AN OPERADIC DESCRIPTION OF ALGEBRAIC QUANTUM FIELD THEORIES

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Abstract. Algebraic quantum field theories assign algebras to regions of spacetime such that the algebras describe the observables on that spacetime region. Operads provide a parametrization of these algebra operations in a way that allows us to consider the observables on a spacetime region from a more abstract perspective. In this paper, we follow the work of Benini, Schenkel, and Woike in assigning to each small category an operad whose category of algebras is identical to the category of possible quantum field theories on that category, regarded as a category of spacetimes.

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1. Introduction

Given an object representing a spacetime, such as a manifold, an algebraic quantum field theory assigns to each region of spacetime an algebra in a manner that respects inclusion of spacetimes and causal locality. This assignment is taken to be functorial, and therefore lends itself nicely to a categorical treatment. Namely, we describe a colored operad for any small category $\mathcal{C}$ representing spacetimes of interest, such that the category of algebras over that operad is the same as the category of algebraic quantum field theories over $\mathcal{C}$. In addition, this operad is capable of building in a $\perp$-commutativity structure, meaning that observables from causally disjoint regions of spacetime commute in any shared algebra.

This paper mainly consists of preliminaries in sections 2, 3, and 4, introducing orthogonality relations on small categories, general colored operads, and algebras over colored operads. Section 5 introduces the relevant quantum field theory operad and constructs it in three different ways. Section 6 describes algebras over the quantum field theory operads, and gives the main result, which is that the category

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of such algebras is equivalent to the category of algebraic quantum field theories over the original category.

2. Orthogonal Categories

In this section we define orthogonal categories and the physically motivated notion of $\perp$-commutativity. We will first define an orthogonality relation on a small category, and then we will define orthogonal functors, which will play an important role in our quantum field theoretic interpretation in the remaining sections.

For the following definition, let $\mathbf{Mor}\,\mathcal{C}_t \times_t \mathbf{Mor}\,\mathcal{C}$ be pairs of morphisms that share a target object.

Definition 2.1. Let $\mathcal{C}$ be a small category. An *orthogonality relation* $\perp$ on $\mathcal{C}$ is $\perp \subset \mathbf{Mor}\,\mathcal{C}_t \times_t \mathbf{Mor}\,\mathcal{C}$ such that

(1) $(f_1, f_2) \in \perp \implies (f_2, f_1) \in \perp$
(2) $(f_1, f_2) \in \perp \implies (gf_1h_1, gf_2h_2) \in \perp$ for all composable g, h_1, h_2

Remark 2.2. Allowing $\perp$ to be the entirety of $\mathbf{Mor}\,\mathcal{C}_t \times_t \mathbf{Mor}\,\mathcal{C}$ is a valid orthogonality relation. In addition, setting $\perp$ to be the empty set is also a valid orthogonality relation. The latter of these two will be useful later.

A pair $(\mathcal{C}, \perp) := \bar{\mathcal{C}}$ is called an orthogonal category. An orthogonal functor is a functor F between two orthogonal categories $(\mathcal{C}, \perp_{\mathcal{C}})$ and $(\mathcal{D}, \perp_{\mathcal{D}})$ such that

$$(f_1, f_2) \in \perp_{\mathcal{C}} \implies (Ff_1, Ff_2) \in \perp_{\mathcal{D}} \,. \tag{2.3}$$

These orthogonal functors are morphisms in the category of orthogonal categories, denoted **OrthCat**.

Notation 2.4. We write the denote the category of associative and unital algebras in a category M by $\mathbf{Alg}(M)$.

Definition 2.5. Let $\bar{\mathcal{C}}$ be an orthogonal category. A functor $\mathfrak{U}\colon \mathcal{C} \to \mathbf{Alg}(M)$ (see Notation 3.5) is $\perp$-commutative over c if for every $(f_1, f_2) \in \perp$ such that f_1 (and therefore also f_2) has target c, multiplication of the images of $\mathfrak{U}(f_1)$ and $\mathfrak{U}(f_2)$ is commutative. That is, the following diagram commutes

$$\begin{tikzcd}
\mathfrak{U}(c_1) \otimes \mathfrak{U}(c_2) \arrow[r, "\mathfrak{U}(f_1)\otimes\mathfrak{U}(f_2)"] \arrow[d, "\mathfrak{U}(f_1)\otimes\mathfrak{U}(f_2)"] & \mathfrak{U}(c) \otimes \mathfrak{U}(c) \arrow[d, "\mu_c^{op}"] \\
\mathfrak{U}(c) \otimes \mathfrak{U}(c) \arrow[r, "\mu_c"] & \mathfrak{U}(c)
\end{tikzcd}$$

where μ_c is the multiplication on $\mathfrak{U}(c)$ and $\mu_c^{op} = \mu_c \tau$, with τ the symmetric braiding of our symmetric monoidal category M.

