THE ROTA-WELSH CONJECTURE FOR REPRESENTABLE MATROIDS

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Abstract. The Rota-Welsh Conjecture claims that the coefficients of specific polynomials associated to certain mathematical objects called matroids form a log concave sequence. This paper will define matroids and the associated polynomials referenced in the conjecture. It will then prove the conjecture for the special class of matroids representable over the complex field and explain how the strategies for this special case can be used to tackle parts of the general case of the conjecture.

Contents

1. Introduction

A matroid is an object that abstractly captures combinatorial information related to independence. Matroids can be constructed from sets of vectors, graphs, and many other situations or exist abstractly on their own unrelated to any underlying structure. Every matroid has an associated characteristic polynomial associated with it, the coefficients of which were conjectured to form a log concave sequence in the Rota-Welsh Conjecture.

Surprisingly, for a special class of matroids the Rota-Welsh Conjecture can be reformulated into a problem in intersection theory, a field that had already been studied for some time before the Rota-Welsh Conjecture and has many classical results that can be utilized to prove the conjecture. More recently, Adiprasito, Huh, and Katz were able to prove that the main result gained from the special case's connection to intersection theory still hold when we are not in the special case. They were not able to use intersection theory to prove it, so the proof is quite

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hard, but the motivation for the theorem they do prove comes from the special case we focus on here.

In this paper, we will rigorously define matroids and the characteristic polynomial, explain the connection of matroids to hyperplane arrangements, leverage this connection to construct the Chow rings that form the basis of intersection theory, and finally prove the Rota-Welsh Conjecture for a special class of matroids using intersection theory and the Chow ring of a matroid.

2. Unimodality and Log Concavity

Unimodality and log concavity are properties that show up frequently for sequences in algebra and combinatorics. The Rota–Welsh Conjecture asserts that certain sequences are log concave, so we will first identify these properties here.

Definition 2.1. A finite sequence $a_0, a_1, ..., a_n$ is *unimodal* if there is some i with $0 \leq i \leq n$ such that $a_0 \leq a_1 \leq \cdots \leq a_i \geq \cdots \geq a_{n-1} \geq a_n$.

Definition 2.2. A finite sequence $a_0, a_1, ..., a_n$ is *log concave* if $a_i^2 \geq a_{i-1}a_{i+1}$ for all $0 < i < n$.

Proposition 2.3. *If a finite, positive sequence $a_0, a_1, ..., a_n$ is log concave, then it is unimodal.*

Proof. If the sequence was not unimodal, then there would be some i such that $a_{i-1} \geq a_i \leq a_{i+1}$ with at least one of the inequalities being strict. This would mean $a_i^2 < a_{i-1}a_{i+1}$, so the sequence would not be log concave. □

Example 2.4. The coefficients of $(x+1)^n$ are log concave. To check for the kth coefficient, we get

$$\frac{(n!)^2}{(k!)^2(n-k!)^2} \geq \frac{(n!)^2}{(k+1)!(k-1)!(n-k+1)!(n-k-1)!}.$$

After moving things around, that becomes

$$1 \geq \frac{k}{k+1}\frac{n-k}{n-k+1},$$

which is true because both fractions on the right are less than one.

3. Algebraic Varieties

Let k be an algebraically closed field. *Affine n-space* over k is the set of n-tuples of elements in k, denoted $\mathbf{A}_k^n$ or $\mathbf{A}^n$. Each point P in $\mathbf{A}_k^n$ has coordinates $(a_1, ..., a_n)$.

For any function $f \in k[x_1, ..., x_n]$, we can assign a member of k to each point of $\mathbf{A}_k^n$ using $f(P) = f(a_1, ..., a_n)$. We let

$$Z(f) = \{P \in \mathbf{A}_k^n | f(P) = 0\}.$$

This is the zero set of f. For some collection of functions $T \subset k[x_1, ..., x_n]$, we let

$$Z(T) = \{P \in \mathbf{A}_k^n | f(P) = 0 \text{ for all } f \in T\}.$$

This forms the set of zeros common to all members of T. $A \subset \mathbf{A}_k^n$ is an *algebraic set* if $A = Z(T)$ for some T.

Proposition 3.1. *We can construct a topology using algebraic sets by having their complements be the open sets. This is called the Zariski topology.*

Proof. First, we have $Z(0) = \mathbf{A}_k^n$ and $Z(1) = \emptyset$, so both the empty set and all of $\mathbf{A}_k^n$ are open. $Z(S) \cup Z(T) = Z(ST)$ where $ST = \{st | s \in S \text{ and } t \in T\}$, so a finite intersection of open sets is open. Finally, $\bigcap Z(T_i) = Z\left(\bigcup T_i\right)$, so an arbitrary union of open sets is open. □

Definition 3.2. An algebraic set A is *irreducible* in the Zariski topology if for any pair of subsets A_1 and A_2 closed in A where $A = A_1 \cup A_2$, either $A = A_1$ or $A = A_2$.

Definition 3.3. An *affine variety* is a subset of $\mathbf{A}_k^n$ which is closed and irreducible in the Zariski topology. An open subset of an affine variety is a *quasi-affine variety.*

In an analogous way, we can define projective varieties over projective space.

Definition 3.4. Let k be an algebraically closed field. *Projective n-space*, denoted $\mathbf{P}_k^n$, is the set $\mathbf{A}_k^n - \{(0, ..., 0)\}$ quotiented by all lines through the origin.

Similarly, for a vector space V, $\mathbf{P}(V)$ is the set $V - \mathbf{0}$ quotiented by all lines through the origin.

The construction of projective varieties and quasi-projective varieties follows a similar path to the affine case, so I will not repeat it here. However, there are two notable differences worth focusing on. First, in order for a polynomial in $k[x_0, .., x_n]$ to be zero on the lines, it must be *homogeneous*, or have each term be the same degree. Second, we can often use *homogeneous coordinates* to represent the lines. This is written as $(x_0 : x_1 : \cdots : x_n)$ to emphasize that we care about the *ratio* of the coordinates and not the exact numbers. For the purposes of this paper, an *algebraic variety* is either an affine or projective variety.

4. Matroids

A matroid is a mathematical object that extends the ideas of independent vector sets to abstract sets. There are several equivalent ways to define a matroid, each of which gives different benefits. We will define it here using flats.

Definition 4.1. Let E be any set. Take $F, F' \in \{F_1, ..., F_n\} \subset 2^E$. We say F' *covers* F if $F \subsetneq F'$ but there is no i such that $F \subsetneq F_i \subsetneq F'$.

Definition 4.2. A *matroid* M is a finite set E and a collection of subsets of E, $\mathcal{F}$ satisfying properties:

(M1) $E \in \mathcal{F}$.
(M2) For all $F, F' \in \mathcal{F}$, $F \cap F' \in \mathcal{F}$.
(M3) If $F \in \mathcal{F}$ and $\{F_1, ..., F_n\}$ is all members of $\mathcal{F}$ which cover F, then the set $\{F_1 \setminus F, ..., F_n \setminus F\}$ partitions $E \setminus F$.

We call the members of $\mathcal{F}$ *flats*.

