Stochastic Calculus and Volatility Models

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Abstract

Stochastic processes are essential for asset pricing. First, we introduce the theory of stochastic processes and stochastic calculus so that we can discuss the Black-Scholes model. As an application of this theory, we use Ito's lemma to derive the Black-Scholes equations. Finally, we examine the limitations of the Black-Scholes Model and introduce a class of extensions to this model, stochastic volatility models, that improve the Black-Scholes Model.

Contents

1 Introduction

Financial derivatives are instruments whose value depends on, or derives from, the value of other, more basic, underlying securities. For example, forwards and futures allow investors to purchase or sell a basic stock at a fixed price to avoid risk. Options give investors the opportunity to buy or sell a security at a fixed price in the future. All of these examples of derivatives provide some kind of protection against future price movements; they protect the investors from uncertainty and risk. Because of the insurance effect of derivatives, it is essential to price derivatives properly, relative to the level of insurance they provide to the investors, so that people have the incentive to trade derivatives.

If we assume that the returns yielded by a stock follow a stochastic process (more precisely, a Brownian motion), then the stock price can be modeled by the following process:

$$\frac{dS}{S} = \mu dt + \sigma dz,$$

where z is a standard Brownian motion, μ is the expected return, and σ is the volatility.

Intuitively, the derivative of a security will depend on the price of the underlying security. Denote the price of an option to be c. Then c is a function of S, specifically,

$$dc = f(dS) = f(\mu S dt + \sigma S dz).$$

Therefore, the price of a derivative depends on a stochastic process z. In non-stochastic calculus, if we can determine the relationship between the price of the security and the price of the derivative, we would be able to solve this equation and obtain the appropriate price. However, we cannot solve the equation using Riemann or Lebesgue integration because a stochastic process is involved. We will alleviate this issue by introducing Ito's process and Ito's lemma, essential theorems of stochastic calculus. They will allow us to calculate integrals involving stochastic processes.

Given the movement of stock prices ds, we can already see that derivatives depend on the expected return and the volatility of the underlying stock. Besides the price of the underlying stock, the price of a derivative should also depend on time to expiration, since less time means less opportunity for price movements. Since the return of a riskless security should equal the risk-free rate, the price of a derivative will also depend on the risk-free rate. If there were riskless derivatives with higher yield, it would result in people buying the higher-yield security and selling the lowing-yield security to make riskless profit, which is called arbitrage. Given all these factors, we will introduce the Black-Scholes differential equation, which will provide the relationship between any security and its derivatives. We will obtain a special solution to this differential equation specifically in the context of options pricing.

The Black-Scholes formula for options pricing requires a lot of assumptions, and many of them usually do not hold in the real world. For example, it assumes the underlying stock has constant volatility throughout the lifetime of the option. Given the real-life option prices of a particular underlying security, we can inversely obtain some value of volatility called implied volatility. Comparing volatility of options with different strike price or time to expiration, we can see that volatility is barely ever constant. Below is a plot of the implied volatility of options with the same time to expiration as a function of its strike price. The result is known as the **volatility smile**.

