THE FUNDAMENTAL THEOREM OF ALGEBRA OVER THE TROPICAL SEMIRING

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Abstract. Tropical geometry, or min-plus geometry, is a relatively young field of mathematics that resembles a bridge between algebraic geometry and combinatorics. By replacing the standard operations of addition and multiplication with the minimum function and addition, respectively, we find that complicated equations become surprisingly manageable and results of standard algebraic geometry can be proven more simply by combinatorial arguments. In this paper, we will begin by introducing the tropical semiring and go on to prove the Fundamental Theorem of Algebra over the tropical semiring.

Contents

1. Introduction

Tropical geometry was founded by Brazilian mathematician and computer scientist Imre Simon. The name "tropical" was given by French mathematicians in honor of Simon's work in the application of min-plus algebra to optimization theory [1].

The motivation for working with the tropical semiring is in both its simplicity and its analogy with algebraic geometry. We find that objects in classical mathematics, such as polynomials, can be degenerated in a well-defined way to get an analogous object in this tropical semiring. Due to the simplicity of working with min-plus algebra, which we will see in the next section, we are able to approach difficult problems from an easier angle to find a result that often resembles that of classical algebra [3]. In addition, there are many applications of tropical geometry to problems of optimization, such as optimizing train departure times or finding the shortest path in a weighted directed graph [1].

In this paper, we will focus strictly on tropical polynomials in one variable and demonstrate the simplicity and analogy of tropical geometry by proving the Fundamental Theorem of Algebra.

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2. The Tropical Semiring

We will begin by defining a semiring so we may better understand the tropical semiring.

Definition 2.1. A *semiring*[1] is a set R along with two operations addition $(+)$ and multiplication $(\cdot)$, which satisfy the following axioms:

(1) Addition is associative and commutative.
(2) Multiplication is associative.
(3) There exists an additive identity, which we usually call 0. Additionally, 0 annihilates R–for all $x \in R$, $0 \cdot x = x \cdot 0 = 0$.
(4) There exists a multiplicative identity, which we usually call 1.
(5) Multiplication left and right distributes over addition.

Definition 2.2. The *tropical semiring* is the set of the real numbers with the additional term of infinity, written $\mathbb{R}\cup\{\infty\}$ (or just $\overline{\mathbb{R}}$), together with the operations $\oplus$ and $\odot$, where, for all $a, b \in \overline{\mathbb{R}}$:

$$a \oplus b \coloneqq \min(a, b) \quad \text{and} \quad a \odot b \coloneqq a + b.$$

For the remainder of this paper will refer to $(\overline{\mathbb{R}}, \oplus, \odot)$ as $\mathbb{T}$.

We can quickly verify that $\mathbb{T}$ satisfies the axioms of a semiring from Definition 2.1 and see why the set is not a ring. A point of interest is that $\mathbb{T}$ actually satisfies one more axiom than is required for it to be a semiring, namely the commutativity of multiplication. Consequently, $\mathbb{T}$ satisfies every axiom of a field except for the existence of an additive inverse.

In this paper we use the min-plus convention. The max-plus convention is sometimes used, which replaces ∞ with $-\infty$ and $\oplus$ is defined as being the maximum function instead of the minimum. These two conventions are isomorphic, but min-plus algebra is more readily applicable to algebraic geometry while max-plus algebra is easier to draw dualities from [2].

Before we begin proving the Fundamental Theorem of Algebra, we will first become better acquainted with the semiring. If we are to extend the semiring and create an analogue to polynomials, we must first look at exponents. We find quite naturally that, for all $a \in \mathbb{T}$, $n \in \mathbb{N}$,

$$a^{\odot n} \coloneqq \overbrace{a + a + \cdots + a}^{n \text{ times}} = na. \tag{2.3}$$

This definition can then be extended to all $n \in \mathbb{R}$, but for our purposes we usually restrict n to integer values.

Now we find another interesting result that illustrates the simplicity of tropical algebra, referred to in [1] as the "Freshman's Dream."

Proposition 2.4 (Freshman's Dream)**.** *Let $a, b \in \mathbb{T}$ and $n \in \mathbb{R}^+ \cup \{0\}$. Then,*

$$(a \oplus b)^{\odot n} = a^{\odot n} \oplus b^{\odot n}.$$

Proof. We see

$$\begin{aligned} (a \oplus b)^{\odot n} &\coloneqq n \cdot \min(a, b) \\ &= \min(na, nb) \\ &\eqqcolon a^{\odot n} \oplus b^{\odot n}. \end{aligned}$$

[1] Semirings are sometimes referred to as "rigs", as they are rings without the "n" for negatives.