It is useful to define a closure operation on any subset of $X \subset E$ as

$$cl(X) = \bigcap_{X \subset F \in \mathcal{F}} F. \tag{4.3}$$

There will always be some F containing X because E is a flat, and $cl(X)$ is a flat itself by (M2). This closure operation gives the minimal flat containing X.

Example 4.4. The *uniform matroid* $U_{r,m}$ is the matroid over the set $E = \{1, ..., m\}$ such that all flats $F \subset E$ have either $|F| < r$ or $F = E$.

Example 4.5. Let $E = \{1, 2, 3\}$ and $\mathcal{F} = \{\{\}, \{1\}, \{2\}, \{3\}, \{1, 2, 3\}\}$. This forms the uniform matroid $U_{2,3}$. We can also construct this matroid in another way using a set of vectors in some ambient vector space if we let $E = \{\langle 1, 0\rangle, \langle 0, 1\rangle, \langle 1, 1\rangle\} \subset \mathbf{R}^2$ and flats be the subsets such that no elements outside the subset are contained in the span of vectors in the subset. We can see that the flat structure is the same as the abstract construction we started with.

Example 4.6. A *partition matroid* is a direct sum of uniform matroids.

For example, we could take the direct sum of $U_{2,3}$ on $\{1, 2, 3\}$ and $U_{1,2}$ on $\{4, 5\}$ to get a partition matroid over $\{1, 2, 3, 4, 5\}$. The flats of this matroid would be $\{\}, \{1\}, \{2\}, \{3\}, \{1, 2, 3\}, \{4, 5\}, \{1, 4, 5\}, \{2, 4, 5\}, \{3, 4, 5\}, \{1, 2, 3, 4, 5\}$.

Definition 4.7. A matroid M is *simple* if every subset of E with cardinality less than 2 is a flat.

Proposition 4.8. *Every matroid has a minimal flat.*

Proof. By (M2), $\bigcap_{F\in\mathcal{F}} F$ is a flat contained in every other flat. □

If the empty set is not a flat, then there exist some elements that are members of every flat. These elements are called *loops*. A matroid is *loopless* if it has no loops or equivalently if the empty set is a flat.

Proposition 4.9. *If the empty set is a flat and F covers the empty set, then any flat containing one member of F contains all members of F.*

Proof. Suppose F' contains one element of F but $F \not\subset F'$. We have $\emptyset \subsetneq F \cap F' \subsetneq F$, which is a contradiction to F covering the empty set. □

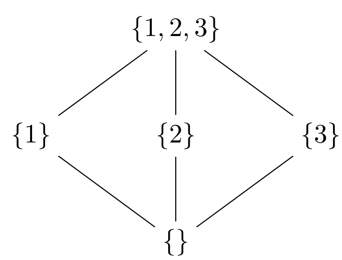
Elements contained in the same flat covering the empty set are *parallel*.

For any matroid M, we can constrict its *simplification* $\hat{M}$ by first removing all loops and then combining all parallel elements into a single element, making appropriate changes to the flats.

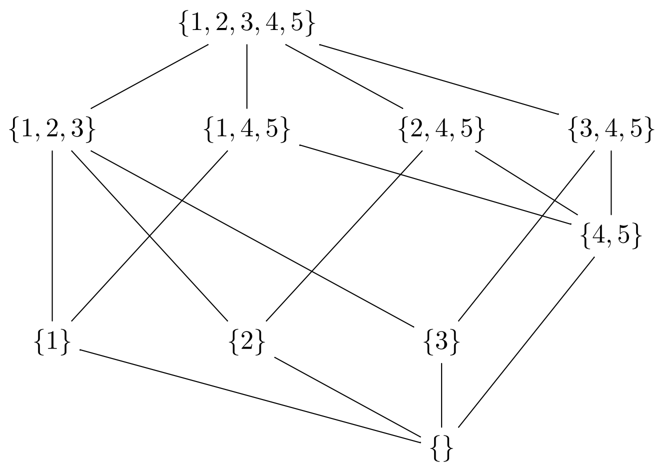
Definition 4.10. A *lattice* is a partially ordered set such that for any x, y in the lattice, they have a least upper bound $x \wedge y$ and a greatest lower bound $x \vee y$, which are called *meet* and *join*, respectively.

For a matroid M, the flats form a poset using containment. This poset becomes the *lattice of flats*, $\mathcal{L}(M)$, when we define $X \wedge Y = X \cap Y$ and $X \vee Y = cl\,(X \cup Y)$.

Example 4.11. We can represent the lattice structure of matroids using diagrams showing containment. For the matroid $U_{2,3}$, we have the lattice:



For the partition matroid $U_{2,3} \oplus U_{1,2}$ described in Example 4.6, we have a more complicated lattice of flats:



This matroid is not simple because $\{4\}$ and $\{5\}$ are not flats. If we simplify this matroid, we obtain an isomorphic lattice (the choice to place $\{4, 5\}$ higher up than the single element flats is purely cosmetic). This is the power of simplifying a matroid in general: the lattice of flats keeps the same structural information.

Every finite lattice has a minimal element that is the meet of all its members. Elements that cover this minimal element are called *atoms*.

Definition 4.12. A lattice is *geometric* if it has both of these properties

(1) Every member of the lattice is a join of atoms
(2) If x and y cover $x \vee y$, then $x \wedge y$ covers x and y.

Proposition 4.13. *The lattice of flats for any simple matroid is a geometric lattice.*

Proof. From our definition of simple lattices, every single element set is a flat and will be and atom. Thus, any flat is a join of the atoms of each of its members.

Suppose x and y cover $x \vee y$. If $x \subset y$, then $x \vee y = x$, but x does not cover itself. Thus, we have some $e \in x$ such that $e \notin y$. From (M3), there is a flat $\bar{y}$ which covers y and contains e. We have $e \in x \cap \bar{y}$ and $y \subset \bar{y}$, so $x \vee y = x \cap y \subsetneq x \cap \bar{y} \subset x$. Because x covers $x \vee y$, we must have $x \cap \bar{y} = x$, which means $x \subset \bar{y}$. This means $x \wedge y \subset \bar{y}$, but because $\bar{y}$ covers y and x is not contained in y, $\bar{y} = x \wedge y$ and $x \wedge y$ covers y. Swapping x and y shows $x \wedge y$ covers x. □

Because the lattice of flats for a matroid is isomorphic to the lattice of its simplification, this shows that all matroids have a geometric lattice of flats.

Definition 4.14. If F is a flat in a matroid, the *rank* of F, $r(F)$, is the maximal l for which their is a sequence of nested flats $F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_l = F$. The rank of the whole matroid, $r(M)$, is the rank of the maximal flat E.

The Möbius function is a useful combinatorial function that can be defined on any poset. We will define it here only on a lattice of flats. The existence and uniqueness of this function is beyond the scope of this paper, but working with the function in the context we require only takes straightforward computations.

Definition 4.15. The *Möbius function* is the function $\mu_{\mathcal{L}(M)} : \mathcal{L}(M) \times \mathcal{L}(M) \to \mathbf{Z}$ satisfying $\mu_{\mathcal{L}(M)}(x, x) = 1$, $\mu_{\mathcal{L}(M)}(x, y) = 0$ if $x \not\subset y$, and

$$\sum_{x \subset z \subset y} \mu_{\mathcal{L}(M)}(x, z) = 0$$

if $x \subset y$.

