EQUIVARIANT K-THEORY AND THE ATIYAH-SEGAL COMPLETION THEOREM

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Abstract. In this paper we present the Atiyah-Segal Completion theorem and its proof. We will begin by introducing equivariant K-theory starting from the definition of a vector bundle. Then we will state some important theorems (Bott periodicity, Thom isomorphism) that are needed to understand the statement of the Atiyah-Segal Completion theorem as well as its proof.

Contents

1. Introduction

In this section, we introduce some preliminaries to K-theory.

1.1. Vector Bundle.

A vector bundle is intuitively a continuous family of vector spaces parametrized by another topological space pointwise. We associate a vector space V_x to every point x of the base space X such that several axioms are satisfied. We will introduce the idea of complex vector bundles, and the definition of real vector bundles is completely analogous. Unless otherwise specified, all bundles are assumed to be complex.

Definition 1.1. A *complex vector bundle* E is a topological space equipped with a continuous map $p : E \to X$ called the *projection map*, that satisfies the following:

(1) Each fiber $E_x = p^{-1}(x), x \in X$ has a finite dimensional complex vector space structure.

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(2) (*Local triviality*) There is an open covering $\{U_\alpha\}$ of X such that there exists a homeomorphism $h_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^n$ taking $p^{-1}(x) \to x \times \mathbb{C}^n$ by a vector space isomorphism for each $x \in X$. This h_α is called the local trivialization and $p^{-1}(x)$ is called the fiber. If n is a constant on all such maps, we say that the vector bundle has dimension n.

In other words, we can view E as a space equipped with an addition map $E \times_X E \to E$ and an action of $\mathbb{C}$ on E, that satisfies the conditions listed above.

We give a few examples of vector bundles.

Example 1.2. The *trivial bundle* $E = X \times \mathbb{C}^n$ with p the projection map onto the first factor.

Example 1.3. The *tangent bundle* over a manifold has the tangent space of a point on the manifold as its fibers. This is a real vector bundle.

Example 1.4. The *canonical line bundle* $p : E \to \mathbb{RP}^n$ has its total space the subspace of $\mathbb{RP}^n \times \mathbb{R}^{n+1}$ consisting of pairs (l, v) with $v \in l$ and projection map is the projection to the first factor. Trivialization can be defined by orthogonal projection. Note that this is actually a real vector bundle.

Example 1.5. Let E be a complex vector bundle over X with fibers $\mathbb{C}^n$. Then there is an associated projective bundle $p : P(E) \to X$ with fibers $\mathbb{CP}^{n-1}$, where $P(E)$ is the space of lines in E. Note that over $P(E)$ there is a canonical line bundle $H \to P(E)$ consisting of the vectors in the lines of $P(E)$. This is in some sense a generalization of the complex version of Example 1.4.

Definition 1.6. A *section* of a vector bundle $p : E \to X$ is a map $s : X \to E$ such that $ps(x) = x$ for all $x \in X$.

An *isomorphism* between vector bundles $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ over the same base space X is a homeomorphism between the two total spaces respecting the structure map to X that restricts to an linear isomorphism on each fiber. We then denote the isomorphism classes of vector bundles over the base space X as $Vect_{\mathbb{C}}(X)$, and let $Vect^n_{\mathbb{C}}(X)$ be the subset of isomorphism classes of n-dimensional vector bundles over X. Note that under the direct sum operation which is defined fiberwise, $Vect_{\mathbb{C}}(X)$ is an abelian semi-group. We can then define the tensor product operation fiberwise on this semi-group structure and make $Vect_{\mathbb{C}}(X)$ into a commutative semi-ring.

If given a continuous map $f : Y \to X$ and a vector bundle $p : E \to X$, then we can form a *pullback* vector bundle f^*E over X, shown in the following diagram:

$$\begin{tikzcd} f^*E = E \times_X Y \arrow[r, dashed] \arrow[d, dashed] & E \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

Remark 1.7. Vector bundles are homotopy invariant. In particular, if f_0 and f_1 are homotopic maps from Y to X, then the pullback bundles $f_0^*(E)$ and $f_1^*(E)$ are isomorphic. Therefore, a homotopy equivalence $f : Y \to X$ of compact spaces induces a bijection $f^* : Vect_{\mathbb{C}}(X) \to Vect_{\mathbb{C}}(Y)$.

The notion of vector bundles can be generalized to the equivariant world.

1.2. G-equivariant vector bundle.

Definition 1.8. Let G be a compact Lie group. A *G-vector bundle* E is a G-space over another G-space X together with a G-map $p : E \to X$ (i.e. $p(g.\zeta) = g.p(\zeta)$) that satisfies the following:

1. $p : E \to X$ has a vector bundle structure.
2. for any $g \in G$ and $x \in X$ the group action $g : E_x \to E_{gx}$ is a homomorphism of vector spaces.

If G is a group of one element, then any vector bundle is a G-vector bundle.

On the other hand, if X is a space of one point, then G-vector bundles are simply representations of G.

We give an example of a G-vector bundle.

Example 1.9. If E is any vector bundle on a space X, then the k-fold tensor product $E \otimes E \otimes \cdots \otimes E$ is naturally a S_k-vector bundle on X where S_k is the symmetric group permuting the factors of the product and X is regarded as a trivial S_k space. We can also do this over X^k where the vector bundle is the k-fold external tensor product of E. This bundle is then a S_k-equivariant vector bundle when we use the S_k action to permute the factors of X^k.

2. K-Theory

Roughly speaking, K-theory is a cohomology theory built from vector bundles on topological spaces. In some regard, K-theory is largely about doing linear algebra fiberwise over a base space; therefore it is often easier to do calculations in K-theory than in normal generalized cohomology theory. Furthermore, it is a very nice construction due to the fact many results of it can be easily generalized to the equivariant world. In this section, we will define both the ordinary and the equivariant K-theory.

Definition 2.1. Let X be a compact topological space. $K(X)$ is the Grothendieck group of $Vect_{\mathbb{C}}(X)$, *i.e.* the group obtained by formally adding inverses to the abelian monoid $Vect_{\mathbb{C}}(X)$. Categorically, this is the initial abelian group that the abelian monoid $Vect_{\mathbb{C}}(X)$ maps to. Since $Vect_{\mathbb{C}}(X)$ is a semi-ring by tensor product, $K(X)$ is a commutative ring.

We use $[E]$ to denote the isomorphism class of vector bundles in $K(X)$ represented by the vector bundle E. Note that by definition, every element in $K(X)$ is of the form $[E] - [F]$ where E, F are bundles over X.

