CHANGING GROUND CATEGORY IN OPERAD THEORY

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Abstract. Developed by J. Michael Boardman, Rainer M. Vogt, and J. Peter May, operads describe algebraic structure and abstract operations in symmetric monoidal categories. This expository paper aims to introduce operads and how the functors on ground categories induce maps among the operads. Changing ground category in operad theory has many applications such as the homology operads and the rise of some interesting spectra.

Contents

1. Introduction

The concept of operads has arisen from the study of iterated loop spaces in algebraic topology. One of the first developers of operad theory, J Peter May created the word "operad" as a portmanteau of the words "operation" and "monad". An operad can be visualized with computational trees and interpreted as a collection of composable abstract operations each with a fixed number of inputs and a single output under conditions of associativity, unitality, and equivariance.

We are going to define operads as collections of objects and maps in a ground category satisfying particular associativity, unitality, and commutativity conditions. We will be concerned with operads in topological spaces, simplicial sets, simplicial abelian groups, and chain complexes. There exist functors mapping between these underlying categories, but it is essential to show they preserve the property of operads to show that the functors induce operads on the respective categories. The singular complex functor maps topological spaces to simplicial sets; the free abelian group functor maps simplicial sets to simplicial abelian groups. These two functors can be shown to preserve symmetric monoidal structures up to isomorphisms. On the other hand, in order to show the passage to chain complexes induces an operad on chain complexes from an operad on simplicial abelian groups, we will introduce the Eilenberg-Zilber Map whose associativity, unitality, and commutativity induce the preserving of symmetric monoidal structures up to isomorphisms.

$$\begin{array}{c}
\text{Operads on topological spaces} \\
\big\downarrow \scriptstyle\text{singular complex functor} \\
\text{Operads on simplicial sets} \\
\big\downarrow \scriptstyle\text{free abelian group functor} \\
\text{Operads on simplicial abelian groups} \\
\big\downarrow \scriptstyle\text{passage to normalized chains} \\
\text{Operads on chain complexes}
\end{array}$$

Changing the ground category for operads has many applications such as in homology operads. The theory also gives rise to interesting spectra including the sphere spectrum and the spectra that give algebraic K-theory of rings.

2. Definition of Operads

This section will introduce the definitions of operads and how operads can be represented by computational tree diagrams.

Definition 2.1. (Operads) Fix an arbitrary symmetric monoidal category D. An operad $\mathscr{C}$ consists of objects $\mathscr{C}(j)$ of D for all $j \geq 0$, a unit map $\eta : \kappa \longrightarrow \mathscr{C}(1)$, a right action by the symmetric group Σ_j on $\mathscr{C}(j)$ for each j, and for any $k \geq 1, j_s \geq 0, \Sigma j_s = j$, maps

$$\gamma : \mathscr{C}(k) \bigotimes \bigotimes_{s=1}^{k} \mathscr{C}(j_s) \longrightarrow \mathscr{C}(j)$$

that are associative, unital, and equivariant, as illustrated by diagrams in the following cases:

(1) Associative
The associativity diagrams commute where $\Sigma j_s = j, \Sigma i_t = i$.

$$\begin{array}{ccc}
\mathscr{C}(k) \bigotimes (\bigotimes_{s=1}^{k} \mathscr{C}(j_s)) \bigotimes (\bigotimes_{r=1}^{j} \mathscr{C}(i_r)) & \xrightarrow{\gamma \bigotimes Id} & \mathscr{C}(j) \bigotimes (\bigotimes_{r=1}^{j} \mathscr{C}(i_r)) \\
\Big\downarrow {\scriptstyle permutation/shuffle} & & \Big\downarrow \gamma \\
\mathscr{C}(k) \bigotimes (\bigotimes_{s=1}^{k} (\mathscr{C}(j_s) \bigotimes (\bigotimes_{q=1}^{j_s} \mathscr{C}(j_{s-1+q})))) & & \mathscr{C}(i) \\
& \searrow {\scriptstyle Id \bigotimes (\bigotimes_s \gamma)} & \Big\uparrow \gamma \\
& & \mathscr{C}(k) \bigotimes (\bigotimes_{s=1}^{k} \mathscr{C}(i_s))
\end{array}$$

(2) Unital
The following unit diagrams commute.