Due to a theorem of Rota, a matroid's lattice of flats being geometric implies

$$(-1)^{r(y)-r(x)} \mu_{\mathcal{L}(M)}(x, y) > 0.$$

Definition 4.16. Let M be a loopless matroid. The characteristic polynomial of M is given by

$$\chi_M(t) = \sum_{F \in \mathcal{L}(M)} \mu_{\mathcal{L}(M)}(\emptyset, F) t^{r(M)-r(F)}$$

Proposition 4.17. *For any matroid, $\chi_M(1) = 0$.*

Proof. We have

$$\chi_M(1) = \sum_{F \in \mathcal{L}(M)} \mu_{\mathcal{L}(M)}(\emptyset, F) 1^{r(M)-r(F)} = \sum_{\emptyset \subset F \subset E} \mu_{\mathcal{L}(M)}(\emptyset, F) = 0. \qquad \square$$

This polynomial captures information about the matroid and is deeply connected to chromatic polynomials of graphs for matroids constructed from them. Importantly, it is a monic polynomial of degree $r(M)$. If we write $\chi_M(t) = a_0 + a_1 t + \cdots + a_{r(M)} t^{r(M)}$, the coefficients alternate sign. The Rota–Welsh conjecture states that the sequence $|a_0|, |a_1|, ..., |a_{r(M)}|$ is log concave.

Example 4.18. (The Rota–Welsh Conjecture on $U_{2,3}$) Let $M = U_{2,3}$ and let $L = \mathcal{L}(M)$. From the definition of the Möbius function, we can see if x covers $\emptyset$, then $\mu_L(\emptyset, x) = -1$. To compute the characteristic polynomial, we now only need $\mu_L(\emptyset, \{1, 2, 3\})$, which must equal 2 from the identity defining the Möbius function. We can now evaluate the sum to obtain the characteristic polynomial, which we simplify by combining the three equivalent single-element flats:

$$\begin{aligned}\chi_M(t) &= \mu_L(\emptyset, \emptyset) t^{r(M)-r(\emptyset)} + 3\mu_L(\emptyset, \{1\}) t^{r(M)-r(\{1\})} + \mu_L(\emptyset, \{1, 2, 3\}) t^{r(M)-r(\{1,2,3\})} \\ &= t^2 - 3t + 2\end{aligned}$$

Because $3^2 \geq 1 \cdot 2$, the Rota–Welsh Conjecture holds.

Example 4.19. (The Rota–Welsh Conjecture on a Partition Matroid) Let $M = U_{2,3} \oplus U_{1,2}$ described in Example 4.6 and let $\hat{M}$ be its simplification by turning $\{4, 5\} \mapsto \{4\}$. Let L be the lattice of flats of $\hat{M}$. We can work out values of $\mu_L(\emptyset, x)$ by starting from the bottom of the lattice and moving upwards to get

$$\mu_L(\emptyset, \emptyset) = 1,$$
$$\mu_L(\emptyset, \{1\}) = \mu_L(\emptyset, \{2\}) = \mu_L(\emptyset, \{3\}) = \mu_L(\emptyset, \{4\}) = -1,$$
$$\mu_L(\emptyset, \{1, 2, 3\}) = 2,$$
$$\mu_L(\emptyset, \{1, 4\}) = \mu_L(\emptyset, \{2, 4\}) = \mu_L(\emptyset, \{3, 4\}) = 1,$$
$$\mu_L(\emptyset, \{1, 2, 3, 4\}) = -2.$$

Again, grouping by equivalent flats, we evaluate the characteristic polynomial:

$$\chi_{\hat{M}}(t) = \mu_L(\emptyset, \emptyset) t^{r(\hat{M})-r(\emptyset)} + 4\mu_L(\emptyset, \{1\}) t^{r(\hat{M})-r(\{1\})} + \mu_L(\emptyset, \{1, 2, 3\}) t^{r(\hat{M})-r(\{1,2,3\})}$$

$$+3\mu_L(\emptyset, \{1,4,5\})t^{r(\hat{M})-r(\{1,4,5\})} + \mu_L(\emptyset, \{1,2,3,4,5\})t^{r(\hat{M})-r(\{1,2,3,4,5\})}$$

$$= t^3 - 4t^2 + 5t - 2$$

This time, we get two inequalities for the Rota–Welsh Conjecture, $4^2 > 1 \cdot 5$ and $5^2 > 4 \cdot 2$, both of which hold.

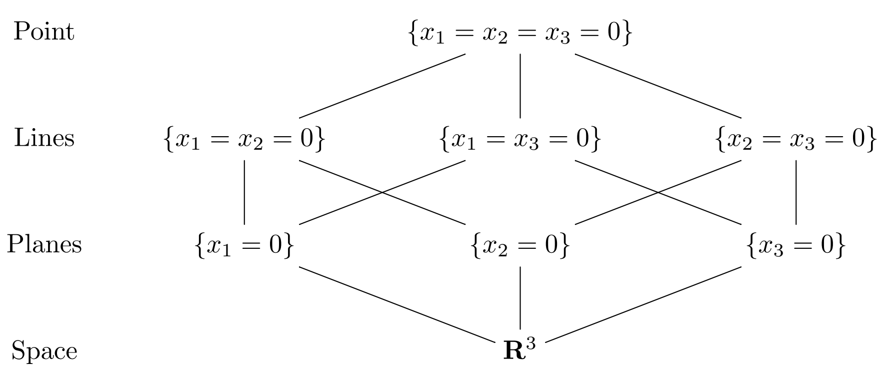
5. Hyperplane Arrangements and Complements

Definition 5.1. Let V be an n dimensional real or complex vector space. A *hyperplane arrangement* $\mathcal{A}$ in V is a finite set of $n-1$ dimensional subspaces of V. We have an associated topological space $\mathcal{M}(\mathcal{A}) := V \setminus \cup_{H \in \mathcal{A}} H$ which is the complement of the arrangement.

A significant amount of information about a hyperplane arrangement is captured in the way the subspaces intersect. For example, three distinct hyperplanes in $\mathbf{R}^3$ differ only by a change of basis except when the hyperplanes have a mutual intersection of an entire line. Like for matroids, one way to remember most of the important structure of an arrangement while ignoring unimportant aspects is to form a lattice.

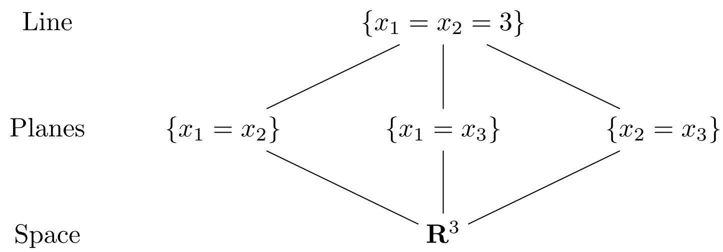
Definition 5.2. For a hyperplane arrangement $\mathcal{A}$, the *intersection lattice*, $\mathcal{L}(\mathcal{A})$, is the lattice on the partially ordered set of all intersections of hyperplanes ordered by reverse inclusion. For two elements x and y, their join is their intersection and their meet is the span of their union. The atoms of this lattice are the hyperplanes and each element is a join (intersection) of these hyperplanes.