Definition 2.2. The *reduced K-theory* of a pointed compact space X, denoted $\widetilde{K}(X)$, is the kernel of the homomorphism $\alpha^* : K(X) \to K(*)$ where α^* is the induced map of the inclusion from the base point $*$ into the space X. Note that $K(X) = \widetilde{K}(X_+)$ where X_+ is the union of X with a disjoint base point. Let $(X, *)$ denote a pointed space X with its base point $*$.

Then for $n \in \mathbb{N}$, we define the negative K-theory

$$\widetilde{K}^{-n}((X,*)) = \widetilde{K}(S^n(X,*))$$
$$K^{-n}(X) = \widetilde{K}(S^n X_+)$$
$$K^{-n}(X,Y) = \widetilde{K}(S^n(X/Y))$$

where S^n denote the n-th suspension and Y is a closed pointed subset of a compact Hausdorff space X.

Since the K-theory of a point is simply $\mathbb{Z}$ (the dimension of the vector bundle), $K(X) \cong \widetilde{K}(X) \oplus \mathbb{Z}$ by the isomorphism theorem.

To extend the definition of K-groups to positive integers, we have the periodicity theorem.

Theorem 2.3. *(Bott Periodicity) let L be a line bundle (bundle of dimension 1) over X. Then as a $K(X)$-algebra, $K(P(L \oplus 1))$ is generated by $[H]$ which is subject to the single relation $([H]-1)([L][H]-1) = 0$. Here $[H]$ is the isomorphism class of the canonical line bundle over $P(L \oplus 1)$.*

The proof is given in [1] Chapter 2.2. The general idea is to develop a correspondence between homotopy classes of clutching functions and isomorphism classes of vector bundles. Then we can use an analysis argument to reduce the clutching function down to a Laurent clutching function, and then further down to a linear clutching function. Finally, relating it back to isomorphism classes of vector bundles, we get the Bott periodicity.

The most commonly used formulation of this theorem is the following:

Proposition 2.4. *For a compact space X and any $n \geq 0$, the map $K^{-2}(*) \otimes K^{-n}(X) \to K^{-n-2}(X)$ induces an isomorphism $\beta : K^{-n}(X) \to K^{-n-2}(X)$.*

Hence it is natural to define the positive K-theory groups as $K^n(X) = K^0(X)$ if $2|n$, and $K^n(X) = K^1(X)$ otherwise. Furthermore, we have the following statement on K-theory.

Proposition 2.5. *The following sequence is exact:*

$$\begin{aligned} \cdots \to K^{-n}(X,A) \to K^{-n}(X) \to K^{-n}(A) \to K^{-n+1}(X,A) \to \dots \\ \to K(X,A) \to K(X) \to K(A) \end{aligned}$$

Remark 2.6. A detailed proof is given in [1] Chapter 2. It is worth noting that the functor K is indeed a representable functor [6] in the homotopy category, and we have the following equivalence

$$K(X) \cong [X_+, BU \times \mathbb{Z}] \cong \mathrm{colim}[X_+, BU(n) \times \mathbb{Z}]$$

where X is a compact space, $\mathbb{Z}$ is given the discrete topology.

The Bott Periodicity then reduces the sequence in Proposition 2.5 to the following diagram:

$$\begin{tikzcd} K^0(X,Y) \arrow[r] & K^0(X) \arrow[r] & K^0(Y) \arrow[d] \\ K^1(Y) \arrow[u] & K^1(X) \arrow[l] & K^1(X,Y) \arrow[l] \end{tikzcd}$$

As stated in remark 1.7, vector bundles are homotopy invariant, which implies that K-theory is also homotopy invariant. This is one of the axioms of the Eilenberg-Steenrod axioms for an ordinary cohomology theory. In fact, it can be proved that K-theory actually satisfies all of them except for the dimension axiom. [5] gives a detailed proof of this, and we will take this result for granted.

2.1. **Equivariant K-theory.** Analogous to the non-equivariant case, we denote $Vect_G(X)$ to be the isomorphism classes of G-vector bundles over X. This is an abelian semi-group under the direct sum operation. We can then form the associated abelian group by taking the Grothendieck group of this semi-group. Denote it as $K_G(X)$. By definition the elements in this abelian group take on the form $[E_1] - [E_2]$ where $[E_1]$ and $[E_2]$ are isomorphism classes of G-vector bundles over X. The tensor product of G-vector bundles then induces a commutative ring structure on $K_G(X)$.

If $\phi : X \to Y$ is a G-map of compact G-spaces, then there is an induced map $\phi^* : K_G(Y) \to K_G(X)$. Therefore, K_G can be viewed as a contravariant functor from the category of G-spaces to the category of commutative rings.

If G is the trivial group, then $K_G(X) = K(X)$. If X is the space of one point, then $K_G(X) = R(G)$, the representation ring of the group G.

We then prove a few important propositions of equivariant K-theory that will be used later in the proof of the Atiyah-Segal Completion theorem.

Proposition 2.7. *If X is a compact H-space, we can form a compact G-space $\overline{X} = (G \times X)/H = G \times_H X$, and $K_H(X) \cong K_G(\overline{X})$.*

Proof. For any H-vector bundle E on X, we can always identify it with a G-vector bundle on $\overline{X}$ by $E \mapsto G \times_H E$. On the other hand, consider the inclusion $\phi : X \to \overline{X}$ that sends $x \mapsto \overline{(1, x)}$. ϕ^*, the induced map of ϕ, then pulls back a G-vector bundle over $\overline{X}$ to a H-vector bundle over X. These maps are mutually inverse, and thus define an isomorphism between the K-theory groups. □

Proposition 2.8. *If X is a compact G-space with a base point, and A is a closed G-subspace (with the same base point), then we have the following exact sequence*

$$\widetilde{K}_G(X \cup_A CA) \to \widetilde{K}_G(X) \to \widetilde{K}_G(A)$$

The proof of this proposition is given in [9] and is more or less the same as the non-equivariant case. Note that if $A \to X$ is a cofibration, then here we can identify the equivariant K-theory of the mapping cone with the equivariant K-theory of the quotient by homotopy invariance.

With this result, we can define the negative K-theory groups as the following.

Definition 2.9. If X is a compact G-space with a base point, A is a closed G-subspace that contains the base point, define for any $n \in \mathbb{N}$

$$\widetilde{K}_G^{-n}(X) = \widetilde{K}_G(S^n X)$$
$$\widetilde{K}_G^{-n}(X, A) = \widetilde{K}_G(S^n(X \cup_A CA))$$

For a locally compact G-space X not necessarily compact, we denote X_+ as its one-point compactification, which is a G-space with a base point. If X is already compact, then define $X_+ = X \cup *$, the sum of X and a base point.