□

In Lemma 2.4, the nonnegativity of n is crucial. This is because $-\min(a, b) \neq \min(-a, -b)$, and in fact $-\min(-a, -b) = \max(a, b)$.

Having defined tropical exponents, we may begin defining and working with tropical polynomials.

Definition 2.5. Let $n \in \mathbb{N}$ and $d \in \mathbb{N}_0$. Let $\mathcal{I} = \{(i_1, \dots, i_n) \in \mathbb{N}_0^n \mid i_1 + \dots + i_n \leq d\}$. A *tropical polynomial* of degree d is a function $P \colon \mathbb{T}^n \to \mathbb{T}$ such that, for $\mathbf{x} \in \mathbb{T}^n$,

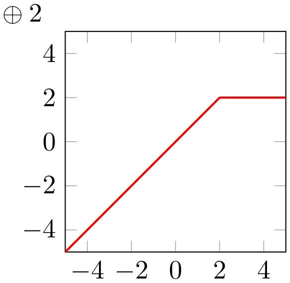
$$P(\mathbf{x}) = \bigoplus_{\mathbf{i} \in \mathcal{I}} a_{\mathbf{i}} \odot x_1^{\odot i_1} \odot \dots \odot x_n^{\odot i_n},$$

where all $a_{\mathbf{i}} \in \mathbb{T}$ and, for some $\mathbf{i}$ satisfying $i_1 + \dots + \imath_n = d$, $a_{\mathbf{i}} \neq \infty$. Note that if any $a_{\mathbf{i}} = \infty$, the term $a_{\mathbf{i}} \odot x_1^{\odot i_1} \odot \dots \odot x_n^{\odot i_n}$ is negligible.

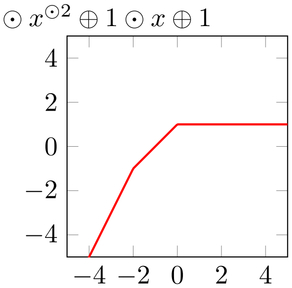
Although we have defined tropical polynomials for n dimensions, we may simplify for the sake of illustration to one variable without losing generality. Let us look at some examples of tropical polynomials in one dimension, along with their graphs.

Examples 2.6. The following are tropical polynomials in one variable:

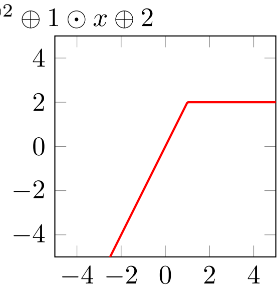
(a) $x \oplus 2$



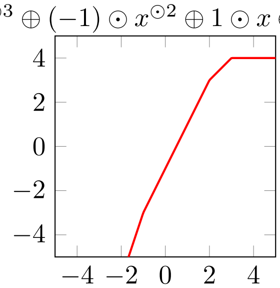
(b) $3 \odot x^{\odot 2} \oplus 1 \odot x \oplus 1$



(c) $x^{\odot 2} \oplus 1 \odot x \oplus 2$



(d) $x^{\odot 3} \oplus (-1) \odot x^{\odot 2} \oplus 1 \odot x \oplus 4$.



Note that in Example 2.6c,

$$x^{\odot 2} \oplus (1 \odot x) \oplus 2 = x^{\odot 2} \oplus 2 = (x \oplus 1)^{\odot 2}.$$

In fact, we find that if we replace the second coefficient, 1, with any number greater the resulting function would be equivalent. Later on in this paper when we prove the *Fundamental Theorem of Algebra*, we will work with this phenomenon which is not present in classical polynomials; polynomials with different coefficients for some terms can still be equivalent as functions. Because of this, we must be very careful how we describe tropical polynomials.

We also see that tropical polynomials appear to be piecewise linear functions. From [1], we have the following lemma.

Lemma 2.7. *Let P be a tropical polynomial in n variables. Then, P is continuous, piecewise linear with finite pieces, and concave.*

We would also like to define some form of tropical root, and we see there exists a well-defined tropical definition analogous to classical geometry. We define $\lambda \in \mathbb{T}$ to be a *tropical root* of P if there exists a tropical polynomial Q such that, for all $x \in \mathbb{T}$,

$$P(x) = (x \oplus \lambda) \odot Q(x).$$

The roots of a classical polynomial C may also equivalently be defined as the values of x such that $C(x)$ is equal to the additive identity, but these definitions are not equivalent for tropical roots. We choose to use Definition 3.11 because it produces the most interesting results.

Having defined tropical polynomials and roots, we may now prove the *Fundamental Theorem of Algebra* in tropical geometry.