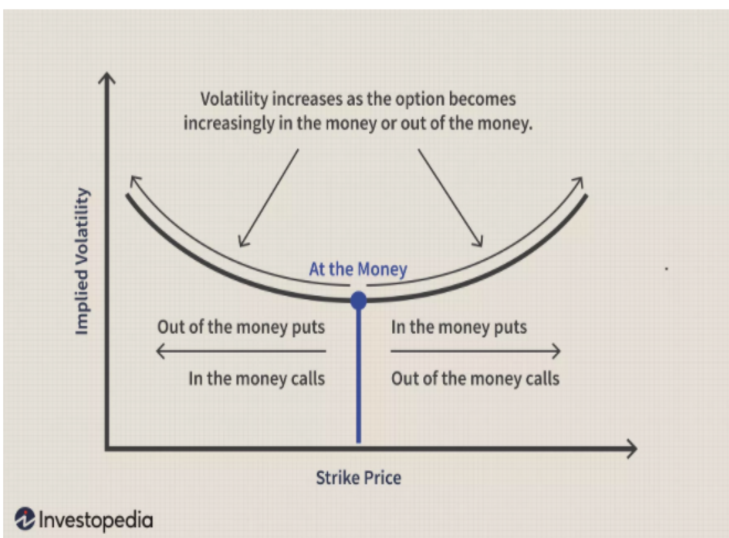


Image by Julie Bang © Investopedia 2019

*https://www.investopedia.com/terms/v/volatilitysmile.asp

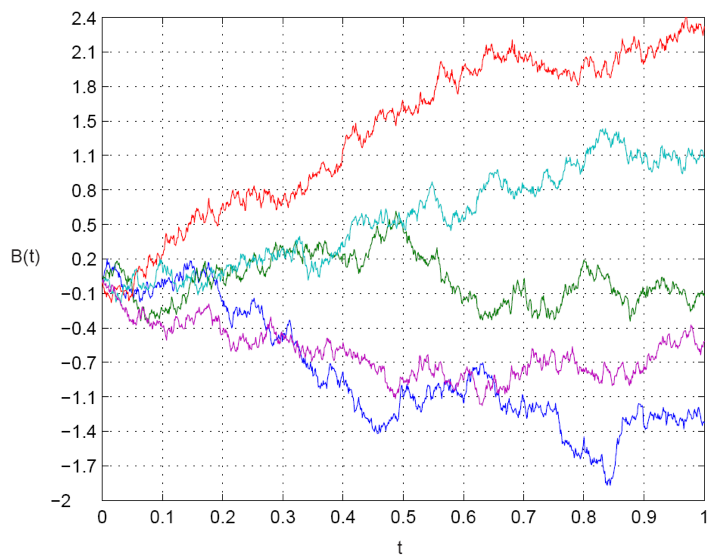
We will conclude by introducing other volatility models that address this issue of constant volatility and improve the performance of the Black-Scholes model.

2 Stochastic Process

Definition 2.1. A **Stochastic Process** is a collection of random variables X_t indexed by time.

Definition 2.2. A **Brownian Motion (Wiener Process)** with variance σ^2 is a stochastic process X_t taking values in the real numbers satisfying:
(i) $X_0 = 0$;
(ii) For any $a < b \le c < d$, $X_b - X_a$ and $X_d - X_c$ are independent;
(iii) For any $a < b$, $X_b - X_a$ follows a Normal Distribution with mean 0 and variance $(b-a)\sigma^2$;
(iv) The paths are continuous.

For example, the graph below gives some examples of Brownian motions. A typical stock return might follow a path like these.



*https://tex.stackexchange.com/questions/59926/how-to-draw-brownian-motions-in-tikz-pgf

Remark 2.3. A Brownian Motion with variance 1 is called a **Standard Brownian Motion**

Definition 2.4. A random variable X following a Brownian Motion with mean μ and variation σ^2 satisfies $dX_t = \mu dt + \sigma dB_t$

Definition 2.5. An **Ito process** is a generalized Wiener process in which the parameters μ and σ are functions of the values of the underlying variable x and time t. An Ito process can therefore be written as

$$dx = a(x,t)dt + b(x,t)dz$$

3 Stochastic Calculus and Ito's formula

Next, we will show how to solve differential equations involving stochastic processes. In particular, we will use the next two theorems called Ito's Lemma.

First, we will look at functions that only depend on one variable which is a Brownian motion.

Theorem 3.1. *Suppose f is a C^2 function and B_t is a standard Brownian motion. Then, for every t,*

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)dB_t + \frac{1}{2}\int_0^t f''(B_s)ds$$

or,

$$df(B_t) = f'(B_s)dB_t + \frac{1}{2}f''(B_s)ds$$

Proof. First, we want to show that for $t = 1$,

$$f(B_1) = f(B_0) + \int_0^1 f'(B_s)dB_t + \frac{1}{2}\int_0^1 f''(B_s)ds$$

For any $n \in \mathbb{N}$,

$$f(B_1) - f(B_0) = \sum_{j=1}^{n}[f(B_{j/n}) - f(B_{(j-1)/n})]$$

Using Taylor's Approximation, we get

$$f(B_{j/n}) - f(B_{(j-1)/n}) = f'(B_{(j-1)/n})\Delta_{j,n} + \frac{1}{2}f''B_{(j-1)/n})\Delta_{j,n}^2 + o(\Delta_{j,n}^2)$$

