hopf algebras are assumed to be finite-dimensional.

Hopf and Frobenius algebras are similar in spirit, in that both admit algebra and coalgebra structures. In comparison, FA have a more “topological" set of compatibility conditions, while HA's bialgebra axioms are not meant to be topological at all. They also differ in that Hopf algebras have stricter conditions to fulfill: while FA have the Frobenius relation, HA fulfill both the bialgebra and antipode conditions. Thus, it is natural to conjecture that every Hopf algebra admits a Frobenius algebra structure. However, it is proven that with the same set of algebra and coalgebra maps, Frobenius and Hopf algebras coexist only in very special cases:

Proposition 4.4. *A Frobenius algebra comultiplication $\delta : A \to A \otimes A$ is also a Hopf algebra comultiplication if and only if $A \cong \mathbf{k}$ and $\epsilon = \mathrm{id}_A$ (where $\epsilon : A \to \mathbf{k}$ is the counit).*

The proof is quite straightforward, using FA and HA conditions on comultiplication ([1], page 50-51).

In the general case, we must construct a different coalgebra structure in order to find a FA structure within HA. The key element is *integrals*.

4.1. **Existence of Non-zero Integrals.**

Before proceeding, some notations need to be clarified. We denote multiplication $\mu(a, b)$ as ab, and we often omit writing η since η is the identity map in most contexts. The Sweedler notation may be used for comultiplication:

$$\delta(c) = \sum_c c_1 \otimes c_2,$$
$$(\mathrm{id}_C \otimes \delta)\delta(c) = (\delta \otimes \mathrm{id}_C)\delta(c) = \sum_c c_1 \otimes c_2 \otimes c_3.$$

With this notation, the counit condition is reflected by the equation

$$\sum_c \epsilon(c_1) \otimes c_2 = c = \sum_c \epsilon(c_2) \otimes c_2.$$

Definition 4.5. Let H be a finite-dimensional Hopf algebra with counit $\epsilon : H \to \mathbf{k}$. An element $x \in H$ is called a *left integral* in H if

$$hx = \epsilon(h)x, \quad \forall h \in H.$$

Right integrals are defined symmetrically. We denote the spaces of integrals as:

$$\int_H^\ell = \{x \in H : hx = \epsilon(h)x \quad \forall h \in H\},$$
$$\int_H^r = \{x \in H : xh = \epsilon(h)x \quad \forall h \in H\}.$$

Remark 4.6. Not every Hopf algebra has a non-zero left (or right) integral, but as we will show later, every finite-dimensional Hopf does have a non-zero right (or left) integral, and it is unique up to a scalar.

Example 4.7. Recall the group algebra $\mathbf{k}[G]$ from Example 4.3. It can be checked that the element

$$x = \sum_{g \in G} g$$

is both a right and left integral in $\mathbf{k}[G]$.

Theorem 4.8. *If f in the dual space H^* is a non-zero left (or right) integral in H^*, then the bilinear form it defines*

$$\beta : H \otimes H \to \mathbf{k}, \ \beta(h \otimes k) = f(hk)$$

is associative and non-degenerate.

We will save the proof to later, after we have proven some results.

The theorem shows the importance of non-zero integrals: a Hopf algebra is automatically a Frobenius algebra with a non-degenerate pairing defined by a non-trivial integral. In fact, this integral is also the counit in the new coalgebra structure.

We now aim to show finite-dimensional Hopf algebras have non-zero integrals. The idea is to use the Fundamental Theorem of Hopf Modules [3], which provides an isomorphism between Hopf algebras and its modules. We first show the dual space H^* is a Hopf module over H, and by applying the Fundamental Theorem, we prove $\dim(\int_H^\ell) = 1$.

Definition 4.9. A *left H-module* is a vector space M equipped with a linear map, called the *left action* of H on M,

$$\alpha : H \otimes M \to M$$

that respects multiplication and unit maps:

$$\begin{array}{ccc} H \otimes H \otimes M & \xrightarrow{\mathrm{id}_H \otimes \alpha} & H \otimes M \\ \downarrow{\scriptstyle \mu \otimes \mathrm{id}_M} & & \downarrow{\scriptstyle \alpha} \\ H \otimes M & \xrightarrow[\alpha]{} & M \end{array} \qquad \begin{array}{ccc} H \otimes M & \xleftarrow{\eta \otimes \mathrm{id}_M} & \mathbf{k} \otimes M \\ \downarrow{\scriptstyle \alpha} & \swarrow & \\ M & & \end{array}$$

The *right H-module* is defined symmetrically with a right action. The *left (or right) H-comodule* is dually defined with a *left (or right) co-action* of H on M:

$$\rho : M \to H \otimes M.$$

Definition 4.10. Let M, N be two left H-modules. A linear map $\varphi : M \to N$ is called a *left H-module homomorphism* if it satisfies:

$$\begin{array}{ccc} H \otimes M & \xrightarrow{\mathrm{id}_H \otimes \varphi} & H \otimes N \\ \big\downarrow & & \big\downarrow \\ M & \xrightarrow[\varphi]{} & N. \end{array}$$

A *right H-module homomorphism* is defined symmetrically, and *H-comodule homomorphism* is instead defined with co-actions.