3. The Fundamental Theorem of Algebra over the Tropical Semiring

A somewhat surprising result we can prove in tropical geometry is the *Fundamental Theorem of Algebra* for one dimension. Essentially, we would like to find a well-defined algorithm such that, for any tropical polynomial P of degree d, there exist unique $\lambda_1, \ldots, \lambda_n \in \mathbb{T}$, $m_1, \ldots, m_n \in \mathbb{N}$, and $\alpha \in \mathbb{T}$ such that, for all $x \in \mathbb{T}$,

$$P(x) = \alpha \odot \bigodot_{1 \leq i \leq n} (x \oplus \lambda_i)^{\odot m_i}.$$

Let us look at the tropical polynomials from Examples 2.6. We see

(a) $x \oplus 2$ is already in the correct form,
(b) $3 \odot x^{\odot 2} \oplus 1 \odot x \oplus 1 = 3 \odot (x \oplus -2) \odot (x \oplus 0)$,
(c) $x^{\odot 2} \oplus 1 \odot x \oplus 2 = (x \oplus 1)^{\odot 2}$, and
(d) $x^{\odot 3} \oplus (-1) \odot x^{\odot 2} \oplus 1 \odot x \oplus 4 = (x \oplus -1) \odot (x \oplus 2) \odot (x \oplus 3)$.

More generally, we may look at all tropical polynomials of degree 2.

Example 3.1. Consider the tropical polynomial $P(x) = c_2 \odot x^{\odot 2} \oplus c_1 \odot x \oplus c_0$, where $c_2 \neq \infty$.

(i) If $c_1 - c_2 \leq c_0 - c_1$,

$$P(x) = c_2 \odot (x \oplus (c_1 - c_2)) \odot (x \oplus (c_0 - c_1)).$$

(ii) If $c_1 - c_2 \geq c_0 - c_1$,

$$P(x) = c_2 \odot \left(x \oplus \frac{c_0 - c_2}{2}\right)^{\odot 2}.$$

Proof. Consider the case that $c_1 - c_2 \leq c_0 - c_1$. We see

$$\begin{aligned} c_2 \odot (x \oplus (c_1 - c_2)) \odot (x \oplus (c_0 - c_1)) &= c_2 \odot (x \odot x \oplus (c_1 - c_2) \odot x \oplus x \odot (c_0 - c_1) \oplus (c_1 - c_2) \odot (c_0 - c_1)) \\ &= c_2 \odot (x^{\odot 2} \oplus (c_1 - c_2) \odot x \oplus (c_0 - c_2) \\ &= P(x). \end{aligned}$$

Consider the case that $c_1 - c_2 \geq c_0 - c_1$. Assume for the purpose of contradiction that for some $a \in \mathbb{T}$,

$$P(a) \neq c_2 \odot x^{\odot 2} \oplus c_0.$$

Thus, $a + c_1 < 2a + c_2$ and $a + c_1 < c_0$. However,

$$a + c_1 < 2a + c_2 \implies a > c_1 - c_2,$$

and

$$a + c_1 < c_0 \implies a < c_0 - c_1,$$

which is a contradiction. Thus, we conclude by Proposition 2.4,

$$P(x) = c_2 \odot \left(x \oplus \frac{c_0 - c_2}{2} \right)^{\odot 2}.$$

□

Worthy of note is that the above proposition holds even at the edge cases where b or c equal ∞. We can observe that Example 2.6b satisfies the condition of Example 3.1i, while Example 2.6c satisfies the condition of Example 3.1ii. However, Example 2.6d factors in a similar way to Example 2.6b. Let $c_3 = 0, c_2 = -1, c_1 = 1$, and $c_0 = 4$. In the case of Example 2.6d,

$$c_3 \odot x^{\odot 3} \oplus c_2 \odot x^{\odot 2} \oplus 1 \odot x \oplus 4 = (x \oplus (c_2 - c_3)) \odot (x \oplus (c_1 - c_2)) \odot (x \oplus (c_0 - c_1)).$$

These solutions for λ_i are, however, not always correct, as illustrated by Example 2.6c. We find that it is possible to reduce every tropical polynomial to a form in which we may deduce its roots in a generalized version of these "differences of coefficients."

We now come to the fundamental condition that allows us to reduce a tropical polynomial.

