MULTIPLICATIVE STRUCTURES INDUCED BY N_∞ OPERADS

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Abstract. We survey the basic properties related to N_∞ operads, equivariant generalizations of E_∞ operads. We show that the admissible sets of an N_∞ operad capture the data of the norm maps of its algebra. The $\underline{\pi_0}$ of such N_∞ ring spectra are in general incomplete Tambara functors. We present the detailed structures of these incomplete Tambara functors, both explicitly and in a categorical way. We also show in the appendix that an operad pair induces a self-pairing of the additive operad.

Contents

1. Introduction

In the non-equivariant world, the study of E_∞ operads and their algebras is central to the study of the infinite loop spaces and commutative ring spectra. In particular, May observed that all E_∞ operads are equivalent up to some weak equivalence, and they have homotopically equivalent categories of algebras [6]. A similar story had been studied in the equivariant world. Interestingly, the situation there is much more complicated: there are many different G-operads $\mathcal{O}$ that are non-equivariantly E_∞ but have different categories of algebras. Blumberg and Hill provide a very nice counterexample for this phenomenon in [2, §1]. In particular, we have an operad $\mathcal{E}^{tr}$ with n^{th}-space $E\Sigma_n$ with trivial G-actions (we call it a "naive E_∞ G-operad"), and another operad $\mathcal{E}_G$ with n^{th}-space the universal space for all subgroups of $G \times \Sigma_n$ (we call it a "genuine E_∞ G-operad"). According to Blumberg and Hill, these two E_∞ G-operads have quite different categories of algebras: for almost all positive cofibrant orthogonal G-spectra E,

$$\mathcal{E}^{tr}(n)_+ \wedge_{\Sigma_n} E^{\wedge n} \not\simeq \mathcal{E}_G(n)_+ \wedge_{\Sigma_n} E^{\wedge n}$$

Even more surprisingly, the structure of being a genuine equivariant commutative ring spectra is not necessarily preserved under Bousfield localization, in contrast to the case in the non-equivariant world. A counterexample can be found in Hill and Hopkins' paper [4, 6.1]. We provide another counterexample, suggested by Mike Hill, of Bousfield localization that does not preserve the structure of an equivariant commutative ring spectrum here:

Proposition 1.1. *The commutative C_2-ring spectrum $\mathbb{S} = \Sigma^\infty S^0$ localized with respect to $a_\sigma : S^0 \to S^\sigma$, the Euler class for the sign representation of C_2, is no longer a commutative C_2-ring spectrum.*

Proof. Consider the isotropy separation sequence [1, 2.7]:

$$EC_{2+} \to S^0 \to \widetilde{EC_2}$$

We have in terms of fixed points:

$$\widetilde{EC_2}^{C_2} \simeq S^0, \quad \widetilde{EC_2}^{e} \simeq *$$

Denote the sign representation of C_2 by σ, then we claim that $\widetilde{EC_2} \simeq a_\sigma^{-1} S^0$ where $a_\sigma : S^0 \to S^\sigma$ is the Euler class. In fact, we have

$$a_\sigma^{-1} S^0 \simeq \text{ho} \lim_{\to} (S^0 \overset{-\wedge a_\sigma}{\to} S^\sigma \overset{\wedge a_\sigma}{\to} S^{2\sigma} \cdots) \simeq S^{\infty\sigma}$$

Then

$$(S^{\infty\sigma})^{C_2} \simeq S^0, \quad (S^{\infty\sigma})^e \simeq S^\infty \simeq *$$

We therefore have weak equivalences $\widetilde{EC_2} \simeq S^{\infty\sigma} \simeq a_\sigma^{-1} S^0$. In terms of the corresponding C_2-spectra, we have

$$\Sigma^\infty \widetilde{EC_2} \simeq a_\sigma^{-1} \mathbb{S}$$

On the other hand, via the above analysis of fixed points, we compute that $\underline{\pi_0}(\Sigma^\infty \widetilde{EC_2})$ is the Mackey functor whose value at C_2/C_2 is $\mathbb{Z}$ and whose value at C_2/e is 0. This Mackey functor is a Green functor, but cannot be made into a Tambara functor. In particular, there is no multiplicative unit map $0 \to \mathbb{Z}$. Combining the remark below Theorem 4.21 of this paper, the argument implies that there is no way to give it a genuine equivariant commutative ring structure after this localization. □

A general Bousfield localization result in this context can be found in [4, §6]; however, it is not the primary focus of this paper.

Blumberg and Hill therefore introduced the notion of N_∞ operads. An N_∞ operad has an underlying nonequivariant structure of an E_∞ operad, but equivariantly interpolates between the "naive E_∞ G-operad" and the "genuine E_∞ G-operad" [2]. In terms of the underlying Mackey functor structures, the $\underline{\pi_0}$ of a "genuine" E_∞" G-ring spectrum has all norm maps that make it into a *Tambara functor*, while the $\underline{\pi_0}$ of a "naive E_∞" G-ring spectrum has no such kind of norm maps, and is just a *Green functor*. For N_∞ ring spectra, the structure of $\underline{\pi_0}$ also interpolates between a full set of norms and non-existence of norms. Blumberg and Hill gives a full description of these structures induced by N_∞ operads, called *incomplete Tambara functors*, in [3].

In this paper, mostly expository following [2] and [3], we will survey through the ideas of N_∞ operadic structures. Our goal is to understand how the N_∞ operadic structure induces an incomplete set of "norm" maps as discussed above. We will tell a relatively general story: specifically, what structures an N_∞ operad has in section 2, and what kinds of norm maps on its algebras are induced by a given N_∞ operad in section 3. Section 4 then discusses the structures of incomplete Tambara functors in detail. In particular, the notion of norm maps in section 3 further descends into those purely algebraic structures after applying the functor $\underline{\pi_0}$, which turns out to be an incomplete Tambara functors. There is a further remark about operad pairs and the pairing of operads in the appendix. We will assume that the readers have some knowledge of operads and basic equivariant homotopy theory. (Readers that are only interested to see algebraic structures of incomplete Tambara functors could skip section 3.)

2. N_∞ operads and indexing systems

Definition 2.1. A *family* of subgroups for a group G is a collection of subgroups closed under passage to subgroups and conjugation.

Definition 2.2 ([8], 2.1)**.** A *G-operad* $\mathcal{O}$ consists of a sequence of $G \times \Sigma_n$ spaces $\mathcal{O}(n)$, for $n \geq 0$, such that:

(a) there is a G-fixed identity element $1 \in \mathcal{O}(1)$,
(b) there exist G-equivariant maps:

$$\gamma : \mathcal{O}(k) \times \mathcal{O}(n_1) \times \cdots \times \mathcal{O}(n_k) \to \mathcal{O}(n_1 + ... + n_k)$$

satisfying

$$\gamma(\gamma(c; d_1, ...d_k); e_1, ...e_i) = \gamma(c; f_1, ...f_k) \quad \text{(associativity)}$$

where $i = i_1 + ... + i_k$ and $f_s = \gamma(d_s; e_{i_1+...+i_{s-1}+1}, ..., e_{i_1+...+i_s})$,

$$\gamma(1; d) = d \text{ for } d \in \mathcal{O}_k, \quad \gamma(c; 1^k) = c \text{ for } c \in \mathcal{O}_k \quad \text{(unit)},$$

and for $c \in \mathcal{O}(k), d_s \in \mathcal{O}(i_s), \sigma \in \Sigma_k$ and $\tau_s \in \Sigma_{i_s}$,

$$\gamma(c\sigma; d_{\sigma(1)}, ..., d_{\sigma(k)}) = \gamma(c; d_1, ..., d_k)\sigma(i_1, ..., i_k)$$

$$\gamma(c; d_1\tau_1, ..., d_k\tau_k) = \gamma(c; d_1, ..., d_k)(\tau_1 \oplus \cdots \oplus \tau_k) \quad (\Sigma\text{−equivariant})$$

where if $i = i_1 + ... + i_k$, then $\sigma(i_1, ..., i_k) \in \Sigma_i$ permutes i letters in blocks as σ permutes k letters, and $\tau_1 \oplus \cdots \oplus \tau_k \in \Sigma_i$ is the evident permutation.

A *map* between G-operads $\mathcal{O} \to \mathcal{O}'$ is a sequence of $G \times \Sigma_n$-equivariant maps compatible with the operadic structures. It is a *weak equivalence* if in addition each map $\mathcal{O}(n)^\Gamma \to \mathcal{O}'(n)^\Gamma$ is an equivalence for all subgroups $\Gamma \subset G \times \Sigma_n$ and all $n \geq 0$.

An N_∞ operad is a G-operad whose underlying nonequivariant operad is an E_∞ operad [6, 3.5]. However, to make everything compatible, we need more conditions.