Example 5.3. Let $V = \mathbf{R}^3$ with coordinates (x_1, x_2, x_3). Consider the arrangement of the coordinate hyperplanes $W_1 = \{x_1 = 0\}$, $W_2 = \{x_2 = 0\}$, and $W_3 = \{x_3 = 0\}$. From this, we get the lattice:



Example 5.4. Let $V = \mathbf{R}^3$ with coordinates (x_1, x_2, x_3). Consider the arrangement of the hyperplanes $W_{12} = \{x_1 = x_2\}$, $W_{13} = \{x_1 = X_3\}$, $W_{23} = \{x_2 = x_3\}$. This particular arrangement is called the *real rank 2 braid arrangement* and has

lattice structure:



The lattice of this hyperplane arrangement is identical to the matroid $U_{2,3}$ we investigated in the previous section, giving a hint to a connection between matroids and hyperplane arrangements that we will investigate later.

Hyperplane arrangements exist in affine space, which is fine for some purposes but not for others. In order to take advantage of projective space, we will define De Concini-Procesi arrangement models. This construction pushes our arrangements into projective space while maintaining the topology of $\mathcal{M}(\mathcal{A})$.

Definition 5.5. Let $\mathcal{A}$ be an arrangement of real or complex hyperplanes in V with intersection lattice $\mathcal{L}$. Let $\overline{\mathcal{L}}$ be all members of the lattice with the minimal element removed. We define a map:

$$\Psi : \mathcal{M}(\mathcal{A}) \to V \times \prod_{X \in \overline{\mathcal{L}}} \mathbf{P}(V/X)$$

$$x \mapsto \left(x, (\mathrm{span}(x, X)/X)_{X \in \overline{\mathcal{L}}}\right).$$

This will be an open embedding. The closure of the image is the *De Concini-Procesi arrangement model for* $\mathcal{A}$ denoted $Y_{\mathcal{A}}$.

The effect of this process is recording the relationship between each point of $\mathcal{M}(\mathcal{A})$ and each member of the intersection lattice.

Example 5.6. Consider the arrangement $\mathcal{A}$ of the coordinate planes in $\mathbf{R}^3$ given in Example 5.3. We can ignore factors of $\mathbf{P}^0$ in the codomain because they do not add any additional info, so we are mapping into $\mathbf{R}^3 \times \mathbf{P}^2 \times \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$. For some $(x, y, z) \in \mathcal{M}(\mathcal{A})$, we get

$$(x, y, z) \mapsto ((x, y, z), (x : y : z), (x : y), (x : z), (y : z)).$$

This map is simple to write explicitly, but rather complex for seeing what is actually going on.

Example 5.7. Consider the rank-2 braid arrangement $\mathcal{A}$ given in Example 5.4. Again, we can ignore factors of $\mathbf{P}^0$ in the codomain, so we are mapping into $\mathbf{R}^3 \times \mathbf{P}^1$. Unlike our first example, the map is not as nice or informative to write explicitly, though the process for doing so is still a straightforward computational endeavor. However, the result is easier to parse: the component of the image in $\mathbf{P}^1$ captures the "direction" a point is from the line $x_1 = x_2 = x_3$.

6. Representable Matroids

We saw in Example 5.5 that it is possible to use vectors to capture the structure of a specific matroid. When this happens, we say a matroid is representable over a field.

Definition 6.1. Let V be a finite-degree vector space over some field and let E be some collection of vectors in V. For $F \subset E$, we define the *closure of* F *in* E, $cl(F)$, to be all members of E that lie in the span of F.

Because of its reliance on span, we see that the closure operation preserves containment and that $F \subset cl(F)$ for all F.

Proposition 6.2. *Let* V *be a finite-degree vector space over some field and let* E *be some collection of vectors in* V*. If we let* $\mathcal{F}$ *be the collection of all subsets* $F \subset E$ *such that* $F = cl(F)$*, then the set* E *with flats* $\mathcal{F}$ *forms a matroid.*

Proof. We will go through each property.

(M1) $cl(E) = E$, so $E \in \mathcal{F}$.

(M2) Let A and B be two members of $\mathcal{F}$. $A \cap B \subset A$, so $cl(A \cap B) \subset cl(A) = A$. Similarly, $cl(A \cap B) \subset B$. Thus, we have $A \cap B \subset cl(A \cap B) \subset A \cap B$, which means $A \cap B \in \mathcal{F}$.

(M3) Take $A, B \in \mathcal{F}$. If B covers A, then the subspace spanned by the elements of B must have a degree larger than the degree of A. For $v \in E \setminus A$, $A \cup \{v\}$ will span a subspace with a degree one larger than that of A. Because of this, any strict subspace of that span containing the span of A must be A, so $cl(A \cup \{v\})$ covers A. Thus, all members of $E \setminus A$ are contained in at least one member of $\mathcal{F}$ that covers A.

Now suppose some $v \notin A$ is contained in two distinct members of $\mathcal{F}$ covering A, B and B', which we can choose without loss of generality such that $B \not\subset B'$. Because $v \in B \cap B'$, $B \cap B' \neq A$, so we have $A \subsetneq B \cap B' \subsetneq B$. We have already shown that $B \cap B' \in \mathcal{F}$, so we have a contradiction of B covering A. Therefore, we satisfy the desired partition relation.

□

Can we represent all matroids in this way? Unfortunately, we are not able to do that. While many matroids are representable over any field, some are representable only over fields with specific characteristic and others over no fields. In fact, almost all matroids are representable over no field as proven in [11]. However, we will focus now only on matroids that are representable over $\mathbf{C}$ because of their connection to hyperplane arrangements, so for the remainder of this paper a *representable matroid* will be a matroid that is representable over $\mathbf{C}$. The methods we use for these matroids can be extended to non-representable matroids, but it requires work to build the tools analogous to the ones we get much more readily for the representable case.

Definition 6.3. If v is a nonzero vector in $\mathbf{C}^n$, then we define the *orthogonal complement* of v, $v^\perp$, to be all vectors w such that $v \cdot w = 0$ using the usual dot product. We see $v^\perp$ is an $n-1$ dimensional subspace of $\mathbf{C}^n$, so it is a hyperplane.

Proposition 6.4. *For nonzero vectors* $v, v_1, ..., v_n$ *in a vector space over* $\mathbf{C}$*, the following are equivalent:*

- v is contained in $\text{span}(v_1, ...v_n)$

- $v^\perp$ contains $v_1^\perp \cap \cdots \cap v_n^\perp$

Proof. Suppose v is contained in $\text{span}(v_1, ...v_n)$. We can write $v = \mu_1 v_1 + \cdots + \mu_n v_n$ for some scalars $\mu_1, ..., \mu_n$. For any $w \in v_1^\perp \cap \cdots \cap v_n^\perp$, $w \cdot v_i = 0$ for all i. This gives

$$w \cdot v = \mu_1(w \cdot v_1) + \cdots + \mu_n(w \cdot v_n) = 0.$$

Thus, all such w are contained in $v^\perp$ and $v^\perp$ contains $v_1^\perp \cap \cdots \cap v_n^\perp$.

