KOSZUL DUALITY OF QUADRATIC OPERADS

WENXI YAO

Abstract. In this expository paper, we discuss Koszul duality as a method of explicitly constructing minimal models for quadratic dg operads. We define Koszul operads, which are quadratic operads whose Koszul complexes are acyclic. Our main result is that the minimal model of a Koszul operad is the cobar complex of its Koszul dual cooperad.

Contents

1. Introduction

Operads describe algebraic operations with arbitrarily many arguments that compose in specific ways and respect constraints of associativity, unitality and equivariance [1]. For a differential graded (dg) operad $\mathcal{P}$, the minimal model of $\mathcal{P}$ is defined as a free operad quasi-isomorphic to $\mathcal{P}$ with a decomposable differential. Intuitively, the minimal model of $\mathcal{P}$ encodes the same homological information as $\mathcal{P}$ but reduces the "size" of $\mathcal{P}$ to the minimum. Similar definitions can be made for dg algebras; we point to [2] for the connection between two concepts.

Minimal models have important applications in rational homotopy theory [3], where a dg algebra can be assigned to every 1-connected topological space. It has been shown that every homologically connected dg algebra and dg operad admits a unique minimal model, which furthers our understanding about rational spaces. However, the explicit construction of such minimal model is often nontrivial. In this paper, we show that for the minimal model of a class of dg operads called Koszul operads can be constructed from their Koszul dual cooperads.

This paper is structured as follows. In Section 2, we provide basic definitions related to operads and cooperads. In Section 3, we present important constructions of operadic homological algebra such as the operadic bar and cobar constructions. In Section 4, we define Koszul duality for quadratic operads using constructions in Section 3 and give the explicit expressions of Koszul duality for binary quadratic operads. We also present the Koszul duality relations among three widely known operads **Assoc**, **Com** and **Lie**: **Assoc** is self-dual while **Com** and **Lie** are Koszul duals of each other. In Section 5, we define Koszul operads and present the main theorem, which builds an equivalence between the acyclicity of Koszul complexes and the existence of a quasi-isomorphism between the operad and the cobar complex of its Koszul dual cooperad. It is straightforward that such cobar complex is the minimal model of the operad.

2. Algebraic operads

2.1. **Basic definitions.** Operads can be defined in many different ways. We use the monoidal definition of operads so that cooperads can be defined in a similar fashion. Throughout the paper, $\mathbb{K}$ denotes a field of characteristic 0 and $\mathbb{S}_n$ is the symmetric group over n elements.

Definition 2.1. $\mathbb{S}$-module. An $\mathbb{S}$-*module* is a collection $M = \{M(n)\}_{n\geq 0}$ where $M(n)$ is a right $\mathbb{K}[\mathbb{S}_n]$-module for each n. A morphism $f : M \to N$ of $\mathbb{S}$-modules is a collection of $\mathbb{S}_n$-equivariant maps $f_n : M(n) \to N(n)$.

In other words, each $M(n)$ is a $\mathbb{K}$-linear representation of $\mathbb{S}_n$. Constructions on vectors spaces can be extended to $\mathbb{S}$-modules. The *direct sum* of $\mathbb{S}$-modules is defined component-wise as

$$(M \oplus N)(n) := M(n) \oplus N(n)$$

The *tensor product* of $\mathbb{S}$-modules is defined as

$$(M \otimes N)(n) := \bigoplus_{i+j=n} \mathrm{Ind}_{\mathbb{S}_i\times\mathbb{S}_j}^{\mathbb{S}_n}\big(M(i)\otimes N(j)\big)$$

where $\mathrm{Ind}_{\mathbb{S}_i\times\mathbb{S}_j}^{\mathbb{S}_n}$ is the $\mathbb{K}$-linear representation of $\mathbb{S}_n$ induced by the representation of $\mathbb{S}_i \times \mathbb{S}_j$ given by $M(i)$ and $N(j)$.

Definition 2.2. Composite of $\mathbb{S}$-modules. The *composite* of $\mathbb{S}$-modules M and N is the $\mathbb{S}$-module

$$M \circ N := \bigoplus_{k\geq 0} M(k) \otimes_{\mathbb{S}_k} N^{\otimes k}$$

where $N^{\otimes k}$ denotes the tensor product of k copies of N. $\otimes_{\mathbb{S}_k}$ denotes the usual tensor product with the equivalence relation $x \otimes_{\mathbb{S}_k} (y\cdot\sigma) \sim (x\cdot\sigma) \otimes_{\mathbb{S}_k} y$ for $\sigma \in \mathbb{S}_k$. Explicitly, the components of $M \circ N$ are expressed by

$$(M \circ N)(n) = \bigoplus_{k\geq 0} M(k) \otimes_{\mathbb{S}_k} \Bigg[\bigoplus_{i_1+\cdots+i_k=n} \mathrm{Ind}_{\mathbb{S}_{i_1}\times\cdots\times S_{i_k}}^{\mathbb{S}_n}\big(N(i_1)\otimes\cdots\otimes N(i_k)\big)\Bigg]$$

where $\mathrm{Ind}_{\mathbb{S}_{i_1}\times\cdots\times S_{i_k}}^{\mathbb{S}_n}$ is the representation of $\mathbb{S}_n$ induced by its subgroup $\mathbb{S}_{i_1} \times \cdots \times \mathbb{S}_{i_k}$. Meanwhile, the action of $\mathbb{S}_k$ permutes $N(i_1), \cdots, N(i_k)$ and commutes with the action of $\mathbb{S}_{i_j}$ on $N(i_j)$.

Alternatively, we can take invariants of $N^{\otimes k}$ under the action of $\mathbb{S}_k$ instead of coinvariants and obtain another composite

$$M \,\bar{\circ}\, N := \bigoplus_{k\geq 0} \left(M(k)\otimes N^{\otimes k}\right)^{\mathbb{S}_k}$$

Explicitly, the components of $M \,\bar{\circ}\, N$ are expressed by

$$(M \,\bar{\circ}\, N)(n) = \bigoplus_{k\geq 0} M(k)\otimes \left[\bigoplus_{i_1+\ldots+i_k=n} \mathrm{Ind}^{\mathbb{S}_n}_{\mathbb{S}_{i_1}\times\cdots\times S_{i_k}} \left(N(i_1)\otimes\cdots\otimes N(i_k)\right)\right]^{\mathbb{S}_k}$$

We point to Section 5.1.21 of [4] for the connection between $\circ$ and $\bar{\circ}$.

Both $\circ$ and $\bar{\circ}$ give a monoidal structure on $\mathbb{S}$-**Mod**, the category of $\mathbb{S}$-modules. For both structures, the unit object is given by $I = (0, \mathbb{K}, 0, 0, \cdots)$. As we shall see, operads are essentially monoids in the monoidal category $(\mathbb{S}\text{-}\mathbf{Mod}, \circ, I)$ while cooperads are comonoids in $(\mathbb{S}\text{-}\mathbf{Mod}, \bar{\circ}, I)$.

Definition 2.3. Monoidal definition of an operad. A *symmetric operad* $\mathcal{P} = (\mathcal{P}, \gamma, \eta)$ is an $\mathbb{S}$-module $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 0}$ such that

(a) there exists a *composition map* $\gamma : \mathcal{P}\circ\mathcal{P} \to \mathcal{P}$ and a *unit map* $\eta : I \to \mathcal{P}$ that are both $\mathbb{S}$-module morphisms;

(b) γ and η satisfy the classical axiom of monoids.

We can view an operad $\mathcal{P}$ as a collection of operations with arbitrarily many arguments where $\mathcal{P}(n)$ consists of all n-ary operations in $\mathcal{P}$. Operations in $\mathcal{P}$ compose through γ in the following way:

$$\gamma : \mathcal{P}(k)\otimes\mathcal{P}(i_1)\otimes ... \otimes\mathcal{P}(i_k) \to \mathcal{P}(i_1 + ... + \mathcal{P}_k)$$

where $\mathcal{P}(k)$ comes from the first copy of $\mathcal{P}$ in $\mathcal{P}\circ\mathcal{P}$ and the rest come from the k-fold tensor product $\mathcal{P}^{\otimes k}$. γ is visualized by the "grafting" of trees in Figure 1.