Definition 2.3. An N_∞ *operad* is a G-operad such that

(a) $\mathcal{O}(0)$ is G-contractible,
(b) For $n \geq 1$, Σ_n acts on $\mathcal{O}(n)$ freely,
(c) For $n \geq 1$, $\mathcal{O}(n)$ is a universal space for a *family* $\mathcal{F}_n(\mathcal{O})$ of subgroups of $G \times \Sigma_n$ which contains all subgroups of the form $H \times \{1\}$, that is, for any subgroup $\Gamma \subset G \times \Sigma_n$,

$$\mathcal{O}(n)^\Gamma = \begin{cases} * & \Gamma \in \mathcal{F}_n(\mathcal{O}) \\ \emptyset & \Gamma \notin \mathcal{F}_n(\mathcal{O}) \end{cases}$$

Remark 2.4. It is immediate from the definition that the underlying nonequivariant operad for any N_∞ operad is an E_∞ operad. Moreover, The condition that $\mathcal{O}(n)$ is Σ_n-free implies if $\Gamma \in \mathcal{F}_n(\mathcal{O})$, then $\Gamma \cap (\{1\} \times \Sigma_n) = \{1\}$. There is an observation that $\Gamma \cap (\{1\} \times \Sigma_n) = \{1\}$ if and only if there is a subgroup $H \subset G$ and a homomorphism $H \to \Sigma_n$ such that Γ is the graph of f [2, 4.2]. This is equivalent to imposing an H-set structure on an n-element set. Further analysis with indexing systems heavily relies on this fact.

Definition 2.5. Let $\mathcal{O}$ be an N_∞ operad. An *algebra* over $\mathcal{O}$, either in $G\mathcal{T}op$ (the category of G-spaces) or $\mathcal{S}p^G$ (the category of orthogonal G-spectra), is an object X together with maps:

$$\mathcal{O}(n) \otimes_{\Sigma_n} X^{\otimes n} \to X$$

compatible with the operadic structure. "$\otimes$" denotes the symmetric monoidal product in $G\mathcal{T}op$ or $\mathcal{S}p^G$.

Examples of N_∞ operads and their algebras can be found in [2] and [9]. For the purpose of this paper, we focus on the general structural theory of N_∞ operads.

2.1. **Indexing systems.** We turn to the study of indexing systems. There is a nice correspondence [2, 5.6] between N_∞ operads and indexing systems, in that there is a faithful embedding

$$\underline{\mathcal{C}} : \mathrm{Ho}(\mathcal{N}_\infty\text{–}Op) \to \mathcal{I}$$

where the weak equivalence of N_∞ operads are level-wise $G \times \Sigma_n$-equivalences and $\mathcal{I}$ denotes the poset of all indexing systems. Blumberg and Hill made a conjecture that in fact this is an equivalence of categories, and it is later resolved by J.Rubin [9], J.J. Gutiérrez and D. White [14], and P. Bonventre and L. A. Pereira in [15]. In particular, the indexing system extracts the precise information about the family of subgroups of $G \times \Sigma_n$ an N_∞ operad encodes, which pinpoints the work of figuring out the structures of norm maps.

Definition 2.6. A *symmetric monoidal coefficient system* (SMCS) is a contravariant functor $\underline{\mathcal{C}}$ from the orbit category of G to the category of symmetric monoidal categories with morphisms the strong symmetric monoidal functors.

Definition 2.7. Let $\underline{\mathcal{S}et}$ be the SMCS of finite sets, whose value at $H \subset G$ is $\underline{\mathcal{S}et}(G/H) := \mathcal{S}et^H$, the category of finite H-sets and H-maps. The symmetric monoidal product is given by disjoint union of H-sets.

Definition 2.8. A full sub-SMCS $\underline{\mathcal{F}}$ of $\underline{\mathcal{S}et}$ is *closed under self-induction* if whenever $H/K \in \underline{\mathcal{F}}(G/H)$ and $T \in \underline{\mathcal{F}}(G/K)$, then we have $H \times_K T \in \underline{\mathcal{F}}(G/H)$.

Remark 2.9. Here "sub" means the value at each $H \subset G$ is a subcategory of $\mathcal{S}et^H$. Therefore, we have a natural poset structure on all sub-SMCS of $\underline{\mathcal{S}et}$ ordered by inclusions of subcategories at each orbit.

Definition 2.10. An *indexing system* is a sub-SMCS of $\underline{\mathcal{S}et}$ that

(a) contains all trivial H-sets for all $H \subset G$
(b) is closed under passage to sub-objects (also called truncation subcategory)
(c) is closed under self-induction
(d) is closed under Cartesian product

We now describe the functor $\underline{\mathcal{C}}$ assigning an indexing system to a given N_∞ operad. First notice that for any H-set T with $|T| = n$ and a chosen ordering of its elements, the H-set structure can be rewritten as a homomorphism $H \to \Sigma_n$. Let Γ_T denote the graph of this map.

Definition 2.11. Given any N_∞ operad $\mathcal{O}$, an H-set T is *admissible* if $\Gamma_T \in \mathcal{F}_n(\mathcal{O})$ where $\mathcal{O}(n)$ is the universal space for the family $\mathcal{F}_n(\mathcal{O})$ as in Definition 2.3.

Remark 2.12. This is independent of the choice of ordering of T.

Theorem 2.13. *Let $\underline{\mathcal{C}}(\mathcal{O})$ denote the full sub-SMCS of $\underline{Set}$ whose value at H is the full subcategory of $\mathcal{S}et^H$ spanned by only the admissible H-sets. Then $\underline{\mathcal{C}}(\mathcal{O})$ defines an indexing system.*

The detailed proof checking those four properties can be found in [2, §4]. It is immediate to see it contains all trivial sets and is a truncation subcategory. The other two are shown via careful analysis of operad structures and symmetric group actions.

Theorem 2.14. *Suppose there is a map of N_∞ operads $\mathcal{O} \to \mathcal{O}'$. Then $\underline{\mathcal{C}}(\mathcal{O}) \subset \underline{\mathcal{C}}(\mathcal{O}')$. If in addition the map is a weak equivalence, then $\underline{\mathcal{C}}(\mathcal{O}) = \underline{\mathcal{C}}(\mathcal{O}')$*

Proof. For any admissible set T of $\mathcal{O}$ with $|T| = n$, we have $\mathcal{O}(n)^{\Gamma_T} \simeq * \not\simeq \emptyset$. Since the map $\mathcal{O}(n) \to \mathcal{O}'(n)$ is $G \times \Sigma_n$-equivariant, we have in particular that $\mathcal{O}'(n)^{\Gamma_T} \not\simeq \emptyset$ which concludes the first statement. Now the weak equivalence between operads gives an equivalence $\mathcal{O}(n) \simeq \mathcal{O}'(n)$, and thus they have the same admissible sets. □

Corollary 2.15. *We have a well-defined functor*

$$\underline{\mathcal{C}} : \mathrm{Ho}(\mathcal{N}_\infty\text{–}Op) \to \mathcal{I}$$

Via the indexing system, we can also make precise what we mean by an N_∞ operad interpolates between "naive" E_∞ operads and "genuine" E_∞ operads.

Proposition 2.16. *The indexing system of the "naive E_∞ G-operad" $\underline{\mathcal{C}}(\mathcal{E}^{tr})$ is the initial object of $\mathcal{I}$, and the indexing system of the "genuine E_∞ G-operad" $\underline{\mathcal{C}}(\mathcal{E}_G)$ is the terminal object of $\mathcal{I}$.*

Proof. Since $\mathcal{I}$ is ordered via inclusion, by Corollary 2.15, it is equivalent to show that $\mathcal{E}^{tr}$ has the fewest admissible sets among all N_∞ operads, and $\mathcal{E}_G$ has the most. By definition, the admissible sets for $\mathcal{E}$ are only the sets of the form $H \times \{1\} \subset G \times \Sigma_n$, while the admissible sets for $\mathcal{E}_G$ are all possible subsets $\Gamma \subset G \times \Sigma_n$ with $\Gamma \cap (\{1\} \times \Sigma_n = \{1\})$. The claim then follows. □

3. Norm maps of N_∞ algebras

We now turn to the question about the extra structural maps an N_∞ operad induces on its algebras. For our purpose, we will consider the algebras on the category of orthogonal G-spectra throughout the rest of this paper.

Proposition 2.16 provides an interpretation of N_∞ operads interpolating between the "naive" E_∞ G-operad and the "genuine" E_∞ G-operad. A similar interpolation also appears in the structure of N_∞ algebras in the following sense: there is a notion of "norm" maps induced on N_∞ algebras [2, §6] (not the same as but definitely related to the norm maps of a Tambara functor). A "genuine" E_∞ G-algebra possesses all possible norms, while a "naive" E_∞ G-algebra has no nontrivial norms.