Suppose $v^\perp$ contains $v_1^\perp \cap \cdots \cap v_n^\perp$. We can choose an orthonormal basis of our vector space $w_1, .., w_k$ such that $w_1, ..., w_i$ span the span of $v, v_1, ..., v_n$ and $w_{i+1}, ..., w_k$ span the intersection of their orthogonal compliments. Because that intersection is contained in $v^\perp$, $v \cdot w_j = 0$ for $i < j \leq k$. We can write

$$v = \sum_{1 \leq j \leq k} (v \cdot w_j) w_j = \sum_{1 \leq j \leq i} (v \cdot w_j) w_j,$$

so we see v is in $\text{span}(v_1, ...v_n)$. □

Proposition 6.5. *If M is a representable matroid over $\mathbf{C}$ with nonzero vectors $v_1, ..., v_n$, then its lattice of flats is isomorphic to the intersection lattice of the hyperplane arrangement $v_1^\perp, ..., v_n^\perp$.*

Proof. Both lattices have atoms for which all other elements are the join of them, and these atoms are $v_1, ..., v_n$ and $v_1^\perp, ..., v_n^\perp$. The join of atoms in the former being the same as a join of corresponding elements in the latter is a direct consequence of Proposition 6.4. □

The benefit of this construction is that we have encoded the meaningful combinatorial information relevant to the characteristic polynomial of a representable matroid into an algebraic object that has additional structure that has already been studied. We can exploit this connection to learn more about the matroid. This fact also patches one small issue in this construction: vectors must be nonzero. The zero vector is a loop, and we care only about loopless matroids for the Rota-Welsh conjecture, so the omission causes no problems.

7. Chow Rings

The most helpful consequence we get from the connection between matroids and hyperplane arrangements is a set of strong results from the field of intersection theory. To fully appreciate this connection, we must define two types of Chow rings.

Definition 7.1. Let $F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k$ be a sequence of nested flats of some matroid. The collection of such flats, $\mathscr{F} = \{F_0, F_1, ..., F_k\}$ is called a *flag*.

Definition 7.2. Let M be a loopless matroid over E and let $\overline{\mathcal{F}}$ be the non-empty proper flats of M. The *Chow ring* of M, $A^* = A^*(M)$, is the graded ring given by the quotient of $\mathbf{Z}[x_F]_{F \in \overline{\mathcal{F}}}$ by the ideals associated to two relations:

- $$\sum_{a \in F \in \overline{\mathcal{F}}} x_F = \sum_{b \in F' \in \overline{\mathcal{F}}} x_{F'}.$$
- $x_F x_{F'} = 0$ when the flats F and F' are not comparable in the poset lattice.

If $r = r(M) - 1$, we get a unique isomorphism $\deg : A^r \to \mathbf{Z}$ if we define $\deg(x_{F_1} x_{F_2} ... x_{F_r}) = 1$ when $\{F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_r\}$ is a flag in $\overline{\mathcal{F}}$.

Example 7.3. Let $M = U_{2,3}$. We have $\overline{\mathcal{F}}$ containing the three single-element flats, so we will work with $\mathbf{Z}[x_1, x_2, x_3]$. The quotient relations give

- $x_1 = x_2 = x_3$,
- $x_i x_j = 0$ for all $i \neq j$, which with the first relation means $x_i^2 = 0$ for all i.

This quotient is rather straightforward, giving $A^*(M) = \mathbf{Z}[x]/(x^2)$.

This Chow ring is a concrete structure but does not offer much motivation on its own. However, our second type of Chow ring is more understood by another field.

Definition 7.4. Let X be an algebraic variety. The *group of cycles*, $Z(X)$, is the free abelian group generated by subvarieties of X.

Definition 7.5. Let $t_0, t_1 \in \mathbf{P}^1$ and Φ be a subvariety of $\mathbf{P}^1 \times X$ which is not contained in any $\{t\} \times X$ for fixed t. We define $\mathrm{Rat}(X)$ to be the subgroup generated by

$$\langle \Phi \cap \{t_0\} \times X \rangle - \langle \Phi \cap \{t_1\} \times X \rangle$$

Two cycles $Z_0, Z_1 \in Z(X)$ are rationally equivalent if $Z_0 - Z_1 \in \mathrm{Rat}(X)$.

The meaning of this equivalence is that we are able to slice our variety in $\mathbf{P}^1 \times X$ along two fixed places in $\mathbf{P}^1$ to get each desired cycle in X. We are transforming one cycle to another using a strict method of transformation.

Definition 7.6. Quotienting the group of cycles by this equivalence relation forms the *Chow group on* X, $A(X)$.

If Y is a subvariety of X, then $[Y]$ is the class in the Chow group associated to Y.

The best way to define multiplication for this type of structure is to take intersections of these classes, but we run into a few issues. This operation is currently not well defined, but we are able patch up the problems with a few fixes.

Definition 7.7. Let X be an algebraic variety with subvarieties A and B. We say A and B intersect *transversely* at p if A, B, and X are smooth at p and the tangent spaces[1] of A and B at p together span the tangent space of X at p.

We say A and B are *generically transverse* if A and B intersect transversely at p for all $p \in A \cap B$.

Subvarieties being generically transverse captures a sense of maximal intersection. In Figure 1, the tangent spaces of A and B at $A \cap B$ will only span A, so they do not intersect transversely. The tangent spaces of A' and B at either intersection point will span all of $\mathbf{A}^2$, so A' and B are generically transverse.

Lemma 7.8 (Moving Lemma)**.** *Let X be a smooth quasi-projective variety.*

(1) For all classes $\alpha, \beta \in A(X)$, there are generically transverse cycles $A, B \in Z(X)$ such that $[A] = \alpha$ and $[B] = \beta$.
(2) The class $[A \cap B]$ is independent of choice of such cycles A and B.

The proof of this lemma is hard and is done by Fulton in [8]. As a consequence, we now have enough to define the Chow *ring* rather than just the Chow group.

[1]The tangent space of X at p is informally the set of all lines through p tangent to the X. This can be defined rigorously and only using algebra as is done in section 2.1.2 of [12], though the informal notion is fine for our limited use of tangent spaces.

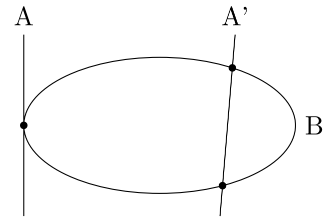


FIGURE 1. Affine varieties A and A' intersecting B in $\mathbf{A}^2$.

Theorem 7.9. *Let X be a smooth quasi-projective variety. Equipping the Chow group with multiplication given by*

$$[A][B] = [A \cap B]$$

for generically transverse A and B creates a commutative, graded ring called the Chow ring.

Again, this is also proven by Fulton. The Chow ring for algebraic varieties is the basis for the field of intersection theory, a field that has been studied deeply on its own. Feichtner and Yazvinsky establish in [7] that the Chow ring $A^*(M)$ is the same as the chow ring for the De Concini-Procesi arrangement model of the hyperplane arrangement with corresponding intersection lattice we defined earlier. Because these coincide, we are able to take parts of the Kähler package, a collection of three classical result in intersection theory, and apply their consequences to matroids. This method was first established by Adiprasito, Huh, and Katz.

We must define a few ideas in intersection theory to take advantage of the desired results.