where $\Delta_{j,n} = B_{j/n} - B_{(j-1)/n}$

Therefore,

$$f(B_1) - f(B_0) = \lim_{n\to\infty}\sum_{j=1}^{n} f'(B_{(j-1)/n})[f(B_{j/n}) - f(B_{(j-1)/n})] \tag{3.0.1}$$

$$+\frac{1}{2}\lim_{n\to\infty}\sum_{j=1}^{n} f''(B_{(j-1)/n})[f(B_{j/n}) - f(B_{(j-1)/n})]^2 \tag{3.0.2}$$

$$+\lim_{n\to\infty}\sum_{j=1}^{n} o(f(B_{j/n}) - f(B_{(j-1)/n})]^2) \tag{3.0.3}$$

For the expression in (3.0.1), we can see that

$$\lim_{n\to\infty}\sum_{j=1}^{n} f'(B_{(j-1)/n})[f(B_{j/n}) - f(B_{(j-1)/n})] = \int_0^1 f'(b_t)dB_t$$

For the expression in (3.0.2), since $f \in C^2$, we can define h by continuous function.

Thus, for every $\epsilon > 0$, there exists a step function $h_\epsilon(t)$ such that $|h(t) - h_\epsilon(t)| < \epsilon$ for every t. Fix $\epsilon > 0$, we get

$$\lim_{n\to\infty} \sum_{j=1}^{n} h_\epsilon(t)[f(B_{j/n}) - f(B_{(j-1)/n})]^2 = \int_0^1 h_\epsilon(t)dt$$

$$|\sum_{j=1}^{n}[h(t) - h_\epsilon(t)][B_{j/n} - B_{(j-1)/n}]^2| \leq \epsilon \sum_{j=1}^{n}[B_{j/n} - B_{(j-1)/n}]^2 \to \epsilon$$

Therefore,

$$\lim_{\epsilon\to 0} \frac{1}{2}\int_0^1 h_\epsilon(t)dt = \frac{1}{2}\int_0^1 h(t)dt = \frac{1}{2}\int_0^1 f''(B_t)dt$$

For any other t, because of the Markov Property of Brownian Motion, partitioning the interval $[0, t]$ as we did for $[0, 1]$, the proof still holds.

Therefore, for every t,

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)dB_t + \frac{1}{2}\int_0^t f''(B_s)ds$$

□

Now, we will look at functions that depend on both time and a Brownian motion variable. This is closer to a stock return with a constant expected return and some variance.

Theorem 3.2. *Let $f(t, B_t)$ be a function that is C^1 in t and C^2 in B_t. For simplicity, we write $x = B_t$. If B_t is a standard Brownian motion, then,*

$$f(t, B_t) = f(0, B_0) + \int_0^t \partial_x f(s, B_s)dB_s + \int_0^t [\partial_s f(s, B_s) + \frac{1}{2}\partial_{xx} f(s, B_s)]ds$$

or, in another form,

$$df(t, B_t) = \partial_x f(t, B_t)dB_t + [\partial_t f(t, B_t) + \frac{1}{2}\partial_{xx} f(t, B_t)]dt$$

Proof. By Taylor's approximation,

$$df(t, x_t) = \partial_x f(t, x)\Delta_t + o(\Delta_t) + \partial_x f(t, x)\Delta_x + \frac{1}{2}\partial_{xx} f(t, x)(\Delta_x)^2 + o((\Delta_x)^2)$$

For simplicity, we consider $t = 1$ and $\Delta_t = \frac{1}{n}$. Then, similar to the previous theorem, we get

$$f(1, B_1) - f(0, B_0) = \lim_{n\to\infty} \sum_{j=1}^{n} \partial_x f((j-1)/n, B_{(j-1)/n})[f((j-1)/n, B_{j/n}) - f((j-1)/n, B_{(j-1)/n})] \tag{3.0.4}$$

$$+\frac{1}{2}\lim_{n\to\infty} \sum_{j=1}^{n} \partial_{xx} f((j-1)/n, B_{(j-1)/n})[f((j-1)/n, B_{j/n}) - f((j-1)/n, B_{(j-1)/n})]^2 \tag{3.0.5}$$