In general, an N_∞ algebra may not have a full set of norms a "genuine" E_∞ G-algebra has, but should bear a sub-collection of norms that distinguishes it from a "naive" E_∞ G-algebra. We now discuss precisely how these norm maps naturally arise and what this sub-collection of norms should be. Our central questions are for which H-set T, and how, we can construct a norm map $N^T E \to E$ on a given N_∞ algebra E. Later our construction will be transformed appropriately to become a multiplicative norm in a Tambara functor.

As previously mentioned, we here consider the N_∞-algebra over orthogonal G-spectra. Note in this case, the "naive" E_∞ G-algebra structure at least guarantees that there is a homotopy coherent multiplication on these equivariant spectra.

Definition 3.1. Given any H-set T with $|T| = n$, H a subgroup of G, and E an orthogonal G-spectrum, we define the G-spectrum

$$N^T E := (G \times \Sigma_n/\Gamma_{T+}) \wedge_{\Sigma_n} E^{\wedge n}$$

Remark 3.2. We have a potential conflict of notation here – $N^T E$ could also refer to the Hill-Hopkins-Ravenel norm [1, 4.6.1]. The following fact resolves the ambiguity [2, 6.2]: for $T = \coprod_i H/K_i$,

$$(G \times \Sigma_n/\Gamma_{T+}) \wedge_{\Sigma_n} E^{\wedge n} \cong G_+ \wedge_H \bigwedge_i N_{K_i}^H res_{K_i}^* E$$

where $N_{K_i}^H$ is the Hill-Hopkins-Ravenel norm. We will not discuss it in detail since it is not our primary focus.

A *norm* on a G-spectrum E is a map

$$N^T E \to E$$

For general T it does not necessarily exist. The exact solution is that for admissible H-sets we can construct such norm maps.

Construction 3.3. Consider T, an admissible H-set for an N_∞ operad $\mathcal{O}$ with $|T| = n$. This means by definition

$$\mathcal{O}(n)^{\Gamma_T} \simeq *$$

where Γ_T is the graph of the H-set structure of T. We also have a canonical identification since an orbit space co-represents the fixed point space [1, 1.1.3]

$$\mathcal{O}(n)^{\Gamma_T} \cong F_{G\times\Sigma_n}((G \times \Sigma_n)/\Gamma_T, \mathcal{O}(n))$$

Therefore we have a contractible space of $G \times \Sigma_n$ maps

$$(G \times \Sigma_n)/\Gamma_T \to \mathcal{O}(n)$$

For E an orthogonal G-spectrum, smashing over Σ_n with $E^{\wedge n}$ gives a contractible space of G-maps

$$(G \times \Sigma_n/\Gamma_{T+}) \wedge_{\Sigma_n} E^{\wedge n} = N^T E \to \mathcal{O}(n)_+ \wedge_{\Sigma_n} E^{\wedge n}$$

Then via the operation of $\mathcal{O}$ on E, $\mathcal{O}(n)_+ \wedge_{\Sigma_n} E^{\wedge n} \to E$, we have a contractible space of maps

$$N^T E \to E$$

In general, for admissible G-sets T, S and a G-map $T \to S$, then for any $\mathcal{O}$ algebra E, we can construct a contractible space of maps $N^T E \to N^S E$. The proof proceeds by decomposing the G-map into orbits and then reducing to the case of $T = G/H$, $S = G/K$, $H \subset K$. Since we have done for the case of $K/H \to K/K$ (Construction 3.3 replacing G by K), we apply $N^{G/K}$ to get $N^{G/H}E \to N^{G/K}E$ (detailed proof in [2, 6.8]).

Here is a punchline for the $\mathcal{O}$-algebras over orthogonal G spectra.

Theorem 3.4. *An $\mathcal{O}$-algebra E has the same data as a G-spectrum with maps*

$$(G \times \Sigma_n/\Gamma_{T+}) \wedge_{\Sigma_n} E^{\wedge n} = N^T E \to E$$

for all admissible H-sets T such that the following condition holds: for all admissible G-sets S and T, we have homotopy commutative diagrams

$$\begin{array}{ccc} N^{S \amalg T}E \simeq N^S E \times N^T E & \longrightarrow & E \times E \\ & \searrow & \downarrow \\ & & E \end{array}$$

$$\begin{array}{ccc} N^{S \times T}E \simeq N^S N^T E & \longrightarrow & N^S E \\ & \searrow & \downarrow \\ & & E \end{array}$$

and if in addition there is some $K \subset G$, such that $i_K^*(S) = i_K^*(T)$, then we have a homotopy commutative diagram

$$\begin{array}{ccc} i_K^* N^S E \simeq N^{i_K^* S} i_K^* E & \longleftrightarrow & i_K^* N^T E \simeq N^{i_K^* T} i_K^* E \\ \searrow & & \swarrow \\ & i_K^* E & \end{array}$$

where $i_K^* : \mathcal{S}et^G \to \mathcal{S}et^K$ denotes the restriction functor.

The idea is that given the above norm maps on X, we can define an $\mathcal{O}$-algebra on X as follows: if we decompose the G-spaces $\mathcal{O}(n)$ into orbits, then the structure map on each orbit is induced from $N^T E \to E$. The first two diagrams ensure the compatibility with the multiplication on E and other norms. The third one shows that the structure is well-behaved upon passage to fixed points [2, 6.11].

4. Norm maps of $\underline{\pi_0}$ of N_∞ ring spectra

In this section, we get into the central part of the discussions of this paper – the structure of *incomplete Tambara functors.*

For any equivariant orthogonal spectrum, by applying $\underline{\pi_0}$ we get a natural *Mackey functor* structure. Conversely, any Mackey functor $\underline{M}$ has a corresponding equivariant Eilenberg-MacLane spectrum $H\underline{M}$ that realizes it [1, 3.3]. A *Green functor* is a commutative monoid in Mackey functors with respect to the symmetric monoidal *box product* [1, 4.2]. Correspondingly, any E_∞ G-ring spectrum R has the underlying structure of a Green functor, where the commutative multiplication is induced by the multiplication of R. There is a special class of Green functors, called the *Tambara functors*, which have the "richest" structures. (We will review its definition in section 4.1.) The reason we single out this class is that Tambara functors can be realized by a "genuine" E_∞ G-ring spectrum, and vice versa [4, 5.12].

The extra structures Tambara functors, but not Green functors, have are the "norm" maps, a multiplicative version of "transfers" in contrast to the additive version of transfer maps in the usual Mackey functor structure. By analogy, the $\underline{\pi}_0$ of an N_∞ ring spectrum should similarly "interpolate" between a Green functor that has no such multiplicative transfers and a Tambara functor that has a full set of multiplicative transfers. In other words, the $\underline{\pi}_0$ of an N_∞ ring spectrum only has a sub-collection of norm maps, determined by the admissible sets. This is summarized in the following theorem:

Theorem 4.1. *If $\mathcal{O}$ is an N_∞ operad and R is an $\mathcal{O}$ algebra of orthogonal G-spectra, then $\underline{\pi_0(R)}$ is a commutative Green functor. If moreover the action $\mathcal{O}$ interchange with itself (Appendix A.5), then for any admissible H-set H/K we have a norm map*

$$N_K^H : \underline{\pi_0(R)}(G/K) \to \underline{\pi_0(R)}(G/H)$$

which is a homomorphism of commutative multiplicative monoids and N_K^H satisfies the multiplicative version of double coset formula.

Remark 4.2. The proof of the above theorem is provided in [2, 7.12]. In particular, the norm maps

$$N_K^H : \underline{\pi_0(R)}(G/K) \to \underline{\pi_0(R)}(G/H)$$

are induced by the canonical projection $\pi_K^H : H/K \to H/H$: for admissible H-set H/K, we have induced map

$$N_K^H R \to R$$

Since $N_K^H R \cong G_+ \wedge_H N_K^H res_K^* R$ by Remark 3.2, applying $\underline{\pi}_0$ gives the desired homomorphisms of commutative multiplicative monoids. This points out the relationship between the norm maps of an N_∞ G-spectrum defined in Section 3 and the norm maps of its corresponding $\underline{\pi}_0$ structure.

Later it will turn out that the information above essentially characterized an incomplete Tambara functor – a Green functor having a partial collection of norm maps. Later in Section 4.2, we will have an alternative description of the collection of norm maps in Theorem 4.1.

4.1. **Tambara functors.** We give a formal review of Tambara functors first. Let us begin with several relevant definitions.

Definition 4.3. (a) A *locally Cartesian closed* category is a category $\mathscr{C}$ whose slice categories $\mathscr{C}/X$ are all Cartesian closed. In particular, this means for any morphism $f : X \to Y$, the pullback functor $f^* : \mathscr{C}/Y \to \mathscr{C}/X$ has both adjoints

$$\Sigma_f \dashv f^* \dashv \Pi_f$$

the left adjoint is called the *dependent sum* and the right adjoint *the dependent product*. In the case of $\mathcal{S}et^G$, the dependent sum is simply "disjoint union of the fibers over y", while the dependent product is the "product of the fibers over y" [3, §2].