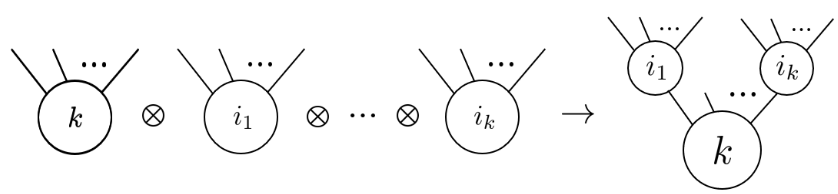


FIGURE 1. Schematic of operad composition.

Definition 2.4. Cooperad. A *cooperad* is an $\mathbb{S}$-module $\mathcal{C}$ with a *decomposition map* $\triangle : \mathcal{C} \to \mathcal{C}\,\bar{\circ}\,\mathcal{C}$ and a *counit map* $\epsilon : \mathcal{C} \to I$ that are morphisms of $\mathbb{S}$-modules and satisfy the axioms of comonoids.

As examples, we now introduce endomorphism operads and cooperads that are closely connected to the definition of algebras over operads. They also play an important role in defining operadic suspension in Section 3.10 and Section 3.11.

Example 2.5. Endomorphism operad. The *endomorphism operad* of any vector space V is given by

$$\mathrm{End}_V(n) := \mathrm{Hom}(V^{\otimes n}, V)$$

where $V^{\otimes 0} = \mathbb{K}$. The right action of $\mathbb{S}_n$ is induced by the left action of $\mathbb{S}_n$ on $V^{\otimes n}$. The composition map γ is given by composition of homomorphisms:

$$\gamma(f; f_1, \cdots, f_k) = f(f_1 \otimes \cdots \otimes f_k)$$

Example 2.6. Endomorphism cooperad. The *endomorphism cooperad* of V is given by

$$\mathrm{Coend}_V(n) := \mathrm{Hom}(V, V^{\otimes n})$$

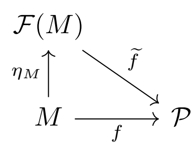
where $V^{\otimes 0} = \mathbb{K}$. The right action of $\mathbb{S}_n$ is induced by the right action on $V^{\otimes n}$ and the composition map is similarly given by composition of homomorphisms.

Recall that the representation of a group G can be defined as a group homomorphism $f : G \to \mathrm{End}(V)$ where $\mathrm{End}(V)$ is the space of endomorphisms for a vector space V. Algebras over operads have an analogous definition, which enables us to view them as "representations" of operads.

Definition 2.7. Algebra over an operad. An *algebra over an operad* $\mathcal{P}$ is a vector space V together with a morphism of operads $f : \mathcal{P} \to \mathrm{End}_V$, which is a collection of maps $f_n : \mathcal{P}(n) \to \mathrm{End}_V(n)$ compatible with symmetric group action, unit elements and compositions.

2.2. **Free operads and cofree cooperads.** We now move on to present definitions of free operads and cofree cooperads which appear in the operadic bar and cobar constructions.

Definition 2.8. Free operad. The *free operad* over an $\mathbb{S}$-module M is an operad $\mathcal{F}(M)$ with an $\mathbb{S}$-module morphism $\eta_M : M \to \mathcal{F}(M)$ and the universal property expressed by the following diagram:



In other words, there exists a unique lift $\widetilde{f} : M \to \mathcal{F}(M)$ for any $\mathbb{S}$-module morphism $f : M \to \mathcal{P}$ where $\mathcal{P}$ is an operad.

Explicitly, $\mathcal{F}(M)$ is the colimit $\mathcal{F}(M) := \mathcal{T}(M) = \mathrm{colim}\mathcal{T}_n M$ where $\mathcal{T}_n M$ is defined inductively by $\mathcal{T}_0 M := I$ and $\mathcal{T}_n M := I \oplus (M \circ \mathcal{T}_{n-1} M)$. We point to Section 5.4.1 of [4] for detailed exposition.

We now introduce a weight grading on the free operad defined as the number of generating operations needed in constructing a given operation in the free operad.

Definition 2.9. Weight-grading of the free operad. Given a $\mathbb{S}$-module M, the *weight* $w(\mu)$ of any operation $\mu \in \mathcal{T}(M)$ is defined by

$$w(\mathrm{Id}) = 0, \quad w(\nu) = 1 \text{ for } \nu \in M(n)$$

and more generally $w(\nu; \nu_1, ..., \nu_n) = w(\nu) + w(\nu_1) + ... + w(\nu_n)$.

We denote by $\mathcal{T}(M)^{(r)}$ the operations in $\mathcal{T}(M)$ of weight r. $\mathcal{T}(M)^{(r)}$ has the structure of an $\mathbb{S}$-module where the nth component consists of n-ary operations generated by r elements of M.

Cofree cooperads can be defined analogously with an additional constraint called conilpotency. Essentially, conilpotency of a cooperad $\mathcal{C}$ means that every element in $\mathcal{C}$ goes to 0 after applying the decomposition map sufficiently many times.

Definition 2.10. Conilpotent cooperad. Let $\mathcal{C} = \mathbb{K}1 \oplus \overline{\mathcal{C}}$ be a coaugmented cooperad and $\bar{\triangle}$ be the corresponding decomposition map on $\overline{\mathcal{C}}$. $\mathcal{C}$ is a *conilpotent cooperad* if for any $x \in \overline{\mathcal{C}}$ there exists n such that $\bar{\triangle}^m(x) = 0$ for any $m \geq n$.

Definition 2.11. Cofree cooperad. The *cofree cooperad* $\mathcal{F}^c(M)$ on M is a conilpotent cooperad with an $\mathbb{S}$-module morphism $\pi_M : \mathcal{F}^c(M) \to M$ called the projection and the universal property expressed by the following diagram:

$$\begin{array}{ccc} \mathcal{F}^c(M) & & \\ {\scriptstyle \pi_M}\downarrow & \overset{\widetilde{f}}{\searrow} & \\ M & \xrightarrow[f]{} & \mathcal{C} \end{array}$$

In other words, any $\mathbb{S}$-module morphism $f : M \to \mathcal{C}$ from M to a cooperad $\mathcal{C}$ can be uniquely lifted to a cooperad morphism $\widetilde{f} : \mathcal{F}^c(M) \to \mathcal{C}$. Again, we point to Section 5.7.7 of [4] for the explicit form of $\mathcal{F}^c(M)$.

2.3. **Quadratic operads.** Operads can be constructed by quotienting out operadic ideals from free operads. Elements in such operadic ideals are called relations. Quadratic operads are obtained when relations only involve two applications of some generating operation of the free operad.

Definition 2.12. Operadic ideal. An *ideal* I of an operad $\mathcal{P}$ is a sequence of $\mathbb{S}_n$-submodules $I(n) \subset \mathcal{P}(n)$ such that $\gamma(\mu \otimes \nu_1 \otimes ... \otimes \nu_k) \in I$ if μ or any of $\nu_1, ..., \nu_k$ is in I. The *quotient* $\mathcal{P}/I$ is defined degree-wise as

$$(\mathcal{P}/I)(n) = \mathcal{P}(n)/I(n)$$

Meanwhile, the ideal (R) generated by a subset $R \subset \mathcal{P}$ is the smallest ideal of $\mathcal{P}$ containing R. We are now ready to define quadratic operads.

Definition 2.13. Operadic quadratic data. An **operadic quadratic data** (E, R) is a graded $\mathbb{S}$-module E together with a graded $\mathbb{S}$-submodule $R \subset \mathcal{T}(E)^{(2)}$. Elements in E and R are called *generating operations* and *relations*, respectively.

Definition 2.14. Quadratic operad. The *quadratic operad* $\mathcal{P}(E, R)$ associated with quadratic data (E, R) is defined as the operadic quotient

$$\mathcal{P}(E, R) := \mathcal{T}(E)/(R)$$

where (R) is the operadic ideal of $\mathcal{T}(E)$ generated by R. If $E = (E(0), E(1), E(2) \cdots)$,

$$\begin{aligned} \mathcal{T}(E)^{(0)} &= I = (0, \mathbb{K}, 0, 0, \cdots) \\ \mathcal{T}(E)^{(1)} &= E = (0, E(1), E(2).E(3), \cdots) \\ \mathcal{T}(E)^{(2)} &= (0, E(1)^{\otimes 2}, \cdots) \end{aligned}$$

Let $(R) = (R(0), R(1), R(2), \cdots)$. Then, the quadratic operad $\mathcal{P} = \mathcal{P}(E, R)$ is explicitly expressed by $\mathcal{P}_0 = I$, $\mathcal{P}_1 = E$ and

$$\mathcal{P}_2(E, R) = (0, E(1)^{\otimes 2}/R(1), \cdots)$$

If E is concentrated in degree 2 and R is concentrated in degree 3, we recover the simplified definition of quadratic operads by Ginzburg and Kapranov [5]. We call it a *binary quadratic operad* in this paper. Intuitively, such operads are generated by binary operations while the relations involve three variables.