$$+\frac{1}{2}\lim_{n\to\infty}\sum_{j=1}^{n}\partial_t f((j-1)/n, B_{(j-1)/n})(1/n) \tag{3.0.6}$$

$$+\lim_{n\to\infty}\sum_{j=1}^{n} o(1/n) \tag{3.0.7}$$

$$+\lim_{n\to\infty}\sum_{j=1}^{n} o((B_{j/n}) - B_{(j-1)/n})^2) \tag{3.0.8}$$

Similar to the proof in Theorem 3.1, 3.0.4 converges to $\int_0^1 \partial_x f(s, B_s)dB_s$, 3.0.5 converges to $\frac{1}{2}\int_0^1 \partial_{xx} f(s, B_s)ds$, 3.0.6 converges to $\int_0^t \partial_s f(s, B_s)ds$, while the other two terms converge to 0.

The proof can also be generalized to any $t > 0$. □

Now we will generalize Ito's lemma to any function dependent on time and an Ito process. This differential equation closely describes movements in derivative prices, which depend on time and the underlying stock price.

Corollary 3.3. *Let $f(t, X_t)$ be an Ito Process which satisfies*

$$dX_t = Z_t dt + y_t dB_t$$

if B_t is a standard Brownian motion and f is a C^2 function, then $f(t, X_t)$ is also an Ito Process satisfying

$$df(t, X_t) = \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial X_t} Z_t + \frac{1}{2}\frac{\partial^2 f}{\partial (X_t)^2} y_t^2\right] dt + \frac{\partial f}{\partial X_t} y_t dB_t$$

Proof. By the previous theorem, we have

$$df(t, X_t) = \partial_X f(t, X_t)dX_t + [\partial_t f(t, X_t) + \frac{1}{2}\partial_{XX} f(t, X_t)y_t^2]dt$$

$$= \partial_X f(t, X_t)(Z_t dt + y_t dB_t) + [\partial_t f(t, X_t) + \frac{1}{2}\partial_{XX} f(t, X_t)y_t^2]dt,$$

which gives us

$$df(t, X_t) = \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial X_t} Z_t + \frac{1}{2}\frac{\partial^2 f}{\partial (X_t)^2} y_t^2\right] dt + \frac{\partial f}{\partial X_t} y_t dB_t$$

□

4 Financial Derivatives and Black-Scholes Formula

A **derivative** can be defined as a financial instrument whose value depends on (or derives from) the values of other, more basic, underlying variables.

A **call option** gives the holder of the option the right to buy an asset by a certain date for a certain price. A **put option** gives the holder of the option the right to sell an asset by a certain date for a certain price. The date specified in the contract is known as the **expiration date** or **maturity date**. The price specified in the contract is known as the **exercise price** or the **strike**

price. The two mostly common kinds of options are American options and European options. **American options** can be exercised at any time up to the expiration date, whereas **European options** can be exercised only on the expiration date. The Black-Scholes formula will consider non-dividend-paying European options for simplicity.

Theorem 4.1. *Let r be the risk-free interest rate, S be a stock price following the stochastic process $dS = \mu S dt + \sigma S dz$, f be the price of a call option or other derivative contingent on S. Then*

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \tag{4.0.1}$$

(This equation is called the Black-Scholes-Merton differential equation.)

Proof. Because f is a function of t and S, which follows a Ito's process described by

$$dS = \mu S dt + \sigma S dz, \tag{4.0.2}$$

where μ and σ are constants, by Ito's lemma, we get

$$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right) dt + \frac{\partial f}{\partial S}\sigma S dz \tag{4.0.3}$$

Now, we construct a portfolio of the underlying stock and the derivative with -1 position in derivative and $\frac{\partial f}{\partial S}$ position in stock. Define Π as the value of the portfolio. By definition,

$$\Pi = -f + \frac{\partial f}{\partial S}S$$

Therefore,

$$d\Pi = -df + \frac{\partial f}{\partial S}dS \tag{4.0.4}$$

Substituting 4.0.2 and 4.0.3 into 4.0.4, we get

$$d\Pi = (-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2)dt \tag{4.0.5}$$