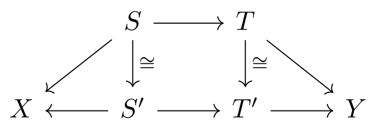
(b) Specifically, in the case of $\mathcal{S}et^G$, for $f : X \to Y$, if $q : A \to X$ is an object of $\mathcal{S}et^G/X$, then its *dependent product* is $q' : \Pi_f A \to Y$ where

$$\Pi_f A := \{(y, s) | y \in Y, s : f^{-1}(y) \to A \text{ such that } q \circ s = id\}$$

(c) If $\mathscr{C}$ is a locally Cartesian closed category, define $\mathscr{P}^{\mathscr{C}}$ to be the category of *"polynomials" in* $\mathscr{C}$ with objects the objects of $\mathscr{C}$ and morphisms the isomorphism classes of polynomials (or "bispans")

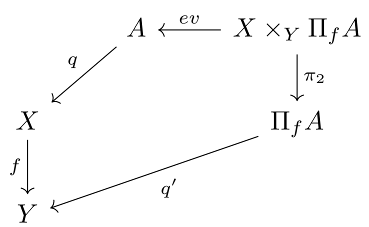
$$X \leftarrow S \rightarrow T \rightarrow Y$$

where the isomorphisms are given by commutative diagrams



The compositions will be discussed in detail later.

(d) An *exponential diagram* in a locally Cartesian closed category $\mathscr{C}$ is an diagram isomorphic to the one of the form



Later a Tambara functor will be a contravariant functor out of $\mathscr{P}^{\mathscr{C}}$. Of the bispan,

$$X \leftarrow S \rightarrow T \rightarrow Y$$

the restriction "comes" from $X \leftarrow S$, the norm "comes" from $S \rightarrow T$, and the transfer "comes" from $T \rightarrow Y$. We will make this precise later in defining compositions. In particular, recall we want transfers to encode additivity and norms to encode multiplicativity. The compatibility of restrictions and transfers, as in Mackey functors, is via the additive version of the double coset formula, and the compatibility of restrictions and norms is via the multiplicative version of the double coset formula (Proposition 4.8(b)).

We need another compatibility condition (Proposition 4.8(c)) between transfers and norms, and this motivates the definition of the *exponential diagram* as above. Though there is no clue to see this at first sight, what an exponential diagram actually packs together is a "distributivity" law. We intentionally write up the diagram in such a way that the horizontal maps will later turn out to be restrictions, vertical maps to be norms, and tilted maps to be transfers.

We exhibit an example, suggested by Foling Zou, explaining why this exponential diagram looks like a "distributivity" law before going into the definition of composition of "bispans".

Example 4.4. Consider $X = \{x_1, x_2\}, Y = \{y\}$, $A = X \coprod \{*\}$. $q : A \rightarrow X$ is the identity on X and maps $*$ to x_1. $f : X \rightarrow Y$ is the only choice. By definition, we see that

$$\Pi_f A = \{(d, s) | s : \{x_1, x_2\} \rightarrow \{x_1, x_2, *\} \text{ such that } q \circ s = id\} \cong \{\bar{q}, \bar{*}\} \cong Y \coprod \{*\}$$

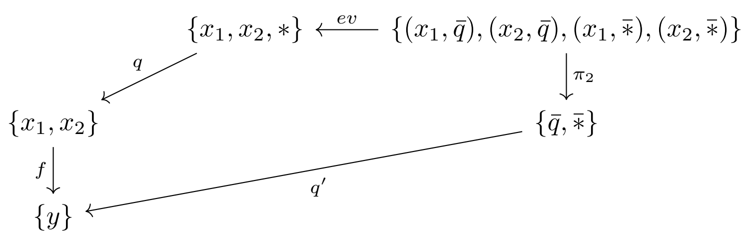
where $\bar{q}, \bar{*} : \{x_1, x_2\} \rightarrow \{x_1, x_2, *\}$ is such that $\bar{q}$ sends $x_1 \mapsto x_1$ and $x_2 \mapsto x_2$, $\bar{*}$ sends $x_1 \mapsto *$ and $x_2 \mapsto x_2$. Then we compute

$$X \times_Y \Pi_f A \cong \{(x_1, \bar{q}), (x_2, \bar{q}), (x_1, \bar{*}), (x_2, \bar{*})\}$$

where the map $ev : X \times_Y \Pi_f A \to A$ is

$$ev(x_1, \bar{q}) = x_1, ev(x_2, \bar{q}) = x_2, ev(x_1, \bar{*}) = *, ev(x_1, \bar{*}) = x_2$$

The vertical map $\pi_2 : X \times_Y \Pi_f A \to \Pi_f A$ is the obvious map.



We now interpret the above diagram as follows (unrigorous but illuminating): the tilted maps q, q' will be denoted as Tr and thought as additive maps. The vertical maps π_2, f will be denoted as N and thought as multiplicative maps. We have

$$N^{-1}(d) = \{x_1, x_2\},\ or\ d \sim x_1 \cdot x_2, \quad Tr^{-1}(d) = \{\bar{q}, \bar{*}\},\ or\ d \sim \bar{q} + \bar{*}$$

$$Tr^{-1}(x_1) = \{x_1, *\},\ or\ x_1 \sim x_1 + *, Tr^{-1}(x_2) = \{x_2\}$$

$$N^{-1}(\bar{q}) = \{(x_1, \bar{q}), (x_2, \bar{q})\},\ or\ \bar{q} \sim x_1 \cdot x_2 \text{ upon restriction } (ev)$$

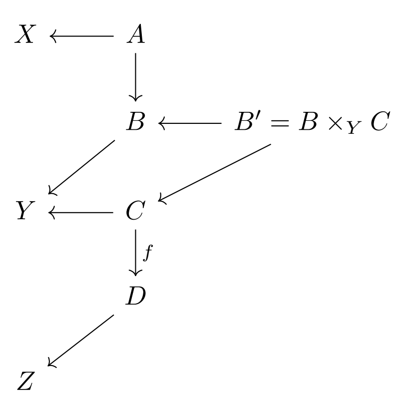
$$N^{-1}(\bar{*}) = \{(x_1, \bar{*}), (x_2, \bar{*})\},\ or\ \bar{*} \sim * \cdot x_2 \text{ upon restriction } (ev)$$

The final relation we get is

$$(x_1 + *) \cdot x_2 \sim d \sim x_1 \cdot x_2 + * \cdot x_2$$

which is the distributivity law we usually see.

Definition 4.5 (Composition in $\mathscr{P}^{\mathscr{C}}$)**.** We follow Blumberg's notes for a concrete exhibition: Given any two classes of bispans $[X \leftarrow A \to B \to Y]$ and $[Y \leftarrow C \to D \to Z]$, we first form pullbacks $B' = B \times_Y C$



Then we can form an exponential diagram on the right

$$\begin{array}{ccccccc}
X & \longleftarrow & A & & & & \\
 & & \big\downarrow & & & & \\
 & & B & \longleftarrow & B' & \longleftarrow & C \times_D \Pi_f B' \\
 & \swarrow & & \swarrow & & & \big\downarrow \\
Y & \longleftarrow & C & & & & \Pi_f B' \\
 & & \big\downarrow{\scriptstyle f} & & & \swarrow & \\
 & & D & & & & \\
 & \swarrow & & & & & \\
Z & & & & & &
\end{array}$$

Finally we take the pullback

$$\begin{array}{ccccccc}
X & \longleftarrow & A & \longleftarrow & & & A' = A \times_B (C \times_D \Pi_f B') \\
 & & \big\downarrow & & & & \big\downarrow \\
 & & B & \longleftarrow & B' = B \times_Y C & \longleftarrow & C \times_D \Pi_f B' \\
 & \swarrow & & \swarrow & & & \big\downarrow \\
Y & \longleftarrow & C & & & & \Pi_f B' \\
 & & \big\downarrow{\scriptstyle f} & & & \swarrow & \\
 & & D & & & & \\
 & \swarrow & & & & & \\
Z & & & & & &
\end{array}$$

and the composition is defined as

$$[Y \leftarrow C \rightarrow D \rightarrow Z] \circ [X \leftarrow A \rightarrow B \rightarrow Y] = [X \leftarrow A' \rightarrow \Pi_f B' \rightarrow Z]$$

It is routine to check it is well-defined and satisfies the associativity and identity law of compositions. Hopefully, the reader are not overwhelmed by these giant diagrams. Remember that the horizontal maps represent restrictions, vertical maps represent norms, and tilted maps represent transfers. It may be helpful to see the proof of Lemma 4.7 below to get a concrete sense.

