PERMUTATION REPRESENTATIONS

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ABSTRACT. This paper discusses permutation representations, culminating in the decomposition of the left regular representation of S_n into irreducibles each associated to a partition of n. We study the dimensions of intertwiner spaces as a computational means to decompose partition representations. Examples illuminate the more general combinatorical resolution theorem, whose power manifests when combined with the RSK correspondence.

CONTENTS

1. INTRODUCTION

The investigations in this paper are driven by a desire to work out the character table of the symmetric group; this in turn is driven by the significance of irreducible representations in composing all other representations and the simple way that characters add up. The regular representation is the container of these irreducibles and thus the key object of study.

It turns out that characters have a simple combinatorical interpretation for permutation representations, and the left regular representation is equivalent to a particular form of partition representation. The aim of this paper is for the reader to appreciate both the framework of representation theory (of finite groups) and its elegant intersection with combinatorics in the study of permutation representations.

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2. Group Representations and Maschke's Theorem

Definitions 2.1. A ***representation*** of a group G is a homomorphism $\varphi : G \to GL(V)$, where V is a vector space over a field K. Throughout this paper, G is finite, and V is a finite-dimensional vector space over $\mathbb{C}$. The dimension of V is called the ***degree*** of φ. Sometimes we write φ_g for $\varphi(g)$ and $\varphi_g v$ for $\varphi(g)(v)$.

Remark 2.2. When the homomorphism φ is clear from the context, we sometimes refer to V itself as a representation of G. Since $\varphi(g)$ is a linear transformation, we can write φ in matrix form as n^2 scalar functions $\varphi_{ij} : G \to \mathbb{C}$.

Example 2.3. For any group G we have the degree-one ***trivial representation*** given by $\varphi : G \to \mathbb{C}^*$, $\varphi(g) = 1$ for all $g \in G$.

To classify representations of a given group G, the first step we take is to define a notion of equivalence.

Definition 2.4. Two representations $\varphi : G \to GL(V)$ and $\rho : G \to GL(W)$ are ***equivalent*** or ***isomorphic*** if there exists an isomorphism $T : V \to W$, such that for all $g \in G$, $\rho_g = T\varphi_g T^{-1}$. If such an isomorphism exists, we denote $\varphi \sim \rho$, or $V \sim W$ when the context is clear.

We can then translate some linear algebra concepts into this new context.

Definition 2.5. Let $\varphi : G \to GL(V)$ be a representation. A subspace $W \leq V$ is ***G-invariant*** if $\varphi_g w \in W$ for all $g \in G$ and $w \in W$.

Definition 2.6. Let $\varphi_1 : G \to GL_n(V_1)$ and $\varphi_2 : G \to GL_m(V_2)$ be two representations of a group G. Their ***direct sum*** is defined as $\varphi_1 \oplus \varphi_2 : G \to GL(V_1 \oplus V_2)$, $(\varphi_1 \oplus \varphi_2)(g)(v_1, v_2) = (\varphi_1(g)(v_1), \varphi_2(g)(v_2))$. Written in block matrix form, we have

$$(\varphi_1 \oplus \varphi_2)(g) = \begin{pmatrix} \varphi_1(g) & 0 \\ 0 & \varphi_2(g) \end{pmatrix}$$

In representation theory of finite groups there is a notion of irreducibility analogous to primes in number theory.

Definitions 2.7. Let $\varphi : G \to GL(V)$ be a non-zero representation. φ is ***irreducible*** or ***simple*** if the only G-invariant subspaces of V are $\{0\}$ and V. φ is ***completely reducible*** if $V = V_1 \oplus \cdots \oplus V_n$, where for all $i \in [n]$, V_i is G-invariant and $\varphi|_{V_i}$ is irreducible. Equivalently, φ is completely reducible if $\varphi \sim \varphi_1 \oplus \cdots \oplus \varphi_n$, where for all $i \in [n]$, φ_i is irreducible. φ is ***decomposable*** if $V = V_1 \oplus V_2$ for some nonzero G-invariant subspaces V_1 and V_2.

Using some routine linear algebra, one can check that the three properties above are defined up to equivalence of representations.

Lemma 2.8. *If $\varphi : G \to GL(V)$ is equivalent to an irreducible (resp. completely reducible)(resp. decomposable) representation, then φ is irreducible (resp. completely reducible)(resp. decomposable).*

An integer is either irreducible or decomposable; the same is true for representations, and we can show this by exploiting a particularly useful class of representations.

Definition 2.9. Let $\varphi : G \to GL(V)$ be a representation, where V is an inner product space. φ is ***unitary*** if for all $g \in G$, φ_g is unitary; i.e., $\langle \varphi_g v, \varphi_g w\rangle = \langle v, w\rangle$ for all $v, w \in V$. Note that when $V = \mathbb{C}^n$, $\langle v, w\rangle = \sum_{i=1}^n v_i \overline{w_i}$ is the standard inner product.

Proposition 2.10. *If $\varphi : G \to GL(V)$ is a unitary representation, then φ is either irreducible or decomposable.*

Proof. Suppose φ is not irreducible. Then V has a nonzero proper invariant subspace W, and we can write $V = W \oplus W^\perp$ as a direct sum of vector spaces (instead of representations). It suffices to show that $W^\perp$ is G-invariant. Fix $g \in G$. For any $w \in W, w' \in W^\perp$, we have

$$\langle \varphi_g(w'), w\rangle = \langle \varphi_{g^{-1}}\varphi_g(w'), \varphi_{g^{-1}}(w)\rangle = \langle w', \varphi_{g^{-1}}(w)\rangle = 0$$

where the first equality follows from the definition of unitary representations. The third equality holds because W is G-invariant and $\varphi_{g^{-1}}(w) \in W$. This shows that $\varphi_g(w') \in W^\perp$. □

Proposition 2.11. *If $\varphi : G \to GL(V)$ is a representation of a finite group G, then φ is equivalent to a unitary representation.*

Proof. Let $\dim V = n$. Then we can define an isomorphism $T : V \to \mathbb{C}^n$ and obtain a representation $\rho : G \to GL_n(\mathbb{C})$ equivalent to φ by setting $\rho_g := T\varphi_g T^{-1}$. Let $\langle \cdot, \cdot\rangle$ be the standard inner product on $\mathbb{C}^n$. We construct a new inner product $(\cdot, \cdot)$ on $\mathbb{C}^n$ with $(v, w) = \sum_{g\in G}\langle \rho_g v, \rho_g w\rangle$. One can check that $(\cdot, \cdot)$ is indeed an inner product and ρ is a unitary representation with respect to it. □

Corollary 2.12. *If $\varphi : G \to GL(V)$ is a nonzero representation of a finite group, then φ is either irreducible or decomposable.*

The following theorem is the analog of prime factorization in the representation theory of finite groups.