Definition 2.15. Quadratic cooperad. The *quadratic cooperad* $\mathcal{C}(E,R)$ associated (E,R) is the sub-cooperad of the cofree cooperad $\mathcal{T}^c(E)$ that is *universal* among sub-cooperads of $\mathcal{T}^c(E)$ such that the following composite is 0:

$$\mathcal{C} \rightarrowtail \mathcal{T}^c(E) \twoheadrightarrow \mathcal{T}^c(E)^{(2)}/(R)$$

Intuitively, quadratic cooperads are the "largest" cooperads that are "orthogonal" to $\mathcal{T}^c(E)^{(2)}/(R)$ and they are defined differently from quadratic operads.

Now, we present the "three graces" of quadratic operads as examples: the (symmetric) associative operad **Assoc**, the commutative operad **Com** and the Lie operad **Lie**. We first briefly recall the definitions of associative, commutative and Lie algebras. Commutative and associative algebras appearing in this section are assumed to be unital.

Definition 2.16. Associative algebra. An *associative algebra* over $\mathbb{K}$ is a vector space A with binary operation $(x,y) \mapsto xy$ and a unit map $\epsilon : \mathbb{K} \to A$ satisfying the associativity and unitality axioms

$$(xy)z = x(yz), \quad 1_A x = x 1_A = x$$

where $1_A = \epsilon(1_{\mathbb{K}})$ is the *unit* in A.

Example 2.17. The symmetric operad Assoc. **Assoc** encodes associative algebras. Each component $\mathbf{Assoc}(n)$ is isomorphic to the group ring $\mathbb{K}[\mathbb{S}_n]$. The composition map

$$\gamma : \mathbb{K}[\mathbb{S}_k] \otimes \mathbb{K}[\mathbb{S}_{i_1}] \otimes \cdots \otimes \mathbb{K}[\mathbb{S}_{i_k}] \to \mathbb{K}[\mathbb{S}_{i_1+\cdots+i_k}]$$

is given by the inclusion of $\mathbb{S}_{i_1} \times \cdots \times \mathbb{S}_{i_k}$ in $\mathbb{S}_{i_k}$ where $\mathbb{S}_k$ permutes the order of $\mathbb{S}_{i_j}$ in the inclusion for $j = 1, \cdots, k$.

All algebras over **Assoc** have the structure of an associative algebra where the binary operation corresponds to an element in $\mathbf{Assoc}(2)$. Meanwhile, all associative algebras naturally arise as operadic algebras over **Assoc**. Specifically, the action of $\mu = k_1\sigma_1 + ... + k_m\sigma_m \in \mathbf{Assoc}(n)$ on an associative algebra A is given by the polynomial

$$\mu(x_1, \cdots, x_n) = k_1 x_{\sigma_1^{-1}(1)} \cdots x_{\sigma_1^{-1}(n)} + \cdots + k_m x_{\sigma_m^{-1}(1)} \cdots x_{\sigma_m^{-1}(n)}$$

The multiplication in the polynomial is given by the binary operation in A, which is associative but not necessarily commutative.

Assoc also admits a quadratic presentation as $\mathcal{T}(E_A, R_A)$ where $E_A \cong \mathbb{K}[\mathbb{S}_2]$ and R_A is the $\mathbb{S}_3$-submodule of $\mathcal{T}(E_A)^{(2)}$ generated by all elements of the form $(x_1x_2)x_3 - x_1(x_2x_3)$. Intuitively, **Assoc** is generated by $\mathbb{K}[\mathbb{S}_2]$ as $\mathbb{S}_n$ is generated by transpositions of two elements.

Definition 2.18. Commutative algebra. A *commutative algebra* over $\mathbb{K}$ is an associative algebra A such that $xy = yx$.

Example 2.19. Com. The operad **Com** encodes commutative algebras. For each n, $\mathbf{Com}(n) = \mathbb{K}\mu_n \cong \mathbb{K}$ where μ_n is invariant under $\sigma \in \mathbb{S}_n$. The composition map γ is trivially defined by

$$\gamma(\mu_k \otimes \mu_{i_1} \otimes \cdots \otimes \mu_{i_k}) = \mu_{i_1+\cdots+i_k}$$

Similar to **Assoc**, algebras over **Com** have the structure of unital commutative algebras where the symmetric binary operation is given by $\mu_2 \in \mathbf{Com}(2)$. Meanwhile, each unital commutative algebra A can be considered as an algebra over **Com**. The action of $\mu_n \in \mathbf{Com}(n)$ is given by

$$\mu_n(x_1, \cdots, x_n) = x_1 \cdots x_n$$

where $x_1 \cdots x_n$ is a monomial with multiplication given by the binary operation in A.

As a quadratic operad, **Com** is presented quadratically as $\mathbf{Com} = \mathcal{T}(E_C, R_C)$. E_C is concentrated in degree 2 with $E_C(2) = \mathbb{K}$. R_C is the $\mathbb{S}_3$-subspace of $\mathcal{T}(E_C)^{(2)}$ generated by the associator $(xy)z - x(yz)$.

Definition 2.20. Lie algebra. A *Lie algebra* $\mathfrak{g}$ is a vector space equipped with a binary operation $[\cdot, \cdot]$ called the *Lie bracket* that is skew-symmetric

$$[x, y] = -[y, x]$$

and satisfies the Jacobi identity

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$$

Example 2.21. Lie. As a quadratic operad, the operad **Lie** encoding Lie algebras is the quotient $\mathbf{Lie} = \mathcal{T}(E_L)/(R_L)$. $E_L = (0, 0, \mathbb{K}c, 0, \cdots)$ where c is an antisymmetric operation; R_L is the subspace of $T(E_L)^{(2)}$ generated by the Jacobians.

Later, we shall show that **Assoc** is self-dual under Koszul duality and **Com** is the Koszul dual of **Lie**.

2.4. **Dg-operads and cooperads.** Now, we present differential graded (dg) operads and cooperads where each component is differential graded. Dg operads play an important role in defining homological algebra for operads.

Definition 2.22. Graded $\mathbb{S}$-module. A *graded $\mathbb{S}$-module* M is a family of $\mathbb{S}$-modules $\{M_p\}_{p\in\mathbb{Z}}$. A morphism $f : M \to N$ of degree r is a family of maps $\{f_p(n) : M_p(n) \to N_{p+r}(n)\}_{p\in\mathbb{Z}, n\geq 0}$ that are $\mathbb{S}_n$-equivariant.

The tensor product of $\mathbb{S}$-modules can be extended to graded $\mathbb{S}$-modules:

$$(M \otimes N)_p(n) := \bigoplus_{i+j=n, q+r=p} \mathrm{Ind}_{\mathbb{S}_i \times \mathbb{S}_j}^{\mathbb{S}_n} M_q(i) \otimes N_r(j)$$

The composite product of two $\mathbb{S}$-modules can be extended to graded $\mathbb{S}$-modules:

$$(M \circ N)_p(n) := \bigoplus_{k\geq 0} M_q(k) \otimes_{\mathbb{S}_k} \left[\bigoplus_{i_1+\cdots+i_k=n} \mathrm{Ind}_{\mathbb{S}_{i_1} \times \cdots \times \mathbb{S}_{i_k}}^{\mathbb{S}_n} \left(N_{r_1}(i_1) \otimes \cdots \otimes N_{r_k}(i_k)\right) \right]$$

where $q, r_1, \cdots, r_k$ is summed over all combinations satisfying $q + r_1 + \cdots r_k = p$.

Now, we construct dg $\mathbb{S}$-modules from graded $\mathbb{S}$-modules by adding a differential d compatible with the grading and the composite product.

Definition 2.23. Dg $\mathbb{S}$-module. A *dg $\mathbb{S}$-module* (M, d) is a graded $\mathbb{S}$-module M equipped with a differential of $\mathbb{S}$-modules

$$\cdots \xrightarrow{d} M_p(n) \xrightarrow{d} M_{p-1}(n) \xrightarrow{d} \cdots \xrightarrow{d} M_0(n) \xrightarrow{d} \cdots$$

such that $d^2 = 0$ for each n. A morphism $f : (M, d_M) \to (N, d_N)$ of dg $\mathbb{S}$-modules is a degree 0 morphism of graded $\mathbb{S}$-modules such that $d_N \circ f = f \circ d_M$.