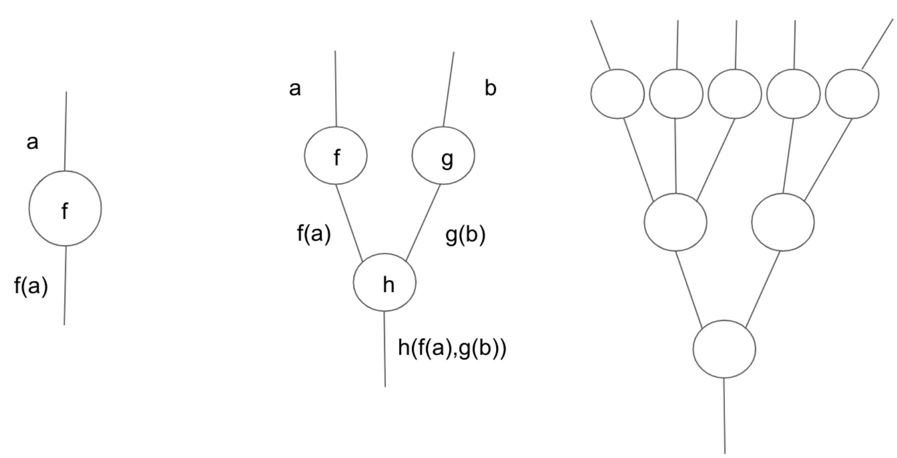
$$\begin{tikzcd} \mathscr{C}(k)\bigotimes(\kappa)^k \arrow[r, "\simeq"] \arrow[d, "Id\otimes\eta^k"'] & \mathscr{C}(k) \\ \mathscr{C}(k)\bigotimes(\mathscr{C}(1))^k \arrow[ur, "\gamma"'] & \end{tikzcd} \qquad \begin{tikzcd} \kappa\bigotimes\mathscr{C}(j) \arrow[r, "\simeq"] \arrow[d, "\eta\otimes Id"'] & \mathscr{C}(j) \\ \mathscr{C}(1)\bigotimes\mathscr{C}(j) \arrow[ur, "\gamma"'] & \end{tikzcd}$$

(3) Equivariant
The following equivariance diagrams commute.
$\sigma \in \Sigma_k,\ \tau_s \in \Sigma_{j_s},\ \sigma(j_1, ..., j_k) \in \Sigma_k$ permutes k blocks of letters as σ permutes k letters, and $\bigoplus_{i=1}^{k}\tau_i$ is the block sum:

$$\begin{tikzcd} \mathscr{C}(k)\bigotimes\bigotimes_{s=1}^{k}\mathscr{C}(j_s) \arrow[r, "\sigma\otimes\sigma^{-1}"] \arrow[d, "\gamma"'] & \mathscr{C}(k)\bigotimes\bigotimes_{s=1}^{k}\mathscr{C}(j_{\sigma(s)}) \arrow[d, "\gamma"] \\ \mathscr{C}(j) \arrow[r, "\sigma(j_{\sigma(1)},\dots,j_{\sigma(k)})"'] & \mathscr{C}(j) \end{tikzcd}$$

$$\begin{tikzcd} \mathscr{C}(k)\bigotimes\bigotimes_{s=1}^{k}\mathscr{C}(j_s) \arrow[r, "Id\otimes\bigotimes_{i=1}^{k}\tau_i"] \arrow[d, "\gamma"'] & \mathscr{C}(k)\bigotimes\bigotimes_{s=1}^{k}\mathscr{C}(j_s) \arrow[d, "\gamma"] \\ \mathscr{C}(j) \arrow[r, "\bigotimes_{i=1}^{k}\tau_i"'] & \mathscr{C}(j) \end{tikzcd}$$

Remark 2.2. Operads can be represented by tree diagrams. Each node is an action. The value k in $\mathscr{C}(k)$ specifies the number of branches taken in by each node. Passing down the tree, we will get compositions of actions.