Theorem 2.13 (Maschke's Theorem)**.** *Every representation of a finite group is completely reducible.*

Proof. Let $\varphi : G \to GL(V)$ be a representation of a finite group G. One can use the previous corollary and proceed by induction on the dimension of V. Take $\dim V = 1$ as the base case, where φ is irreducible since the only subspaces of a one-dimensional vector space are $\{0\}$ and itself. Suppose the claim holds true for all V with $\dim V \le n$, and consider the case when $\dim V = n + 1$. If φ is irreducible, then we are done; if it is not, by the previous corollary it must be decomposable, so $V = W_1 \oplus W_2$ where W_1, W_2 are G-invariant and $\dim W_1, \dim W_2 \le n$. By the induction hypothesis the claim holds for $\dim V = n + 1$. □

3. Schur's Lemma and Intertwiners

We are interested in a particular class of linear maps called intertwiners between two representations. With complete reducibility, we can investigate intertwiner spaces by first looking at intertwiners between irreducible representations. Schur's lemma provides a simple characterization of them.

Definition 3.1. Let $\varphi : G \to GL(V)$ and $\rho : G \to GL(W)$ be representations. An ***intertwiner*** or ***morphism*** from φ to ρ is a linear transformation $T : V \to W$ such that for all $g \in G, T\varphi_g = \rho_g T$. $\mathrm{Hom}_G(V, W)$ denotes the space of such intertwiners.

Remark 3.2. If $T \in \operatorname{Hom}_G(V, W)$ is invertible, then $\varphi \sim \rho$.

Theorem 3.3 (Schur's Lemma)**.** *If $\varphi : G \to GL(V), \rho : G \to GL(W)$ are irreducible representations of G and $T \in \operatorname{Hom}_G(V, W)$, then T is either invertible or $T = 0$. As a result:*

(1) If $\varphi \nsim \rho$, then $\operatorname{Hom}_G(V, W) = 0$
(2) If $\varphi = \rho$, then $T = \lambda I$ for some $\lambda \in \mathbb{C}$
(3) If $\varphi \sim \rho$, then $\dim \operatorname{Hom}_G(V, W) = 1$

Proof. For the main claim, suppose $T \neq 0$. Since $\ker T$ is a G-invariant subspace of V and φ is irreducible, $\ker T = V$ or $\{0\}$; but $T \neq 0$, so $\ker T = \{0\}$ and T is injective. Similarly, $\operatorname{Im} T = W$ and T is surjective, thus T must be invertible.

For (1), If $\varphi \nsim \rho$, then by the definition of intertwiners and equivalence, no linear map in $\operatorname{Hom}_G(V, W)$ is invertible. It follows from the main claim that $\operatorname{Hom}_G(V, W) = \{0\}$.

For (2), we use the fact that $\mathbb{C}$ is algebraically closed. Consequently, the characteristic polynomial has a root, i.e., every linear transformation over $\mathbb{C}$ has an eigenvalue λ. Notice that $\lambda I - T$ is not invertible. If $\varphi = \rho$, then $I \in \operatorname{Hom}_G(V, W) = \operatorname{End}_G V$. Since $\operatorname{Hom}_G(V, W)$ is a subspace of $\operatorname{Hom}(V, W)$, $(\lambda I - T) \in \operatorname{Hom}_G(V, W)$. By the main claim $\lambda I - T = 0$, i.e., $T = \lambda I$.

For (3), if $\varphi \sim \rho$, then there exists an isomorphism T between V and W such that $T \in \operatorname{Hom}_G(V, W)$. Suppose $S \in \operatorname{Hom}_G(V, W)$, then since T is invertible, we can let $X := ST^{-1}$. so that $S = XT$. One can show that $X \in \operatorname{End}_G W$, so by (2) $X = \lambda I$ for some $\lambda \in \mathbb{C}$. That is, any $S \in \operatorname{Hom}_G(V, W)$ is a scalar multiple of T. □

The following theorem is a consequence of Schur's Lemma.

Theorem 3.4. *Let $\varphi : G \to GL(V)$ and $\rho : G \to GL(W)$ be representations. Let $V_1, V_2, \cdots, V_r$ be a complete collection of pairwise inequivalent irreducible subspaces of V and/or W. We can write*

$$V \sim V_1^{\oplus n_1} \oplus V_2^{\oplus n_2} \oplus \cdots \oplus V_r^{\oplus n_r} \tag{3.5}$$

$$W \sim V_1^{\oplus m_1} \oplus V_2^{\oplus m_2} \oplus \cdots \oplus V_r^{\oplus m_r}$$

where $n_k, m_k \geq 0$, and n_k is called the ***multiplicity*** *of V_k in V. Then:*

$$\operatorname{Hom}_G(V, W) = \bigoplus_{k=1}^{r} M_{m_k \times n_k}(\mathbb{C})$$

where $M_{m_k \times n_k}(\mathbb{C})$ denotes the set of $m_k \times n_k$ matrices over $\mathbb{C}$.

Corollary 3.6. *Let φ and ρ be representations as in Theorem 3.4, then*

$$\dim \operatorname{Hom}_G(V, W) = \dim \operatorname{Hom}_G(W, V) = \sum_{k=1}^{r} m_k n_k$$

Remark 3.7. Using Corollary 3.6, we can extract the multiplicities of simple representations in the decomposition of a given representation: If φ is a representation and φ_k is an irreducible representation, then the multiplicity of φ_k in φ is given by $\dim \operatorname{Hom}_G(V, V_k)$.

In particular, the decomposition of a representation is unique.

We also obtain a useful criterion for irreducible representations: a representation φ is irreducible if and only if $\dim \operatorname{End}_G V = 1$.

4. Schur Orthogonality Relations

Now we proceed to show Schur orthogonality relations (Theorem 4.6), another important consequence of Schur's Lemma. It says the matrix-form entries of inequivalent irreducible and unitary representations are orthogonal. The proof of Theorem 4.6 is somewhat tedious, but it paves a convenient way for the orthogonality of characters presented in Section 5. Since each matrix entry is a one-dimensional representation, we make use of the following inner product space.

Definition 4.1. Let G be a finite group, then the ***group algebra*** of G, denoted $\mathbb{C}[G]$, is the set of functions $\{f \mid f : G \to \mathbb{C}\}$. We can make $\mathbb{C}[G]$ into an inner product space with

$$\begin{aligned}(f_1 + f_2)(g) &= f_1(g) + f_2(g)\\ (cf)(g) &= c \cdot f(g)\\ \langle f_1, f_2\rangle &= \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{f_2(g)}\end{aligned}$$

where $\overline{f_2(g)}$ indicates complex conjugation.

Proposition 4.2. *Let $\varphi : G \to GL(V)$ and $\rho : G \to GL(W)$ be representations and $T : V \to W$ a linear transformation. We define a map $P : \operatorname{Hom}(V, W) \to \operatorname{Hom}(V, W)$ and denote it by "$\sharp$" such that $P(T) = T^\sharp = \frac{1}{|G|}\sum_{g\in G} \rho_{g^{-1}} T\varphi_g$. Then:*
(1) $T^\sharp \in \operatorname{Hom}_G(V, W)$
(2) If $T \in \operatorname{Hom}_G(V, W)$, then $T^\sharp = T$
(3) $P : \operatorname{Hom}(V, W) \to \operatorname{Hom}_G(V, W)$ is a surjective linear map.

Proof. For (1), we can do direct calculation and apply a change of variables to show that $T^\sharp\varphi_g = \rho_g T^\sharp$ for all $g \in G$.