A functor $\mathfrak{U}\colon \mathcal{C} \to \mathbf{Alg}(M)$ is $\perp$-commutative if it is $\perp$-commutative on all $c \in \mathbf{Ob}\,\mathcal{C}$.

Recall that our goal will later be to describe algebraic quantum field theories, which can be thought of imprecisely as functors from manifolds to algebras in some category respecting physically-motivated axioms such as inclusion and causal locality. The causal locality characteristic is where the notion of orthogonal categories and functors enters the picture. Intuitively, we want to ensure that observables from causally disjoint regions of spacetime satisfy some notion of independence from one another in their algebras. This notion of independence is commutativity.

Therefore, if on a small category we define an orthogonality relation that suitably describes some notion of causal disjointedness, orthogonal functors into some category of algebras will obey the causal locality we require of algebraic quantum field theories. So we want to build algebraic quantum field theories not only as functors from spacetime regions into a category of algebras, but as *orthogonal* functors from an orthogonal category of spacetime regions into a category of algebras. This idea is made more precise in Section 6.

3. Colored Operads

Here, we define and characterize the category of colored operads, which we will use to encode in our algebraic quantum field theory functors the orthogonality described above.

Definition 3.1. Let $\mathfrak{C}$ be a set. A *colored operad* (in *Set*) with colors $\mathfrak{C}$ has, for all $n \geq 0$, for all $\underline{c} \in \mathfrak{C}^n$ and $t \in \mathfrak{C}$:

(1) a set $\mathcal{O}\binom{t}{\underline{c}}$
(2) (Symmetric group action) for all $\sigma \in S_n$ the symmetric group on n letters, a map
$$\mathcal{O}(\sigma) : \mathcal{O}\binom{t}{\underline{c}} \to \mathcal{O}\binom{t}{\underline{c}\sigma},$$
where $\underline{c}\sigma = (c_{\sigma(1)}, \dots, c_{\sigma(n)})$
(3) (Composition) for $k_i \geq 0$, $b_i \in \mathfrak{C}^{k_i}$, $i = 1, \dots, n$, a map
$$\gamma : \mathcal{O}\binom{t}{\underline{c}} \times \prod_{i=1}^{n} \mathcal{O}\binom{c_i}{\underline{b_i}} \to \mathcal{O}\binom{t}{(b_{11}, \dots, b_{1k_1}, \dots, b_{nk_n})}$$
(4) A unit element $\mathbb{1} \in \mathcal{O}\binom{t}{t}$

One can think of the set $\mathcal{O}\binom{t}{\underline{c}}$ as a set of operations from the colors in $\underline{c}$ to the color t, and so the maps γ are maps performing composition of these operations. Accordingly, those sets and maps are subject to unitality, associativity, and S_n-equivariance axioms. Namely,

(1) (Unitality) The unit $\mathbb{1}$ acts as a unit in the appropriate composition maps (pre-composition and post-composition)
(2) (Associativity) The composition maps are associative
(3) (S_n-equivariance) By converting permutations to block permutations where appropriate, the composition maps are equivariant with respect to the S_n-action on the $\mathcal{O}\binom{t}{\underline{c}}$

For a more complete definition of colored operads, see [6] Definition 11.2.1. If the reader is familiar with the notion of a multicategory, notice that a colored operad is the same as a symmetric multicategory, up to relabeling colors as objects and the sets $\mathcal{O}\binom{t}{\underline{c}}$ as multimorphisms. However, we will not need this perspective on colored operads for the contents of this paper.

Example 3.2. Consider the case in which the set of colors $\mathfrak{C}$ is a singleton, $\{*\}$. This gives an uncolored operad, and so we label the sets only by the "arity" of their operations, as it indicates the only color. There is an *associative operad Assoc*, whose n-ary operations $Assoc(n)$ are the underlying sets of S_n. The symmetric group action is given by right multiplication in S_n, and the unit is given by the unique element of S_1.