Definition 4.5 is good for giving a feeling of what this category $\mathscr{P}^{\mathscr{C}}$ should be like. For our purpose, there is a more concise way of describing the compositions and figuring out the relationship with restrictions, transfers, and norms, as promised.

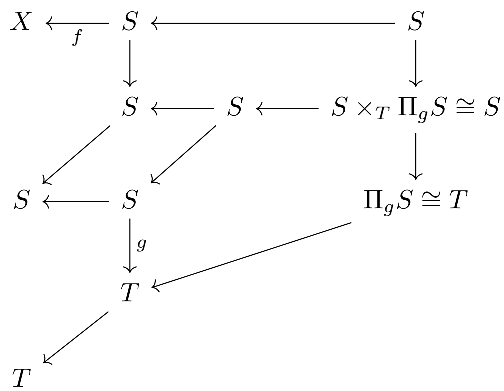
Definition 4.6 (Composition in $\mathscr{P}^{\mathscr{C}}$, alternative)**.** There are three kinds of generating morphisms, called *basic polynomials*: if $f : S \to T$ is a map in $\mathscr{C}$, let

(a) $R_f := [T \overset{f}{\leftarrow} S \overset{1}{\rightarrow} S \overset{1}{\rightarrow} S] \in \mathscr{P}^{\mathscr{C}}(T, S)$,
(b) $N_f := [S \overset{1}{\leftarrow} S \overset{f}{\rightarrow} T \overset{1}{\rightarrow} T] \in \mathscr{P}^{\mathscr{C}}(S, T)$,
(c) $T_f := [S \overset{1}{\leftarrow} S \overset{1}{\rightarrow} S \overset{f}{\rightarrow} T] \in \mathscr{P}^{\mathscr{C}}(S, T)$.

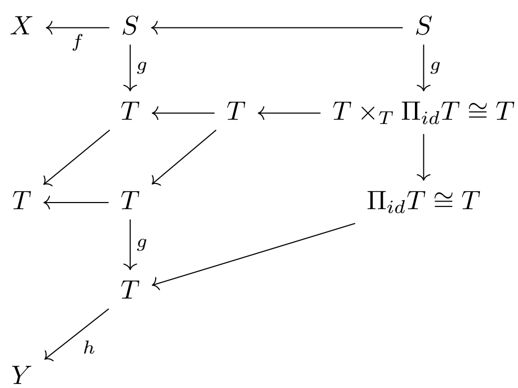
Then according to Lemma 4.7 below, every "polynomial" is a composite of these generating morphisms; Proposition 4.8 tells the rules of composing such generating morphisms.

Lemma 4.7. $[X \xleftarrow{f} S \xrightarrow{g} T \xrightarrow{h} Y] = T_h \circ N_g \circ R_f$

Proof. We first show that $[X \xleftarrow{f} S \xrightarrow{g} T \xrightarrow{id} T] = N_g \circ R_f$. Consider the below diagram, we claim that $\Pi_g S \cong T$. In particular, $\Pi_g S$ is the space of functions $s : g^{-1}(t) \to S$ such that $id \circ s = id$. This implies $s = id$, and it follows that $\Pi_g S \cong T$, and the claim follows.



Then we show that $T_h \circ [X \xleftarrow{f} S \xrightarrow{g} T \xrightarrow{id} T] = [X \xleftarrow{f} S \xrightarrow{g} T \xrightarrow{h} Y]$. Consider the following diagram:



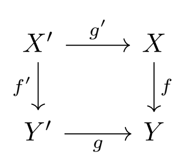
The result follows since $\Pi_{id} T \cong T$. □

Proposition 4.8. *We have further properties of T, N, R that are routine checks:*

(a) *(Composition of res/norm/tr)*

$$N_g \circ N_{g'} = N_{g \circ g'}, \quad T_h \circ T_{h'} = T_{h \circ h'}, \quad R_f \circ R_{f'} = R_{f' \circ f}$$

(b) *(Double coset formula) If*



is a pullback diagram, then

$$R_f \circ N_g = N_{g'} \circ R_{f'}, \quad R_f \circ T_g = T_{g'} \circ R_{f'}$$

(c) *(Distributivity of exponential diagram) If*

$$\begin{array}{ccc} & A & \xleftarrow{f'} & X \times_Y \Pi_f A \\ {\scriptstyle h}\swarrow & & & \downarrow{\scriptstyle g'} \\ X & & & \Pi_f A \\ \downarrow{\scriptstyle g} & & \swarrow{\scriptstyle h'} & \\ Y & & & \end{array}$$

is an exponential diagram, then we have

$$N_g \circ T_h = T_{h'} \circ N_{g'} \circ R_{f'}$$

With the above set up, we can finally get into the definition of Tambara functors.

Definition 4.9. Let $\mathscr{C} = \mathcal{S}et^G$, the category of finite G-sets. We take the Grothendieck completion of $\mathscr{P}^{\mathscr{C}}(S,T)$ [1, 4.3.19] to obtain a new category $\widetilde{\mathscr{P}^{\mathscr{C}}}$. A *Tambara* functor is a functor

$$\widetilde{\mathscr{P}^{\mathscr{C}}}^{op} \to \mathcal{A}b$$

4.2. **Incomplete Tambara functors.** We have the following definitions as set-up.

Definition 4.10. (a) A subcategory $\mathscr{D}$ of a locally Cartesian closed category $\mathscr{C}$ is *wide* if it contains all of the objects.

(b) If $\mathscr{C}$ is a category that admits pullbacks, then a subcategory $\mathscr{D}$ is *pullback stable category* if whenever we have a pullback

$$\begin{array}{ccc} W & \longrightarrow & Y \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ Z & \longrightarrow & X \end{array}$$

with $g \in \mathscr{D}$, then $f \in \mathscr{D}$.

(c) If $\mathscr{D}$ is a wide subcategory, the *polynomial in $\mathscr{C}$ with exponents in $\mathscr{D}$*, denoted $\mathscr{P}^{\mathscr{C}}_{\mathscr{D}}$, is the wide subcategory of $\mathscr{P}^{\mathscr{C}}$ with morphisms the isomorphism classes of bispans

$$X \leftarrow S \xrightarrow{g} T \to Y$$

where $g \in \mathscr{D}$. The partial choices of g allude to the "incompleteness".

Lemma 4.11. *If $\mathscr{D}$ is a wide, pullback stable subcategory, then $\mathscr{P}^{\mathscr{C}}_{\mathscr{D}}$ is a subcategory of $\mathscr{P}^{\mathscr{C}}$.*

Proof. By decomposing morphisms into generating morphisms R_f, N_g, T_h, it suffices to show that the composite of any morphism is of the form

$$T_h \circ N_g \circ R_f$$

with $g \in \mathscr{D}$. Notice that R_f, T_h are in $\mathscr{P}^{\mathscr{C}}_{\mathscr{D}}$ for any morphisms f, h in $\mathscr{C}$. Therefore, we only need to show any composites with N_g for $g \in \mathscr{D}$ are again in $\mathscr{P}^{\mathscr{C}}_{\mathscr{D}}$.

Since $\mathscr{D}$ is a subcategory, $N_g \circ N_{g'} = N_{g \circ g'}$ is again in $\mathscr{P}^{\mathscr{C}}_{\mathscr{D}}$ provided $g, g' \in \mathscr{D}$.

$N_g \circ T_h = T_{h'} \circ N_{g'} \circ R_{f'}$ where by pullback stability, $g' \in \mathscr{D}$ if $g \in \mathscr{D}$.

$R_f \circ N_g = N_{g'} \circ R_{f'}$ where g' is the pullback of g along f, and is therefore in $\mathscr{D}$ provided $g \in \mathscr{D}$. This concludes the proof. ◻

We have an immediate corollary:

Corollary 4.12. *If $\mathscr{D}_1 \subset \mathscr{D}_2$ are wide, pullback stable subcategories of $\mathscr{C}$, then we have an inclusion of subcategories $\mathscr{P}^{\mathscr{C}}_{\mathscr{D}_1} \subset \mathscr{P}^{\mathscr{C}}_{\mathscr{D}_2}$.*

The following lemma will be used later in Proposition 4.19.