For (2), if $T \in \operatorname{Hom}_G(\varphi, \rho)$, then we can write $T\varphi_g = \rho_g T$ and show the claim by direct calculation. (3) follows immediately from (2). □

Proposition 4.3. *If $\varphi : G \to GL(V)$ and $\rho : G \to GL(W)$ are irreducible representations and $T : V \to W$ is a linear transformation, then:*
(1) If $\varphi \nsim \rho$, then $T^\sharp = 0$
(2) If $\varphi = \rho$, then $T^\sharp = \frac{\operatorname{Tr}(T)}{\deg\varphi} I$

Proof. Since $T^\sharp \in \operatorname{Hom}_G(V, W)$, (1) follows directly from Theorem 3.3 (1).

If $\varphi = \rho$, then Theorem 3.3 (2) says $T^\sharp = \lambda I$ for some $\lambda \in \mathbb{C}$, so $\operatorname{Tr}(T^\sharp) = \operatorname{Tr}(\lambda I) = \lambda \operatorname{Tr}(I) = \lambda \dim V = \lambda \deg\varphi$. Also, we can use $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$ to show that $\operatorname{Tr}(T) = \operatorname{Tr}(T^\sharp)$. As a result, $\lambda = \frac{\operatorname{Tr}(T)}{\deg\varphi}$, and (2) follows. □

Lemma 4.4. *Let $A = (a_{ij}) \in M_{rm}(\mathbb{C})$, $B = (b_{ij}) \in M_{ns}(\mathbb{C})$, and $E_{ki} \in M_{mn}(\mathbb{C})$, where E_{ki} is an element of the standard basis of $M_{mn}(\mathbb{C})$ with the ki-th entry 1 and all other entries 0. Then $(AE_{ki}B)_{\ell j} = a_{\ell k}b_{ij}$.*

Lemma 4.5. *Let $U_n(\mathbb{C})$ denote the group of unitary $n \times n$ matrices. If $\varphi : G \to U_n(\mathbb{C})$ and $\rho : G \to U_m(\mathbb{C})$ are unitary representations and $E_{ki} \in M_{mn}(\mathbb{C})$, then $(E_{ki})^\sharp_{\ell j} = \langle\varphi_{ij}, \rho_{k\ell}\rangle$, where the inner product is taken as in Definition 4.1.*

Proof. Since ρ is unitary, $\rho_{g^{-1}} = \rho_g^{-1} = \rho_g^*, i.e., \rho_{\ell k}(g^{-1}) = \overline{\rho_{k\ell}(g)}$. Using Lemma 4.4, we can compute:

$$\begin{aligned}(E_{ki})^\sharp_{\ell j} = \frac{1}{|G|}(\rho_{g^{-1}} E_{ki} \varphi_g)_{\ell j} &= \frac{1}{G}\sum_{g\in G} \rho_{\ell k}(g^{-1})\varphi_{ij}(g) \\ &= \frac{1}{G}\sum_{g\in G} \overline{\rho_{k\ell}(g)}\varphi_{ij}(g) = \langle \varphi_{ij}, \rho_{k\ell}\rangle\end{aligned}$$

□

Theorem 4.6 (Schur orthogonality relations)**.** *If $\varphi : G \to U_n(\mathbb{C})$ and $\rho : G \to U_m(\mathbb{C})$ are inequivalent irreducible unitary representations, then:*

(1) $\langle \varphi_{ij}, \rho_{k\ell}\rangle = 0$

(2) $\langle \varphi_{ij}, \varphi_{k\ell}\rangle = \begin{cases} \frac{1}{n} \ if \ i = k \ and \ j = \ell \\ 0 \ otherwise \end{cases}$

Proof. Since $\varphi \nsim \rho$, for all $k \in [m]$ and $i \in [n], (E_{ki})^\sharp = 0$ by Proposition 4.3 (1). Thus every entry of the matrix $(E_{ki})^\sharp$ is 0, and it follows from Lemma 4.5 that for all $i, j \in [n]$ and $k, \ell \in [m]$, $\langle \varphi_{ij}, \rho_{k\ell}\rangle = (E_{ki})^\sharp_{\ell j} = 0$.

By Lemma 4.5 and Proposition 4.3 (2), $\langle \varphi_{ij}, \varphi_{k\ell}\rangle = (E_{ki})^\sharp_{\ell j} = \frac{\mathrm{Tr}(E_{ki})}{n} I_{\ell j}$. Suppose $i \neq k$, then since the only nonzero entry of E_{ki} is off the diagonal, $\mathrm{Tr}(E_{ki}) = 0$. Suppose $j \neq \ell$, then $I_{\ell j} = 0$. If $i = k$ and $j = \ell$, then $\mathrm{Tr}(E_{ii}) = I_{jj} = 1$, proving (2). □

5. Characters

Traces of representations called characters play an important role in representation theory. Notably, they determine the decomposition of representations, and we exploit their orthonormality, a consequence of Theorem 4.6, to decompose the regular representation in section 6.

Definition 5.1. Let $\varphi : G \to GL(V)$ be a representation. The ***character*** of φ is a function $\chi_\varphi : G \to \mathbb{C}$, with $\chi_\varphi(g) = \mathrm{Tr}(\varphi_g) = \sum_{i=1}^{\deg \varphi} \varphi_{ii}$, where the underlying basis of V can be arbitrarily chosen, as the trace of $(\varphi_{ij}(g))$ is invariant under conjugation by change of basis matrices. The character of an irreducible representation is called an ***irreducible character***.

Definition 5.2. Let $f : G \to \mathbb{C}$ be a function. f is called a ***class function*** if f is constant on conjugacy classes of G, i.e., $(\forall g, h \in G)(f(g) = f(hgh^{-1})$.

Notation 5.3. We denote the space of class functions as $Z(\mathbb{C}[G])$; and the set of conjugacy classes of G as $Cl(G)$

One can check the following statements. For (2) and (4), one can use the fact that the trace is invariant under conjugation.

Proposition 5.4. *Let φ and ρ be representations of G. Then:*

(1) $\chi_\varphi(1_G) = \deg \varphi$.
(2) If $\varphi \sim \rho$, then $\chi_\varphi = \chi_\rho$.
(3) If $\varphi = \rho \oplus \psi$, then $\chi_\varphi = \chi_\rho + \chi_\psi$.
(4) Characters are class functions.
(5) $Z(\mathbb{C}[G])$ is a subspace of $\mathbb{C}[G]$.

Proposition 5.5. *For $C \in Cl(G)$, define functions $\delta_C : G \to \mathbb{C}$ by*

$$\delta_C = \begin{cases} 1 & g \in C \\ 0 & g \notin C \end{cases}$$

Then, the set $B := \{\delta_C \mid C \in Cl(G)\}$ is a basis for $Z(L(G))$. As a result, $\dim Z(\mathbb{C}[G]) = |Cl(G)|$.