Definition 2.24. Composite of dg $\mathbb{S}$-modules. The *composite* of dg $\mathbb{S}$-modules M and N is the graded $\mathbb{S}$-module $M \circ N$ with differential $d_{M \circ N}$ defined as

$$d_{M\circ N}(\mu;\nu_1,\cdots,\nu_k) = (d_M(\nu);\nu_1,\cdots,\nu_k) + \sum_{i=1}^{k}(-1)^{\epsilon_i}(\mu;\nu_1,\cdots,d_N(\nu_i),\cdots,\nu_k)$$

where $\epsilon_i = |\mu| + |\nu_1| + \cdots + |\nu_i|$ by the Koszul sign rule in Section 6.3.1 of [4].

Proposition 2.25. *Under the composite product, dg $\mathbb{S}$-modules form a monoidal category* $(\mathbf{dg}\,\mathbb{S}\text{-}\mathbf{Mod}, \circ, I)$ *where* $I = (0, \mathbb{K}, 0, 0, ...)$ *is considered as a graded $\mathbb{S}$-module concentrated in degree* 0.

Definition 2.26. Homology and cohomology of dg $\mathbb{S}$-modules. The homology $H_\bullet(M)$ of a dg $\mathbb{S}$-module M is the graded $\mathbb{S}$-module whose component with degree p and "arity" n is

$$H_p(M)(n) := \mathrm{Ker}\{d : M_p(n) \to M_{p-1}(n)\}/\mathrm{Im}\{M_{p+1}(n) \to M_p(n)\}$$

If the differential d has degree $+1$, M is said to have *cohomological grading* and the cohomology $H^\bullet(M)$ can be defined as

$$H^p(M)(n) := \mathrm{Ker}\{d : M_p(n) \to M_{p+1}(n)\}/\mathrm{Im}\{M_{p-1}(n) \to M_p(n)\}$$

Now, we present the explicit definitions of dg operads and cooperads. Essentially, dg operads and cooperads are monoids and comonoids in $(\mathbf{dg}\,\mathbb{S}\text{-}\mathbf{Mod}, \circ, I)$.

Definition 2.27. Dg operad. A *dg operad* $(\mathcal{P}, \gamma, \eta)$ is a dg $\mathbb{S}$-module $\mathcal{P}$ with a *composition map* $\gamma : \mathcal{P} \circ \mathcal{P} \to \mathcal{P}$ and a *unit map* $\eta : I \to \mathcal{P}$ that are both dg $\mathbb{S}$-module morphisms of degree 0. γ and d are compatible such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\gamma} & \mathcal{P}\circ\mathcal{P} \\ {\scriptstyle d_{\mathcal{P}\circ\mathcal{P}}}\downarrow & & \downarrow{\scriptstyle d_{\mathcal{P}}} \\ \mathcal{P} & \xrightarrow{\gamma} & \mathcal{P}\circ\mathcal{P} \end{array}$$

Explicitly, γ and d satisfy

$$d_{\mathcal{P}}(\gamma(\mu;\nu_1,\cdots,\nu_k)) := \gamma(d_{\mathcal{P}}(\mu);\nu_1,\cdots,\nu_k) + \sum_{i=1}^{k}(-1)^{\epsilon_i}\gamma(\mu;\nu_1,\cdots,\mathcal{P}(\nu_i),\cdots,\nu_k)$$

Definition 2.28. Dg cooperad. A *dg cooperad* $(\mathcal{C}, \triangle, \epsilon)$ is a dg $\mathbb{S}$-module $\mathcal{C}$ with a *decomposition map* $\triangle : \mathcal{C} \to \mathcal{C} \circ \mathcal{C}$ and a *counit map* $\epsilon : \mathcal{C} \to I$ are dg $\mathbb{S}$-modules morphisms of degree 0. $\triangle$ and ϵ commute with the *coderivation* $d_{\mathcal{C}}$ on $\mathcal{C}$:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\triangle} & \mathcal{C}\circ\mathcal{C} \\ \downarrow{\scriptstyle d_{\mathcal{C}}} & & \downarrow{\scriptstyle d_{\mathcal{C}\circ\mathcal{C}}} \\ \mathcal{C} & \xrightarrow{\triangle} & \mathcal{C}\circ\mathcal{C} \end{array}$$

Explicitly, the coderivation $d_{\mathcal{C}}$ needs to satisfy

$$\triangle(d_{\mathcal{C}}(c)) = \sum_{k}(d_{\mathcal{C}}(c);c_1,\cdots,c_k) + \sum_{k}\sum_{i=1}^{k}(-1)^{\nu_i}(c;c_1,\cdots,d_{\mathcal{C}}(c_i),\cdots,c_k)$$

Remark 2.29. **Homological degree, "arity" and weight grading.** We now distinguish the three possible notions of degree on a dg operad $\mathcal{P}$. First, we have the homological degree inherited from the differential grading of each $\mathcal{P}(n)$. Second, each operation $\mu \in \mathcal{P}(n)$ has "arity" n, which is the number of arguments involved in μ. Third, $\mathcal{P}$ is weight-graded as defined in Definition 2.8.

Finally, we present the definition of minimal models of dg operads. Intuitively, minimal models are "smallest" versions of dg operads that preserve all homological information. Minimal models were proposed by Markl in 1994 [6]. It was also shown that every dg operad admits a unique minimal model up to isomorphism.

Definition 2.30. Models for operads. Let $\mathcal{P}$ be a dg operad. M is a *model* of $\mathcal{P}$ if there exists a *quasi-isomorphism* $f : M \to \mathcal{P}$ which is a morphism of dg operads that induces an isomorphism on homology.

Definition 2.31. Minimal models. M is a *minimal model* of $\mathcal{P}$ if M is a model of $\mathcal{P}$ and (a) M is a free operad; (b) the differential d of $M = \mathcal{T}(E)$ is *decomposable*, which means that $d(E) \subset \mathcal{T}(E)^{\geq 2}$.

Theorem 2.32. *Let $\mathcal{P}$ be a dg operad with $\mathcal{P}(1) = 0$. Then, $\mathcal{P}$ admits a minimal model that is unique up to isomorphism.*

3. Operadic homological algebra

3.1. **Infinitesimal composite.** The operad composite product $\circ$ is left-linear but not right-linear since $\mathcal{P}_1 \circ \mathcal{P}_2$ contains multiple copies of $\mathcal{P}_2$. In this section, we present the "linearized" composite product $\circ_{(1)}$ of operads called the *infinitesimal composite* that is both left-linear and right-linear. The infinitesimal composite motivates the construction of twisting complexes and further the Koszul criterion.

Definition 3.1. Infinitesimal composite product of $\mathbb{S}$-modules. M, N_1 and N_2 are $\mathbb{S}$-modules. The *triple composite* $M \circ (N_1; N_2)$ is the sub-$\mathbb{S}$-module of $\bigoplus_n M(n) \otimes_{\mathbb{S}_n} (N_1 \oplus N_2)^{\otimes n}$ where N_2 appears exactly one time in each summand. Pictorially, an element of the $M \circ (N_1; N_2)$ is represented by

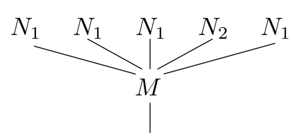


FIGURE 2. A typical additive term of the triple composite.

The *infinitesimal composite product* of two $\mathbb{S}$-modules M and N is defined as

$$M \circ_{(1)} N = M \circ (I; N)$$

Elements of $M \circ_{(1)} N$ are of the form $(\mu; \mathrm{Id}, \cdots, \nu, \cdots, \mathrm{Id})$. Note that the infinitesimal composite product is both left-linear and right-linear:

$$(M \oplus M') \circ_{(1)} N = M \circ_{(1)} N \oplus M' \circ_{(1)} N$$

$$M \circ_{(1)} (N \oplus N') = M \circ_{(1)} N \oplus M \circ_{(1)} N'$$

The structure map of an operad $\mathcal{P}$ naturally lifts to the infinite composite $\mathcal{P} \circ_{(1)} \mathcal{P}$.