Definition 2.10. If X is a locally compact G-space and A is a closed subspace, define $K_G^{-n}(X) = \widetilde{K}_G^{-n}(X_+)$ and $K_G^{-n}(X, A) = \widetilde{K}_G^{-n}(X_+, A_+)$.

By Proposition 2.9, we have the following long exact sequence

$$\cdots \to \widetilde{K}_G^{-n}(X, A) \to \widetilde{K}_G^{-n}(X) \to \widetilde{K}_G^{-n}(A) \to \widetilde{K}_G^{-n+1}(X, A) \to \dots$$
$$\to \widetilde{K}_G(X, A) \to \widetilde{K}_G(X) \to \widetilde{K}_G(A)$$

Analogous to the non-equivariant case, there is a periodicity theorem that reduces this long exact sequence to a six term exact diagram.

Proposition 2.11. *If X is a G-space and L a G-line bundle over X, then the map $t \mapsto [H]$ induces an isomorphism of $K_G(X)$-modules:*

$$K_G(X)[t]/(t[L]-1)(t-1) \to K_G(P(L \oplus 1))$$

The proof is given in [1]. Notice that we could have assumed a G-action on everything in the proof of the non-equivariant periodicity theorem and the arguments will still hold.

3. The Thom Isomorphism

In a general cohomology theory, we have the notion of an orientable bundle and the Thom space of a bundle.

Definition 3.1. Let V be a vector bundle over X. If we choose a metric on the vector bundle, then $D(V)$ is the sub-space of elements of norm ≤ 1 and $S(V)$ is the sub-space of elements of norm $= 1$. A bundle V over X is *orientable* in the cohomology theory E^* if there exists a class $\mu \in E^*(D(V), S(V))$ such that $\mu|_p$, the class restricting to the fiber at p, is a generator of $E^*(D(V_p), S(V_p))$ for each $p \in X$. This is called the *orientation class* or the *Thom class*.

Definition 3.2. Given a vector bundle E over a compact space X, let $Th(E) = D(E)/S(E) = X^E$ be the one point compactification of the vector bundle E. Note that the Thom space is canonically a pointed space since it is a quotient.

The Thom space can also be identified with $P(E \oplus 1)/P(E)$ since $P(E \oplus 1)$ amounts to compactifying fibers of E by gluing in projective hyperplanes at ∞, and quotienting out $P(E)$ sends all these hyperplanes to a point. This is exactly the one point compactification (then the base point is the infinity point) of E.

In general, we have the following Thom Isomorphism theorem for general cohomology theories.

Theorem 3.3. *Let E^* be a generalized cohomology theory and let $p : V \to X$ be an E-oriented n-dimensional vector bundle. If X can be covered by finitely many open subsets on which the vector bundle V is trivial, then there exists an isomorphism*

$$\Phi : E^i(X) \to \widetilde{E}^{i+n}(D(V), S(V))$$

given by $\Phi(b) = p^*(b) \cup c$, where c is the Thom class.

Proof. We can first prove this for the case where V is a trivial vector bundle over X, then use a Mayer-Vietoris argument to extend to the general case. [8]

Lemma 3.4. *Let A, B be open subsets of X on which the vector bundle is trivial (local triviality of vector bundles). If the theorem is true on A, B, and $A \cap B$, then the theorem is also true on $A \cup B$.*

Proof. This can be proved using the Mayer-Vietoris sequence. Consider the following commutative diagram (commutativity by the naturality of Mayer-Vietoris sequence),

$$\begin{tikzcd}
E^{i-1}(A) \oplus E^{i-1}(B) \arrow[r] \arrow[d, "\Phi"] & E^{i-1}(A \cap B) \arrow[r] \arrow[d, "\Phi"] & E^{i-1}(A \cup B) \arrow[r] \arrow[d, "\Phi"] & E^{i}(A) \oplus E^{i}(B) \arrow[r] \arrow[d, "\Phi"] & E^{i}(A \cap B) \arrow[d, "\Phi"] \\
\widetilde{E}^{i+n-1}(Th(V_A)) \oplus \widetilde{E}^{i+n-1}(Th(V_B)) \arrow[r] & \widetilde{E}^{i+n-1}(Th(V_{A\cap B})) \arrow[r] & \widetilde{E}^{i+n-1}(Th(V_{A\cup B})) \arrow[r] & \widetilde{E}^{i+n}(Th(V_A)) \oplus \widetilde{E}^{i+n}(Th(V_B)) \arrow[r] & \widetilde{E}^{i+n}(Th(V_{A\cap B}))
\end{tikzcd}$$

where V_U denote the pullback of V to U, $U \subset X$. By the assumption, we have that the vertical maps are isomorphisms except possibly for the middle one. The five lemma then tells us that the middle map $E^{i-1}(A\cup B)\to \widetilde{E}^{i+n-1}(Th(V_{A\cup B}))$ is also an isomorphism, which completes the proof of our lemma. $\square$

Let $\{U_1,\dots,U_m\}$ denote an open covering of X such that E is trivial on all of U_i. When $m=1$ this is the trivial bundle case. Then suppose the statement is true for $m-1$, we can then use Lemma 3.4 on $U_1\cup U_2\cup\cdots\cup U_{m-1}$ and U_m. This proves the case of m, and hence the proof is completed. $\square$

We now consider the case for K-theory. Atiyah-Bott-Shapiro proved in [3] that a vector bundle is K-orientable if and only if it admits a *spin*^{c} structure and that every complex vector bundle admits a *spin*^{c} structure. Hence every complex vector bundle is K-orientable. This is equivalent to the following statement.

Proposition 3.5. *For a compact space X, there is a canonical orientation class $\lambda_E\in\widetilde{K}(Th(E))$ that is compatible with the direct sum operation and the pull back operation. We call this class the Thom class.*

Proof. We can construct this Thom class explicitly using the Koszul complex.

First, define the *support* of a complex of vector bundles $E^{\cdot}$ on X to be the closed subset of X consisting of the points x for which $E^{\cdot}_x$ is not exact. We then give the following definition.

Definition 3.6. Let A be a closed subset of a compact space X. Let $L(X,A)$ be the set of isomorphism classes of complexes of vector bundles $E^{\cdot}$ on X whose support is a subset of $X-A$. This set is a semi-group under direct sum, and two elements $E^{\cdot}_0$ and $E^{\cdot}_1$ of $L(X,A)$ are called homotopic, $\simeq$, if there is an object $E^{\cdot}$ of $L(X\times[0,1],A\times[0,1])$ such that $E^{\cdot}_0=E^{\cdot}|(X\times 0)$ and $E^{\cdot}_0=E^{\cdot}|(X\times 1)$. We then introduce the equivalence relation $\sim$ in $L(X,A)$ defined by

$$E^{\cdot}_0\sim E^{\cdot}_1 \iff E^{\cdot}_0\oplus F^{\cdot}_0\simeq E^{\cdot}_1\oplus F^{\cdot}_1$$

for some acyclic complexes $F^{\cdot}_0$ and $F^{\cdot}_1$ on X.