Theorem 5.6 (Orthogonality of Irreducible Characters)**.** *If φ, ρ are irreducible representations of G, then*

$$\langle \chi_\varphi, \chi_\rho \rangle = \begin{cases} 1 & \varphi \sim \rho \\ 0 & \varphi \nsim \rho \end{cases}$$

Proof. By Proposition 2.11, every representation is equivalent to a unitary representation, and by Proposition 5.4 (2), equivalent representations have the same character, so we may assume without loss of generality that φ and ρ are unitary. Moreover, the case when $\varphi \sim \rho$ can be simplified by setting $\rho = \varphi$. Let $\varphi : G \to U_n(\mathbb{C}), \rho : G \to U_m(\mathbb{C})$. It follows that

$$\begin{aligned}
\langle \chi_\varphi, \chi_\rho \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi_\varphi(g) \overline{\chi_\rho(g)} \\
&= \frac{1}{|G|} \sum_{g \in G} \sum_{i=1}^{n} \varphi_{ii}(g) \sum_{j=1}^{m} \overline{\rho_{jj}(g)} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{1}{|G|} \sum_{g \in G} \varphi_{ii}(g) \overline{\rho_{jj}(g)} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \langle \varphi_{ii}(g), \rho_{jj}(g) \rangle \\
&= \begin{cases} \sum_{i=1}^{n} \langle \varphi_{ii}(g), \varphi_{ii}(g) \rangle = 1 & \varphi \sim \rho \\ 0 & \varphi \nsim \rho \end{cases}
\end{aligned}$$

where the last step follows from Schur Orthogonality relations (Theorem 4.6). □

The following corollary is a result of the above theorem and Corollary 3.6.

Corollary 5.7. *Let $\varphi : G \to GL(V)$ and $\rho : G \to GL(W)$ have the decomposition as in (3.5). Then*

$$\langle \chi_\varphi, \chi_\rho \rangle = \sum_{k=1}^{r} m_k n_k = \dim \operatorname{Hom}_G(\varphi, \rho)$$

Remark 5.8. The previous theorem allows us to use characters to compute the dimensions of intertwiner spaces. It also establishes applications of characters corresponding to the discussion in Remark 3.7. Namely, if φ is a representation and φ_k is an irreducible representation, then the multiplicity of φ_k in φ is given by $\langle \chi_\varphi, \chi_{\varphi_k} \rangle$. Also, a representation φ is irreducible if and only if $\langle \chi_\varphi, \chi_\rho \rangle = 1$.

6. The Regular Representation

The regular representation is a special case of permutation representation, to be introduced in section 7. Significantly, the regular representation contains all irreducible representations of a group G up to equivalence, each with multiplicity equal to its degree. Thus understanding all irreducible representations amounts to understanding the regular representation. The work in the last section allows us to prove this crucial fact.

Definition 6.1. Let $\mathbb{C}[G]$ denote the vector space of all $\mathbb{C}$-valued functions on G. The standard basis of this space is $\{1_g \mid g \in G\}$, where 1_g denotes the function that takes value 1 at g and 0 elsewhere. Thus any $f \in \mathbb{C}[G]$ may be written as

$$f = \sum_{g \in G} c_g 1_g$$

Definition 6.2. The ***regular representation*** of a group G is the representation $L : G \to GL(\mathbb{C}[G])$ such that the action of G is given on basis elements by $L_g 1_h = 1_{gh}$ for all $g, h \in G$.

One can check the following:

Proposition 6.3. $\chi_L(g) = \begin{cases} |G| & g = 1 \\ 0 & g \neq 1 \end{cases}$

Theorem 6.4. *Let $S = \{\varphi_1, \varphi_2, \cdots\}$ be a complete set of inequivalent irreducible representations of G, and let $d_k := \deg \varphi_k$, $\chi_k := \chi_{\varphi_k}$. Then S is finite. Writing S as $\{\varphi_1, \varphi_2, \cdots \varphi_r\}$, we have*

$$L \sim \varphi_1^{\oplus d_1} \oplus \varphi_2^{\oplus d_2} \oplus \cdots \oplus \varphi_r^{\oplus d_r}.$$

Proof. We use the the values of χ_L in Proposition 6.3 to compute

$$\langle \chi_L, \chi_k \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_L(g) \overline{\chi_k(g)} = \frac{1}{|G|} |G| \chi_k(1) = d_k$$

where the last equality follows from proposition 5.4 (1). By the discussion in Remark 5.8, φ_k has multiplicity d_k in L. By Proposition 5.4(3), $\chi_L = d_1\chi_1 + d_2\chi_2 + \cdots$. Evaluate both sides at $g = 1$ to get $|G| = d_1^2 + d_2^2 + \cdots$. Since G is finite, S must be finite. □

Corollary 6.5. $|G| = d_1^2 + d_2^2 + \cdots + d_r^2$

Theorem 6.6. *The set $B = \{(\varphi_k)_{ij} \mid k \in [r], i, j \in [d_k]\}$ is a basis for $\mathbb{C}[G]$.*

Proof. Schur orthogonality relations (Theorem 4.6) shows that B is an orthogonal and hence linearly independent set. On the other hand, $|B| = d_1^2 + d_2^2 + \cdots + d_r^2 = |G| = \dim \mathbb{C}[G]$. □

Theorem 6.7. *$\{\chi_1, \chi_2, \cdots, \chi_r\}$ is an orthornormal basis for $Z(\mathbb{C}[G])$.*

Proof. In light of the orthogonality of irreducible characters (Theorem 5.6), we only need to show that the irreducible characters span $Z(\mathbb{C}[G])$. By the previous theorem, we can write $f \in Z(\mathbb{C}[G]) \leq \mathbb{C}[G]$ as

$$f = \sum_{k,i,j} c_{kij} (\varphi_k)_{ij}$$

Moreover, since f is constant on conjugacy classes, we can write for any $x \in G$

$$\begin{aligned} f(x) &= \frac{1}{|G|} \sum_{g \in G} f(g^{-1}xg) \\ &= \frac{1}{|G|} \sum_{g \in G} \sum_{k,i,j} c_{kij} (\varphi_k)_{ij} (g^{-1}xg) \\ &= \sum_{k,i,j} c_{kij} \Big(\frac{1}{|G|} \sum_{g \in G} \varphi_k(g^{-1}) \varphi_k(x) \varphi_k(g)\Big)_{ij} \\ &= \sum_{k,i,j} c_{kij} [(\varphi_k(x))^\sharp]_{ij} \\ &= \sum_{k,i,j} c_{kij} \frac{\mathrm{Tr}(\varphi_k(x))}{d_k} I_{ij} \\ &= \sum_{k,i} c_{kii} \frac{1}{d_k} \chi_k(x) \end{aligned}$$

where we used Proposition 4.3(2). □

Corollary 6.8. *The number of equivalence classes of irreducible representations of G equals the number of conjugacy classes of G.*

Proof. Combining the previous theorem with Proposition 5.5, we get

$$r = \dim Z(\mathbb{C}[G]) = |Cl(G)|.$$

□

Now we can introduce a useful tool to document information of a representation.

Definition 6.9. Recall that characters are class functions. The ***character table*** of G is an $r \times r$ matrix whose row is indexed by the inequivalent irreducible characters and whose column is indexed by the conjugacy classes of G. The ij-th entry is the value χ_i takes at C_j.

7. Group Actions and Permutation Representation

Definitions 7.1. For a group G, a ***G-set*** is a finite set X with a ***group action*** of G on X, which is a homomorphism $\sigma : G \to \mathrm{Aut}(X)$, where $\mathrm{Aut}(X)$ is the group of all bijections $X \to X$ under composition. For $g \in G$ and $x \in X$, we denote the group action of g by $g \cdot x := \sigma(g)(x)$.