Definition 4.11. A *Hopf module* over H is a vector space M that has both a left H-module and a left H-comodule structure such that the co-action $M \to H \otimes M$ is a H-module homomorphism. Note, $H \otimes M$ is trivially a H-module and comodule induced by the maps: $\mu \otimes \mathrm{id}_P : H \otimes H \otimes M \to H \otimes M$ and $\delta \otimes \mathrm{id}_P : H \otimes M \to H \otimes H \otimes M$, making $H \otimes M$ a Hopf module.

Proposition 4.12. *The dual space H^* of the finite-dimensional Hopf algebra H is:*

(1) *a Hopf algebra;*
(2) *a Hopf module of H.*

Proof. The first observation is made by constructing the following maps:

$$\mu^*(f \otimes g)(h) = \sum_h f(h_1) g(h_2),$$
$$\delta^*(f) = \sum_f f_1 \otimes f_2 \iff \sum_f f_1(a) f_2(b) = f(ab) \quad \forall a, b \in H,$$
$$\epsilon^*(f) = f(1_H),$$
$$\chi^*(f) = f \circ \chi.$$

It can be verified that these maps indeed make H^* a Hopf algebra. Furthermore, replacing H by H^*, we get back the original Hopf algebra as $(H^*)^* \cong H$.

The second observation is made defining the following action and co-action on H:

$$(\cdot) : H \otimes H^* \to H^*, \quad (h \cdot f)(x) = f(\chi(h) x),$$
$$\nabla : H^* \to H \otimes H^*, \quad \nabla(f) = \sum_f f_0 \otimes f_1 \iff$$
$$fg = \mu^*(f \otimes g) = \sum_f g(f_0) f_1 \quad \forall g \in H^*.$$

It can be checked that the co-action is indeed a H-module homomorphism, making H^* a Hopf module. □

Now, we are ready to prove the existence of non-zero right integrals using the Fundamental Theorem of Hopf Modules [3][4]:

Theorem 4.13 (Fundamental Theorem of Hopf Modules)**.** *Let H be finite-dimensional Hopf algebra, and M a Hopf module over H with co-action $\rho : M \to H \otimes M$. There exists an isomorphism $H \otimes M^{coH} \cong M$ as Hopf modules, where the coinvariant of H in M is defined by:*

$$M^{coH} = \{x \in M : \rho(x) = 1_H \otimes x\}.$$

Theorem 4.14. *If H is a finite-dimensional Hopf algebra, then the followings are true:*

(1) *The space of right (or left) integrals is one-dimensional:* $\dim(\int_H^r) = 1$;
(2) *The antipode $\chi : H \to H$ is bijective, and $\chi(\int_H^\ell) = \int_H^r$.*

Proof. To show (1), since H^* is a Hopf module of H, we can apply the Fundamental Theorem and use the co-action to induce an isomorphism:

$$\Psi : H \otimes (H^*)^{coH} \to H^*, \quad h \otimes f \mapsto h \cdot f.$$

We want to show $(H^*)^{coH} = \int_{H^*}^r$:

(1) For any $f \in (H^*)^{coH}$, we have $\nabla(f) = 1_H \otimes f$, which implies $fg = \sum_f g(f_0) f_1 = g(1_H) f = \epsilon^*(g) f$ for any $g \in H^*$. Thus, f is a right integral in the Hopf algebra H^*.
(2) For any $f \in \int_{H^*}^r$, since $fg = \epsilon^*(g) f = g(1_H) f$ for all g, we have $\nabla(f) = 1_H \otimes f$ and $f \in (H^*)^{coH}$.

Since H and H^* have the same dimension, and Ψ is an isomorphism, we must have $(H^*)^{coH} = \int_{H^*}^r$ to be one-dimensional. Repeat the above procedure for H^*, we get $\dim(\int_H^r) = 1$ from $(H^*)^* \cong H$.