Proposition 3.7. *$L(X,A)/\sim$ is an abelian group naturally isomorphic to $K(X,A)$.*

Proof. This is proved in [1]. We note that when $A=\emptyset$, the desired isomorphism is given by $E^{\cdot}\mapsto\Sigma_k(-1)^kE^k$.

If $E^{\cdot}$ and $F^{\cdot}$ are complexes on X, then one can form their tensor product $E^{\cdot}\otimes F^{\cdot}$ by $(E^{\cdot}\otimes F^{\cdot})^k=\oplus_{p+q=k}E^p\otimes F^q$; this naturally gives a product structure in the ring $K(X)$, which can then be extended to make $K^*(X)$ a graded ring. $\square$

Lemma 3.8. *If V is a finite dimensional vector space, then for $v\in V$, the following sequence is exact if $v\neq 0$,*

$$0\to\mathbb{C}\xrightarrow{v}V\xrightarrow{\wedge v}\bigwedge^2 V\xrightarrow{\wedge v}\bigwedge^3 V\to\dots$$

This is a standard linear algebra fact which leads to the following definition.

Definition 3.9. If E is a vector bundle on X and s is a section of E, we can form the *Koszul complex*

$$\cdots\to 0\to\mathbb{C}\xrightarrow{d}\bigwedge^1 E\xrightarrow{d}\bigwedge^2 E\xrightarrow{d}\dots$$

where d is defined by $d(\xi) = \xi \wedge s(x)$ if $\xi \in \bigwedge^i E_x$. By Lemma 3.6, this complex is acyclic at all points x at which $s(x) \neq 0$ and thus its support is the set of zeros of s.

Note that this definition can be applied when we have a vector bundle and a section. However, if the required section is not given, there is also always a canonical space on which we do have have a section given by pulling back the vector bundle. Specifically, we consider the projection $p : E \to X$ and pull back the vector bundle E along p, then we have a vector bundle p^*E with a canonical section. Namely, there is a map $\delta : E \to E \times_X E = p^*E$ vanishing on the zero section of E. We denote $\bigwedge_E^\cdot$ the Koszul complex on E formed from p^*E and δ, namely

$$\cdots \to 0 \to \mathbb{C} \xrightarrow{\delta} p^* \bigwedge^1 E \xrightarrow{\delta} p^* \bigwedge^2 E \xrightarrow{\delta} \ldots$$

Now if we have a complex $F^\cdot$ on X with compact support on X, then we have $\bigwedge_E^\cdot \otimes p^*F^\cdot$ is a complex on $D(E)$ with compact support on $D(E) - S(E)$. We then have that the assignment $F^\cdot \mapsto \bigwedge_E^\cdot \otimes p^*F^\cdot$ induces an additive homomorphism $\phi_* : K(X) \to \widetilde{K}(D(E), S(E))$ by proposition 3.7.

Considering the zero-section $\phi : X \to E$, we then have $\phi^*\phi_*(F^\cdot)$ is the alternating sum of $\bigwedge^i E \otimes F^\cdot$, and we define $\lambda_{-1}(E)$ to be $\phi^*\phi_*(\xi) = \xi \cdot \lambda_{-1}(E)$ for any $\xi \in K_G(X)$. Here, $\lambda_{-1}(E)$ is given by

$$\lambda_{-1}(E) = \Sigma(-1)^i \lambda^i[E]$$

The desired Thom class is given by $\phi_*(I) = \lambda_E$, which is equivalently $\bigwedge_E^\cdot$.

We also have the following proposition.

Proposition 3.10. *If E and F are bundles on X, and $p : E \oplus F \to E$, $q : E \oplus F \to F$ are the projections, then* $\bigwedge_{E\oplus F}^\cdot \cong p^* \bigwedge_E^\cdot \otimes q^* \bigwedge_F^\cdot$

Proof. This follows directly from definition. Thus the direct sum of vector bundles gives a tensor product of their Kozul complex. □

Now, we want to show that the Thom class we just defined indeed gives a generator when restricted to each fiber. This is proved in [1] Chapter 2.6 and 2.7.

Since we are considering the fibers, we can reduce the case down to when X is a point. Let V be a complex vector bundle over X. Then V is just a complex vector space. We consider the complex $\bigwedge_V^\cdot$ defined as

$$0 \to \mathbb{C} \xrightarrow{v} V \xrightarrow{\wedge v} \bigwedge^2 V \xrightarrow{\wedge v} \bigwedge^3 V \to \cdots \to 0$$

for some v in given by a fiber. When V is one-dimensional, this complex reduces to $0 \to V \times \mathbb{C} \to V \times V \to 0$, which gives us an element in $K(D(V), S(V)) = \widetilde{K}(S^2)$. By Bott Periodicity, this is the canonical generator of $\widetilde{K}(S^2)$ up to a sign. Then by Proposition 3.10, $\bigwedge_V^\cdot$ gives a canonical generator of $\widetilde{K}(S^{2n})$ up to a sign.

In general, we can relate our Kozul complex construction with the projective bundle construction. We consider the Thom space as $P(V \oplus 1)/P(V)$. Let H be the canonical line bundle over $P(V)$, and H^* the dual of H. Note that $\pi : P(V \oplus 1) \to X$ is the projection, and we identify V with $P(V \oplus 1) - P(V)$. Since H is a sub-bundle of $\pi^*(V \oplus 1)$, taking the tensor product with H^*, we get a canonical morphism $1 \to \pi^*V \otimes H^*$. Restricted to $V \subset P(V \oplus 1)$, the morphism gives the Thom class $\bigwedge_V^\cdot$ since H restricting to $P(V \oplus 1) - P(V) = V$ is canonically trivial.

$\square$

With the existence of this Thom class, we have the following Thom Isomorphism theorem for K-theory. Here we give the more general equivariant version of the theorem on locally compact spaces.

Theorem 3.11. *(Thom Isomorphism) If X is a locally compact G-space, E is a complex G-vector bundle over the G-space X, then $\widetilde{K}^*_G(Th(E))$ is a rank 1 module over $K^*_G(X)$ with a generator λ_E. In other words, the map*

$$\phi : K^*_G(X) \to \widetilde{K}^*_G(Th(E))$$

which is given by the multiplication by λ_E, is an isomorphism.