Example 7.2. Let X and Y be G-sets. Then we can make $X \times Y$ into a G-set with the diagonal action: $g \cdot (x, y) = (g \cdot x, g \cdot y)$.

Definition 7.3. Two G-sets X and Y are ***isomorphic*** if there exists a bijection $\phi : X \to Y$ that is compatible with the action of G, i.e., for all $g \in G$, $x \in X$

$$\phi(g \cdot x) = g \cdot \phi(x)$$

Definitions 7.4. Let X be a G-set. For $x \in X$, the ***G-orbit*** of x is the set

$$G \cdot x := \{g \cdot x \mid g \in G\}$$

A subset $O \subseteq X$ is a G-orbit if it is the G-orbit of some $x \in X$. One can check that G-orbits partition G. We write $G\backslash X$ for the set of all G-orbits in X. A G-set X is ***transitive*** if it has only one G-orbit.

Example 7.5. Let G be a group, H a subgroup of G, and G/H the set of left cosets of H in G. Let $X = G/H$, $x \in G$. G acts on G/H by left multiplication:

$$g \cdot xH = gxH.$$

Then G/H is a transitive G-set.

The permutation representation is defined using group action.

Definition 7.6. Let $\mathbb{C}[X]$ denote the vector space of all $\mathbb{C}$-valued functions on a G-set X. Let $g \in G$, $x \in X$, $f \in \mathbb{C}[X]$. The ***permutation representation*** of G associated to the G-set X is a homomorphism $\rho_X : G \to GL(\mathbb{C}[X])$ defined by

$$\rho_X(g)f(x) = f(g^{-1} \cdot x)$$

Remark 7.7. For $x \in X$, let 1_x denote the $\mathbb{C}$-valued function on X which takes value 1 at x and 0 elsewhere. Then one can check that $\rho_X(g)1_x = 1_{g \cdot x}$. From this it is clear that the regular representation is a special case of permutation representation, where we let $X = G$ and $\mathbb{C}[X]$ becomes $\mathbb{C}[G]$.

The next statement is a natural result of the definition of permutation representations.

Proposition 7.8. *If as an G-set X is isomorphic to Y, then $\rho_X \sim \rho_Y$.*

Thanks to the following theorem, we can easily calculate the character of a permutation representation by counting the number of fixed points.

Theorem 7.9. *Let X^g denote the set $\{x \in X \mid g \cdot x = x\}$ of points in X that are fixed by $g \in G$. Then*

$$\chi_{\rho_X}(g) = |X^g|$$

8. Permutations and the Intertwining Number Theorem

In this section we prove two important theorems that prepare for the decomposition of subset and partition representations into irreducibles and the calculation of their character tables in the next section.

In view of Theorem 8.3, we shall study permutations through their cycle decomposition. For a permutation σ of $[n]$, we can pick an element $i_1 \in [n]$ and obtain a cycle $i_1 \xrightarrow{\sigma} i_2 \xrightarrow{\sigma} \cdots \xrightarrow{\sigma} i_k \xrightarrow{\sigma} i_1$. We denote this cycle of length k, or a k-cycle, by $(i_1, i_2, \cdots, i_k)$. The cycle decomposition of σ is the cycles of σ listed in conjunction. For example, (1,2)(3) denotes the element in S_3 that takes 1 to 2, 2 to 1, and fixes 3. Note that cycles of σ partition $[n]$. Since the order in which we list the cycles is arbitrary, we shall list them in the form of a partition, as defined below.

Definitions 8.1. A ***partition*** λ of $[n]$ is a sequence $\lambda_1 \geq \cdots \geq \lambda_l$ of positive integers such that $\lambda_1 + \cdots + \lambda_l = n$. We write $\lambda = (\lambda_1, \cdots, \lambda_l)$. The ***cycle type*** of $\sigma \in S_n$ is the partition of n obtained by listing the cycle lengths of σ in decreasing order. We sometimes use the exponential notation and write, for instance, the partition $(2, 2, 1)$ as $(2^2 1)$.

Remark 8.2. Note the different use of brackets in denoting the cycle decomposition and the cycle type of σ. For example, the permutation in S_3 with cycle decomposition (1,2)(3) has cycle type (2,1).

We saw in previous sections that the conjugacy classes of a group G are of particular interest in representation theory, as their number equals the number of inequivalent irreducible representations of G (Corollary 6.8). One can check that the conjugacy classes of the symmetric group are completely and conveniently determined by cycle types.

Theorem 8.3. *Two permutations are conjugate if and only if they have the same cycle type. Consequently, the number of conjugacy classes in S_n is equal to the number of partitions of n.*

In order to prove the Intertwining number theorem, we need the following linear transformation called an integral operator. Let X and Y be two finite sets. A function $k : X \times Y \to \mathbb{C}$ gives rise to a linear transformation $T_k : \mathbb{C}[Y] \to \mathbb{C}[X]$ such that for $f \in \mathbb{C}[Y]$

$$(T_k f)(x) = \sum_{y \in Y} k(x, y) f(y) \tag{8.4}$$

For the following lemma, prove injection and notice that the two vector spaces have equal dimensions.

Lemma 8.5. *The map $\mathbb{C}[X \times Y] \to \mathrm{Hom}(\mathbb{C}[Y], \mathbb{C}[X])$ given by $k \mapsto T_k$ is an isomorphism of vector spaces.*

Theorem 8.6 (Intertwining number theorem)**.** *Let X and Y be finite G-sets and $(\rho_X, \mathbb{C}[X])$, $(\rho_Y, \mathbb{C}[Y])$ the corresponding permutation representations of G. Then $T_k \in \mathrm{Hom}_G(\mathbb{C}[X], \mathbb{C}[Y])$ if and only if k is constant on the orbits of $X \times Y$, where G acts diagonally as in Example 7.2. Consequently,*

$$\dim \mathrm{Hom}_G(\mathbb{C}[Y], \mathbb{C}[X]) = |G \backslash (X \times Y)|$$

Proof. By definition, T_k is an intertwiner if and only if for all $g \in G$ and $x \in X$

$$(\rho_X(g)^{-1} \circ T_k \circ \rho_Y(g) f)(x) = (T_k f)(x) \tag{8.7}$$

We compute the left hand side

$$\begin{aligned}
(\rho_X(g)^{-1} \circ T_k \circ \rho_Y(g) f)(x) &= (T_k \circ \rho_Y(g) f)(g \cdot x) \\
&= \sum_{y \in Y} k(g \cdot x, y)(\rho_Y(g) f)(y) \\
&= \sum_{y \in Y} k(g \cdot x, y) f(g^{-1} \cdot y) \\
&= \sum_{y \in Y} k(g \cdot x, g \cdot y) f(y)
\end{aligned}$$

and the right hand side is as in (8.4). Therefore, (8.7) holds if and only if $k(x, y) = k(g \cdot x, g \cdot y)$ for all $x \in X, y \in Y, g \in G$. In other words, if and only if k takes the same value on an orbit. Together with Lemma 8.5, we see that the dimension of $\mathrm{Hom}_G(\mathbb{C}[Y], \mathbb{C}[X])$ is the dimension of the subspace of functions k that satisfy (8.7), which in turn is the number of orbits of $X \times Y$. $\square$

9. Partition Representations and the Combinatorical Resolution Theorem

In this section we study partition representations by first decomposing subset representations – a subclass of partition representations – with the intertwining number theorem (Theorem 8.6). As another example, we decompose the partition representations for S_3, and from these examples we derive the more general combinatorical resolution theorem. The significance of partition representations is that they contain the regular representation as a special case. Thus decomposing partition representations of S_3 simultaneously decomposes the regular representation and results in a full classification of irreducible representations of S_3 and the calculation of the character table. We will be able to do this for the general S_n when we combine the combinatorical resolution theorem and the RSK correspondence to derive Young's rule at the end of this paper.