Lemma 3.2. *Let P be a tropical polynomial of degree d, denoted*

$$P(x) = c_d \odot x^{\odot d} \oplus c_{d-1} \odot x^{\odot d-1} \oplus \ldots \oplus c_0.$$

Let $j \in \mathbb{N}_0$ such that $j \leq d$. For all $x \in \mathbb{T}$,

$$P(x) = c_d \odot x^{\odot d} \oplus \ldots \oplus \widehat{c_j \odot x^{\odot j}} \oplus \ldots \oplus c_0,$$

if, and only if, for some $i, k \in \{n \in \mathbb{N}_0 \mid n \leq d\}$ such that $i < j < k$,

$$(j - i)c_k + (k - j)c_i \leq (k - i)c_j. \tag{3.3}$$

Proof. Suppose for all $x \in \mathbb{T}$,

$$P(x) = c_d \odot x^{\odot d} \oplus \ldots \oplus \widehat{c_j \odot x^{\odot j}} \oplus \ldots \oplus c_0.$$

Assume for the purpose of contradiction that for all k, i such that $k > j > i$ and $x \in \mathbb{T}$,

$$\min(kx + c_k, jx + c_j, ix + c_i) \neq \min(kx + c_k, ix + c_i)$$

and therefore, there exists $a_{ki} \in \mathbb{T}$ such that

$$ja_{ki} + c_j < ka_{ki} + c_k \quad \text{and} \quad ja_{ki} + c_j < ia_{ki} + c_i.$$

Let $\mathcal{A}$ be the set of all such a_{ki}. Because $\mathcal{A}$ is finite, there exists a least element, a. For all $1 \leq x \leq d$,

$$ja + c_j < xa + c_x,$$

and so

$$P(a) \neq c_d \odot a^{\odot d} \oplus \ldots \oplus \widehat{c_j \odot a^{\odot j}} \oplus \ldots \oplus c_0,$$

and we reach a contradiction. We conclude there exist k, i such that $k > j > i$ and, for all $x \in \mathbb{T}$,

$$\min(kx + c_k, jx + c_j, ix + c_i) = \min(kx + c_k, ix + c_i).$$

Assume for the purpose of contradiction

$$(j - i)c_k + (k - j)c_i > (k - i)c_j.$$

Let $a = \dfrac{c_i - c_k}{k - i}$. We see

$$ka + c_k = ia + c_i = \frac{kc_i - ic_k}{k - i},$$

and

$$ja + c_j = \frac{j(c_i - c_k)}{k - i} + \frac{(k - i)c_j}{k - i} < \frac{j(c_i - c_k)}{k - i} + \frac{(k - j)c_i + (j - i)c_k}{k - i} = \frac{kc_i - ic_k}{k - i},$$

and so

$$\min(kx + c_k, jx + c_j, ix + c_i) \neq \min(kx + c_k, ix + c_i),$$

which is a contradiction. Thus, we conclude Equation 3.3 holds.

Assume for the purpose of contradiction there exists $a \in \mathbb{T}$ such that

$$P(a) \neq c_d \odot a^{\odot d} \oplus \ldots \oplus \widehat{c_j \odot a^{\odot j}} \oplus \ldots \oplus c_0.$$

Thus, $ja + c_j < da + c_d, ja + c_j < (d - 1)a + c_{d-1}, \ldots, ja + c_j < c_0$. We see

$$ja + c_j < ka + c_k \implies \frac{c_j - c_k}{k - j} < a$$

and

$$ja + c_j < ia + c_i \implies a < \frac{c_i - c_j}{j - i},$$

so

$$\frac{c_j - c_k}{k - j} < \frac{c_i - c_j}{j - i}.$$

This implies

$$(j - i)c_k + (k - j)c_i > (k - i)c_j,$$

and so we reach a contradiction and conclude, for all $x \in \mathbb{T}$,

$$P(x) = c_d \odot x^{\odot d} \oplus \ldots \oplus \widehat{c_j \odot x^{\odot j}} \oplus \ldots \oplus c_0.$$

□

Essentially, Lemma 3.2 states that, in classical form,

$$\min(dx + c_d, \ldots, jx + c_j, \ldots, c_0) = \min(dx + c_d, \ldots, \widehat{jx + c_j}, \ldots, c_0),$$

if, and only if, there exist $i, k \in \mathbb{N}_0$ such that $i < j < k$ and $\min(kx + c_k, jx + c_j, ix + c_i) = \min(kx + c_k, ix + c_i)$, which is true if, and only if, for those same i, k,

$$(j - i)c_k + (k - j)c_i \leq (k - i)c_j.$$

Using this condition, we will make a few definitions that will allow us remove terms of a tropical polynomial that do not contribute to the resulting function so we can work with a well-defined representative of that function.