Proof. Segal proved this theorem in [9]. The idea is that one can first prove this statement for line bundles using the Koszul complex construction, which is just the generalized Bott Periodicity. This step is similar to the projective bundle argument made in the non-equivariant case. Then using Proposition 3.10, we can prove the isomorphism statement for bundles that are locally a sum of G-line bundles. Note that when G is abelian, all G-vector bundles are locally a sum of G-line bundles. We can always embed a compact Lie group in a unitary group. Hence, by appealing to the proposition that there exists a wrong way map from $K^*_T(X) \to K^*_U(X)$ where T is the maximal torus and U is unitary group [9], we can reduce the statement down to the unitary group, and then to a torus where all vector bundles are local sums of line bundles. Hence the statement is proved. $\square$

We give an example of using the equivariant Thom Isomorphism as a computational tool. We calculate the K-theory structure of the real projective space.

Example 3.12. Take $G = \mathbb{Z}/2$, $X = *$, and $E = \mathbb{C}^n$ with the -1 action. Then we have

$$S(E)/G = \mathbb{RP}^{2n-1}$$

We know that if X is a compact G-space on which G acts freely, then $K^*_G(X) = K^*(X/G)$. G acts freely on $S(E) = S(\mathbb{C}^n)$, thus we have $K^*_G(S(E)) = K^*(S(E)/G)$. We know that $K^0_G(*) = R(G)$, and the representation ring of $\mathbb{Z}/2$ is $\mathbb{Z}[\rho]/\rho^2 - 1$ where ρ is the standard representation of dimension 1, and $K^1_G(*) = 0$. Thus by the Thom Isomorphism, we can reduce the long exact sequence of the pair $(D(E), S(E))$ to the following

$$0 \to K^1_G(S(E)) \to R(G) \xrightarrow{\times\lambda_{-1}(E)} R(G) \to K^0_G(S(E)) \to 0$$

We know that $\lambda_{-1}(\mathbb{C}^n) = (1-\rho)^n = \zeta^n$ where ρ is the standard 1-dimensional representation and $\zeta = 1 - \rho$. Thus we have $\zeta^2 = -2\zeta$ and $\lambda_{-1}(E) = (-\zeta)^n$. This gives us that $\widetilde{K}^0(\mathbb{RP}^{2n-1}) = \mathbb{Z}_{2^{n-1}}$ and $K^1(\mathbb{RP}^{2n-1})$ is infinite cyclic. Comparing the sequence of n and $n+1$, we get the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K^1(\mathbb{RP}^{2n+1}) \arrow[r] \arrow[d] & R(G) \arrow[r, "(-\zeta)^{n+1}"] \arrow[d, "-\zeta"] & R(G) \arrow[d, "1"] \\
0 \arrow[r] & K^1(\mathbb{RP}^{2n-1}) \arrow[r] & R(G) \arrow[r, "(-\zeta)^n"] & R(G)
\end{tikzcd}$$

Note that since $\zeta^2 = -2\zeta$, the kernel of the map $(-\zeta)^i$ is $(2-\zeta)$ for any $i \geq 1$, therefore by diagram chasing, we have that the middle rectangle gives a zero map, which means that the left vertical map must be zero. That is

$$K^1(\mathbb{RP}^{2n+1}) \xrightarrow{0} K^1(\mathbb{RP}^{2n-1})$$

We can then consider the long exact sequences of the pairs $(\mathbb{RP}^{2n+1}, \mathbb{RP}^{2n})$ and $(\mathbb{RP}^{2n}, \mathbb{RP}^{2n-1})$ and make suitable replacement in the sequences, which will give us

$$K^1(\mathbb{RP}^{2n+1}) \twoheadrightarrow K^1(\mathbb{RP}^{2n}) \hookrightarrow K^1(\mathbb{RP}^{2n-1})$$

Thus $K^1(\mathbb{RP}^{2n}) = 0$, which then gives us an isomorphism between $K^0(\mathbb{RP}^{2n+1})$ and $K^0(\mathbb{RP}^{2n})$. Thus, we have obtained the $K^*(\mathbb{RP}^n)$ structure

$$K^1(\mathbb{RP}^{2n+1}) = \mathbb{Z}$$

$$K^1(\mathbb{RP}^{2n}) = 0$$

$$K^0(\mathbb{RP}^{2n+1}) = K^0(\mathbb{RP}^{2n}) = \mathbb{Z} \oplus \mathbb{Z}_{2^n}$$

4. The Atiyah-Segal Completion Theorem [4]

4.1. **The Statement of the Theorem.** The motivation of this theorem is to study the G-equivariant K-theory of EG, or, equivalently, the ordinary K-theory of the space BG since G acts freely on EG. Unfortunately this space is generally not compact, so the general definition of K-theory does not extend to this space. One workaround is to formulate this problem in a way such that we only need to work with finite $G-CW$ complexes. Thus, instead of considering the G-equivariant K-theory of EG which so far we have not yet defined, we can consider the skeleta EG^n. Using Milnor's model [7] of EG, which is defined to be the infinite join of G with itself, we have that each level EG_n is a finite $G-CW$ complex. Formally, $EG^n = G * \cdots * G$ and EG is the colimit of $\{EG^n\}$. In this way, we can relate $K_G^*(X)$ with $\{K^*(X \times EG^n/G)\}$ using the following statement.

Theorem 4.1. *(Atiyah-Segal Completion Theorem) Let X be a compact G-space such that $K_G^*(X)$ is finite over $R(G)$. Then,*

$$K_G^*(X)_{\hat{I}} \cong \varprojlim K^*(X \times EG^n/G)$$

where I is the usual augmentation ideal of the representation ring.

Remark 4.2. Note that there is actually a stronger statement given in Theorem 4.6 which is formulated in terms of pro-rings.

One commonly considered special case is when $X = *$. Then Theorem 4.1 restricts to

$$R(G)_{\hat{I}} \cong \varprojlim K(BG^n) \tag{4.3}$$

Before going into the proof of the theorem, we give an example of the application of it.

Example 4.4. Let the space X be a point and take $G = S^1$. Then $BS^1 = \mathbb{CP}^\infty = \varinjlim \mathbb{CP}^n$. By the Thom Isomorphism (or simply Bott periodicity), we have that $K^0(\mathbb{CP}^n) = \mathbb{Z}[t]/t^{n+1}$. In the limit, $K^0(\mathbb{CP}^\infty) = \mathbb{Z}[[t]]$, where the definition of $K^0(\mathbb{CP}^\infty)$ will be given in section 4.3 (K-theory on non-compact spaces). On the other hand, the representation ring of S^1 is $R(S^1) = \mathbb{Z}[X, X^{-1}]$ where X is the standard representation of S^1. Completing it at the augmentation ideal $(X-1)$, we have that $R(S^1)_{\hat{I}} = \mathbb{Z}[[X-1]]$. Thus, we have

$$R(S^1)_{\hat{I}} = \mathbb{Z}[[X-1]] \cong \mathbb{Z}[[t]] = K(\mathbb{CP}^\infty) = K(BS^1)$$

The isomorphism given by (5.2) takes the class of t to the class of $X-1$.