Because this equation does not involve Δz, the portfolio must be riskless during time Δt. Therefore, it must be earning the same rate as any other riskless securities by the no-arbitrage principle. Therefore,

$$d\Pi = r\Pi dt$$

Combining 4.0.4 and 4.0.5, we get

$$(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2)dt = r(\frac{\partial f}{\partial S}S - f)dt$$

which leads to

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

□

Example 4.2. *A forward contract on a non-dividend-paying stock is a derivative dependent on the stock. It should satisfy (4.0.1). By the non-arbitrage argument, we know that the value of a forward contract, f, at time t, given the stock price S at that time is*

$$f = S - Ke^{-r(T-t)}$$

where K is the delivery price. This means that

$$\frac{\partial f}{\partial t} = -rKe^{-r(T-t)}, \frac{\partial f}{\partial S} = 1, \frac{\partial^2 f}{\partial S^2} = 0$$

Substituting into (4.0.1), we get

$$-rKe^{-r(T-t)} + rS = rf$$

We can see that the Black-Scholes-Merton differential equation holds.

Theorem 4.3. *With current price as S_0, strike price K, risk-free interest rate r, time to expiration T, volatility of the stock σ, if the option satisfies the following assumptions:*

1. *The short-term interest is constant through the time period of the option.*
2. *The stock price S follows a Wiener process with $dS = \mu S dt + \sigma S dz$.*
3. *The stock pays no dividends*
4. *It is a European stock.*
5. *There is no transaction cost.*
6. *Borrowing at the current interest rate for any amount is possible.*
7. *Short selling is allowed at no additional cost.*

then the prices of European call and put options can be calculated by the Black-Scholes-Merton options pricing formulas described by:

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

and

$$p = Ke^{-rT} N(-d_2) - S_0 N(-d_1)$$

where

$$d_1 = \frac{ln(S_0/k) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

and

$$d_2 = \frac{ln(S_0/k) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

Proof. We will prove the derivation of the call options pricing formula as an example.

With the stock price at expiration as S_T, the expected value of the option at maturity in a risk-neutral world is

$$E[max(S_T - K, 0)]$$

Now we write V as S_T for simplicity, define $g(V)$ as the probability density function of V. It follows that

$$E[max(V - K, 0)] = \int_k^\infty (V - K)g(V)dV \tag{4.0.6}$$

Since lnV is normally distributed with standard deviation σ, the mean of lnV is m (by properties of Log-normal distribution) where

$$m = ln[E(V)] - \sigma^2/2$$

Define

$$Q = \frac{lnV - m}{\sigma},$$

then Q follows standard normal distribution. Denote the density function of Q by $h(Q)$ so that

$$h(Q) = \frac{1}{\sqrt{2\pi}} e^{-Q^2/2} \tag{4.0.7}$$

Therefore, the right-hand side of (4.0.6) becomes

$$E[max(V - K, 0)] = \int_{(lnK-m)/\sigma)}^\infty (e^{Q\sigma+m} - K)h(Q)dQ \tag{4.0.8}$$

$$\begin{aligned}
(e^{Q\sigma+m} - K)h(Q) &= \frac{1}{\sqrt{2\pi}} e^{(-Q^2+2Q\sigma+2m)/2} \\
&= \frac{1}{\sqrt{2\pi}} e^{[-(q-\sigma)^2+2m+\sigma^2]/2} \\
&= \frac{e^{m+\sigma^2/2}}{\sqrt{2\pi}} e^{[-(q-\sigma)^2]/2} \\
&= \frac{e^{m+\sigma^2/2}}{\sqrt{2\pi}} h(Q - \sigma)
\end{aligned}$$

Therefore, (4.0.8) becomes

$$E[max(V - K, 0)] = e^{m+\sigma^2/2} \int_{(lnK-m)/\sigma)}^\infty h(Q - \sigma)dQ - k \int_{(lnK-m)/\sigma)}^\infty h(Q)dQ \tag{4.0.9}$$

Define N(x) as the probability that a standard normal variable is less than x, then

$$\int_{(lnK-m)/\sigma)}^\infty h(Q - \sigma)dQ = 1 - N[(lnK - m)/\sigma - \sigma] = N[(-lnK + m)/\sigma + \sigma)]$$