To prove (2), let $h \in H$ be such that $\chi(h) = 0$, and we want to show $h = 0$. From (1), there exists some $f \in \int_{H^*}^r$ such that $f \neq 0$. Then,

$$\Psi(h \otimes f)(x) = f(\chi(h)x) = 0 \quad \forall x \in H,$$

which by bijectivity of Ψ implies $h = 0$. Since $\chi : H \to H$ is finite-dimensional and injective, it is bijective.

To see $\chi(\int_H^\ell) = \int_H^r$, we show $\int_H^\ell = \chi^{-1}(\int_H^r)$ instead for convenience. Let x be a non-zero element in $\int_H^r$, and for any $h \in H$, we have

$$\begin{aligned} h\chi^{-1}(x) &= \chi^{-1}(x\chi(h)) && \text{by 4.2} \\ &= \chi^{-1}(\epsilon(\chi(h))x) && \text{by } x \in \int_H^r \\ &= \chi^{-1}(\epsilon(h)x) && \text{by 4.2} \\ &= \epsilon(h)\chi^{-1}(x). \end{aligned}$$

□

Now, we are ready to prove the correspondence between non-zero integrals and non-degenerate pairings:

Proof of Theorem 4.8. Let f be a non-zero left integral in H^*. We want to show $\beta(a,b) = f(ab)$ is an associative non-degenerate bilinear pairing. Clearly, the pairing is bilinear since f is linear, and associative since multiplication is associative: $\beta(ab,c) = f(abc) = \beta(a,bc)$.

Define a map $\Pi : H \to H^*$ as

$$\Pi(h)(k) = f(kh) \quad \forall h,k \in H.$$

(Note the order of the multiplication!) We want to show Π is injective. Since $\chi^* : H^* \to H^*$ is bijective, we can find some $g \in \int_{H^*}^r \setminus \{0\}$ such that $\chi^*(f) = g$. Recall from Theorem 4.14 that

$$\Psi : H \otimes \int_{H^*}^r \to H^*, \quad h \otimes \psi \mapsto h \cdot \psi$$

is an isomorphism, and $\dim(\int^r_{H^*}) = 1$, so the map

$$\Psi_g : H \to H^*, \quad h \mapsto h \cdot g$$

is injective. Let $x \in H$. Suppose $\Pi(x)(h) = f(hx) = 0$ for all $h \in H$. We can write:

$$\begin{aligned}\Psi_g(\chi^{-2}(x))(h) &= g(\chi(\chi^{-2}(x))h) \\ &= \chi^*(f)(\chi^{-1}(x)h) \\ &= f(\chi(\chi^{-1}(x)h)) \\ &= f(\chi(h)x) = 0 \quad \forall h \in H.\end{aligned}$$

This implies $x = 0$ by the injectivity of both Ψ_g and χ^{-1}. Thus, Π is injective. Now, we are ready to show the bilinear pairing is non-degenerate. Here we use an equivalent definition of non-degeneracy in finite dimensions: $b : A \otimes A \to \mathbf{k}$ is non-degenerate if and only if

$$b(x \otimes y) = 0 \text{ for all } x \in A \text{ implies } y = 0.$$

Then, having

$$f(ab) = \Pi(b)(a) = 0 \quad \text{for all } a \in H$$

implies $\Pi(b) = 0$, which gives us $b = 0$. □

Finally, we conclude the main theorem of this section, using Theorem 4.8 and 4.14:

Theorem 4.15. *A finite-dimensional Hopf algebra H admits a Frobenius algebra structure, using the same algebra structure and an associative non-degenerate bilinear pairing:*

$$\beta : H \otimes H \to \mathbf{k}, \ \ \beta(a, b) = f(ab),$$

where f is a non-zero left (or right) integral in H^*.

4.2. **The Converse direction.** To end this section, we briefly discuss the converse direction of Theorem 4.15. Does an arbitrary Frobenius algebra A admit a finite-dimensional Hopf algebra structure?

The quick answer is no, and in fact, not much is understood in this direction. The difficulty lies in constructing the "non-topological" bialgebra structure and antipode map. Arbitrary Frobenius algebras simply do not have enough structure, so one has to weaken the Hopf conditions or strengthen special Frobenius structure to make a connection. For example, using the braided monoidal categories, it is proven that if a Frobenius monoid A is separable, meaning

$$\mu \circ \delta = \mathrm{id}_A,$$

then $A \otimes A$ is a weak Hopf bimonoid [7]. Here, the Hopf structure is weakened by loosening the bialgebra/bimonoid conditions.