Now we introduce the notion of pro-objects which gives a convenient formulation of Theorem 4.1.

Definition 4.5. If C is any category, $Pro(C)$ is a new category whose objects are inverse-systems $\{A_\alpha\}_{\alpha \in S}$ of objects of C indexed by directed sets S. The morphism between $\{A_\alpha\}_{\alpha\in S}$ and $\{B_\beta\}_{\beta\in T}$ is the the collection of maps $f_\beta : A_{\theta\beta} \to B_\beta$ for each $\beta \in T$, where $\theta : T \to S$ (not necessarily order-preserving). The maps f_β are subject to the condition that if $\beta \leq \beta'$ in T then for some $\alpha \in S$ such that $\alpha \geq \theta\beta$ and $\alpha \geq \theta\beta'$, the diagram

$$\begin{tikzcd}
& A_{\theta\beta} \arrow[r, "f_\beta"] & B_\beta \\
A_\alpha \arrow[ur, "a_{\alpha,\theta\beta}"] \arrow[r, "a_{\alpha,\theta\beta'}"'] & A_{\theta\beta'} \arrow[r, "f_{\beta'}"'] & B_{\beta'} \arrow[u, "b_{\beta',\beta}"']
\end{tikzcd}$$

commutes. One identifies the morphisms $(\theta; f_\beta)$ and $(\theta'; f_{\beta'})$ if for each β there is an $\alpha \in S$ such that $\alpha \geq \theta\beta$, $\alpha \geq \theta'\beta$, and $f_\beta a_{\alpha,\theta\beta} = f'_\beta a_{\alpha\theta\beta'}$

Approximately, we can think of pro-objects as the inverse limit with its topology. For example, if we consider the pro-object $\{\mathbb{Z}/p^n\}$, the equivalent data is given by the p-adic integers $\mathbb{Z}_p$ together with the p-adic topology. Isomorphisms between pro-objects can be understood as an isomorphism between the inverse limits that respects the topology but not necessarily the exact tower.

We also give an example to demostrate how a pro-object is different from the inverse limit. Consider the pro-object: all objects in the tower are $\mathbb{Z}$ and the maps between them are multiplication by 2. So we can think of it as

$$\dots \subset 2^n \cdot \mathbb{Z} \subset 2^{n-1} \cdot \mathbb{Z} \subset \dots 2\mathbb{Z} \subset \mathbb{Z}$$

This is a tower of abelian groups whose inverse limit is 0 since there is no integer divisible by 2^n for all n. However in the category of pro-objects this does not give us 0.

With this formulation, the theorem is saying that there is an isomorphism between $K^*_G(X)_{\hat{I}}$ and $K^*(X \times EG/G)$ as pro-rings. Specifically, the LHS of Theorem 4.1 is obtained by an I-adic completion, which can be identified with $\varprojlim K^*_G(X)/I^n_G \cdot K^*_G(X)$. We can view the system $\{K^*_G(X)/I^n_G \cdot K^*_G(X)\}$ as a pro-ring construction. Similarly, the RHS of Theorem 4.1 can be constructed via the pro-system of rings $\{K^*_G(X \times EG^n)\}$. Our goal now is to show that there is an isomorphism between the pro-objects. This is done by the following.

Note that $BG^n = EG^n/G$ is the union of the n contractible subsets U_i, U_i being the set where the i-th join-coordinate does not vanish. Therefore the product of any n elements of the reduced group $\widetilde{K}^*(BG^n)$ is zero. Thus, we have that $(\widetilde{K}^*(BG^n))^n = 0$. Let $\epsilon : K^*(BG^n) \to \mathbb{Z}$ denote the usual augmentation, we know that $\widetilde{K}^*(BG^n)$ is the kernel of this map, which gives us that $\widetilde{K}^*(BG^n) \hookrightarrow K^*(BG^n) \xrightarrow{\epsilon} \mathbb{Z}$ is zero. Denote $\alpha_n : R(G) = K^*_G(\text{point}) \to K^*_G(EG^n)$ as the induced map of the map $EG^n \to (\text{point})$. Note that we have the composite homomorphism $R(G) = K^*_G(\text{point}) \xrightarrow{\alpha_n} K^*_G(EG^n) = K^*(BG^n) \xrightarrow{\epsilon} \mathbb{Z}$ being the usual augmentation of $R(G)$ and the kernel being the usual augmentation ideal I_G. Thus we have

$$\begin{tikzcd} I_G \arrow[r, hook] \arrow[dr, dashed] & R(G) \arrow[r, "\alpha_n"] & K^*(BG^n) \\ & \widetilde{K}^*(BG^n) \arrow[ur, hook] & \end{tikzcd}$$

factors through $\widetilde{K}^*(BG^n)$. This implies that I^n_G factors through $(\widetilde{K}^*(BG^n))^n = 0$, (i.e. $I^n_G \to R(G) \to K^*(BG^n)$ is zero). Thus we have that α_n factors through $R(G)/I^n_G R(G)$.

By naturality this implies that the map induced by the projection $X \times EG^n \to X$ factors through the submodule generated by I^n_G, which gives us the map:

$$\alpha_n : K^*_G(X)/I^n_G \cdot K^*_G(X) \to K^*_G(X \times EG^n)$$

We will prove that this map induces an isomorphism on the pro-rings. Note that this is in fact a stronger statement than our original theorem given in Theorem 4.1. Formally, our main theorem is

Theorem 4.6. *Let X be a compact G-space such that $K^*_G(X)$ is finite over $R(G)$. Then the homomorphisms*

$$\alpha_n : K^*_G(X)/I^n_G \cdot K^*_G(X) \to K^*_G(X \times EG^n)$$

induce an isomorphism of pro-rings. Specifically, for each n one can find k and a homomorphism $\beta_n : K^*_G(X \times EG^{n+k}) \to K^*_G(X)/I^n_G \cdot K^*_G(X)$ such that the diagram

$$\begin{tikzcd} K^*_G(X)/I^{n+k}_G \cdot K^*_G(X) \arrow[r, "\alpha_{n+k}"] \arrow[d] & K^*_G(X \times EG^{n+k}) \arrow[d] \arrow[dl, "\beta_n"'] \\ K^*_G(X)/I^n_G \cdot K^*_G(X) \arrow[r, "\alpha_n"'] & K^*_G(X \times EG^n) \end{tikzcd}$$

commutes.