Definition 3.4. Let $d \in \mathbb{N}$, and let $j \in \mathbb{N}_0$ such that $j \leq d$. Define a map $\mathrm{red}_j \colon \mathbb{T}[x]_{\deg \leq d} \to \mathbb{T}[x]_{\deg \leq d}$ such that, for all tropical polynomials $P \in \mathbb{T}[x]_{\deg \leq d}$,

$$\mathrm{red}_j(P) = \begin{cases} c_d \odot x^{\odot d} \oplus \ldots \oplus \widehat{c_j \odot x^{\odot j}} \oplus \ldots \oplus c_0 & \text{if there exist } i,k \text{ such that} \\ & \quad (j-i)c_k - (k-j)c_i \leq (k-i)c_j, \\ P(x) & \text{otherwise.} \end{cases}$$

We call $\mathrm{red}_j(P)$ the *reduction of P at j*. Note that $\mathrm{red}_j = \mathrm{red}_j \circ \mathrm{red}_j$.

Corollary 3.5. *Compositions of reductions commute. For a tropical polynomial P of degree d, for all $0 \leq i, j \leq d$,*

$$(red_j \circ red_i)(P) = (red_i \circ red_j)(P).$$

Proof. If $j = i$, the Lemma is trivial. Also, if $\mathrm{red}_j(P) = P$ or $\mathrm{red}_i(P) = P$, we see the Lemma is trivial, because red can only remove indices.

Suppose $\mathrm{red}_j(P) \neq P$ and $\mathrm{red}_i(P) \neq P$, and therefore

$$P(x) = c_d \odot x^{\odot d} \oplus \ldots \oplus \widehat{c_j \odot x^{\odot j}} \oplus \ldots \oplus c_0 = c_d \odot x^{\odot d} \oplus \ldots \oplus \widehat{c_i \odot x^{\odot i}} \oplus \ldots \oplus c_0.$$

Without loss of generality, assume $i < j$. Assume for the purpose of contradiction,

$$P(x) \neq c_d \odot x^{\odot d} \oplus \ldots \oplus \widehat{c_j \odot x^{\odot j}} \oplus \ldots \oplus c_i \odot x^{\odot i} \oplus \ldots \oplus c_0.$$

By Lemma 2.7, P is continuous and piecewise. Since the j term or the i term can be removed but not both, there exist infinitely many points $x \in \mathbb{T}$ such that, for all $0 \leq n \leq d$, when $n \neq j$,

$$jx + c_j < nx + c_n,$$

and when $n \neq i$,

$$ix + c_i < nx + c_n.$$

Thus, there exist unique $a, b \in \mathbb{T}$ such that when $n \neq j$, $ja + c_j < na + c_n$, and when $n \neq i$, $ib + c_i < nb + c_n$. However, because for all $x \in \mathbb{T}, \mathrm{red}_j(P)(x) = \mathrm{red}_i(P)(x) = P(x)$,

$$ja + c_j = ia + c_i \quad \text{and} \quad ib + c_i = jb + c_j,$$

which implies $a = b$, so we reach a contradiction and conclude reductions commute. □

More intuitively, Corollary 3.4 states that if $ix + c_i$ and $jx + c_j$ do not contribute to the resulting function of $\min(dx + c_d, \ldots, c_0)$, we may remove the terms from the expression in either order, and the removal of one does not affect the removal of the other.

Definition 3.6. Let $d \in \mathbb{N}$. Define a map $\mathrm{Red} \colon \mathbb{T}[x]_{\deg \leq d} \to \mathbb{T}[x]_{\deg \leq d}$ such that, for all tropical polynomials $P \in \mathbb{T}[x]_{\deg \leq d}$,

$$\mathrm{Red}(P) = (\mathrm{red}_1 \circ \mathrm{red}_2 \circ \cdots \circ \mathrm{red}_d)(P)$$

We call $\mathrm{Red}(P)$ the *reduction of P*. If the coefficient of each term of $\mathrm{Red}(P)$ is equivalent to that of P, we say P is *reduced*.

Corollary 3.7. *Let P be a tropical polynomial. Then, $Red(P)$ is reduced.*

Proof. Let $d = \deg(P)$. By Definition 3.6,

$$\mathrm{Red}(P) = (\mathrm{red}_1 \circ \mathrm{red}_2 \circ \cdots \circ \mathrm{red}_d)(P)$$

and

$$\mathrm{Red}(\mathrm{Red}(P)) = (\mathrm{red}_1 \circ \mathrm{red}_2 \circ \cdots \circ \mathrm{red}_d) \circ (\mathrm{red}_1 \circ \mathrm{red}_2 \circ \cdots \circ \mathrm{red}_d)(P)$$

By Corollary 3.5, reductions commute, so

$$\begin{aligned}\mathrm{Red}(\mathrm{Red}(P)) &= (\mathrm{red}_1 \circ \mathrm{red}_1) \circ (\mathrm{red}_2 \circ \mathrm{red}_2) \circ \cdots \circ (\mathrm{red}_d \circ \mathrm{red}_d)(P) \\ &= (\mathrm{red}_1 \circ \mathrm{red}_2 \circ \cdots \circ \mathrm{red}_d)(P) \\ &= \mathrm{Red}(P).\end{aligned}$$