3. Simplicial Sets and Singular Complex Functor

We begin by introducing the notion of an simplicial set. We will give two equivalent definitions of a simplicial set.

Definition 3.1. A ***simplicial set*** consists of a sequence of sets $X_0, X_1, ...$ for each $n \geq 0$, functions $d_i : X_n \longrightarrow X_{n-1}$ and $s_i : X_n \longrightarrow X_{n+1}$ for each i with $0 \leq i \leq n$ such that:

$d_i d_j = d_{j-1} d_i$ if $i < j$
$d_i s_j = s_{j-1} d_i$ if $i < j$
$d_j s_j = d_{j+1} s_j = id$
$d_i s_j = s_j d_{j-1}$ if $i > j+1$
$s_i s_j = s_{j+1} s_i$ if $i \leq j$

Definition 3.2. The category Δ has as objects the finite ordered sets $[n] = 0, 1, 2, ..., n$. The morphisms of Δ are order-preserving functions $[m] \longrightarrow [n]$.

Definition 3.3. (Category-Theoretical Definition) A simplicial set is a contravariant functor $X : \Delta \longrightarrow Set$. Equivalently, a simplicial set is a covariant functor $X : \Delta^{op} \longrightarrow Set$.

We then introduce the functor that maps from the category of topological spaces to the category of simplicial sets.

Definition 3.4. Let X be any topological space. Define the singular complex functor S : Top $\rightarrow$ sSet by sending a topological space X to the simplicial set SX by letting $SX_n = \textbf{Top}(\Delta_n, X)$ be the set of continuous maps from the standard topological n-simplex to X.

Consider the set of morphisms $d^i : \Delta_{n-1} \rightarrow \Delta_n$ and morphisms $s^i : \Delta_{n+1} \rightarrow \Delta_n$ where d^i inserts a 0 in the i^{th} coordinate and s^i adds the i^{th} and $(i+1)^{th}$ coordinates.

Applying Top$(-, X)$ to d^i and s^i induce maps of sets:

$d_i : \textbf{Top}(\Delta_{n+1}, X) \longrightarrow \textbf{Top}(\Delta_n, X)$.

$s_i : \textbf{Top}(\Delta_{n-1}, X) \longrightarrow \textbf{Top}(\Delta_n, X)$

It is not too hard to check that SX satisfies the definition of a simplicial set.

The important property of the functor S that shows that it induces a functor from operads of topological spaces to operads of simplicial sets is that the singular complex functor S preserves symmetric monoidal structures up to isomorphisms. We show this quality by showing that S has a left adjoint, namely the geometric realization G.

Definition 3.5. There exists a functor from the category of simplicial sets to the category of topological spaces named the geometric realization. The **geometric realization** of a simplicial set X is the space

$$GX = \frac{\sqcup X_n \times \Delta^n}{(\partial_i x, u) \sim (x, \partial_i u), (s_i x, u) \sim (x, \sigma_i u)}$$

Proposition 3.6. *The geometric realization G is left adjoint to the singular complex functor.*

Proof. See [5] for a proof. □

Proposition 3.7. *Right adjoints preserve products.*

Proof. Let G, H be two functors such that H is the right adjoint to G. Then:

$$\begin{aligned} Hom(A, H(C_1 \times C_2)) &\cong Hom(G(A), C_1 \times C_2) \\ &\cong Hom(G(A), C_1) \times Hom(G(A), C_2) \\ &\cong Hom(A, H(C_1)) \times Hom(A, H(C_2)) \\ &\cong Hom(A, H(C_1) \times H(C_2)) \end{aligned}$$

If $Hom(A, H(C_1 \times C_2)) \cong Hom(A, H(C_1) \times H(C_2))$, then there holds $H(C_1 \times C_2) \cong H(C_1) \times H(C_2)$. In other words, H preserves products up to isomorphisms. □

We conclude from Propositions 3.6 and 3.7 that the singular complex functor preserves products so that

$$S(\mathscr{C}(k) \times \mathop{\times}_{s=1}^{k} \mathscr{C}(j_s)) = S(\mathscr{C}(k)) \times \mathop{\times}_{s=1}^{k} S\mathscr{C}(j_s).$$

With this, we can conclude the following result by simply checking the definition of an operad in section 2.