4.2. **The Proof of the Main Theorem.** The proof of the main theorem consists of four steps.

Step1 : The case of $S^1 = T$

Since in step 2 we will inductively prove the theorem for T^n, here we prove a slightly more general statement.

Lemma 4.7. *Let G be a compact Lie group, and X a compact G-space such that $K^*_G(X)$ is finite over $R(G)$. Let $\theta : G \to T$ be a homomorphism by which G acts on ET. Then*

$$\alpha_n : K^*_G(X)/I^n_T \cdot K^*_G(X) \to K^*_G(X \times ET^n),$$

induces an isomorphism of pro-rings.

Proof. We know that $ET^n = T * \cdots * T$, which is just S^{2n-1} where T acts on S^{2n-1} via scalar multiplication as a subgroup of $\mathbb{C}^\times$.

Consider the exact sequence of the pair $(X \times D(V_n), X \times S(V_n))$, where V_n is the direct sum of n copies of the 1-dimensional standard representation.

$$\cdots \to K^0_G(X \times D(V_n), X \times S(V_n)) \to K^0_G(X \times D(V_n)) \to K^0_G(X \times S(V_n)) \to \dots$$

$D(V_n)$ is contractible; thus we have $K_G^*(X \times D(V_n)) \cong K_G^*(X)$. Also, $K_G^*(X \times D(V_n), X \times S(V_n)) = \widetilde{K}_G^*(Th(X \times V_n))$. By the Thom Isomorphism, we can reduce the long exact sequence of the pair to the following exact diagram

$$\begin{tikzcd}
\widetilde{K}_G^0(Th(X \times V_n)) \arrow[r, "\times\lambda_{-1}(V_n)"] & K_G^0(X) \arrow[r] & K_G^0(X \times S(V_n)) \arrow[d] \\
K_G^1(X \times S(V_n)) \arrow[u] & K_G^1(X) \arrow[l] & \widetilde{K}_G^1(Th(X \times V_n)) \arrow[l, "\times\lambda_{-1}(V_n)"']
\end{tikzcd}$$

Note that $\lambda_{-1}(V_n) = (1-\rho)^n = \zeta^n$ where ρ is the standard one-dimensional representation of T and $\zeta = 1-\rho$ (We used the fact that direct sum of bundles gives a product of the Thom classes). We know that the augmentation ideal of $R(T)$ is also generated by ζ, thus we can further reduce the exact diagram to

$$0 \to K^0/\zeta^n \cdot K^0 \xrightarrow{\alpha_n} K_G^0(X \times ET^n) \to {}_{\zeta^n}K^1 \to 0$$

where $K^0 = K_G^0(X)$, $K^1 = K_G^1(X)$ and ${}_{\zeta^n}K^1 = \{x \in K^1 : \zeta^n x = 0\}$

Note that K^* is a Noetherian $R(G)$-module; thus there must exist some $k \in \mathbb{N}$ such that ${}_{\zeta^k}K^1 = {}_{\zeta^{k+1}}K^1 = \dots$, which means that ζ^k annihilates all of ${}_{\zeta^{k+i}}K^1$ for all $i \geq 1$. Therefore, we have the following commutative diagram,

$$\begin{tikzcd}
0 \arrow[r] & K^0/\zeta^{n+k} \cdot K^0 \arrow[r, "\alpha_{n+k}"] \arrow[d] & K_G^0(X \times S^{2n+2k-1}) \arrow[r] \arrow[d] \arrow[dl, "\beta_n"] & {}_{\zeta^{n+k}}K^1 \arrow[r] \arrow[d, "\times\zeta^k=0"] & 0 \\
0 \arrow[r] & K^0/\zeta^n \cdot K^0 \arrow[r, "\alpha_n"'] & K_G^0(X \times S^{2n-1}) \arrow[r] & {}_{\zeta^n}K^1 \arrow[r] & 0
\end{tikzcd}$$

The existence of such a β_n is due to the fact that the right vertical map is the zero map, then by diagram chasing, the middle map factors through $K^0/\zeta^n \cdot K^0$. This concludes the proof of Lemma 4.7. $\square$

Step2: Proof for T^m

We will prove by induction on m. The main lemma that we are trying to prove in this step is

Lemma 4.8. *Let G be a compact Lie group and X a compact G-space such that $K_G^*(X)$ is finite over $R(G)$. If G acts on ET^m by a homomorphism $\theta : G \to T^m$, then the homomorphism*

$$\alpha_n : K_G^*(X)/I_{T^m}^n \cdot K_G^*(X) \to K_G^*(X \times (ET^m)^n)$$

induces an isomorphism between pro-rings.

Proof. Write $T^m = T \times T^{m-1}$. Step 1 proved the base case $m = 1$, and we assume the statement is true for T^i for any $i < m$. Let $H = T^{m-1}$ for notational simplicity. One can identify ET^m with $ET \times EH$ by the axioms of universal spaces, which means that they are G-homotopy equivalent. This induces an isomorphism between $\{K_G^*(X \times (ET^m)^n)\}$ and $\{K_G^*(X \times ET^n \times EH^m)\}$ as pro-rings. We can then use the cofinal subfamily $\{ET^p \times EH^q\}$ of compact subspaces to model the pro-ring. Let I_T and I_H denote the ideals of $R(T^m)$ generated by the respective augmentation ideals of the two groups. Then we have $I_{T^m} = I_T + I_H$. For notational simplicity, let $K = K_G^*(X)$. Since $I_T^n + I_H^n \subset (I_T + I_H)^n = (I_{T^m})^n$, and $I_{T^m}^{p+q-1} = (I_T + I_H)^{p+q-1} \subset I_T^p + I_H^q$, we have that the pro-rings $\{K/I_{T^m}^n \cdot K\} \cong \{K/(I_T^p + I_H^q) \cdot K\}$. In other words, the I_{T^m}-adic topology is the same as the one induced by $I_T^p + I_H^q$. Therefore, we can use $\{K/(I_T^p + I_H^q) \cdot K\}$ to compute the completion of the LHS of Lemma 4.8.