Definition 3.2. Infinitesimal composition map of operads. The *infinitesimal composition map* $\gamma_{(1)} : \mathcal{P} \circ_{(1)} \mathcal{P} \to \mathcal{P}$ of an operad $\mathcal{P}$ is given by

$$\gamma_{(1)} : \mathcal{P} \circ_{(1)} \mathcal{P} = \mathcal{P} \circ (I; \mathcal{P}) \rightarrowtail \mathcal{P} \circ (I \oplus \mathcal{P}) \xrightarrow{\mathrm{Id}_{\mathcal{P}} \circ (\eta + \mathrm{Id}_{\mathcal{P}})} \mathcal{P} \circ \mathcal{P} \xrightarrow{\gamma} \mathcal{P}$$

where the second map is the natural embedding of $\mathcal{P} \circ (I; \mathcal{P})$ into $\mathcal{P} \circ (I \oplus \mathcal{P})$. Intuitively, $\gamma_{(1)}$ is the restriction of γ where we only compose two operations of $\mathcal{P}$.

Similarly, the decomposition map of cooperads can be linearized. However, before defining the infinitesimal decomposition of cooperads, we need to define the infinitesimal composite of morphisms.

Definition 3.3. Infinitesimal composite of morphisms. The *infinitesimal composite* of two $\mathbb{S}$-module morphisms $f : M_1 \to M_2$ and $g : N_1 \to N_2$ is a morphism

$$f \circ' g : M_1 \circ N_1 \to M_2 \circ (N_1; N_2)$$

defined as

$$f \circ' g := \sum_i f \otimes (\mathrm{Id}_{N_1} \otimes \cdots \otimes g \otimes \cdots \otimes \mathrm{Id}_{N_1})$$

where g is inserted in the ith position.

Definition 3.4. Infinitesimal decomposition map of cooperads. Dually, the *infinitesimal decomposition map* $\triangle_{(1)} : \mathcal{C} \to \mathcal{C} \circ_{(1)} \mathcal{C}$ of a cooperad $\mathcal{C}$ is defined by the composite

$$\triangle_{(1)} : \mathcal{C} \xrightarrow{\triangle} \mathcal{C} \circ \mathcal{C} \xrightarrow{\mathrm{Id}_{\mathcal{C}} \circ' \mathrm{Id}_{\mathcal{C}}} \mathcal{C} \circ (\mathcal{C}; \mathcal{C}) \xrightarrow{\mathrm{Id}_{\mathcal{C}} \circ (\eta; \mathrm{Id}_{\mathcal{C}})} \mathcal{C} \circ (I; \mathcal{C}) = \mathcal{C} \circ_{(1)} \mathcal{C}$$

3.2. **Twisting morphisms for operads.** In this section, we define twisting morphisms as solutions to the Maurer-Cartan equation and construct the twisted complex that is important in determining Koszulity.

Definition 3.5. Convolution operad. Let $(\mathcal{C}, \triangle_{\mathcal{C}}, \epsilon)$ be a cooperad and $(\mathcal{P}, \gamma_{\mathcal{P}}, \eta)$ be an operad. We consider the right $\mathbb{S}$-module

$$\mathrm{Hom}(\mathcal{C}, \mathcal{P}) := \{\mathrm{Hom}_{\mathbb{K}}(\mathcal{C}(n), \mathcal{P}(n))\}_{n \geq 0}$$

where the right $\mathbb{S}_n$-action is defined by conjugation

$$\sigma(f)(x) = \sigma(f(\sigma^{-1}x))$$

$\mathrm{Hom}(\mathcal{C}, \mathcal{P})$ can be made into an operad with a composition map. For $f_k \in \mathrm{Hom}_{\mathbb{K}}(\mathcal{C}(k), \mathcal{P}(k))$ and $g_l \in \mathrm{Hom}_{\mathbb{K}}(\mathcal{C}(i_l), \mathcal{P}(i_l))$ where $l = 1, \cdots, k$, the composition map γ is expressed by

$$\mathcal{C}(n) \xrightarrow{\triangle_{\mathcal{C}}} (\mathcal{C} \circ \mathcal{C})(n) \twoheadrightarrow \mathcal{C}(k) \otimes \mathcal{C}(i_1) \otimes \cdots \otimes \mathcal{C}(i_k) \otimes \mathbb{K}[\mathbb{S}_n] \xrightarrow{f \otimes g_1 \otimes \cdots \otimes g_k \otimes \mathrm{Id}}$$

$$\mathcal{P}(k) \otimes \mathcal{P}(i_1) \otimes \cdots \otimes \mathcal{P}(i_k) \otimes \mathbb{K}[\mathbb{S}_n] \to (\mathcal{P} \circ \mathcal{P})(n) \xrightarrow{\gamma_{\mathcal{P}}} \mathcal{P}(n)$$

where $n = i_1 + \cdots + i_k$. $\mathrm{Hom}(\mathcal{C}, \mathcal{P})$ with γ is called the *convolution operad.*

The convolution operad can also be made into a dg operad by defining a differential d. For a homogeneous morphism $f : \mathcal{C} \to \mathcal{P}$ of dg $\mathbb{S}$-modules of degree $|f|$, the *derivative* of f is defined as

$$\partial(f) = d_{\mathcal{P}} \circ f - (-1)^{|f|} f \circ d_{\mathcal{C}}$$

where $d_{\mathcal{P}}$ and $d_{\mathcal{C}}$ are the differential on $\mathcal{P}$ and $\mathcal{C}$, respectively. The convolution operad $(\mathrm{Hom}(\mathcal{C}, \mathcal{P}), \partial)$ with differential is a dg operad.

We now state the Maurer-Cartan equation and define operadic twisting morphisms. Using the notion of infinitesimal composites in Section 3.1, we construct the twisted composite product.

Definition 3.6. Maurer-Cartan equation. The Maurer-Cartan equation in $\mathrm{Hom}_{\mathbb{S}}(\mathcal{C}, \mathcal{P})$ is

$$\partial(\alpha) + \alpha \star \alpha = 0$$

where the *convolution product* $\star$ is defined by

$$f \star g = \mu \circ (f \otimes g) \circ \triangle$$

for $f, g \in \mathrm{Hom}_{\mathbb{S}}(\mathcal{C}, \mathcal{P})$. Note that $\mathrm{Hom}_{\mathbb{S}}(\mathcal{C}, \mathcal{P})$ is the subspace of $\mathrm{Hom}(\mathcal{C}, \mathcal{P})$ consisting of $\mathbb{S}_n$-equivariant maps.

Definition 3.7. Operadic twisting morphism. An *operadic twisting morphism* is a map $\alpha : \mathcal{C} \to \mathcal{P}$ of degree -1 that is a solution to the Maurer-Cartan equation.

3.8. Twisted composite product. The operadic composite product $\mathcal{C} \circ \mathcal{P}$ can be made into a dg $\mathbb{S}$-module $\mathcal{C} \circ_\alpha \mathcal{P}$ by defining a differential d_α corresponding to a twisting morphism $\alpha : \mathcal{C} \to \mathcal{P}$.

We now define d_α on $\mathcal{C} \circ_\alpha \mathcal{P}$. Consider the map $f_\alpha : \mathcal{C} \to \mathcal{C} \circ \mathcal{P}$ defined by

$$f_\alpha : \mathcal{C} \xrightarrow{\triangle_{(1)}} \mathcal{C} \circ_{(1)} \mathcal{C} \xrightarrow{\mathrm{Id}_{\mathcal{C}} \circ_{(1)} \alpha} \mathcal{C} \circ_{(1)} \mathcal{P} \longrightarrow \mathcal{C} \circ \mathcal{P}$$

The map d^r_α extending f_α to $\mathcal{C} \circ \mathcal{P}$ is the "twisting term" of the differential on $\mathcal{C} \circ_\alpha \mathcal{P}$. d^r_α is expressed by

$$d^r_\alpha : \mathcal{C} \circ \mathcal{P} \xrightarrow{\triangle_{(1)} \circ \mathrm{Id}_{\mathcal{P}}} (\mathcal{C} \circ_{(1)} \mathcal{C}) \circ \mathcal{P} \xrightarrow{(\mathrm{Id}_{\mathcal{C}} \circ_{(1)} \alpha) \circ \mathrm{Id}_{\mathcal{P}}} (\mathcal{C} \circ_{(1)} \mathcal{P}) \circ \mathcal{P} \cong$$

$$\mathcal{C} \circ (\mathcal{P}; \mathcal{P} \circ \mathcal{P}) \xrightarrow{\mathrm{Id}_{\mathcal{C}} \circ (\mathrm{Id}_{\mathcal{P}}; \gamma)} \mathcal{C} \circ (\mathcal{P}; \mathcal{P}) \cong \mathcal{C} \circ \mathcal{P}$$

The two isomorphisms can be seen from writing out a "typical" term of the triple composites. The full derivation on $\mathcal{C} \circ \mathcal{P}$ is

$$d_\alpha = d_{\mathcal{C}} \circ \mathrm{Id}_{\mathcal{P}} + \mathrm{Id}_{\mathcal{C}} \circ' d_{\mathcal{P}} + d^r_\alpha$$

The left twisted composite product $\mathcal{P} \circ_\alpha \mathcal{C}$ can be defined dually as a dg $\mathbb{S}$-module with differential degree $+1$.