Corollary 3.8. *The singular complex functor induces a functor mapping from operads of topological spaces to operads of simplicial sets.*

Proof. The proof requires only straightforward checks from the definition of an operad and hence is omitted and left to the readers. □

4. Simplicial Abelian Groups and The Free Abelian Group Functor

The previous section gave a functor from the category of operads of topological spaces to the category of operads of simplicial sets. This section will introduce the free abelian group functor and show that it induces a functor from the operads of simplicial sets to operads of simplicial abelian groups. It suffices to show that the free abelian group functor preserves symmetric monoidal structures up to isomorphisms.

First, we will begin by introducing the functor that maps from the category of sets to the category of abelian groups.

Definition 4.1. The free abelian group functor K assigns to a set S the abelian group $\mathbb{Z}[S]$ of formal linear combinations $\sum_{s \in S} c_s s$, $c_s \in \mathbb{Z}$ and assigns to a function $f : S \to T$ the homomorphism $\mathbb{Z}[f]$ sending $\sum c_s s$ to $\sum c_s f(s)$.

Remark 4.2. The free abelian group functor induces a functor from simplicial sets to simplicial abelian groups.

A simplicial set is a covariant functor $X : \Delta^{op} \to Set$, and the free abelian group functor is a functor mapping from $Set \to Ab$. The composition of the two functors yields a functor from $\Delta^{op} \to Ab$, which is a simplicial abelian group.

Proposition 4.3. *The free abelian group functor K preserves symmetric monoidal structures up to isomorphisms.*

Proof. Consider two sets A, B. Notice that $K(A \times B)$ and $K(A) \bigotimes K(B)$ are generated by the same basis $A \times B$. Hence, $K(A \times B) \cong K(A) \bigotimes K(B)$. □

Remark 4.4. For a commutative ring k, we have an analogous functor from simplicial sets to simplicial k-modules. This result can be generalized to the simplicial free k-module by passing through the free k-module functor.

5. Passage to Chain Complexes

When passing from simplicial abelian groups to chain complexes, applying the Eilenberg-Zilber Map gives preservation of symmetric monoidal structures. We will begin by introducing the notion of a chain complex and the functor that maps from the category of simplicial abelian groups to the category of chain complexes.

Definition 5.1. A **chain complex** is a sequence of abelian groups C_i and maps $d_i : C_i \to C_{i-1}$ such that $d_{i-1} \circ d_i = 0$. The d_i's are called the boundary operators or the differentials.

Definition 5.2. Let sAb be the category of simplicial abelian groups. Let $A : sAb \to Ch$ be a functor from the category of simplicial abelian groups to the category of chain complexes. The differentials of the resulting chain complex are the alternating sum of the face maps of the simplicial abelian group, $d_n = \sum_{i=0}^{n}(-1)\partial_i$.

In order for A to induce a mapping from operads on simplicial abelian groups to operads on chain complexes, it is necessary to find an equivalence of symmetric monoidal structures in the two categories, for which we will introduce the Eilenberg-Zilber Theorem.

Theorem 5.3. *(Eilenberg-Zilber Theorem) There exist chain maps:*

$$f : A(X \bigotimes Y) \longrightarrow A(X) \bigotimes A(Y)$$
$$g : A(X) \bigotimes A(Y) \longrightarrow A(X \bigotimes Y)$$

which are unique up to chain homotopy, are natural in X and Y, and such that $f \circ g$ *and* $g \circ f$ *are each chain homotopic to the identity.*

The proof of the Eilenberg-Zilber Theorem involves the method of acyclic models. Acyclic models were developed by Eilenberg and Maclane to generalize proofs that establish equivalence of homology theories.

Definition 5.4. Let C be an arbitrary category and O be a collection of objects of C. Let $F : C \to Ch$ be a functor mapping from C to the category of chain complexes. F is **acyclic** for O if for all $n > 0, X \in O$, there is $H_n(F(X)) = 0$.