Definition 9.1. For each $0 \leq k \leq n$, let X_k denote the set of all subsets of $[n]$ of size k. A ***subset representation*** of S_n on X_k is a permutation representation where $\sigma \in S_n$ takes one k-subset of $[n]$ to another.

Theorem 9.2. *For $0 \leq k, l \leq n$, two pairs (S, T) and (S', T') in $X_k \times X_l$ are in the same $G-$orbit if and only if $|S \cap T| = |S' \cap T'|$.*

Proof. If (S, T) and (S', T') are in the same G-orbit, then there exists $\sigma \in S_n$ such that $\sigma(S) = S'$ and $\sigma(T) = T'$. It follows that $\sigma(S \cap T) = S' \cap T'$ and $|S \cap T| = |S' \cap T'|$.

Conversely, if $|S \cap T| = |S' \cap T'|$, then any bijection from $S \cap T$ to $S' \cap T'$ extends to a permutation σ such that $\sigma(S) = S'$ and $\sigma(T) = T'$. □

Corollary 9.3. *If $k, l \leq \frac{n}{2}$, then*

$$\dim \operatorname{Hom}_{S_n}(\mathbb{C}[X_k], \mathbb{C}[X_l]) = \min\{k, l\} + 1$$

Proof. By the intertwining number theorem (Theorem 8.6), $\dim \operatorname{Hom}_{S_n}(\mathbb{C}[X_k], \mathbb{C}[X_l])$ is the number of orbits of $X_k \times X_l$, which by the above theorem is the values that $|S \cap T|$ can take for $S \in X_k$ and $T \in X_l$. If $k, l \leq \frac{n}{2}$, $|S \cap T|$ can be any integer between 0 and $\min\{k, l\}$. □

Theorem 9.4 (Decomposition of subset representations)**.** *There exist irreducible representations $V_0, \cdots, V_{\lfloor \frac{n}{2} \rfloor}$ such that for integers $0 \leq k \leq \frac{n}{2}$,*

$$\mathbb{C}[X_k] \sim V_0 \oplus \cdots \oplus V_k.$$

Proof. We make use of Remark 3.7 in this proof. By the previous corollary, $\dim \operatorname{End}_{S_n} \mathbb{C}[X_0] = 1$, so $\mathbb{C}[X_0]$ is an irreducible representation. Let $V_0 = \mathbb{C}[X_0]$. Since $\dim \operatorname{Hom}_{S_n}(\mathbb{C}[X_0], \mathbb{C}[X_l]) = 1$ for every $l \leq 1$, V_0 occurs with multiplicity one in $\mathbb{C}[X_l]$. Let $\mathbb{C}[X_1] \sim V_0 \oplus \mathbb{C}[X_1]_0$ for some $\mathbb{C}[X_1]_0$. Then $\dim \operatorname{End}_{S_n} \mathbb{C}[X_l]_0 = \dim \operatorname{End} \mathbb{C}[X_1] - 1 \cdot 1 = 2 - 1 = 1$. As a result, $\mathbb{C}[X_1]_0$ is irreducible, and we set $V_1 = \mathbb{C}[X_1]_0$. Note that V_1 also occurs in every subsequent subset representations with multiplicity one. We can continue this process and construct representations V_2, V_3, etc.. Their irreducbility is likewise deduced by noting that their spaces of self-intertwiners always have dimension one. □

Definitions 9.5. An ***ordered partition*** of $[n]$ with l parts is a decomposition of $[n]$ into nonempty pairwise disjoint subsets:

$$[n] = S_1 \amalg S_2 \amalg \cdots \amalg S_l$$

Let $\lambda_i = |S_i|$. Then the sequence

$$\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$$

is the ***shape*** of the ordered partition $S = (S_1, S_2, \cdots, S_l)$. In more general contexts, a ***weak composition*** of n into l parts is a sequence of positive integers $\lambda = (\lambda_1, \cdots, \lambda_l)$ such that $\lambda_1 + \cdots + \lambda_l = n$.

Let X denote the set of all ordered partitions of $[n]$. We define an action of $\sigma \in S_n$ on X in the following manner:

$$\sigma \cdot (S_1, S_2, \cdots, S_l) = (\sigma(S_1), \sigma(S_2), \cdots, \sigma(S_l))$$

Definition 9.6. Denote by X_λ the set of all ordered partitions of $[n]$ of shape λ, then we may form a permutation representation corresponding to the action of S_n on X_λ. For a weak composition λ of n, the ***partition representation*** of S_n of shape λ is the permutation representation $\mathbb{C}[X_\lambda]$.

Remark 9.7. In the notations of Definition 9.1 and Definition 9.6, the S_n-set X_k is isomorphic to the S_n-set $X_{(k,n-k)}$ by a canonical embedding. Thus $\mathbb{C}[X_k]$ and $\mathbb{C}[X_{(k,n-k)}]$ are equivalent representations by Proposition 7.8. This shows that subset representations are a special case of partition representations.

Proposition 9.8. *The partition representation $\mathbb{C}[X_{(1^n)}]$ is equivalent to the regular representation of S_n.*

Proof. To show $\mathbb{C}[X_{(1^n)}] \sim \mathbb{C}[S_n]$, we simply need an identification of $X_{(1^n)}$ with S_n. This is provided by viewing the ordered partition $(\{x_1\}, \{x_2\}, \cdots, \{x_n\})$ as the permutation which takes $(1, 2, \cdots, n)$ to $(x_1, x_2, \cdots, x_n)$. □

Proposition 9.9. *Every partition representation is equivalent to a partition representation whose shape is a partition.*

Proof. Suppose we have a weak composition $\mu = (\mu_1, \cdots, \mu_l)$ of $[n]$, then we can find a permutation $\sigma \in S_l$ such that $\lambda = (\mu_{\sigma(1)}, \cdots, \mu_{\sigma(l)})$ is a partition of $[n]$. Then note that the map $X_\mu \to X_\lambda$ defined by $(S_1, \cdots, S_l) \mapsto (S_{\sigma(1)}, \cdots, S_{\sigma(l)})$ is an isomorphism of S_n-sets. By Proposition 7.8, $\mathbb{C}[X_\mu] \sim \mathbb{C}[X_\lambda]$. □

Example 9.10. We want to compute a character table of partition representations of S_n (This is not *the* character table of S_n). The row of a character table is indexed by characters associated to inequivalent representations, and the column is indexed by the conjugacy classes of the represented group. In view of the previous proposition, we only need to consider all partitions of $[n]$ for the row index. Thanks to the fact that conjugacy classes of S_n are fully determined by cycle types (Theorem 8.3), the column is also indexed by partitions of $[n]$. When $n = 3$, the three partitions of $[3]$ are $(3), (2, 1)$, and $(1, 1, 1)$. The character of a permutation representation at a group element is the number of fixed points of that group element (Theorem 7.9), so we can easily compute the following table.