Now, we show that the twisted derivation d_α is a differential on $\mathcal{C} \circ \mathcal{P}$ iff α is a twisting morphism. The complex $\mathcal{C} \circ_\alpha \mathcal{P} := (\mathcal{C} \circ \mathcal{P}, d_\alpha)$ is called the *right twisted composite product.*

Proposition 3.9. *The derivation d_α on $\mathcal{C} \circ \mathcal{P}$ satisfies*

$$d^2_\alpha = d^r_{\partial(\alpha) + \alpha \star \alpha}$$

In other words, d_α is a differential on $\mathcal{C} \circ \mathcal{P}$ if and only if α is a solution to the Maurer-Cartan equation i.e. a twisting morphism.

3.3. **Operadic bar and cobar construction.** In this section, we briefly introduce the operadic bar and cobar functors which are adjoint to each other. We start by defining the operadic suspension that appears in the bar and cobar constructions.

Definition 3.10. Operadic suspension. The *operadic suspension* of an operad $\mathcal{P}$ is the operad given by

$$\mathcal{P}_s := \mathcal{S} \otimes_H \mathcal{P}$$

$\mathcal{S}$ is the endomorphism operad over $s\mathbb{K}$ where s shifts the degree of any graded $\mathbb{S}$-module by $+1$. We also define $\mathcal{S}^{-1} = \mathrm{End}_{s^{-1}\mathbb{K}}$ and define the *operadic desuspension* of $\mathcal{P}$ as $\mathcal{P}_{s^{-1}} := \mathcal{S}^{-1} \otimes_H \mathcal{P}$. $\otimes_H$ is the Hadamard product of $\mathbb{S}$-modules defined by

$$(\mathcal{P} \otimes_H \mathcal{Q})(n) := \mathcal{P}(n) \otimes \mathcal{Q}(n)$$

Definition 3.11. Cooperadic suspension. The *cooperadic suspension* (see [7]) of a cooperad $\mathcal{P}$ is given by

$$\mathcal{C}_s = \mathcal{S}^c \otimes_H \mathcal{C}$$

where $\mathcal{S}^c$ is the endomorphism cooperad of $s\mathbb{K}$.

3.12. Bar construction. The *bar construction* is a functor from the category of augmented dg operads to the category of conilpotent dg cooperads

$$B : \{\text{augmented dg operads}\} \to \{\text{conilpotent dg cooperads}\}$$

The precise construction is the following. Consider an augmented operad (non-dg) $(\mathcal{P}, \gamma, \eta, \epsilon)$ where $\epsilon : \mathcal{P} \to I$ is the augmentation map. The *augmentation ideal* $\bar{\mathcal{P}}$ of $\mathcal{P}$ is defined as $\bar{\mathcal{P}} = \ker \epsilon$. The bar construction $B\mathcal{P}$ of $\mathcal{P}$ is a dg cooperad constructed from the free cooperad $\mathcal{T}^c(s\bar{\mathcal{P}})$ by adding a suitably defined differential. To define such differential, consider the map $\widetilde{d_2}$

$$\widetilde{d_2} : \mathcal{T}^c(s\bar{\mathcal{P}}) \twoheadrightarrow \mathcal{T}^c(s\bar{\mathcal{P}})^{(2)} \cong (\mathbb{K}s \otimes \mathcal{P}) \circ_{(1)} (\mathbb{K}s \otimes \mathcal{P}) \xrightarrow{\mathrm{Id} \otimes \tau \otimes \mathrm{Id}}$$

$$(\mathbb{K}s \otimes \mathbb{K}s) \otimes (\bar{\mathcal{P}} \circ_{(1)} \bar{\mathcal{P}}) \xrightarrow{\gamma_s \otimes \gamma_{(1)}} \mathbb{K}s \otimes \bar{\mathcal{P}} = s\bar{\mathcal{P}}$$

By definition of the cofree cooperad $\mathcal{T}^c(s\bar{\mathcal{P}})$, there exists a map $d_2 : \mathcal{T}^c(s\bar{\mathcal{P}}) \to \mathcal{T}^c(s\bar{\mathcal{P}})$ that extends $\widetilde{d_2}$. We can check that $d_2^2 = 0$; in other words, d_2 is a differential on $\mathcal{T}^c(s\bar{\mathcal{P}})$. The *bar construction* of an augmented non-dg operad $\mathcal{P}$ is defined as the conilpotent dg cooperad $B\mathcal{P} := (\mathcal{T}^c(s\bar{\mathcal{P}}), d_2)$.

If $\mathcal{P} = (\mathcal{P}, d_{\mathcal{P}})$ is a dg operad, d_2 can be defined analogously on $\mathcal{T}^c(s\mathcal{P})$. Meanwhile, the differential $d_{\mathcal{P}}$ on $\mathcal{P}$ induces a differential d_1 on $\mathcal{T}^c(s\bar{\mathcal{P}})$ that anti-commutes with d_2:

$$d_1 \circ d_2 + d_2 \circ d_1 = 0$$

The bar construction of a dg operad $\mathcal{P}$ is the total complex

$$B\mathcal{P} := (\mathcal{T}^c(s\bar{\mathcal{P}}), d = d_1 + d_2)$$

3.13. Cobar construction. The *cobar construction* is a functor

$$\Omega : \{\text{coaugmented dg cooperads}\} \to \{\text{augmented dg operads}\}$$

We first define the cobar construction of non-dg cooperads. Consider the non-dg cooperad $\mathcal{C} = (\mathcal{C}, \triangle, \epsilon, \eta)$ with coaugmentation ideal $\overline{\mathcal{C}} = \text{coker } \eta$. The cobar construction $\Omega\mathcal{C}$ of $\mathcal{C}$ is the free operad $\mathcal{T}(s^{-1}\overline{\mathcal{C}})$ with differential d_2 defined as the unique extension of $\widetilde{d_2}$ to $\mathcal{T}(s^{-1}\overline{\mathcal{C}})$. $\widetilde{d_2} : s^{-1}\mathcal{C} \to \mathcal{T}(s^{-1}\overline{\mathcal{C}})$ is given by

$$\widetilde{d_2} : s^{-1}\mathcal{C} = \mathbb{K}s^{-1} \otimes \overline{\mathcal{C}} \xrightarrow{\triangle_s \otimes \triangle_{(1)}} (\mathbb{K}s^{-1} \otimes \mathbb{K}s^{-1}) \otimes (\mathcal{C} \circ_{(1)} \mathcal{C}) \xrightarrow{\mathrm{Id}\otimes\tau\otimes\mathrm{Id}}$$

$$(\mathbb{K}s^{-1} \otimes \overline{\mathcal{C}}) \circ_{(1)} (\mathbb{K}s^{-1} \otimes \overline{\mathcal{C}}) \cong \mathcal{T}(s^{-1}\overline{\mathcal{C}})^{(2)} \rightarrowtail \mathcal{T}(s^{-1}\overline{\mathcal{C}})$$

The *cobar construction* of $\mathcal{C}$ is defined as the dg operad $\Omega\mathcal{C} := (\mathcal{T}(s^{-1}\overline{\mathcal{C}}), d_2)$.

When $\mathcal{C} = (\mathcal{C}, d_{\mathcal{C}})$ is a dg cooperad, d_2 can be defined analogously. Meanwhile, there exists a differential d_1 on $\mathcal{T}(s^{-1}\overline{\mathcal{C}})$ induced by the coderivative $d_{\mathcal{C}}$ on $\mathcal{C}$ that anticommutes with d_2. The cobar construction of $\mathcal{C}$ is the complex

$$\Omega\mathcal{C} := (\mathcal{T}(s^{-1}\mathcal{C}), d = d_1 + d_2)$$

In fact, the bar and cobar functors are adjoint with the adjunction given by twisting morphisms.