Definition 7.10. A function $c : 2^E \to \mathbf{R}_{\geq 0}$ is *strictly submodular* if $c(E) = c(\emptyset) = 0$ and $C(A \cup B) + c(A \cap B) < c(A) + c(B)$ whenever the subsets A, B are incomparable.

Submodular functions exist. We can define a function l mapping submodular functions into the chow ring:

$$l(c) = \sum_{F \in \overline{\mathcal{F}}} c(F) x_F \in A^1(M)_{\mathbf{R}}$$

The elements of the form $l(c)$ are called *ample* and make up the *ample classes* in $A^1(M)_{\mathbf{R}}$. From this, we get the two results from intersection theory that are essential for proving the Rota-Welsh conjecture.

Theorem 7.11. *Let M be a representable matroid of rank $r = d+1$, let $l \in A^1(M)_{\mathbf{R}}$ be ample, and let $0 \leq k \leq \frac{d}{2}$. We :*

(1) Hard Lefschetz theorem: Multiplication by l^{d-2k} gives an isomorphism

$$L_l^k : A^k(M)_{\mathbf{R}} \to A^{d-k}(M)_{\mathbf{R}}.$$

(2) Hodge-Riemann relations: The bilinear form

$$Q_l^k : A^k(M)_{\mathbf{R}} \times A^k(M)_{\mathbf{R}} \to \mathbf{R}$$

defined by $Q_l^k(a, b) = (-1)^k a \cdot L_l^k b$ is positive definite on the kernel of $l \cdot L_l^k$.

Intersection theory was a well developed field with analogous results to theorem 7.11 known for Chow rings of projective varieties well before their connection to representable matroids was made. Because representable matroids have Chow rings isomorphic to that of the projective variety constructed in Proposition 6.5, the proof of this theorem by by Adiprasito, Huh, and Katz in [1] is mostly just reframing the classical results in intersection theory into the language of the matroid's Chow ring.

8. Proving the Rota-Welsh Conjecture for Representable Matroids

Even though we want to prove something about the coefficients of the characteristic polynomials of matroids, we will find it convent to work with a related reduced characteristic polynomial.

Definition 8.1. Let M be some matroid with characteristic polynomial. $\chi_M(t)$. The *reduced characteristic polynomial*, $\overline{\chi_M}(t)$, is the quotient $\chi_M(t)/(t-1)$. The binomial $(t-1)$ always divides $\chi_M(t)$ by Proposition 4.17.

For some matroid, we will define $\chi_M(t) = a_0 + a_1 t + ... + a_d t^d$ and $\overline{\chi_M}(t) = b_0 + b_1 t + ... + b_{d-1} t^{d-1}$. We will also extend the second sequence using two extra terms: $b_{-1} = b_d = 0$. We now get two very simple facts about the relationship of these coefficients using a few algebraic computations

$$a_i = b_{i-1} - b_i, \tag{8.2}$$

$$b_i = \sum_{j=0}^{d-i-1} (-1)^j a_{1+j}. \tag{8.3}$$

From (8.3), we can see the terms of $b_0, ..., b_{d-1}$ alternate sign. If $b_0, ..., b_{d-1}$ is log concave, the extended sequence will be as well because $b_0^2 \geq 0$ and $b_{d-1}^2 \geq 0$. We can use (8.2) to get a useful result.

Proposition 8.4. *If $b_0, ..., b_{d-1}$ is log concave, then $a_0, ..., a_d$ is log concave.*

Proof. We need to show

$$(b_{i-1} - b_i)^2 \geq (b_{i-2} - b_{i-1})(b_i - b_{i+1})$$

for $0 < i < d$. Expanding, we get

$$b_{i-1}^2 - 2b_{i-1}b_i + b_i^2 \geq b_{i-2}b_i - b_i b_{i-1} - b_{i-2}b_{i+1} + b_{i+1}b_{i-1}.$$

We can break this into four smaller inequalities

$$b_{i-1}^2 \geq b_{i-2}b_i,$$
$$b_{i-1}^2 \geq b_{i-2}b_i,$$
$$b_{i-1}b_i \leq b_{i-1}b_i,$$
$$b_{i-1}b_i \leq b_{i-2}b_{i+1}.$$

The first two are a direct consequence of the log concavity of $b_0, ..., b_{d-1}$ and the third is just an equality. The fourth one can be seen as true by multiplying both sides by $b_{i-1}b_i$, which is a non-positive value so we flip the sign, giving us

$$(b_{i-1}^2)(b_i^2) \geq (b_{i-2}b_i)(b_{i-1}b_{i+1})$$

which again is true because $b_{i-1}^2 \geq b_{i-2}b_i$ and $b_{i-1}^2 \geq b_{i-2}b_i$ from log concavity. □

Finally, we will define a sequence $\mu_0, ..., \mu_{d-1}$ where $\mu_i = |b_{d-i-1}|$. When necessary, we will specify the matroid these are associated to by writing $\mu_i(M)$. Log concavity of the positive, reversed sequence implies log concavity of the coefficients and will reduce some notation later.

We also able to take advantage on induction on a matroid by using truncation to reduce the rank of a matroid without changing its characteristic polynomial too much.

Definition 8.5. The *truncation of a matroid* M over E with lattice of flats L, written $\mathrm{tr}(M)$, is the matroid over E with lattice of flats $L \setminus \{F \in L | E \text{ covers } F\}$. If the rank of M is $r > 1$, then the rank of $\mathrm{tr}(M)$ is $r - 1$.

Because this only changes the very top of the matroid, the coefficients of the characteristic polynomial will only change slightly: the terms will all be decreased by a degree and the linear coefficient in the truncated matroid's characteristic polynomial might not match the linear coefficient in the untruncated matroid's characteristic polynomial. The good thing about this is that (8.3) ignores the linear coefficient, so we get an equality

$$\mu_i(\mathrm{tr}(M)) = \mu_i(M) \text{ for } i < r(M).$$

We will now define a function on the flats of a matroid that is related to the sequence μ_i.

Definition 8.6. Let M be a loopless matroid over $\{1, ..., n\}$. We define the function $D_k(M)$ to be the number of flags $\mathscr{F} = \{F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k\}$ satisfying three properties:

(1) F_i is a proper, non-empty subset of E for all i
(2) $\mathscr{F}$ is *initial*, or $r(F_i) = i$ for all i
(3) $\mathscr{F}$ is *descending*, or $\min(F_1) > \min(F_2) > \cdots > \min(F_k) > 1$, where $\min(F)$ is the smallest integer in the flat F.

Lemma 8.7. *For all k, $D_k(M) = \mu_k$.*

The proof of this done by Adiprasito, Huh, and Katz is too technical to explain in this paper, but boils down showing equality holds for the constant term using a combinatorial argument and then repeatedly truncating the matroid to prove this for each other μ_i. The full proof can be found in [1]. We will calculate and verify this fact for a few matroids.

Example 8.8. Let $M = U_{2,3}$. It has reduced characteristic polynomial $\overline{\chi_M}(X) = x - 2$. For $\mu_0 = 1$, we have the one empty flag. For $\mu_1 = 2$, we have three rank 1 flats, but $\min(\{1\}) \not> 1$, leaving the two single-flat flags $\{2\}$ and $\{3\}$.