TABLE 1. Characters of Partition Representations of S_3

	(3)	(2,1)	(1,1,1)
$\mathbb{C}[X_{(3)}]$	1	1	1
$\mathbb{C}[X_{(2,1)}]$	0	1	3
$\mathbb{C}[X_{(1,1,1)}]$	0	0	6

We now proceed to decompose partition representations of S_3 using the intertwining number theorem (Theorem 8.6). First we need a combinatorical interpretation of $S_n\backslash(X_\lambda \times X_\mu)$, where $\lambda = (\lambda_1, \cdots, \lambda_l)$ and $\mu = (\mu_1, \cdots, \mu_m)$ are weak compositions of n.

Definition 9.11. With λ and μ as above, a ***$\lambda \times \mu$ matrix*** is an $l \times m$ matrix with non-negative integer entries, such that the sum of the i-th row is λ_i and the sum of the j-th column is μ_j. We use $\mathbf{M}_{\lambda\mu}$ to denote the set of $\lambda \times \mu$ matrices and $M_{\lambda\mu}$ the cardinality of $\mathbf{M}_{\lambda\mu}$.

Let $S = (S_1, \cdots, S_l) \in X_\lambda$ and $T = (T_1, \cdots, T_m) \in X_\mu$. We can define a map $r : X_\lambda \times X_\mu \to \mathbf{M}_{\lambda\mu}$, where $r_{ij}(S,T) := [r(S,T)]_{ij} = |S_i \cap T_j|$. Note that the image of this map is indeed a $\lambda \times \mu$ matrix:

$$r_{i1}(S,T) + \cdots + r_{im}(S,T) = |S_i \cap T_1| + \cdots + |S_i \cap T_l| = |S_i| = \lambda_i$$

Similarly, the sum of the j-th column is μ_j.

Theorem 9.12. *The map $(S,T) \mapsto r(S,T)$ induces a bijection $S_n\backslash(X_\lambda \times X_\mu) \to \mathbf{M}_{\lambda\mu}$.*

Proof. For any $\sigma \in S_n$, $r_{ij}(S,T) = |S_i \cap T_j| = |\sigma(S_i \cap T_j)| = |\sigma(S_i) \cap \sigma(T_j)| = r_{ij}(\sigma \cdot (S,T))$, so r descends to a well-defined function $S_n\backslash(X_\lambda \times X_\mu) \to \mathbf{M}_{\lambda\mu}$.

To prove injectivity, suppose $(S,T), (S',T') \in X_\lambda \times X_\mu$ are such that $|S_i \cap T_j| = |S'_i \cap T'_j|$ for all i and j. Then note that $[n] = \amalg_{i.j} S_i \cap T_j$ and $[n] = \amalg_{i,j} S'_i \cap T'_j$ are partitions of $[n]$ with the same shape. Then we can find $\sigma \in S_n$ that maps each $S_i \cap T_j$ to $S'_i \cap T'_j$. It follows that $\sigma(S_i) = S'_i$ and $\sigma(T_j) = T'_j$ for all i and j, so (S,T) and (S',T') are in the same S_n-orbit.

To prove surjectivity, let $r \in \mathbf{M}_{\lambda\mu}$. Then $n = \amalg_{i,j} r_{ij}$. Construct a partition $n = \amalg_{i,j} A_{ij}$ such that $|A_{ij}| = r_{ij}$ for all i and j. Define $S_i = \amalg_j A_{ij}$ and $T_j = \amalg_i A_{ij}$, then $S_i \cap T_j = A_{ij}$, and we find a pair of partitions (S,T) such that $r(S,T) = r$. ☐

Thanks to the previous theorem and the intertwining number theorem (Theorem 8.6), we obtain the following corollary.

Corollary 9.13. *For weak compositions λ and μ of n,*

$$\dim \mathrm{Hom}_{S_n}(\mathbb{C}[X_\lambda], \mathbb{C}[X_\mu]) = M_{\lambda\mu}$$

To complete the investigation of partition representations of S_3, we need the following table.

Theorem 9.14. *There exist irreducible representations $V_{(3)}, V_{(2,1)}$ and $V_{(1,1,1)}$ of S_3 such that*

$$\mathbb{C}[X_{(3)}] \sim V_{(3)}$$
$$\mathbb{C}[X_{(2,1)}] \sim V_{(3)} \oplus V_{(2,1)}$$
$$\mathbb{C}[X_{(1,1,1)}] \sim V_{(3)} \oplus V_{(2,1)}^{\oplus 2} \oplus V_{(1,1,1)}.$$

TABLE 2. $M_{\lambda\mu}$ for partitions of 3

	(3)	(2,1)	(1,1,1)
(3)	1	1	1
(2,1)	1	2	3
(1,1,1)	1	3	6

Proof. We are now prepared to imitate the reasoning in the decomposition of subset representations (Theorem 9.4). Since $\dim \mathrm{End}_{S_n}(\mathbb{C}[X_{(3)}]) = M_{(3)(3)} = 1$, $\mathbb{C}[X_{(3)}]$ is irreducible. Let $V_{(3)} = \mathbb{C}[X_{(3)}]$. Observe from Table 2 that $V_{(3)}$ has multiplicity one in $\mathbb{C}[X_{(2,1)}]$ and $\mathbb{C}[X_{(1,1,1)}]$. Therefore, there exist representations $\mathbb{C}[X_\lambda]_{(3)}$ such that $\mathbb{C}[X_\lambda] \sim \mathbb{C}[X_\lambda]_{(3)} \oplus V_{(3)}$ respectively for $\lambda = (2,1)$ and $(1,1,1)$. We obtain the following updated table:

TABLE 3. $\dim \mathrm{Hom}_{S_n}(\mathbb{C}[X_\lambda]_{(3)}, \mathbb{C}[X_\mu]_{(3)})$

	(2,1)	(1,1,1)
(2,1)	1	2
(1,1,1)	2	5

Observe that $\mathbb{C}[X_{(2,1)}]_{(3)}$ is irreducible; we set $V_{(2,1)} = \mathbb{C}[X_{(2,1)}]_{(3)}$. Table 3 shows that $V_{(2,1)}$ occurs with multiplicity two in $\mathbb{C}[X_{(1,1,1)}]_{(3)}$, so there exists a representation $\mathbb{C}[X_{(1,1,1)}]_{(2,1)}$ such that $\mathbb{C}[X_{(1,1,1)}]_{(3)} \sim V_{(2,1)}^{\oplus 2} \oplus \mathbb{C}[X_{(1,1,1)}]_{(2,1)}$. We easily compute that $\dim \mathrm{End}_{S_n}(\mathbb{C}[X_{(1,1,1)}]_{(2,1)} = 5 - 2^2 = 1$, so $V_{(1,1,1)} := \mathbb{C}[X_{(1,1,1)}]_{(2,1)}$ is also irreducible. □

Remark 9.15. By Proposition 9.8, $\mathbb{C}[X_{(1,1,1)}]$ is the left regular representation of S_3, so $V_{(3)}, V_{(2,1)}$ and $V_{(1,1,1)}$ form a complete set of representatives for the equivalence classes of irreducible representations of S_3, each with dimension equal to its multiplicity in $\mathbb{C}[X_{(1,1,1)}]$ (Theorem 6.4). With the information in Table 1 and Theorem 9.14, we can calculate the character table of S_3 using Proposition 5.4 (3).