Theorem 3.14. *For an augmented dg operad $\mathcal{P}$ and a conilpotent dg cooperad $\mathcal{C}$, there is a natural isomorphism*

$$\mathrm{Hom}_{\mathbf{dg\ Op}}(\Omega\mathcal{C}, \mathcal{P}) \cong \mathrm{Hom}_{\mathbf{dg\ coOp}}(\mathcal{C}, B\mathcal{P})$$

4. Koszul duality of quadratic operads

In this section, we define the Koszul dual (co)operad of a quadratic operad. Then, we show some important examples: the operad **Assoc** is self-dual while **Com** and **Lie** are Koszul dual to each other.

4.1. Koszul dual cooperads and operads.

Recall that quadratic operads and cooperads are generated by graded $\mathbb{S}$-module E and constrained by relations $(R) \subset \mathcal{T}(E)^{(2)}$.

Definition 4.1. Koszul dual cooperad. The *Koszul dual cooperad* of a quadratic operad $\mathcal{P}(E, R)$ is the quadratic cooperad $\mathcal{P}^{\text{i}}$ defined by

$$\mathcal{P}^{\text{i}} := \mathcal{C}(sE, s^2R)$$

where s shifts the homological degree of any graded module by 1. Without considering the homological grading, $\mathcal{P}^{\text{i}}$ can be identified with $\mathcal{C}(E, R)$ as $\mathbb{S}$-modules.

Definition 4.2. Koszul dual operad. The *Koszul dual operad* $\mathcal{P}^!$ of a quadratic operad $\mathcal{P}$ is the linear dual of the cooperadic suspension of the Koszul dual cooperad:

$$\mathcal{P}^! := (\mathcal{S}^c \otimes_H \mathcal{P}^{\text{i}})^*$$

where $(-)^*$ denotes "arity-graded linear dualization" with each arity component dualized individually.

Proposition 4.3. *If E is a graded $\mathbb{S}$-module whose arity components are finite-dimensional, the Koszul dual operad $\mathcal{P}^!$ of $\mathcal{P}(E, R)$ admits the quadratic presentation*

$$\mathcal{P}^! = \mathcal{P}(s^{-1}\mathcal{S}^{-1} \otimes_H E^*, R^\perp)$$

Here E^* is the "arity-graded linear dual" of E and $R^\perp \subset \mathcal{T}(s^{-1}\mathcal{S}^{-1} \otimes_H E^*)^{(2)}$ is the orthogonal complement of R under some inner product $\langle -, - \rangle$ which we make explicit in Section 4.5 for binary quadratic operads.

We call $\mathcal{P} \mapsto \mathcal{P}^!$ a duality since it connects exactly two operads. This conclusion can be obtained by direct inspection of the quadratic data of $\mathcal{P}$, $\mathcal{P}^!$ and $(\mathcal{P}^!)^!$.

Proposition 4.4. *For a quadratic operad $\mathcal{P}(E, R)$ where E is finite-dimensional in each arity, we have*

$$(\mathcal{P}^!)^! \cong \mathcal{P}$$

4.2. **Koszul duality for binary quadratic operads.** Recall that a binary quadratic operad $\mathcal{P}(E, R)$ is generated by operations involving two arguments only. In this section, we present a dualization procedure that gives the Koszul dual of binary quadratic operads explicitly. As examples, we show that $\mathbf{Assoc}^! = \mathbf{Assoc}$ and $\mathbf{Com}^! = \mathbf{Lie}$.

4.5. Explicit form of binary quadratic operads. When $\mathcal{P}(E, R)$ is binary quadratic, E is concentrated in arity degree 2:

$$E = (0, 0, \overline{E}, 0, \cdots)$$

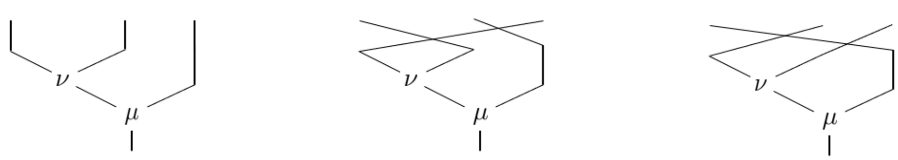
where $\overline{E}$ is a $\mathbb{S}_2$-module. Since the composition of two binary operations involves three arguments, we have $\mathcal{T}(E)^{(2)} = \mathcal{T}(E)(3)$.

Explicitly, $\mathcal{T}(E)(3)$ is isomorphic to the direct sum of three copies of $\overline{E} \otimes \overline{E}$, which we denote by $3\,\overline{E} \otimes \overline{E}$. This isomorphism can be obtained combinatorially. Let μ, ν be two binary operations in $\overline{E}$. After accounting for the action of $\mathbb{S}_2$, μ and ν form three ternary operations:

$$(\mu \circ_{\mathrm{I}} \nu) := \mu(\nu(x, y), z)$$
$$(\mu \circ_{\mathrm{II}} \nu) := \mu(\nu(z, x), y)$$
$$(\mu \circ_{\mathrm{III}} \nu) := \mu(\nu(y, z), x)$$

Each $\mu \circ_u \nu$ provides a copy of $\overline{E} \otimes \overline{E}$ for $u = \mathrm{I}, \mathrm{II}, \mathrm{III}$. With the action of $\mathbb{S}_3$ expressed by $\omega^\sigma(x, y, z) = \omega(\sigma(x, y, z))$, $\mathcal{T}(E)(3)$ is expressed by

$$3\,\overline{E} \otimes \overline{E} = (\overline{E} \otimes \overline{E})_{\mathrm{I}} \oplus (\overline{E} \otimes \overline{E})_{\mathrm{II}} \oplus (\overline{E} \otimes \overline{E})_{\mathrm{III}}$$



$(\mu \circ_{\mathrm{I}} \nu)$ $(\mu \circ_{\mathrm{II}} \nu)$ $(\mu \circ_{\mathrm{III}} \nu)$

FIGURE 3. Schematic of ternary operations in $\mathcal{T}(E)(3)$.

Hence, when $\mathcal{P}(E, R)$ is binary quadratic, R is concentrated in arity 3 and

$$\mathcal{P}(E, R)^{(2)} = (0, 0, 0, 3(\overline{E} \otimes \overline{E})/R(3), \cdots)$$

4.6. Dualizing binary quadratic data. We now present the explicit form of the Koszul dual of binary quadratic operads. For a finite-dimensional right $\mathbb{S}_2$-module E, the linear dual $E^* = \mathrm{Hom}(E, \mathbb{K})$ is a left $\mathbb{S}_2$-module. E^* can be made into a right $\mathbb{S}_2$-module by tensoring the sign representation of $\mathbb{S}_2$:

$$E^\vee = E^* \otimes \mathrm{sgn}_2$$

where the action of $\sigma \in \mathbb{S}_2$ on $f \in \mathrm{Hom}(E, \mathbb{K})$ is defined by

$$\sigma(f)(x) = \mathrm{sgn}(\sigma) f(\sigma^{-1}(x))$$

$\mathcal{T}(E^\vee)(3)$ can be identified with the dual of $\mathcal{T}(E)(3)$ by a scalar product

$$\langle -, - \rangle : \mathcal{T}(E^\vee)(3) \otimes \mathcal{T}(E)(3) \to \mathbb{K}$$

$$\langle f \circ_u g, \mu \circ_v \nu \rangle = \begin{cases} f(\mu) g(\nu) & \text{if } u = v \\ 0 & \text{if } u \neq v \end{cases}$$

where $f, g \in \mathrm{Hom}(E, \mathbb{K})$, $\mu, \nu \in E$ and $u, v \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$.

Theorem 4.7. *The Koszul dual operad of a finitely generated binary quadratic operad $\mathcal{P} = \mathcal{P}(E, R)$ is given by*

$$\mathcal{P}^! = \mathcal{P}(E^\vee, R^\perp)$$

4.8. Com, Assoc and Lie. First, we show that $\mathbf{Com}^! = \mathbf{Lie}$ by finding $E^\vee$ and $R^\perp$ for **Com** explicitly. Recall that $\mathbf{Com} = \mathcal{P}(E, R)$ is generated by $E = \mathbb{K}\mu$ for some binary symmetric operation μ. The space of relations R is spanned by the associators $\mu \otimes_{\mathrm{I}} \mu - \mu \otimes_{\mathrm{II}} \mu$ and $\mu \otimes_{\mathrm{II}} \mu - \mu \otimes_{\mathrm{III}} \mu$.