Substituting m into the equation leads to

$$N(\frac{ln[E(V)/K] + \sigma^2/2}{\sigma}) = N(d_1)$$

Similarly, the second integral in (4.0.9) is $N(d_2)$. Therefore, (4.0.9) becomes

$$E[max(V-K,0)] = e^{m+\sigma^2/2}N(d_1) - KN(d_2)$$

Substituting m into the equation yields

$$c = S_0N(d_1) - Ke^{-rT}N(d_2)$$

and

$$p = Ke^{-rT}N(-d_2) - S_0N(-d_1)$$

□

Example 4.4. *Consider a European non-dividend-paying call option on a stock with current price 50 dollars and strike price 45. The risk-free rate is* 5%, *time to expiration is 1 year. The volatility of the stock is* 20%.

We get

$$d_1 = \frac{ln(50/45) + (0.05 + 0.04/2) * 1}{0.2 * 1} = 0.88$$

and

$$d2 = d1 - 0.2 * 1 = 0.68$$

Therefore,

$$c = 50 * N(0.88) - 45 * e^{-0.05*1}N(0.68) = 40.5 - 32.1 = 8.4,$$

the price of the option is about 8.4 now.

5 Non-Constant and Stochastic Volatility Models

A key assumption for the Black-Scholes pricing formula is that the underlying stock price has a known and constant volatility. However, in most cases, true volatility of a stock price remains unknown and non-constant over time. For example, the volatility of a stock price is likely to increase before the release of earnings report or major news. Historical average may not be representative of the future stock volatility. Therefore, we introduce here other models that capture the characteristics of stock volatility better.

Definition 5.1. The **ARCH(m) model** estimates the volatility based on a long-run average V_L and m recent observations. The older the observation, the less weight it is given. The volatility takes the form

$$\sigma_n^2 = \gamma V_L + \sum_{i=1}^{m} \alpha_i u_{n-1}^2$$

where $\gamma + \sum_{i=1}^{m} \alpha_i = 1$ and $u_i = ln\frac{S_i}{S_{i-1}}$, which is return.

Definition 5.2. The **exponentially weighted moving average(EWMA) model** is a particular case of the basic weighted model

$$\sigma_n^2 = \sum_{i=1}^{m} \alpha_i u_{n-1}^2$$

with $\sum_{i=1}^{m} \alpha_i = 1$, where $\alpha_{i+1} = \lambda\alpha_i$. In other word, the weight for each observation decreases exponentially over time. It can be expressed as

$$\sigma_n^2 = \lambda\sigma_{n-1}^2 + (1-\lambda)u_{n-1}^2$$

One advantage of this model is that only the volatility after the previous observation and the newest observation need to be stored. This increases program efficiency, particularly when dealing with large data sets.

Definition 5.3. The **GARCH(1, 1) model** calculates volatility based on a long-run average variance rate V_L, like in the ARCH model, as well as from σ_{n-1} and μ_{n-1}. Specifically, it is defined as

$$\sigma_n^2 = \gamma V_L + \alpha u_{n-1}^2 + \beta\sigma_{n-1}^2$$

with $\gamma + \alpha + \beta = 1$.

The GARCH(1, 1) model recognizes that over time, the variance tends to get pulled back to a long-run average level V_L. The model is equivalent to a model where the variance V follows the stochastic process

$$dV = \gamma(V_L - V)dt + \lambda V dz$$

where time is measured in days and $\lambda = \alpha\sqrt{2}$

Definition 5.4. The **SABR model** that describes the volatility for valuing European options maturing at time T is

$$dF = \sigma F^\beta dz$$
$$\frac{d\sigma}{\sigma} - vdw$$

where F is the forward interest rate or the forward price of an asset, dz and dw are Wiener processes, σ is the stochastic volatility, β and v are constants.

Definition 5.5. The **Heston Model** suggests that the price of an asset follows the stochastic process

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_t^S$$

where V_t, the instantaneous variance, follows

$$dV_t = \kappa(V_L - V_t)dt + \lambda\sqrt{V_t} dW_t^V$$

Similar to the stochastic process defined in equation 5.0.4, it's also a mean-reversion process.

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