Definition 5.5. Let C, O, and F be defined as in Definition 5.4. F is **free** relative to O if there is an index set I_k for each $k \in \mathbb{N}$, an $e_\alpha \in O$ for each $\alpha \in I_k$, a family $\{O_\alpha\}_{\alpha \in I_k} \subseteq O$ and $f_\alpha \equiv \{f : O_\alpha \to X\}$ such that

$$\bigoplus_{\alpha \in I_k} \bigoplus_{f \in f_\alpha} \mathbb{Z}_{e_\alpha} \overset{\oplus F_k f}{\longrightarrow} F_k X$$

is an isomorphism.

Theorem 5.6. *(Acyclic Model Theorem) Let C and O be defined as in Definition 5.4, $F, G : C \to Ch$ be functors. If F is free relative to O, and G is acyclic for O, then*

(1) For every natural transformation $\phi : H_0F \to H_0G$, there is a natural transformation $T : F \to G$ such that $\phi = H_0T$.

(2) Let $T, T' : F \to G$ be two natural transformations and $H_0T = H_0T'$, then there is a natural chain homotopy $D : T \to T'$.

Proof. The proofs of the two parts are similar, so we will include only the proof for (1).

We define the natural transformation $T_k : F_k \to G_k$ by induction on k. We begin with $k = 0$. Choose $g_\alpha \in G_0(O_\alpha)$ satisfying $[g_\alpha]= \phi([e_\alpha])$. Define $T_0(e_\alpha) = g_\alpha$.

Consider $k > 0$. Suppose for $k \in \mathbb{N}$, there is a chain map $T_j : F_j \to G_j, j < k$ such that $\phi = H_0T$.

Note, $T_{k-1}e_\alpha$ is a cycle. By assumption of $G(O_\alpha)$ being acyclic, there exists $x \in G_kO_\alpha$ satisfying $\partial x = T_{k-1}e_\alpha$. By taking $T_k e\alpha = x$, we complete the proof. □

We now turn to the proof of Theorem 5.3.

Proof. (Theorem 5.3)

Let $C = Ab \times Ab$. Let $F(X, Y) = A(X) \bigotimes A(Y)$, $G(X, Y) = A(X \bigotimes Y)$. Let $O = \{(\Delta^p, \Delta^q)|p, q = 0, 1, 2, ...\}$, $I_n = \{(p, q)|p + q = n\}$ with $e_{p,q} = e_p \bigotimes e_q$.

F is free with respect to O, and G is acyclic for O because $\Delta^p \times \Delta^q$ is contractible.

By Acyclic Model Theorem, there exists a map $g : A(X) \bigotimes A(Y) \to A(X \bigotimes Y)$.

The reverse direction follows similarly. □

Theorem 5.3 ensures the existence of such maps f and g. In fact, we can give an exact definition for the f, g in the Eilenberg-Zilber Theorem.

Definition 5.7. (AW and EZ Maps)

Define $f : A(X \bigotimes Y) \longrightarrow A(X) \bigotimes A(Y)$

by $f(k_n \bigotimes l_n) = \sum_{i=0}^{n} \partial_{i+1}...\partial_n k_n \bigotimes \partial_0^i l_n$.

Define $g : A(X) \bigotimes A(Y) \longrightarrow A(X \bigotimes Y)$

by $g(k_p \bigotimes l_q) = \sum_{(\mu,\nu)} (-1)^{\sigma(\mu)} (s_{\nu_q}...s_{\nu_1} k_p \bigotimes s_{\mu_p}...s_{\mu_1} l_q)$,

where $\sigma(\mu) = \sum_{i=1}^{p} [\mu_i - (i - 1)]$.

f is the **Alexander-Whitney Map**; g is the **Eilenberg-Zilber Map**.

Only the Eilenberg-Zilber Map is commutative. We are most interested in the Eilenberg-Zilber Map and its associativity and commutativity, which will allow us to conclude the passage to a functor between the corresponding categories of operads.