We conclude $\mathrm{Red}(P)$ is reduced. □

Corollary 3.8. *Let P and Q be tropical polynomials such that $Red(P) = Red(Q)$. Then, for all $x \in \mathbb{T}$, $P(x) = Q(x)$.*

Proof. Because red_j only changes the form of a polynomial and not its function, for all $x \in \mathbb{T}$, $\mathrm{Red}(P)(x) = P(x)$ and $\mathrm{Red}(Q)(x) = Q(x)$. Thus, for all $x \in \mathbb{T}$.

$$P(x) = \mathrm{Red}(P)(x) = \mathrm{Red}(Q)(x) = Q(x).$$

□

We are almost able to prove the Fundamental Theorem of Algebra, but must first prove a simple lemma for what will be the tropical roots.

Lemma 3.9. *Let P be a reduced tropical polynomial of degree d, denoted*

$$P(x) = c_d \odot x^{\odot d} \oplus c_{d-1} \odot x^{\odot d-1} \oplus \ldots \oplus c_0.$$

Let I be the set of all $j > 0$ such that $c_j \neq \infty$, along with 0, denoted

$$I = \{i_0, i_1, \ldots, i_n\},$$

where $n \leq d$ and $0 = i_0 < \cdots < i_n = d$. Thus,

$$P(x) = c_{i_n} \odot x^{\odot i_n} \oplus \ldots \oplus c_{i_0}.$$

Then, for all $j \in \mathbb{N}$ such that $0 < j \leq n$,

$$\frac{c_{i_j} - c_{i_{j+1}}}{i_{j+1} - i_j} < \frac{c_{i_{j-1}} - c_{i_j}}{i_j - i_{j-1}}.$$

Proof. Assume for the purpose of contradiction there exists j such that

$$\frac{c_{i_j} - c_{i_{j+1}}}{i_{j+1} - i_j} \geq \frac{c_{i_{j-1}} - c_{i_j}}{i_j - i_{j-1}}$$

and therefore,

$$(i_j - i_{j-1})(c_{i_j} - c_{i_{j+1}}) \geq (c_{i_{j-1}} - c_{i_j})(i_{j+1} - i_j),$$

which implies

$$(i_{j+1} - i_{j-1})c_{i_j} \geq (i_{j+1} - i_j)c_{i_{j-1}} + (i_j - i_{j-1})c_{i_{j+1}},$$

which contradicts P being reduced. Thus, we complete the proof. □

Finally, we may prove the equivalence of every tropical polynomial to the product of roots by showing their reductions are the same.

Theorem 3.10. *Let P be a reduced tropical polynomial of degree d, denoted*

$$P(x) = c_{i_n} \odot x^{\odot i_n} \oplus \ldots \oplus c_{i_0}.$$

Then, for all $x \in \mathbb{T}$,

$$P(x) = c_{i_n} \odot \bigodot_{j=1}^{n} (x \oplus \lambda_j)^{\odot m_j}, \tag{3.11}$$

where $m_j = i_j - i_{j-1}$ and $\lambda_j = \dfrac{c_{i_{j-1}} - c_{i_j}}{i_j - i_{j-1}}$. Furthermore, $\lambda_1, \ldots, \lambda_n$ are the only elements of $\mathbb{T}$ and $m_1, \ldots, m_n$ are the only elements of $\mathbb{N}_0$ that satisfy Equation 3.11.

Proof. Let Q be a tropical polynomial such that, for all $x \in \mathbb{T}$,

$$Q(x) = c_{i_n} \odot \bigodot_{j=1}^{n} (x \oplus \lambda_j)^{\odot m_j}.$$

Let $r_d, \ldots, r_1$ be such that $r_d, \ldots, r_{d-m_n+1} = \lambda_n$ and $r_{d-m_{j+1}}, \ldots, r_{d-m_j+1} = \lambda_j$. Thus,

$$\begin{aligned} Q(x) &= c_{i_n} \odot (x \oplus \lambda_n)^{\odot m_n} \odot \ldots \odot (x \oplus \lambda_1)^{\odot m_1} \\ &= c_{i_n} \odot (x \oplus r_d) \odot \ldots \odot (x \oplus r_1) \\ &= c_{i_n} \odot (x^{\odot d} \oplus (r_d \oplus \ldots \oplus r_1) \odot x^{\odot d-1} \oplus (r_d \odot r_{d-1} \oplus \ldots \oplus r_2 \odot r1) \odot x^{\odot d-2} \oplus \ldots \oplus (r_d \odot \ldots \odot r_1)). \end{aligned}$$