Lemma 4.13 ([3], 3.1)**.** *If $\mathscr{D}$ is a pullback stable, symmetric monoidal subcategory of $\mathcal{S}et^G$ that contains $\emptyset \to *$, then $\mathscr{D}$ contains all monomorphims.*

Proof. Since $\mathscr{D}$ is pullback stable and containing the terminal object $*$, for any object S, we have a pullback diagram

$$\begin{array}{ccc} S & \longrightarrow & * \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle id} \\ S & \longrightarrow & * \end{array}$$

so that by pullback stability, S is in $\mathscr{D}$, i.e. $\mathscr{D}$ is wide. Moreover, any monomorphism $S \to T$ can be written as $\emptyset \coprod S \to (T \setminus S) \coprod S$. Since $\mathscr{D}$ is also symmetric monoidal, it is then enough to show $\emptyset \to S$ is in $\mathscr{D}$ for all S. Via pullback stability of the following diagram,

$$\begin{array}{ccc} \emptyset & \longrightarrow & \emptyset \\ \downarrow & & \downarrow \\ S & \longrightarrow & * \end{array}$$

we conclude the result. □

Our intuition should tell us that $\mathscr{P}^{\mathscr{C}}_{\mathscr{D}}$ exactly picks out the norm maps that originate in $\mathscr{D}$. If given any N_∞ operad $\mathcal{O}$, or equivalently an indexing system (we will use these two terms interchangeably), we can associate $\mathcal{O}$ to a wide, pullback stable subcategory $\mathscr{D}$ of $\mathcal{S}et^G$, then a functor

$$\mathscr{P}^{G^{op}}_{\mathscr{D}} \to \mathcal{A}b$$

should have partial norm maps determined by $\mathcal{O}$. We will head towards this goal.

Definition 4.14 ([3], 3.8)**.** For an indexing system $\mathcal{O}$, let $\mathcal{S}et^G_{\mathcal{O}}$ denote the wide subcategory of $\mathcal{S}et^G$ where $f : S \to T$ is in $\mathcal{S}et^G_{\mathcal{O}}$ if and only if for all $s \in S$,

$$\mathrm{stab}_G(f(s))/\mathrm{stab}_G(s) \in \mathcal{O}(\mathrm{stab}_G(f(s)))$$

where $\mathrm{stab}_G(s)$ is the stabilizer subgroup of s in G. Equivalently, $f : S \to T$ is in $\mathcal{S}et^G_{\mathcal{O}}$ if and only if for all $s \in S$

$$\mathrm{stab}_G(f(s)) \cdot s \in \mathcal{O}(\mathrm{stab}_G(f(s)))$$

We pack all the technical details into the following theorem:

Proposition 4.15 ([3], 3.11, 3.13)**.** *$\mathcal{S}et^G_{\mathcal{O}}$ is a wide, pullback stable and finite coproduct complete subcategory of $\mathcal{S}et^G$, and thus the polynomials with exponents in $\mathcal{S}et^G_{\mathcal{O}}$ is a subcategory of $\mathscr{P}^G$.*

Conversely, any wide, pullback stable and finite coproduct complete subcategory $\mathscr{D}$ of $\mathcal{S}et^G$ would determine an indexing system $\mathcal{O}_{\mathscr{D}}$ by

$$\mathcal{O}_{\mathscr{D}}(G/H) := res^*_H(\mathscr{D}/(G/H))$$

the slice category over G/H in $\mathscr{D}$ then restricted to H-actions. Blumberg and Hill check in detail that it is indeed a sub-SMCS of $\underline{\mathcal{S}et}$ satisfying the four defining properties of indexing systems. We will not do the proof here. Instead, we summarize several big results into the following theorem:

Theorem 4.16 ([3], 3.26, 3.27)**.** *The above gives inverse order preserving functors between the poset of indexing systems and the collection of wide, pullback stable, finite coproduct complete, symmetric monoidal subcategories of $\mathcal{S}et^G$, namely*

$$\mathscr{D} = \mathcal{S}et^G_{\mathcal{O}_\mathscr{D}}, \text{ and } \mathcal{O} = \mathcal{O}_{\mathcal{S}et^G_\mathcal{O}}$$

Proof.

(a) To check $\mathscr{D} = \mathcal{S}et^G_{\mathcal{O}_\mathscr{D}}$, we need the fact that for $\mathscr{D}$ (and similarly $\mathcal{S}et^G_{\mathcal{O}_\mathscr{D}}$) in this case, it is the smallest subcategory having all finite coproducts containing $\mathcal{O}rb_\mathscr{D}$, the full subcategory of $\mathscr{D}$ restricting to objects on G/H that are contained in $\mathscr{D}$ ([2], 3.6). Since $K/H \in \mathcal{O}_\mathscr{D}(G/K)$ if and only if $G/H \to G/K$ is in $\mathcal{O}rb_\mathscr{D}$, and by definition, $K/H \in \mathcal{O}_\mathscr{D}(G/K)$ if and only if $G/H \to G/K$ is in $\mathcal{O}rb_\mathcal{O}$, we have $\mathcal{O}rb_\mathscr{D} = \mathcal{O}rb_\mathcal{O}$, and the result follows.

(b) For the other equation, we note a fact that any object $T \to G/H$ in the slice category is isomorphic to $G \times_H T' \to G/H$ where $T' \to *$ is the canonical map. Using another fact that $f : A \to B$ is in $\mathcal{S}et^H_{res^*_H\mathcal{O}}$ if and only if $G \times_H f$ is in $\mathcal{S}et^G_\mathcal{O}$ ([3], 3.12), we have $T \to G/H$ is in $\mathcal{S}et^G_\mathcal{O}$ if and only if $T' \to *$ is in $\mathcal{S}et^H_{res^*_H\mathcal{O}}$. On the other hand, by definition $T' \to *$ is in $\mathcal{S}et^H_{res^*_H\mathcal{O}}$ if and only if $T' \in res^*_H\mathcal{O}(G/H) = \mathcal{O}(G/H)$. This gives the result.

□

We are finally ready to define the incomplete Tambara functors.

Definition 4.17 (Incomplete Tambara functors)**.** Let $\mathscr{D}$ be a wide pullback stable, symmetric monoidal subcategory of $\mathcal{S}et^G$. We take the Grothendieck completion of $\mathscr{P}^G_\mathscr{D}(S,T)$ to obtain a new category $\widetilde{\mathscr{P}^G_\mathscr{D}}$. A *$\mathscr{D}$-Tambara functor* is an additive product preserving functor

$$\widetilde{\mathscr{P}^G_\mathscr{D}}^{op} \to \mathcal{A}b$$

Remark 4.18. In particular, if $\mathscr{D} = \mathcal{S}et^G$, then the $\mathscr{D}$-Tambara functor is just the usual Tambara functor in Definition 4.9. Moreover, from Theorem 4.16 we see that $\mathcal{S}et^G_{\mathcal{E}_G} = \mathcal{S}et^G$, since the admissible sets of $\mathcal{E}_G$ are all possible subgroups of $G \times \Sigma_n$. This means an $\mathcal{E}_G$-Tambara functor is in fact a Tambara functor.

Proposition 4.19. *We have the following properties of particular $\mathscr{D}$-Tambara functors*

(a) Any $\mathscr{D}$-Tambara functor contains an underlying Mackey functor structure.
(b) A $\mathcal{S}et^G_{mono}$-Tambara functor contains a Mackey functor together with a unit map $\underline{A} \to \underline{M}$, where $\underline{A}$ is the Burnside Mackey functor.
(c) A $\mathscr{D}$-Tambara functor $\underline{R}$ contains a Green functor structure if $\emptyset \to *$ and $* \coprod * \to *$ are in $\mathscr{D}$. Thus for any indexing system, the $\mathcal{O}$-Tambara functor has a Green functor structure.
(d) A $\mathscr{D}$-Tambara functor $\underline{R}$ has a norm map $\underline{R}(G/H) \to \underline{R}(G/K)$ if $G/H \to G/K$ is in $\mathscr{D}$ satisfying the Tambara reciprocity relations [1]. If $G/K \coprod G/K \to G/K$ is in $\mathscr{D}$ as well, then this norm is a map of multiplicative monoids. If $\emptyset \to G/K$ is in $\mathscr{D}$, then it is unital. In particular for any $\mathcal{O}$-Tambara functor, it has norm maps of multiplicative monoids

$$N^K_H : \underline{R}(G/H) \to \underline{R}(G/K)$$

[1] See [13] and ([2] 3.10, 3.11)

for all morphisms $G/H \to G/K \in \mathcal{O}rb_{\mathcal{O}}$ satisfying the Tambara reciprocity relations.

Proof. (a) By Lemma 4.12, for any $\mathscr{D}$, $\mathcal{S}et^G_{Iso} \subset \mathcal{S}et^G_{\mathscr{D}}$. For any polynomials with exponents in $\mathcal{S}et^G_{Iso}$, it has the form

$$X \leftarrow S \xrightarrow{\cong} S \to Y$$

which can be identified with the "span"

$$X \leftarrow S \to Y$$

and thus the incomplete Tambara functor $\mathcal{S}et^G_{Iso} \to \mathcal{A}b$ is the same information as a Mackey functor.

(b) The pointed map comes from the distinguished monomorphism $\emptyset \to T$ for all T, which exists by Lemma 4.13.