Proposition 5.8. *(Associativity) The Eilenberg-Zilber map is associative in the sense that the following diagram commutes:*

$$\begin{array}{ccc}
A(K) \bigotimes A(L) \bigotimes A(M) & \xrightarrow{Id \bigotimes g} & A(K) \bigotimes A(L \bigotimes M) \\
\downarrow{\scriptstyle g \bigotimes Id} & & \downarrow{\scriptstyle g} \\
A(K \bigotimes L) \bigotimes A(M) & \xrightarrow{g} & A(K \bigotimes L \bigotimes M)
\end{array}$$

Proposition 5.9. *(Commutativity) The Eilenberg-Zilber map g is commutative. Let K and L be simplicial abelian groups. Define $t : K \bigotimes L \longrightarrow L \bigotimes K$ by $t(x \bigotimes y) = y \bigotimes x$. The following diagram is commutative:*

$$\begin{array}{ccc} A(K) \bigotimes A(L) & \xrightarrow{T} & A(L) \bigotimes A(K) \\ \downarrow g & & \downarrow g \\ A(K \bigotimes L) & \xrightarrow{t} & A(L \bigotimes K) \end{array}$$

where $T(x \bigotimes y) \equiv (-1)^{deg(x)deg(y)} y \bigotimes x$.

With Propositions 5.8 and Proposition 5.9, we can now proceed to show that A induces a functor from operads of simplicial abelian groups to operads of chain complexes.

Proposition 5.10. *A induces a mapping from operads on simplicial abelian groups to operads on chain complexes.*

Proof. Let $\mathscr{A}$ be an operad of simplicial abelian groups. $\mathscr{A}$ consists of simplicial abelian groups $\mathscr{A}(j)$, a unit map $\eta : \kappa \longrightarrow \mathscr{A}(1)$, a right action by the symmetric group Σ_j on $\mathscr{A}(j)$ and $\gamma : \mathscr{A}(k) \bigotimes \bigotimes_{s=1}^{k} \mathscr{A}(j_s) \longrightarrow \mathscr{A}(j)$ satisfying associativity, unitality, and equivariance.

By applying A to $\mathscr{A}$, $\mathscr{A}(j) \to A(\mathscr{A}(j))$, $\eta \to A\eta : A\kappa \to A(\mathscr{A}(1))$, and right action are preserved.

$A\gamma : A(\mathscr{A}(k)) \bigotimes A(\bigotimes_{s=1}^{k} \mathscr{A}(j_s)) \leftrightarrow A(\mathscr{A}(k) \bigotimes \bigotimes_{s=1}^{k} \mathscr{A}(j_s)) \longrightarrow A(\mathscr{A}(j))$

(1) Associativity

$$\begin{array}{c} A(\mathscr{A}(k)) \bigotimes (\bigotimes_{s=1}^{k} A(\mathscr{A}(j_s))) \bigotimes (\bigotimes_{r=1}^{j} A(\mathscr{A}(i_r))) \\ \downarrow \text{permutation/shuffle} \\ A(\mathscr{A}(k)) \bigotimes (\bigotimes_{s=1}^{k} (A(\mathscr{A}(j_s)) \bigotimes (\bigotimes_{q=1}^{j_s} A(\mathscr{A}(j_{s-1+q}))))) \\ \downarrow Id \bigotimes (\bigotimes_{s} (A\gamma \circ g)) \\ A(\mathscr{A}(k)) \bigotimes (\bigotimes_{s=1}^{k} A(\mathscr{A}(h_s))) \\ \downarrow A\gamma \circ g \\ A(\mathscr{A}(j)) \end{array}$$

$$\begin{array}{c} A(\mathscr{A}(j)) \\ \uparrow {\scriptstyle A\gamma \circ g} \\ A(\mathscr{A}(j)) \bigotimes (\bigotimes_{r=1}^{j} A(\mathscr{A}(i_r))) \\ \uparrow {\scriptstyle A\gamma \circ g \bigotimes Id} \\ A(\mathscr{A}(k)) \bigotimes (\bigotimes_{s=1}^{k} A(\mathscr{A}(j_s))) \bigotimes (\bigotimes_{r=1}^{j} A(\mathscr{A}(i_r))) \end{array}$$