By Lemma 3.9, all $r_j \leq r_{j-1}$, so

$$Q(x) = c_{i_n} \odot (x^{\odot d} \oplus (r_d) \odot x^{\odot d-1} \oplus (r_d \odot r_{d-1}) \odot x^{\odot d-2} \oplus \ldots \oplus (r_d \odot \ldots \odot r_1)),$$

or equivalently,

$$Q(x) = c_d \odot x^{\odot d} \oplus c_{d-1} \odot x^{\odot d-1} \oplus \ldots \oplus c_0,$$

where each $c_x = c_{i_n} \odot r_d \odot \ldots \odot r_{x+1}$. If $x \in I$, there exists a j such that $i_j = x$, and it follows from our construction of each r_y that

$$c_{i_j} = c_d \odot \lambda_n^{\odot m_n} \odot \ldots \odot \lambda_{j+1}^{\odot m_{j+1}}.$$

We may then substitute all $i_j \in I$ to get

$$Q(x) = c_{i_n} \odot x^{\odot i_n} \oplus \ldots \oplus c_{i_{n-1}} \odot x^{\odot i_{n-1}} \oplus \ldots \oplus c_{i_0},$$

where there may exist terms between i_j terms, but all i_j terms are necessarily present.

Consider $\mathrm{Red}(Q)$. Let $j \in \mathbb{N}$ such that $1 \leq j \leq n$. Suppose there exists k such that $i_j > k > i_{j-1}$. Then,

$$c_{i_j} = c_{i_n} \odot \lambda_n^{\odot m_n} \odot \ldots \odot \lambda_{j+1}^{\odot m_{j+1}},$$

$$c_{i_{j-1}} = c_{i_j} \odot \lambda_j^{\odot m_j} := m_j \lambda_j + c_{i_j},$$

and

$$c_k = c_{i_j} \odot \lambda_j^{\odot m_k} := m_k \lambda_j + c_{i_j},$$

where $m_k = i_j - k$. Thus,

$$\begin{aligned} c_{i_j}(k - i_{j-1}) + c_{i_{j-1}}(i_j - k) &= c_{i_j}(k - i_{j-1}) + (m_j\lambda_j + c_{i_j})(i_j - k) \\ &= c_{i_j}(i_j - i_{j-1}) + m_j\lambda_j(i_j - k) \\ &= c_{i_j}(i_j - i_{j-1}) + (i_j - i_{j-1})\lambda_j(i_j - k) \\ &= c_{i_j}(i_j - i_{j-1}) + (i_j - i_{j-1})\lambda_j m_k \\ &= (m_k\lambda_j + c_{i_j})(i_j - i_{j-1}) \\ &= c_k(i_j - i_{j-1}). \end{aligned}$$

Thus, i_j, k, i_{j-1} satisfy Equation 3.3, and by Lemma 3.2,

$$Q(x) = c_d \odot x^{\odot d} \oplus \ldots \oplus \widehat{c_k \odot x^{\odot k}} \oplus \ldots \oplus c_0.$$

Therefore, we may remove all indices x such that $x \notin I$, and by Corollary 3.8, for all $x \in \mathbb{T}$,

$$Q(x) = \mathrm{Red}(Q)(x) = c_d \odot x^{\odot d} \oplus c_{d-1} \odot x^{\odot d-1} \oplus \ldots \oplus c_0 = \mathrm{Red}(P)(x) = P(x).$$

We will now prove uniqueness. First, we will prove that $\lambda_n = \gamma_\nu$. Suppose there exist $p_1, \ldots, p_\nu$ and $\gamma_1, \ldots, \gamma_\nu$, with each $\gamma_j < \gamma_{j-1}$, such that, for all $x \in \mathbb{T}$,

$$P(x) = c_{i_n} \odot \bigodot_{k=1}^{\nu} (x \oplus \gamma_k)^{\odot p_k}.$$

Note that $m_n + \cdots + m_1 = p_\nu + \cdots + p_1 = d$. Because all $\lambda_i \leq \lambda_{i-1}$,

$$P(\lambda_n) = c_{i_n} \odot \bigodot_{j=1}^{n} (\lambda_n \oplus \lambda_j)^{\odot m_j} = c_{i_n} \odot \lambda_n^{\odot d}.$$

By setting this equal to the other form of P we have that

$$P(\lambda_n) = c_{i_n} \odot \bigodot_{k=1}^{\nu} (\lambda_n \oplus \gamma_k)^{\odot p_k} = c_{i_n} \odot \lambda_n^{\odot d}.$$

Because

$$c_{i_n} \odot \bigodot_{k=1}^{\nu} (\lambda_n \oplus \gamma_k)^{\odot p_k} = c_{i_n} \odot \lambda_n^{\odot d},$$

each $\lambda_n \oplus \gamma_k = \lambda_n$, and thus, for all $k \in \mathbb{N}$ such that $k \leq \nu, \lambda_n \leq \gamma_k$. By a similar argument, for all $j \in \mathbb{N}$ such that $j \leq n$, $\gamma_\nu \leq \lambda_j$. Thus, $\lambda_n = \gamma_\nu$.