Example 8.9. Let M be the simplification of the partition matroid $U_{2,3} \oplus U_{1,2}$ over $\{1, 2, 3, 4\}$ as described in Example 4.19. It has reduced characteristic polynomial $\overline{\chi_M}(X) = x^2 - 3x + 2$. For $\mu_0 = 1$, we have the one empty flag again. For $\mu_1 = 3$, we have the three single-flat flags $\{2\}$, $\{3\}$, and $\{4\}$. For $\mu_2 = 2$, there are nine flags satisfying the first two requirements, but the third requirement leaves only $\{\{4\} \subsetneq \{2, 4\}\}$ and $\{\{4\} \subsetneq \{3, 4\}\}$.

After all of this, we are ready to put the Chow ring to use. We will assume all matroids are now over the set $\{1, ..., n\}$ as this is just a relabeling of the elements in the base set.

Definition 8.10. For a matroid M over E. For some $i \in E$, we can define to elements in the chow ring $A^*(M)$ by their preimages before we quotient:

$$\sum_{i \in F} x_F \mapsto \alpha, \quad \sum_{i \notin F} x_F \mapsto \beta.$$

Note that in these summations, i is fixed and we are summing over all flats F that contain or do not contain i. Because of the choice of quotient relations, α and β do not depend on the choice of i.

Definition 8.11. If $\mathscr{F} = \{F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k\}$ is a flag, then the element in the Chow ring denoted $x_{\mathscr{F}}$ is given by $x_{\mathscr{F}} = x_{F_1} x_{F_2} \cdots x_{F_k}$.

Proposition 8.12. *Let M be a representable matroid with $r = r(M) - 1$. For all $k \leq r$, let $\mathcal{F}_k$ be the set of all descending flags of proper nonempty flats of length k. Then*

$$\beta^k = \sum_{\mathscr{F} \in \mathcal{F}_k} x_{\mathscr{F}}$$

Proof. We will prove this by induction. For $k = 1$, we observe that the definition of β is precisely the sum over all single flat descending flags if we let $i = 1$ because a single flat flag is descending if the flat does not contain 1.

Now assume the claim holds for a fixed $k < r$. For some descending flag $\mathscr{F} = \{F_1 \subsetneq \cdots \subsetneq F_k\}$, set $i = \min(F_1)$. Then we have

$$\beta x_{\mathscr{F}} = \sum_{i \notin F} x_F x_{\mathscr{F}}.$$

None of the F from the sum contain F_1 as a subset because $i \in F_1$, so the flats are only comparable, and thus $x_F x_{F_1}$ is nonzero, when $F \subset F_1$. Because $i \notin F$, $i = \min(F_1) < \min(F)$. This gives

$$\beta x_{\mathscr{F}} = \sum_{i \notin F} x_F x_{\mathscr{F}} = \sum_{\mathscr{F} \subset \mathscr{F}' \in \mathcal{F}_{k+1}} x_{\mathscr{F}'}.$$

From our assumption, we get

$$\beta^{k+1} = \sum_{\mathscr{F} \in \mathcal{F}_k} \beta x_{\mathscr{F}} = \sum_{\mathscr{F} \in \mathcal{F}_k} \left(\sum_{\mathscr{F} \subset \mathscr{F}' \in \mathcal{F}_{k+1}} x_{\mathscr{F}'} \right).$$

All descending flag of length $k + 1$ flats will be counted once in the double sum because removing the first flat will result in a unique descending flag of length k. Thus,

$$\beta^{k+1} = \sum_{\mathscr{F} \in \mathcal{F}_k} \left(\sum_{\mathscr{F} \subset \mathscr{F}' \in \mathcal{F}_{k+1}} x_{\mathscr{F}'} \right) = \sum_{\mathscr{F} \in \mathcal{F}_{k+1}} x_{\mathscr{F}}.$$

□

Proposition 8.13. *Let M be a representable matroid with $r = r(M) - 1$. Let $\mathscr{F} = \{F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k\}$ be a flag of length $k < r$.*

1. *If $\mathscr{F}$ is not initial, then $x_{\mathscr{F}} \alpha^{r-k} = 0$.*
2. *If $\mathscr{F}$ is initial, then $x_{\mathscr{F}} \alpha^{r-k} = \alpha^r$.*

Proof. (1) If $k = r - 1$, then the rank of F_k must be r or the flag would be initial. Take some $i \notin F_k$ and consider

$$x_{F_k}\alpha = \sum_{i \in F} x_{F_k} x_F.$$

The only flats containing F_k are itself and the maximal flat in the matroid, neither of which are possible values of F in the sum. Because $i \in F$, $F \not\subset F_k$. Thus, for all F, F and F_k are incomparable and $x_{F_k} x_F = 0$, so the sum is zero.

Now suppose the proposition hold for all $k' > k$. Choose $i \notin F_k$ and consider

$$x_{F_1} x_{F_2} \cdots x_{F_{k-1}} x_{F_k} \alpha^{r-k} = \sum_{i \in F} x_{F_1} x_{F_2} \cdots x_{F_{k-1}} x_{F_k} x_F \alpha^{r-k-1}.$$

Because $i \in F$, $F \not\subset F_k$, $x_{F_k} x_F$ is nonzero only when $F_k \subset F$. However, if this is true, then the flag $\{F_1 \subsetneq \cdots \subsetneq F_k \subsetneq F\}$ is a noninitial flag of length $k+1$, so by assumption

$$x_{F_1} x_{F_2} \cdots x_{F_{k-1}} x_{F_k} x_F \alpha^{r-k-1} = 0.$$

Thus, we have

$$x_{F_1} x_{F_2} \cdots x_{F_{k-1}} x_{F_k} \alpha^{r-k} = \sum_{i \in F} x_{F_1} x_{F_2} \cdots x_{F_{k-1}} x_{F_k} x_F \alpha^{r-k-1} = 0.$$

(2) If $k = 1$, pick any $i \in F_1$, and we have

$$\alpha^r = \sum_{i \in F} x_F \alpha^{r-1}.$$

Every flat that is not F_1 contains it so it will have rank greater than 1 and form a non-initial flag. The product of those flats with α^{r-1} is zero by the first part of this proposition. This leaves $\alpha^r = x_{F_1} \alpha^{r-1}$.

Now suppose the proposition hold for all $k' < k$. Pick $i \in F_k \setminus F_{k-1}$. We can write

$$\alpha^r = x_{F_1} x_{F_2} \cdots x_{F_{k-1}} \alpha^{r-k+1} = \sum_{i \in F} x_{F_1} x_{F_2} \cdots x_{F_{k-1}} x_F \alpha^{r-k}.$$

Because $i \notin F_{k-1}$, $x_{F_{k-1}} x_F$ is only nonzero if $F_{k-1} \subset F$. The only rank k flat containing both F_{k-1} and i is F_k because we could otherwise take an intersection to get a flat with a rank between $k-1$ an k, a contradiction. Thus, when $F \neq F_k$, the flag $\{F_1 \subsetneq \cdots \subsetneq F_{k-1} \subsetneq F\}$ is not initial and its product with α^{r-k} is zero. This gives

$$\alpha^r = \sum_{i \in F} x_{F_1} x_{F_2} \cdots x_{F_{k-1}} x_F \alpha^{r-k} = x_{F_1} x_{F_2} \cdots x_{F_{k-1}} x_{F_k} \alpha^{r-k}.$$ □

Proposition 8.14. *Let M be a representable matroid with $r = r(M) - 1$. Then* $\deg(\alpha^r) = 1$.