TABLE 4. The character table of S_3

	(3)	(2,1)	(1,1,1)
$V_{(3)}$	1	1	1
$V_{(2,1)}$	-1	0	2
$V_{(1,1,1)}$	1	-1	1

The following theorem is a generalization of our work in Theorem 9.4 and Theorem 9.14. Note that in those examples the stated conditions are satisfied.

Theorem 9.16 (Combinatorical resolution theorem)**.** *Let $(P, \leq)$ be a partially ordered set, and $\{U_\lambda\}_{\lambda\in P}$ be a family of completely reducible representations of a group G. Let $M_{\mu\lambda} := \dim \mathrm{Hom}_G(U_\lambda, U_\mu)$. Suppose there exist nonnegative integers $K_{\mu\lambda}$ for all $\mu \leq \lambda$ in P such that $K_{\lambda\lambda} = 1$ for all $\lambda \in P$, and for all $\mu, \lambda \in P$,*

$$M_{\mu\lambda} = \sum_{v\leq\mu, v\leq\lambda} K_{v\mu} K_{v\lambda}. \tag{9.17}$$

Then, for each $\mu \in P$, there exists an irreducible representation V_μ such that for all $\lambda \in P$,

$$U_\lambda \sim \bigoplus_{\mu \leq \lambda} V_\mu^{\oplus K_{\mu\lambda}}$$

Proof. Let λ_0 be a minimal element of P. The condition given in (9.17) gives $M_{\lambda_0\lambda_0} = K^2_{\lambda_0\lambda_0} = 1$, so $V_{\lambda_0} := U_{\lambda_0}$ is irreducible. Also by (9.17), $M_{\lambda_0\lambda} = K_{\lambda_0\lambda_0}K_{\lambda_0\lambda} = K_{\lambda_0\lambda}$ for all $\lambda \in P$. This shows that V_{λ_0} occurs in U_λ $K_{\lambda_0\lambda}$ times, and there exist representations U^0_λ in which V_{λ_0} does not occur, such that $U_\lambda = U^0_\lambda \oplus V_{\lambda_0\lambda}^{\oplus K_{\lambda_0\lambda}}$.

Using a trick we saw previously in this section, let $P^0 = P \backslash \{\lambda_0\}$; for all $\mu, \lambda \in P^0$, let $M^0_{\mu\lambda} = \dim \mathrm{Hom}_G(U^0_\lambda, U^0_\mu)$. Compute to get

$$M^0_{\mu\lambda} = M_{\lambda\mu} - K_{\lambda_0\lambda}K_{\lambda_0\mu} = \sum_{\lambda_0 < v \leq \lambda, \lambda_0 < v \leq \mu} K_{v\lambda}K_{v\mu}$$

Therefore, $\{U^0_\lambda\}_{\lambda \in P^0}$ is a smaller collection of representations of G where the conditions of the theorem hold. Thus the theorem follows by induction on $|P|$. □

10. The RSK Correspondence and Classification of Irreducible Representations of S_n

The Robinson-Schensted-Knuth (RSK) correspondence makes use of the following combinatorical object to find integers $K_{\mu\lambda}$ that satisfy the hypothesis of the combinatorical resolution theorem (Theorem 9.16).

Definitions 10.1. A ***Young diagram*** consists of left justified rows of boxes, such that each row has at most as many boxes as the row above. A ***semistandard Young tableau*** (**SSYT**) is a Young diagram whose boxes are filled in with positive integers such that the entries increase *strictly* down each column and *weakly* along each row.

The ***shape*** of an SSYT is the partition $(\mu_1, \mu_2, \cdots)$, where μ_i is the number of boxes in the i-th row. The ***type*** of an SSYT is the partition $(\lambda_1, \lambda_2, \cdots)$, where λ_j is the number of times the integer j appears in the tableau.

Example 10.2.

1	1	1	2	3
2	2	3		
4	5			

is an SSYT of shape (5,3,2) and type (3,3,2,1,1).

Notation 10.3. Let $K_{\mu\lambda}$ be the number of SSYT of shape μ and type λ.

Definition 10.4. Let μ and λ be partitions of n. Then we say $\mu \leq \lambda$ if $\mu_1 + \cdots + \mu_i \geq \lambda_1 + \cdots + \lambda_i$ for all $1 \leq i \leq \min\{l, m\}$. One can check $' \leq '$ is a partial order on the set of partitions of n, and we call it the ***reverse dominance order***.

We have the following observations.

Lemma 10.5. *(1) $K_{\lambda\lambda} = 1$ for all partitions λ.*
(2) For all partitions μ and λ, $K_{\mu\lambda} > 0$ only if $\mu \leq \lambda$.

We omit the proof of the RSK correspondence, as it is a bit involved and purely combinatorical. See [1] for an elegant proof via the Viennot-RSK algorithm.

Theorem 10.6 (RSK correspondence)**.** *There is a bijection $A \to (P, Q)$ from the set of all $\lambda \times \mu$ matrices to pairs of SSYT P and Q such that P and Q have the same shape v, P is of type μ and Q is of type λ.*

Theorem 10.7 (Young's Rule)**.** *For each partition v of n, there exists an irreducible representation V_v of S_n such that for all partitions λ of n,*

$$\mathbb{C}[X_\lambda] \sim \bigoplus_{v \leq \lambda} V_v^{\oplus K_{v\lambda}}$$

Proof. Let the $M_{\mu\lambda}$ in Theorem 9.17 be the number $M_{\mu\lambda}$ of $\lambda \times \mu$ matrices. By the RSK correspondence we can write

$$M_{\mu\lambda} = \sum_v K_{v\lambda} K_{v\mu} = \sum_{v \leq \lambda, v \leq \mu} K_{v\lambda} K_{v\mu}$$

where Lemma 10.5 (2) allows us to omit the zero summands. Together with Lemma 10.5 (1), the hypotheses in the combinatorical resolution theorem (Theorem 9.17) are satisfied. This completes the proof. □

Note that $v \leq (1^n)$ for all partitions v of n. Recall that $\mathbb{C}[X_{(1^n)}] \sim \mathbb{C}[S_n]$, and the left regular representation contains all irreducible representations up to equivalence. We obtain:

Corollary 10.8.

$$\mathbb{C}[S_n] \sim \mathbb{C}[X_{(1^n)}] \sim \bigoplus_v V_v^{\oplus K_{v,(1^n)}}$$

In particular, $\{V_v\}$ as v runs over all partitions of n is a complete collection of irreducible representations of S_n, each with degree $K_{v,(1^n)}$.

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