(2) Unitality

$$\begin{array}{ccc} A\mathscr{A}(k) \bigotimes (A\kappa)^k & \xrightarrow{\simeq} & A\mathscr{A}(k) \\ \downarrow {\scriptstyle Id \bigotimes A\eta^k} & \nearrow {\scriptstyle A\gamma \circ g} & \\ A\mathscr{A}(k) \bigotimes (A\mathscr{A}(1))^k & & \end{array}$$

and

$$\begin{array}{ccc} A\kappa \bigotimes A\mathscr{A}(j) & \xrightarrow{\simeq} & A\mathscr{A}(j) \\ \downarrow {\scriptstyle A\eta \bigotimes Id} & \nearrow {\scriptstyle A\gamma \circ g} & \\ A\mathscr{A}(1) \bigotimes A\mathscr{A}(j) & & \end{array}$$

(3) Equivariance

$$\begin{array}{ccc} A\mathscr{A}(k) \bigotimes \bigotimes_{s=1}^{k} A\mathscr{A}(j_s) & \xrightarrow{\sigma \bigotimes \sigma^{-1}} & A(k) \bigotimes \bigotimes_{s=1}^{k} A\mathscr{A}(j_{\sigma(s)}) \\ \downarrow {\scriptstyle A\gamma \circ g} & & \downarrow {\scriptstyle A\gamma \circ g} \\ A\mathscr{A}(j) & \xrightarrow[\sigma(j_{\sigma(1)}, \ldots, j_{\sigma(k)})]{} & A\mathscr{A}(j) \end{array}$$

$$\begin{array}{ccc} A\mathscr{A}(k) \bigotimes \bigotimes_{s=1}^{k} A\mathscr{A}(j_s) & \xrightarrow{Id \bigotimes \bigotimes_{i=1}^{k} \tau_i} & A\mathscr{A}(k) \bigotimes \bigotimes_{s=1}^{k} A\mathscr{A}(j_s) \\ \downarrow {\scriptstyle A\gamma \circ g} & & \downarrow {\scriptstyle A\gamma \circ g} \\ A\mathscr{A}(j) & \xrightarrow[\bigotimes_{i=1}^{k} \tau_i]{} & A\mathscr{A}(j) \end{array}$$

□

Proposition 5.10 showed that the functor from simplicial abelian groups to chain complexes induces a functor from operads of simplicial abelian groups to operads of chain complexes. $\mathscr{C}$ is E_∞ if each $\mathscr{C}(j)$ is Σ_j-free and contractible, and then $A\mathscr{C}$ is E_∞. Furthermore, A maps $\mathscr{C}$-algebras to $A\mathscr{C}$-algebras, and the modules over $\mathscr{C}$-algebras to modules over $A\mathscr{C}$-algebras.

Sections 3, 4, 5 have altogether showed that we can build the following commutative diagram:

$$\begin{array}{ccc} \text{Operads on Top} & \longrightarrow & \text{Operads on Chain Complexes} \\ \downarrow & & \uparrow \\ \text{Operads on SSet} & \longrightarrow & \text{Operads on SAb} \end{array}$$

The functor mapping from the category of topological spaces to chain complexes is the composition of the functors we discussed above and named the singular chain functor. It follows that the singular chain functor induces a functor from operads of topological spaces to operads of chain complexes.

Our discussion about the simplicial abelian groups and chain complexes can be easily generalized to the simplicial k-modules and differential graded k-modules, which will give rise to the study of homology operads.

The following section will briefly introduce some of the applications arising from changing the ground category of operads.

6. Applications

6.1. **Homology Operads.** Let $\mathscr{C}$ be an operad of topological spaces and $A\mathscr{C}$ be the induced operad of k-modules, where k is a field.

Definition 6.1. $\mathscr{C}$ is a unital operad of a k-module if it is an operad of k-modules with $\mathscr{C}(0) = k$.