The Koszul dual operad of **Com** is generated by

$$E^\vee = \mathbb{K}\mu^* \otimes \mathrm{sgn}_2 \cong \mathbb{K}\nu$$

where ν is an antisymmetric operation such that $\sigma(\nu) = -\nu$ for $\sigma = (12) \in \mathbb{S}_2$. Then, $\mathcal{T}(E^\vee)(3)$ is spanned by $\nu \circ_{\mathrm{I}} \nu$, $\nu \circ_{\mathrm{II}} \nu$ and $\nu \circ_{\mathrm{III}} \nu$. By definition of the scalar product, $R^\perp$ in $\mathcal{T}(E^\vee)(3)$ is spanned by $\nu \circ_{\mathrm{I}} \nu + \nu \circ_{\mathrm{II}} \nu + \nu \circ_{\mathrm{III}} \nu$. Since ν is antisymmetric, we can denote it by $\nu(x, y) = [x, y]$. The basis of $R^\perp$ then becomes the Jacobi relation:

$$[[x, y], z] + [[z, x], y] + [[y, z], x] = 0$$

Hence, $\mathbf{Com}^!$ has the same generators and relations as **Lie** and $\mathbf{Com}^! = \mathbf{Lie}$.

We now show that $\mathbf{Assoc}^! = \mathbf{Assoc}$. Recall that the operad $\mathbf{Assoc} = \mathcal{P}(E, R)$ is generated by $E = \mathbb{K}\mu \oplus \mathbb{K}\mu'$ where μ is a binary operation and $\mu' = \mu \circ \tau$. $\mathcal{T}(E)(3)$ is spanned by 12 elements $u_1, ..., u_{12}$:

$$\begin{array}{lll} u_1 : \mu \otimes_{\mathrm{I}} \mu \Leftrightarrow (xy)z, & u_2 : \mu' \otimes_{\mathrm{II}} \mu \Leftrightarrow x(yz), & u_3 : \mu' \otimes_{\mathrm{II}} \mu' \Leftrightarrow x(zy) \\ u_4 : \mu \otimes_{\mathrm{III}} \mu' \Leftrightarrow (xz)y, & u_5 : \mu \otimes_{\mathrm{III}} \mu \Leftrightarrow (zx)y & u_6 : \mu' \circ_{\mathrm{I}} \mu \Leftrightarrow (zx)y \\ u_7 : \mu' \circ_{\mathrm{I}} \mu' \Leftrightarrow z(xy) & u_8 : \mu \circ_{\mathrm{II}} \mu' \Leftrightarrow (zy)x & u_9 : \mu \circ_{\mathrm{II}} \mu \Leftrightarrow (yz)x \\ u_{10} : \mu' \circ_{\mathrm{III}} \mu \Leftrightarrow y(zx) & u_{11} : \mu' \circ_{\mathrm{III}} \mu' \Leftrightarrow y(xz) & u_{12} : \mu \circ_{\mathrm{I}} \mu' \Leftrightarrow (yx)z \end{array}$$

The space of relators $R \subset \mathcal{T}(E)(3)$ is spanned by $\{u_i - u_{i+1}, i = 1, 3, 5, \cdots, 11\}$. The Koszul dual operad is generated by

$$E^\vee = (\mathbb{K}\mu^* \oplus \mathbb{K}(\mu)'^*) \otimes \mathrm{sgn}_2 \cong \mathbb{K}\nu \oplus \mathbb{K}\nu'$$

where $\sigma(\nu) = -\nu'$. $\mathcal{T}(E^\vee)(3)$ is generated by $\nu_1, ..., \nu_{12}$ where we substitute μ with ν in the basis of $\mathcal{T}(E)(3)$. The orthogonal complement of R in $\mathcal{T}(E^\vee)(3)$ is spanned by $\{\nu_i + \nu_{i+1}, i = 1, 3, 5, \cdots, 11\}$. Taking $\nu \mapsto \mu$ and $\nu' \mapsto -\mu'$ gives an isomorphism between $\mathbf{Com}^!$ and **Lie**. In other words, $\mathbf{Com}^! = \mathbf{Lie}$ and **Assoc** is Koszul dual to itself.

5. Koszul operads and minimal models

In this section, we define Koszul operads whose minimal models we construct explicitly. The main result is the Koszul criterion, which states the equivalence between Koszulity and a quasi-isomorphism between the operad and the cobar complex of its Koszul dual cooperad.

5.1. Koszul complexes and Koszul operads. We start by defining a twisting morphism κ from the Koszul dual cooperad $\mathcal{P}^{\mathsf{i}}$ to $\mathcal{P}$, which enables us to define the Koszul complex.

Definition 5.2. The natural twisting morphism κ. For a given quadratic data (E, R), the *natural twisting morphism* $\kappa : \mathcal{P}^{\mathsf{i}}(E, R) \to \mathcal{P}(E, R)$ is defined as the composite

$$\kappa : \mathcal{P}^{\mathsf{i}}(E, R) = \mathcal{C}(sE, s^2R) \twoheadrightarrow sE \xrightarrow{s^{-1}} E \rightarrowtail \mathcal{P}(E, R)$$

where $\twoheadrightarrow$ denotes projection from $\mathcal{P}^{\mathsf{i}}$ to $sE \cong \mathcal{C}(sE, s^2R)^{(1)}$ and $\rightarrowtail$ denotes the embedding from $E \cong \mathcal{P}(E, R)^{(1)}$ into $\mathcal{P}$.

Note that κ has degree -1 due to the degree shift in s^{-1}. κ is also a twisting morphism since $\kappa \star \kappa = 0$ and $\partial(\kappa)$ is trivial. Consequently, we can obtain a twisted composite product $\mathcal{P}^{\mathsf{i}} \circ_\kappa \mathcal{P}$ which we call the Koszul complex of $\mathcal{P}$.

Definition 5.3. The Koszul complex. The *(left) Koszul complex* of a quadratic operad is defined as

$$\mathcal{P}^{\mathsf{i}} \circ_\kappa \mathcal{P} := (\mathcal{P}^{\mathsf{i}} \circ \mathcal{P}, d_\kappa)$$

We can also define another Koszul complex $\mathcal{P} \circ_\kappa \mathcal{P}^{\mathsf{i}}$ called the *right Koszul complex*.

Definition 5.4. Koszul operad. A quadratic operad is *Koszul* if its associated Koszul complex $\mathcal{P}^{\mathsf{i}} \circ_\kappa \mathcal{P}$ is acyclic. In other words, $\mathcal{P}$ is a Koszul operad if the homology of $\mathcal{P}^{\mathsf{i}} \circ_\kappa \mathcal{P}$ is completely trivial.

5.5. Minimal models and the Koszul criterion. When $\mathcal{P}$ is Koszul, the Koszul dual construction gives the minimal model of $\mathcal{P}$. We point to Section 7.4.5 of [4] for more detailed exposition.

Theorem 5.6. (The Koszul criterion) *For a quadratic operad $\mathcal{P} := \mathcal{P}(E, R)$ and its Koszul dual cooperad $\mathcal{P}^{i}$. The following assertions are equivalent:*
- (a) *the right Koszul complex $\mathcal{P}^{i} \circ_\kappa \mathcal{P}$ is acyclic;*
- (b) *the left Koszul complex $\mathcal{P} \circ_\kappa \mathcal{P}^{i}$ is acyclic;*
- (c) *the inclusion $i : \mathcal{P}^{i} \rightarrowtail B\mathcal{P}$ is a quasi-isomorphism of dg operads;*
- (d) *the projection $p : \Omega\mathcal{P}^{i} \twoheadrightarrow \mathcal{P}$ is a quasi-isomorphism of dg operads;*

The Koszul criterion directly implies that $\Omega\mathcal{P}^{\mathsf{i}}$ is the minimal model of $\mathcal{P}$ since $\Omega\mathcal{P}^{i}$ is a free operad and the differential on $\Omega\mathcal{P}^{\mathsf{i}}$ is quadratic by construction.

Corollary 5.7. *If $\mathcal{P}$ is Koszul, $\Omega\mathcal{P}^{i}$ is the minimal model of $\mathcal{P}$.*

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