We will now prove $m_n = p_\nu$. Assume for the purpose of contradiction $m_n < p_\nu$. We see

$$P(\lambda_{n-1}) = c_{i_n} \odot \bigodot_{j=1}^{n} (\lambda_{n-1} \oplus \lambda_j)^{\odot m_j} = c_{i_n} \odot \lambda_n^{\odot m_n} \odot \lambda_{n-1}^{\odot m_{n-1}}.$$

Setting this equal to the other form of P, we have that

$$P(\lambda_{n-1}) = c_{i_n} \odot \bigodot_{k=1}^{\nu} (\lambda_{n-1} \oplus \gamma_k)^{\odot p_k} = c_{i_n} \odot \lambda_n^{\odot m_n} \odot \lambda_{n-1}^{\odot m_{n-1}}.$$

For all $k \in \mathbb{N}$ such that $k \leq \nu - 1$, $\lambda_{n-1} \oplus \gamma_k \leq \lambda_{n-1}$. Thus,

$$\begin{aligned} c_{i_n} \odot \bigodot_{k=1}^{\nu} (\lambda_{n-1} \oplus \gamma_k)^{\odot p_k} &= c_{i_n} \odot (\lambda_{n-1} \oplus \lambda_n)^{\odot p_\nu} \odot \bigodot_{k=1}^{\nu-1} (\lambda_{n-1} \oplus \gamma_k)^{\odot p_k} \\ &\leq c_{i_n} \odot \lambda_n^{\odot p_\nu} \odot \lambda_{n-1}^{\odot d - p_\nu}. \end{aligned}$$

However, because $\lambda_n < \lambda_{n-1}$ and $p_\nu > m_n$,

$$c_{i_n} \odot \lambda_n^{\odot p_\nu} \odot \lambda_{n-1}^{\odot d-p_\nu} < c_{i_n} \odot \lambda_n^{\odot m_n} \odot \lambda_{n-1}^{\odot m_{n-1}} = P(\lambda_{n-1}).$$

We reach a contradiction and conclude $m_n \not< p_\nu$.

Assume for the purpose of contradiction $m_n > p_\nu$. Let $s_d, \ldots, s_1$ be defined in the same way as $r_d, \ldots, r_1$, except with γ_i instead of λ_i. Because $m_n \leq d$, $s_{d-p_\nu}, s_{d-p_\nu-1}, \ldots, s_{d-m_n+1}$ exist. Let $R = \bigodot_{j=p_\nu}^{m_n-1} s_{d-j}$. We see

$$c_{i_{n-1}} = c_{i_n} \odot \gamma_\nu^{\odot p_\nu} \odot R = c_{i_n} \odot \lambda_n^{\odot p_\nu} \odot R.$$

Each s_i such that $d - p_\nu > i > d - m_n + 1$ is equal to some γ_k such that $k < \nu$, and since each $\gamma_k < \gamma_{k-1}$,

$$c_{i_n} \odot \lambda_n^{\odot p_\nu} \odot R \geq c_{i_n} \odot \lambda_n^{\odot p_\nu} \odot \gamma_{n-1}^{\odot p_{\nu-1}} > c_{i_n} \odot \lambda_n^{\odot m_n} = c_{i_{n-1}}.$$

Thus, we reach a contradiction and conclude $m_n \not> p_\nu$. Therefore, $m_n = p_\nu$.

We now have that

$$c_{i_n} \odot \bigodot_{k=1}^{\nu} (x \oplus \gamma_k)^{\odot p_k} = c_{i_n} \odot (x \oplus \lambda_n)^{\odot m_n} \odot \bigodot_{k=1}^{\nu-1} (x \oplus \gamma_k)^{\odot p_k}.$$

A simple proof by induction shows $\nu = n$ and for all $k \in \mathbb{N}$ such that $k < n$, $\gamma_k = \lambda_k$ and $p_k = m_k$. Thus, we conclude $\lambda_1, \ldots, \lambda_n$ and $m_1, \ldots, m_n$ are unique. ☐

We have proven the Fundamental Theorem of Algebra over the Tropical Semiring, which serves as an interesting connection between classical and tropical mathematics.

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