Definition 6.2. Let $H_*(\mathscr{C})$ be a unital operad, $H_*(\mathscr{C}(j))$ being its jth k-module. It has maps γ induced by the topological structure maps. Let $H_n(\mathscr{C})$, $n \geq 0$ be a suboperad of $H_*(\mathscr{C}(j))$ such that the jth k-module of $H_n(\mathscr{C})$ is $H_{n(j-1)}(\mathscr{C}(j))$. Keeping the grading, the jth term of the defined $H_n(\mathscr{C})$ is concentrated in degree $n(j-1)$. Regrading so that all terms of $H_n(\mathscr{C})$ are concentrated in degree 0 gives a "degree 0 translate" operad associated to $H_*(\mathscr{C})$.

Operads in characteristic zero and in positive characteristic have different calculational behavior because $H_*(X/\pi)$ is naturally isomorphic to $H_*(X)/\pi$ in a field of characteristic zero for a finite group π.

We will introduce the notion of a monad and its relation to an operad.

Definition 6.3. A monad (T, μ, ν) on a category C is an endofunctor T on C and two natural transformations $\mu : Id_C \to T$, $\nu : TT \to T$, such that the following two diagrams commute:

$$\begin{array}{ccc} T^3 & \xrightarrow{T\mu} & T^2 \\ {\scriptstyle \mu T}\downarrow & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow[\mu]{} & T \end{array}$$

$$\begin{array}{ccc} T & \xrightarrow{T\nu} & T^2 \\ {\scriptstyle \nu T}\downarrow & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow[\mu]{} & T \end{array}$$

With the definition of a monad, we examine the relation between a monad and an operad. In fact, we can construct a monad over an operad $\mathscr{C}$.

Definition 6.4. Let $\mathscr{C}$ be an operad with map γ and unital map η. Define the monad (T, μ, ν) associated to $\mathscr{C}$ by letting $TX = \coprod_{j\geq 0} \mathscr{C}(j) \bigotimes_{\kappa[\Sigma_j]} X^j$, two natural transformations be $\mu = \eta \bigotimes Id : X = \kappa \bigotimes X \to \mathscr{C}(1) \bigotimes X$ and $\nu : TTX \to TX$ induced by:

$$\begin{array}{c} \mathscr{C}(k) \bigotimes \mathscr{C}(j_1) \bigotimes X^{j_1} \bigotimes ... \mathscr{C}(j_k) \bigotimes X^{j_k} \\ {\scriptstyle \text{shuffle}}\downarrow \\ \mathscr{C}(k) \bigotimes \mathscr{C}(j_1) \bigotimes ... \bigotimes \mathscr{C}(j_k) \bigotimes X^j \\ {\scriptstyle \gamma \bigotimes \text{id}}\downarrow \\ \mathscr{C}(j) \bigotimes X^j \end{array}$$

where $j = \sum_{s=1}^{k} j_s$.

Definition 6.4 showed that every operad gives rise to an associated monad. Then, we are ready to introduce the following theorem:

Theorem 6.5. *Suppose $\mathscr{C}$ is an operad of spaces. Let T denote both the monad in the category of spaces associated to $\mathscr{C}$ and the monad in the category of k-modules associated to $H_*(\mathscr{C})$. Let T' denote both the monad in the category of based spaces associated to $\mathscr{C}$ and in the category of k-modules associated to $H_*(\mathscr{C})$.*

Then, there holds:

$$H_*(TX) \cong TH_*(X)$$
$$H_*(T'X) \cong T'(H_*(X))$$

as $H_*(\mathscr{C})$-algebras for all (based) spaces X given that k is a field of characteristic zero.

Proof. See [3] for a proof. □

The fine behavior noted in Theorem 6.5 allows topological realizations of free algebras.

6.2. **With Respect to Spectra.** Changing ground category in operads also gives rise to many interesting spectra. Let us consider the following example.

Let Cat be the category of categories. Suppose B is the classifying space functor from Cat to the category of spaces. Let C be an operad in Cat such that for all j, BC(j) is contractible. Then we get a functor from BC-spaces to spectra, and hence we get a functor from C-categories to spectra.

In fact, many interesting spectra arise from this construction, such as the sphere spectrum and the spectra that give algebraic K-theory of rings.

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