(c) The map $\emptyset \to *$ gives the unit map and $* \coprod * \to *$ encodes the multiplication. Since for any indexing system $\mathcal{O}$, $\mathcal{S}et^G_{\mathcal{O}^{tr}} \subset \mathcal{S}et^G_{\mathcal{O}}$, and the trivial indexing systems by definition contains the above two maps.

(d) Since any map in $\mathcal{S}et^G_{\mathcal{O}}$ can be written as iterated fold maps and disjoint unions of maps of the form $G/H \to G/K$, the results follows.

□

Remark 4.20. There is also a notion of *non-unital* Tambara functor, basically a $\mathcal{S}et^G_{epi}$-Tambara functor. It has a non unital commutative Green functor structure together with norm maps for all $G/H \to G/K$ satisfying the exponential formula.

Proposition 4.19 (d) is a concrete interpretation of the sub-collection of norms we mentioned at the beginning of section 4. Let us also restate Theorem 4.1 here:

Theorem 4.21 ([2], 4.14)**.** *If R is an algebra in orthogonal G spectra over an N_∞ operad $\mathcal{O}$ that interchanges with itself, then $\pi_0(R)$ is an $\mathcal{O}$-Tambara functor.*

Proof. Since $\mathcal{S}et^G_{\mathcal{O}}$ is a wide, pullback stable, coproduct complete subcategory of $\mathcal{S}et^G$, any $\mathcal{O}$- Tambara functor is a Green functor plus norm maps. Then Proposition 4.19 (d) shows that if $G/H \to G/K \in \mathcal{S}et^G_{\mathcal{O}}$, then we have a norm map satisfying the Tambara reciprocity relations. This implies an $\mathcal{O}$- Tambara is essentially a commutative Green functor $\underline{M}$ together with norm maps of multiplicative monoids

$$N^K_H : \underline{M}(G/H) \to \underline{M}(G/K)$$

for each $G/H \to G/K \in \mathcal{S}et^G_{\mathcal{O}}$. Then Theorem 4.1 is saying that $\underline{\pi_0(R)}$ is just an $\mathcal{O}$- Tambara functor since it satisfies these characterizations. □

In particular, since an $\mathcal{E}_G$-Tambara functor is in fact a Tambara functor, Theorem 4.21 further implies $\underline{\pi_0}$ of a "genuine" G-E_∞ ring spectrum is indeed a Tambara functor.

5. Categorical reinterpretations

Our results in Section 4 can be packed into a more categorical description in terms of the $\mathcal{O}$-commutative monoids. We follow mostly [4] and reveal the relationships in Theorem 5.6.

We will focus on the symmetric monoidal structures. There is a symmetric monoidal product on the category of *symmetric monoidal coefficient systems* (SMCS, Definition 2.6) given by

$$\underline{\mathcal{C}_1} \times \underline{\mathcal{C}_2}(G/H) := \underline{\mathcal{C}_1}(G/H) \times \underline{\mathcal{C}_2}(G/H)$$

The SMCS $\underline{\mathcal{S}et}$ has multiplication map with respect to this product

$$\underline{\mathcal{S}et} \times \underline{\mathcal{S}et} \to \underline{\mathcal{S}et}$$

induced by the Cartesian product on $\mathcal{S}et$ itself.

Let $\mathcal{S}et^{Iso}$ denotes the category of finite sets with morphisms only isomorphisms (bijections). We have a SMCS generalization, $\underline{\mathcal{S}et}^{Iso}$, defined by letting $\underline{\mathcal{S}et}^{Iso}(G/H)$ be the category of finite H-sets with morphisms H-equivariant isomorphisms.

Definition 5.1 (([4],3.3), G-symmetric monoidal structure)**.** (a) Classically, a *symmetric monoidal structure* on a category $\mathcal{C}$ is equivalent to a bilinear map

$$-\Box- : \mathcal{S}et^{Iso} \times \mathcal{C} \to \mathcal{C}$$

$$(T, M) \mapsto T \Box M := \bigotimes_T M$$

(b) For a SMCS $\underline{\mathcal{C}}$, define a *G-symmetric monoidal structure* on $\underline{\mathcal{C}}$ to be a bilinear map

$$-\Box- : \underline{\mathcal{S}et}^{Iso} \times \underline{\mathcal{C}} \to \underline{\mathcal{C}}$$

such that
(i) for each $H \subset G$, when restricted to $\mathcal{S}et^{Iso} \subset \underline{\mathcal{S}et}(G/H)$ of trivial objects, then this is just the map

$$(T, M) \mapsto \bigotimes_T M$$

(ii) the following diagram commutes up to natural isomorphism

$$\begin{array}{ccc} \underline{\mathcal{S}et}^{Iso} \times \underline{\mathcal{S}et}^{Iso} \times \underline{\mathcal{C}} & \xrightarrow{1\times\Box} & \underline{\mathcal{S}et}^{Iso} \times \underline{\mathcal{C}} \\ {\scriptstyle (-\times-)\times 1}\downarrow & & \downarrow{\scriptstyle \Box} \\ \underline{\mathcal{S}et}^{Iso} \times \underline{\mathcal{C}} & \xrightarrow{\quad\Box\quad} & \underline{\mathcal{C}} \end{array}$$

Remark 5.2. A G-symmetric monoidal structure is inequivalent to a symmetric monoidal G-category.

Definition 5.3 (G-commutative monoid)**.** (a) [4, 3.7] Classically, A *commutative monoid M* in a category $\mathcal{C}$ is equivalent to an extension:

$$\begin{array}{ccc} \mathcal{S}et^{Iso} & \xrightarrow{-\Box M} & \mathcal{C} \\ \downarrow & \nearrow & \\ \mathcal{S}et & & \end{array}$$

(b) For SMCS, a *G-commutative monoid* is an object $M \in \underline{\mathcal{C}}(G/G)$ together with an extension:

$$\begin{array}{ccc} \underline{\mathcal{S}et}^{Iso} & \xrightarrow{-\Box M} & \underline{\mathcal{C}} \\ \downarrow & \nearrow_{N^{(-)}(M)} & \\ \underline{\mathcal{S}et} & & \end{array}$$

and a *map* between G-commutative monoids is a morphism $f : M \to M' \in \underline{\mathcal{C}}(G/G)$ such that for all $T \in \underline{\mathcal{S}et}(G/H)$, we have

$$\begin{array}{ccc} T\Box M & \xrightarrow{N^T} & M \\ {\scriptstyle T\Box f}\downarrow & & \downarrow{\scriptstyle f} \\ T\Box M' & \xrightarrow[N^T]{} & M' \end{array}$$

(c) Equivalently, a G-commutative monoid is a commutative monoid M in $\underline{\mathcal{C}}(G/G)$ together with commutative monoid maps:

$$N_H^G : G/H\Box M \to *\Box M \cong M$$

By Proposition 2.19, the indexing system of the "naive" E_∞ G-operad $\underline{\mathcal{C}}(\mathcal{E}^{tr})$ is the initial object of all indexing systems. We write $\underline{\mathcal{C}}(\mathcal{E}^{tr}) = \mathcal{E}^{tr}$ as indexing systems here. We can generalize the above characterization from $\underline{\mathcal{S}et}$ to any indexing systems $\mathcal{O}$ as follows.

Definition 5.4 ($\mathcal{O}$-symmetric monoidal structure)**.** An *$\mathcal{O}$-symmetric monoidal structure* on a SMCS $\underline{\mathcal{C}}$ is an extension of the canonical map

$$\mathcal{E}^{tr,Iso} \times \underline{\mathcal{C}} \to \underline{\mathcal{C}}$$

to a bilinear map

$$-\Box- : \mathcal{O}^{Iso} \times \underline{\mathcal{C}} \to \underline{\mathcal{C}}$$

Definition 5.5 ($\mathcal{O}$-commutative monoid)**.** If $\mathcal{O}$ and $\mathcal{O}'$ are indexing systems with $\mathcal{O} \subset \mathcal{O}'$, and if $\underline{\mathcal{C}}$ is an $\mathcal{O}'$-symmetric monoidal category, then an $\mathcal{O}$-commutative monoid is an object $M \in \underline{\mathcal{C}}(G/G)$ with an extension

$$\begin{array}{ccccc} \mathcal{O}^{Iso} & \longrightarrow & \mathcal{O}'^{Iso} & \xrightarrow{-\Box M} & \underline{\mathcal{C}} \\ \downarrow & & & \nearrow_{N^{(-)}(M)} & \\ \mathcal{O} & & & & \end{array}$$

In other words, we have only norm maps for objects of $\mathcal{O}$.

By unpacking Definition 5.3(c) and Definition 5.5, we have the following specific examples of $\mathcal{O}$-commutative monoids:

Theorem 5.6. ([4],§5)

(a) The category of G-Tambara functors is equivalent to the category of G-commutative monoids in Mackey functors.

(b) The category of $\mathcal{O}$-Tambara functors is equivalent to the category of $\mathcal{O}$-commutative monoids in Mackey functors.