Note that the induced projection map $K \to K_G^*(X \times ET^p \times EH^q)$ factors through the quotient module $\{K/(I_T^p + I_H^q) \cdot K\}$, and thus gives us maps

$$\alpha_{p,q} : K/(I_T^p + I_H^q) \cdot K \to K_G^*(X \times ET^p \times EH^q)$$

On completion, we know that these homomorphisms induce the same map as $\{\alpha_n\}$. Thus, we want to prove that these maps define an isomorphism of pro-rings. Let $R = R(T^m)$. Note that

$$K/(I_T^p + I_H^q) \cdot K \cong K \otimes_{R(T^m)} R(T^m)/I_T^p \otimes_{R(T^m)} R(T^m)/I_H^q = K/I_T^p \otimes_R R/I_H^q$$

We can then then factorize $\alpha_{p,q}$ by

$$\alpha_{p,-} \otimes 1 : K/I_T^p \otimes R/I_H^q \xrightarrow{\cong} K_G^*(X \times (ET)^p) \otimes R/I_H^q$$

$$1 \otimes \alpha_{-,q} : K_G^*(X \times (ET)^p) \otimes R/I_H^q \xrightarrow{\cong} K_G^*(X \times (ET)^p \times (EH)^q)$$

where the the first isomorphism is given by step 1, and the second isomorphism is given by the induction hypothesis (the induction hypothesis applies because $K_G^*(X \times ET^p)$ is finite over $R(G)$, as showed in the exact sequence in step 1). Therefore, on completion, $\{\alpha_n\}$ gives an isomorphism between pro-rings. □

Step3: The Proof for $U(m) = U$.

This step requires a blackbox from [2].

Theorem 4.9. *Let $j : T \to U$ be the inclusion of the maximal torus into the unitary group. For any compact U-space X, let $j^* : K_U^*(X) \to K_T^*(X)$ be the induced map. Then there exists a $K_U^*(X)$-module homomorphism*

$$j_* : K_T^*(X) \to K_U^*(X)$$

which is functorial in X and is a left inverse of j^*.

We use the fact that $K_T^*(X)$ is a finite module over $K_U^*(X)$ for any compact U-space X. This tells us that $K_U^*(X)$ is a natural canonical direct summand of $K_T^*(X)$. Therefore, $\alpha_n : K_U^*(X)/I_U^n \cdot K_U^*(X) \to K_U^*(X \times EU^n)$ induces an isomorphism between pro-rings follows from the fact that

$$\eta_n : K_T^*(X)/I_U^n \cdot K_T^*(X) \to K_T^*(X \times EU^n)$$

induces an isomorphism between pro-rings.

We then examine the following commutative rectangle

$$\begin{array}{ccc} K_T^*(X)/I_U^n \cdot K_T^*(X) & \xrightarrow{\eta_n} & K_T^*(X \times EU^n) \\ \downarrow & & \downarrow \\ K_T^*(X)/I_T^n \cdot K_T^*(X) & \xrightarrow{\alpha_n} & K_T^*(X \times ET^n) \end{array}$$

By step 2, $\{\alpha_n\}$ defines an isomorphism of pro-rings. The left vertical map defines an isomorphism of pro-rings because the I_U-adic and I_T-adic topologies coincide on $R(T)$, and thus coincide on any $R(T)$-module. We then consider the objects on the right. Note that $\{EU^n\}$ are U-CW complexes, which are larger T-CW complexes (dimension of U is greater than dimension of T). This means that $\{ET^n\}$ and $\{EU^n\}$ are cofinal families of compact T-spaces, defining the same pro-object of T-equivariant K-theory rings. Thus the right vertical map in the commutative diagram is an isomorphism of pro-rings. This then forces the top map η_n to be an isomorphism of pro-rings, and this completes the proof of the third step.

Step4 : The general case

Embed G in a unitary group U. For a compact G-space X, define $\overline{X} = (U \times X)/G$, which is then an U-space. By Proposition 2.8, $K_G^*(X) \cong K_U^*(\overline{X})$.

Note that $\overline{X} \times EU^n = (U \times_G X) \times EU^n = \overline{X \times EU^n}$ by the associativity of balanced products. Thus we have $K_U^*(\overline{X} \times EU^n) = K_U^*(\overline{X \times EU^n}) \cong K_G^*(X \times EU^n)$. Step 3 gives us an isomorphism of pro-rings

$$\alpha_n : K_U^*(X)/I_U^n \cdot K_U^*(X) \to K_U^*(X \times EU^n)$$

for any compact U-space X. Note that $K_U^*(\overline{X})$ is finite over $R(U)$ because $K_G^*(X)$ is finite over $R(G)$ and $R(G)$ is finite over $R(U)$. Thus we can apply the statement in step 3 to $\overline{X}$ and we get

$$\alpha_n : K_G^*(X)/I_U^n \cdot K_G^*(X) \to K_G^*(X \times EU^n)$$

Note that since EU is a universal space for G, $\{EG^n\}$ and $\{EU^n\}$ are cofinal families. Hence $K_G^*(X \times EU^n)$ is isomorphic to $K_G^*(X \times EG^n)$ as pro-rings. Finally, we know that I_U-adic and I_G-adic topologies coincide on $R(G)$-modules[10]. This means that $\{K_G^*(X)/I_U^n \cdot K_G^*(X)\}$ and $\{K_G^*(X)/I_G^n \cdot K_G^*(X)\}$ give the same pro-object. The proof is then completed.

4.3. **Defining $K_G(X \times EG)$.** We return to the original formulation of our theorem. We define the K-theory for non compact spaces as follows.

$$K^n(X) = [X, F^n]$$

the group of homotopy classes of maps $X \to F^n$, where F^n is a suitable H-space. Note that $F^0 = \mathbb{Z} \times BU$, and F^1 is the infinite unitary group.

This definition is consistent with the previous definition on compact spaces as mentioned in Remark 2.6.

The property that we need in order to formulate the RHS of Theorem 4.1 is the following

Proposition 4.10. *(Milnor) If the space X is the limit of an expanding sequence of compact subspaces X_n, then there is a natural exact sequence*

$$0 \leftarrow R^1 \varprojlim K^{k-1}(X_n) \to K^k(X) \to \varprojlim K^k(X_n) \to 0$$

Proof. A proof of this proposition is given in [4]. □

Note that the space $X \times EG/G$ is the limit of the sequence of compact subspaces $X_n = X \times EG^n/G$. By Theorem 4.1 we know that $\{K^*(X_n)\} = \{K_G^*(X \times EG^n)\}$ satisfies the Mittag-Leffler condition [4]. Hence $R^1 \varprojlim K^*(X_n) = 0$, and so by Proposition 4.10, we have that $K^*(X \times EG/G) \cong \lim K_G^*(X \times EG^n)$, which gives us the other form of Theorem 4.1:

Theorem 4.11. *If X is a compact G-space such that $K_G^*(X)$ is finite over $R(G)$, then we have*

$$\alpha : K_G^*(X) \to K^*(X \times EG/G)$$

induces an isomorphism between the I_G-adic completion of $K_G^*(X)$ and $K^*(X \times EG/G)$

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