5. Frobenius Algebra Structure in Cohomology Ring

Using the non-degenerate pairing definition of Frobenius algebras, we explored its structural similarities with Hopf algebras. In this section, we study how the Frobenius structure lives in cohomology rings with Poincaré duality by constructing a FA pairing using the duality isomorphism.

Definition 5.1. Let X be a topological space and R a coefficient ring. The cap product

$$\cap : C_k(X;R) \times C^\ell(X;R) \to C_{k-\ell}(X;R)$$

for $k \geq \ell$ is defined as

$$\sigma \cap \psi = \psi(\sigma|[v_0, ..., v_\ell])\sigma|[v_\ell, ..., v_k],$$

where $\sigma : \Delta^k \to X$ and $\psi \in C^\ell(X;R)$. This induces a cap product in homology and cohomology $\cap : H_k(X;R) \times H^\ell(X;R) \to H_{k-\ell}(X;R)$ by the formula:

$$\partial(\sigma \cap \psi) = (-1)^\ell(\partial\sigma \cap \psi - \sigma \cap \delta\psi).$$

Theorem 5.2 (Poincaré duality)**.** *If M is an orientable, closed n-manifold (compact and without boundary) with fundamental class $[M] \in H_n(M;R)$, then the map*

$$D : H^k(M;R) \to H_{n-k}(M;R)$$
$$\psi \mapsto [M] \cap \psi$$

is an isomorphism for all k.

Having introduced Poincaré duality, we need a few more maps in order to construct FA pairing in a cohomology ring $H^*(M)$. From now on, we will denote $H_k(M;R)$ as $H_k(M)$ and $H^k(M;R)$ as $H^k(M)$ for simplicity.

First, define the Kronecker index or evaluation map

$$\langle -, - \rangle : H^*(M) \otimes H_*(M) \to R,$$
$$\langle \psi, \sigma \rangle := \begin{cases} \psi(\sigma) & \psi \in H^k, \sigma \in H_\ell, \text{ and } k = \ell \\ 0 & \text{otherwise.} \end{cases}$$

Using the fact that finite-dimensional vector space is naturally isomorphic to its double dual, we have an isomorphism (for each k):

$$**_H : H_{n-k}(M) \to \mathrm{Hom}(\mathrm{Hom}(H_{n-k}(M), R), R)$$
$$\sigma \mapsto \langle -, \sigma \rangle.$$

Given the cap product is adjoint with cup product

$$\cup : H^k(M) \times H^\ell(M) \to H^{k+\ell}(M)$$

with respect to the Kronecker index, we can define an isomorphism (for each k):

$$Ad : \mathrm{Hom}(\mathrm{Hom}(H_{n-k}(M), R), R) \to \mathrm{Hom}(H^{n-k}(M), R)$$
$$\langle -, [M] \cap \psi \rangle \mapsto \langle - \cup \psi, [M] \rangle.$$

Now, we are ready to prove the main theorem:

Theorem 5.3. *A cohomology ring $H^*(M)$ with Poincaré duality is a Frobenius algebra.*

Proof. By composing the above maps, we obtain an isomorphism for each k:

$$Ad \circ **_H \circ D : H^k(M) \to \mathrm{Hom}(H^{n-k}(M), R)$$
$$\psi \mapsto \langle - \cup \psi, [M] \rangle.$$

In a graded manner, these isomorphisms can be composed into a pairing over $H^*(M)$:

$$\beta : H^*(M) \otimes H^*(M) \to R$$
$$\psi \otimes \varphi \mapsto \langle \psi \cup \varphi, [M] \rangle = \langle \psi, [M] \cap \varphi \rangle.$$

Note that this pairing is only non-trivial when the dimensions of ψ and φ add up to n.

We aim to show β is bilinear, associative and non-degenerate in order to conclude that $H^*(M)$ is a Frobenius algebra. Bilinearity and associativity are both inherited from the cup product:

$$\beta(\psi \cup \varphi \otimes \zeta) = \langle \psi \cup \varphi \cup \zeta, [M] \rangle = \beta(\psi \otimes \varphi \cup \zeta).$$

To show it is non-degenerate, since $\varphi \mapsto [M] \cap \varphi$ is an isomorphism, we must have

$$\beta(\psi \otimes \varphi) = \langle \psi, [M] \cap \varphi \rangle = 0 \text{ for all } \varphi \in H^*(M)$$
$$\text{implies } \psi = 0.$$

Thus, with an associative non-degenerate pairing β, the graded cohomology ring $H^*(M)$ with Poincaré duality is a Frobenius algebra. □

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