Appendix A. Operad pairs and pairing of operads

We discuss the relationship between operad pairs and pairing of operads, and exhibit one of its applications, which is used in the proof of Theorem 4.1.

Definition A.1 (([11], 10.1), Pairings of operads)**.** Let $\mathcal{O}, \mathcal{O}', \mathcal{O}''$ be operads in a symmetric monoidal category $(\mathscr{C}, \otimes)$. A *pairing* of operads

$$\boxtimes : (\mathcal{O}, \mathcal{O}') \to \mathcal{O}''$$

consists of maps

$$\boxtimes : \mathcal{O}(j) \otimes \mathcal{O}'(k) \to \mathcal{O}''(jk)$$

with the following properties: Let $c \in \mathcal{O}(j)$ and $d \in \mathcal{O}'(k)$

(a) If $\mu \in \Sigma_j$ and $\nu \in \Sigma_k$, then

$$c\mu \boxtimes d\nu = (c \boxtimes d)(\mu \otimes \nu)$$

where the tensor product $\otimes : (\Sigma_j, \Sigma_k) \to \Sigma_{jk}$ is the standard natural pairings [11, §10].

(b) For $j = k = 1$, $id \boxtimes id = id$.

(c) If $c_q \in \mathcal{O}(h_q)$ for $1 \leq q \leq j$ and $d_r \in \mathcal{O}'(i_r)$ of $1 \leq r \leq k$, then

$$\gamma(c \boxtimes d; \times_{(q,r)} c_q \boxtimes d_r) \cdot \delta = \gamma(c; \times_q c_q) \boxtimes \gamma(d; \times_r d_r)$$

where δ is the isomorphism of distributivity

$$\delta : \coprod_{(q,r)} (h_q \times i_r) \to (\coprod_q h_q) \times (\coprod_r i_r)$$

Definition A.2 (([12], 1.8), Operad pairs)**.** An *action* of an operad $\mathcal{E}$ on $\mathcal{O}$ consists of maps

$$\lambda : \mathcal{E}(k) \times \mathcal{O}(j_1) \times \cdots \mathcal{O}(j_k) \to \mathcal{O}(j_i \cdots j_k)$$

for $k \geq 1, j_r \geq 0$ and $\lambda(*) = id \in \mathcal{O}(1)$ for $k = 0$, where $* \in \mathcal{E}(0)$ is the chosen base point and id is the distinguished element of $\mathcal{O}(1)$. The maps λ satisfy for

$$e \in \mathcal{E}(k), e_r \in \mathcal{E}(j_r), \quad 1 \leq r \leq k$$

$$c \in \mathcal{O}(j), c_r \in \mathcal{O}(j_r), \quad 1 \leq r \leq k$$

$$c_{r,q} \in \mathcal{O}(i_{r,q}) \quad 1 \leq q \leq j_r, \quad 1 \leq r \leq k$$

(i)

$$\lambda(\gamma(g; \times_{r=1}^k g_r); \times_{r=1}^k \times_{q=1}^{j_r} c_{r,q}) = \lambda(g; \times_{r=1}^k \lambda(g_r; \times_{q=1}^{j_r} c_{r,q})) \quad \text{(I)}$$

$$\gamma(\lambda(g; \times_{r=1}^k c_r); \times_Q \lambda(g; \times_{r=1}^k c_{r,q_r})) \cdot \delta = \lambda(g; \times_{r=1}^k \gamma(c_r; \times_{q=1}^{j_r} c_{r,q})) \quad \text{(II)}$$

where Q runs through the lexicographically ordered set of sequneces $(q_1, ..., q_k)$, $1 \leq q_r \leq j_r$, and δ is the isomorphism of distributivity as in Definition 3.13,

(ii)

$$\lambda(id; c) = c \quad \text{(III)}$$

$$\lambda(g; id^k) = id \quad \text{(IV)}$$

(iii)

$$\lambda(g\sigma; \times_{r=1}^k c_r) = \lambda(g; \times_{r=1}^k c_{\sigma^{-1}(r)})\sigma(j_1, ..., j_k) \quad \text{(V)}$$

$$\lambda(g; \times_{r=1}^k c_r \tau_r) = \lambda(g; \times_{r=1}^k c_r)(\tau_1 \otimes \cdots \otimes \tau_k) \quad \text{(VI)}$$

May observed that if we have an operad pair $(\mathcal{O},\mathcal{E})$ and we fix any element $x\in\mathcal{E}(2)$, this induces a map via the operad pair action

$$\mathcal{O}(j)\times\mathcal{O}(k)\to\mathcal{O}(jk)$$

We want to figure out whether it indeed gives a pairing of operad.

Proposition A.3. *Given an operad pair $(\mathcal{O},\mathcal{E})$, fix any $x\in\mathcal{E}(2)$, it induces a self-pairing of operad*

$$\boxtimes := \lambda(x;-,-):(\mathcal{O},\mathcal{O})\to\mathcal{O}$$

Proof. We check the axioms in Definition 3.13 one by one:

(a) If $\mu\in\Sigma_j$ and $\nu\in\Sigma_k$, then

$$\begin{aligned} c\mu\boxtimes d\nu &:= \lambda(x;c\mu,d\nu)\\ &= \lambda(x;c,d)(\mu\otimes\nu) \quad \text{(by VI)}\\ &= (c\boxtimes d)(\mu\otimes\nu)\end{aligned}$$

(b)

$$\begin{aligned} id\boxtimes id &= \lambda(x;id,id)\\ &= id \quad \text{(by IV)}\end{aligned}$$

(c) If $c_q\in\mathcal{C}(h_q)$ for $1\le q\le j$ and $d_r\in\mathcal{D}(i_r)$ of $1\le r\le k$, then

$$\begin{aligned}\gamma(c\boxtimes d;\times_{(q,r)}c_q\boxtimes d_r)\cdot\delta &= \gamma(\lambda(x;c,d);\times_{(q,r)}\lambda(x;c_q,d_r))\cdot\delta\\ &= \lambda(x;\gamma(c;\times_q c_q),\gamma(d;\times_r d_r)) \quad \text{(by II)}\\ &= \gamma(c;\times_q c_q)\boxtimes\gamma(d;\times_r d_r)\end{aligned}$$

□

Remark A.4. The above proof uses only the three of six axioms of the operad pair, namely (II), (IV), (VI). This makes sense since we are fixing an element in $\mathcal{E}$ and considering the induced action on $\mathcal{O}$. Axioms (I), (III), (V) are related to operations in $\mathcal{E}$, which play no role here.

As an application, we show that any opeard that can be made into an additive operad in an opeard pair interchanges with itself.

Definition A.5 (Interchange of operads)**.** Given an object X which is both an $\mathcal{O}$-algebra and an $\mathcal{O}'$-algebra, the two actions *exchange* if for each $o\in\mathcal{O}(n)$, the map $X^{\times n}\xrightarrow{o\cdot}X$ is a map of $\mathcal{O}'$-algebras and vice versa. Diagrammatically,

$$\begin{array}{ccccc}(X^n)^m & \xrightarrow{\cong} & (X^m)^n & \xrightarrow{\beta^n} & X^n\\ \downarrow{\scriptstyle\alpha^m} & & & & \downarrow{\scriptstyle\alpha}\\ X^m & & \xrightarrow{\quad\beta\quad} & & X\end{array}$$

for all $\alpha\in\mathcal{O}(n)$ and $\beta\in\mathcal{O}'(m)$.

Lemma A.6 ([10] §1)**.** *$\mathcal{O}$ and $\mathcal{O}'$ actions interchange if and only if X is an $\mathcal{O}\otimes\mathcal{O}'$-algebra.*

We have the following fact relating pairings of operads and tensor product of operads as follows [11, 6.1]:

Proposition A.7. *The tensor product is the universal recipient for pairings, that is, there is an extension*

$$\begin{array}{ccc} (\mathcal{O}, \mathcal{O}') & \longrightarrow & \mathcal{O}'' \\ \downarrow & \nearrow & \\ \mathcal{O} \otimes \mathcal{O}' & & \end{array}$$

Combining A.3 and A.7, any additive operad $\mathcal{O}$ in an operad pair has a self-pairing, and therefore there is a universal map $\mathcal{O} \otimes \mathcal{O} \to \mathcal{O}$. This implies any $\mathcal{O}$-algebra is automatically an $\mathcal{O} \otimes \mathcal{O}$-algebra, showing that $\mathcal{O}$ interchanges with itself.

Once an N_∞ algebra E has its $\mathcal{O}$-action interchange with itself, then for any surjective maps $T \to S$ of admissible H-sets, the structure maps

$$N^T res_H^* E \to N^S res_H^* E$$

are maps of $N^S res_H^* \mathcal{O}$-algebras [2, 6.28]. This is key to the proof of Theorem 4.1.

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