Proof. Choose any maximal flag of non-empty proper flats $\mathscr{F} = \{F_1 \subsetneq \cdots \subsetneq F_r\}$. Using Proposition 8.13, we have

$$\alpha^r = x_{F_1} \alpha^{r-1} = x_{F_1} x_{F_2} \alpha^{r-2} = x_{F_1} x_{F_2} \cdots x_{F_r} = x_{\mathscr{F}}.$$

From the definition of the deg map, we have $\deg(\alpha^r) = \deg(x_{\mathscr{F}}) = 1$. □

Lemma 8.15. *Let M be a representable matroid with $r = r(M) - 1$. For all $k \leq r(M)$, $\mu_k = \deg(\alpha^{r-k}\beta^k)$.*

Proof. Let $\mathcal{F}_k$ be the set of all descending flags of proper nonempty flats of length k and let $\mathcal{F}'_k \subset \mathcal{F}$ be the subset of $\mathcal{F}$ which are also initial flags. Using Propositions 8.12 and 8.13,

$$\alpha^{r-k}\beta^k = \alpha^{r-k} \sum_{\mathscr{F}, \in \mathcal{F}} F_{\mathscr{F}} = \sum_{\mathscr{F} \in \mathcal{F}'_k} F_{\mathscr{F}} \alpha^{r-k} = \sum_{\mathscr{F} \in \mathcal{F}'_k} \alpha^r.$$

Combining this with Proposition 8.14 and Lemma 8.7 gives

$$\deg(\alpha^{r-k}\beta^k) = \sum_{\mathscr{F} \in \mathcal{F}'_k} \deg(\alpha^r) = \sum_{\mathscr{F} \in \mathcal{F}'_k} 1 = |\mathcal{F}'_k| = D_k(M) = \mu_k.$$ □

Now that we have established such results about α and β, we will prove an inequality using the Chow ring that resembles the inequality of log concavity.

Lemma 8.16. *Let M be a representable matroid and let $r = r(M) - 1$. Let l_1 and l_2 be elements of $A^1(M)_{\boldsymbol{R}}$ with l_2 ample. We have*

$$\deg(l_1 l_1 l_2^{r-2})\deg(l_2 l_2 l_2^{r-2}) \leq \deg(l_1 l_2 l_2^{r-2})^2.$$

Proof. If $l_1 = cl_2$ for $c \in \mathbf{R}$, we get

$$\deg(l_1 l_1 l_2^{r-2})\deg(l_2 l_2 l_2^{r-2}) = \deg(c^2 l_2^r)\deg(l_2^2) = \deg(c l_2^r)^2 = \deg(l_1 l_2 l_2^{r-2})^2,$$

which shows the inequality is true.

Now suppose l_1 is not a multiple of l_2. Because l_2 is ample, we can define the bilinear form

$$Q^1_{l2} : A^1(M)_{\mathbf{R}} \times A^1(M)_{\mathbf{R}} \to \mathbf{R}, \quad (a_1, a_2) \mapsto -\deg(a_1 l_2^{r-2} a_2).$$

The Hodge-Riemann relations tell us that Q^1_{l2} is negative definite on the span of l_2 and positive definite on its orthogonal compliment. Because l_1 is not contained in the span of l_2, the restriction of Q^1_{l2} to their span will be an indeterminate quadratic form. We can choose an basis e_1 and e_2 of the span of l_1 and l_2 that satisfy the relations $Q^1_{l2}(e_1, e_1) = -1$, $Q^1_{l2}(e_2, e_2) = 1$, and $Q^1_{l2}(e_1, e_2) = Q^1_{l2}(e_2, e_1) = 0$. In this case, we can write $l_1 = ae_1 + be_2$ and $l_2 = ce_1$. We have

$$\deg(l_1 l_1 l_2^{r-2})\deg(l_2 l_2 l_2^{r-2}) = Q^1_{l2}(l_1, l_1) Q^1_{l2}(l_2, l_2)$$
$$= Q^1_{l2}(ce_1, ce_1) Q^1_{l2}(ae_1 + be_2, ae_1 + be_2) = -c^2(-a^2 - b^2),$$
$$\deg(l_1 l_2 l_2^{r-2})^2 = Q^1_{l2}(l_1, l_2)^2 = Q^1_{l2}(ae_1 + be_2, ce_1)^2 = a^2 c^2.$$

Combining these gives

$$\deg(l_1 l_1 l_2^{r-2})\deg(l_2 l_2 l_2^{r-2}) = c^2 a^2 - c^2 b^2 \leq c^2 a^2 = \deg(l_1 l_2 l_2^{r-2})^2,$$

which is what we wanted to show. □

Proposition 8.17. *Let M be a representable matroid and let $r = r(M) - 1$. Let l_1 be an element of $A^1(M)_{\boldsymbol{R}}$. We have*

$$\deg(l_1 l_1 \beta^{r-2})\deg(\beta\beta\beta^{r-2}) \leq \deg(l_1 \beta\beta^{r-2})^2.$$

Proof. This proof would be easy if β was ample, but unfortunately it is not. However, in [1], it is proven that for some ample $l \in A^1(M)_{\mathbf{R}}$, $\beta + tl$ is ample for all real $t > 0$. We can put this into the inequality in Lemma 8.16 with $l_2 = \beta + tl$ and take the limit as $t \to 0$ to see the inequality holds as desired. □

Lemma 8.18. *Let M be a representable matroid. For all $0 < k < r(M) - 1$,*

$$\mu_{k-1}\mu_{k+1} \leq \mu_k^2$$

Proof. We will prove this by induction on $r(M)$. If $r(M) = 2$, the claim is vacuously true.

Now suppose the claim is true for all matroids of rank r. For a matroid M of rank $r + 1$, we can consider the rank r matroid $\mathrm{tr}(M)$ to see the claim is true for $0 < k < r - 1$, so we need only show this for $k = r$. Using Proposition 8.17 with $\alpha = l_1$, we get the inequality

$$\deg(\alpha^2\beta^{r-2})\deg(\beta^2\beta^{r-2}) \leq \deg(\alpha^2\beta^{r-2})^2.$$

Using Lemma 8.15, we get

$$\mu_{r-1}\mu_{r+1} \leq \mu_r^2.$$

This completes induction. □

Theorem 8.19. *If M is a representable matroid, then the coefficients of the characteristic polynomial form a log concave sequence.*

Proof. This is a direct consequence of Lemma 8.18 and Proposition 8.4. □

As we saw before, this barely makes a dent in the true Rota-Welsh conjecture because representable matroids are a very special case of matroids. However, to eventually prove it for all matroids, Adiprasito, Huh, and Katz were able to show the content of Theorem 7.8 applies to the chow ring of all matroids in [1]. Nothing else we proved in this section requires the matroid to be representable, so proving the theorem for a general matroid is the same as we have done in this section. The simple representable case gives the motivation for such a technique that is not able to rely on the direct connection to intersection theory.

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