The composition maps are given by block permutations. Consider $\sigma \in S_n$ and $k_i \geq 0$ for $i = 1, \ldots, n$ such that $\sum_{i=1}^{n} k_i = k$. Let $\sigma_i \in S_{k_i}$. Then

$$\gamma : As(n) \otimes \bigotimes_{i=1}^{n} As(k_i) \to As(k) \tag{3.3}$$

is given by multiplication in the group S_k. The first appropriate element of S_k is the direct sum $\sigma_{k_1} \oplus \cdots \oplus \sigma_{k_n}$, which acts by σ_{k_1} on the first k_1 elements, σ_{k_2} on the next k_2, and so on. The composition map multiplies this element on the left by the block permutation given by σ, which divides k letters into blocks of size k_i in order, then permutes those blocks by σ.

The group properties of S_n make these follow the axioms given above.

Now that we have defined the objects in the category of colored operads, we turn to the morphisms. Given two colored operads $\mathcal{O}$ and $\mathcal{P}$ with color sets $\mathfrak{C}$ and $\mathfrak{D}$ respectively, one can define a morphism between them as follows.

Definition 3.4. A *morphism of colored operads* $\phi : \mathcal{O} \to \mathcal{P}$ is given by:

(1) A map of the color sets, $\tilde{\phi} : \mathfrak{C} \to \mathfrak{D}$
(2) For all $t \in \mathfrak{C}$, $n \geq 0$, $\underline{c} \in \mathfrak{C}$, a map of sets $\phi : \mathcal{O}\binom{t}{\underline{c}} \to \mathcal{P}\binom{\tilde{\phi}(t)}{\tilde{\phi}(\underline{c})}$, where $\tilde{\phi}$ is applied componentwise to $\underline{c}$

such that permutation action, composition, and units are respected.

Given a colored operad $\mathcal{O}$, one can forget the S_n-action, the composition, and the unit. Reducing our object to the set of colors and associated sets of operations, one obtains a colored sequence. Let $\mathbf{Op}_{\mathfrak{C}}$ be the category of $\mathfrak{C}$-colored operads, and $\mathbf{Seq}_{\mathfrak{C}}$ the category of $\mathfrak{C}$-colored sequences. Then we let U be the forgetful functor taking a colored operad to the colored sequence with the same colors and sets of operations.

By Theorem 2.3 and Proposition 2.5 of [1], we have the following two statements.

Theorem 3.5. *The forgetful functor U: $\mathbf{Op}_{\mathfrak{C}} \to \mathbf{Seq}_{\mathfrak{C}}$ is right adjoint to a free $\mathfrak{C}$-colored operad functor, F: $\mathbf{Seq}_{\mathfrak{C}} \to \mathbf{Op}_{\mathfrak{C}}$. That is, for a $\mathfrak{C}$-colored sequence S and $\mathfrak{C}$-colored operad $\mathcal{O}$, morphisms of colored operads $\mathbf{Op}_{\mathfrak{C}}(FS, \mathcal{O})$ correspond via natural isomorphism to morphisms of colored sequences $\mathbf{Seq}_{\mathfrak{C}}(S, U\mathcal{O})$.*

Proposition 3.6. *$\mathbf{Seq}_{\mathfrak{C}}$ and $\mathbf{Op}_{\mathfrak{C}}$ are both complete and cocomplete.*

In particular, this implies that diagrams in either of these categories have colimits and coequalizers, which we will later make use of.

4. Algebras over Colored Operads

The definition of colored operads in the previous section noted that the sets $\mathcal{O}\binom{t}{\underline{c}}$ can be thought of as sets of operations between objects of different colors. Considering the operads in isolation, we are mainly concerned with how such operations compose and permute with one another, but here we realize these sets as operations on some objects by defining algebras over operads.

Let M be any symmetric monoidal category with unit I. We further assume that M is bicomplete, and therefore has a Set-tensoring, which we denote by $S \otimes m = \amalg_{s\in S} m$. Notice that this is an object of M.

Definition 4.1. Let $\mathcal{O}$ be a $\mathfrak{C}$-colored operad. An $\mathcal{O}$-algebra A in M consists of:

(1) For each $t \in \mathfrak{C}$, an object A_t in M
(2) ($\mathcal{O}$-action) For each $t \in \mathfrak{C}$, $n \geq 0$, and $\underline{c} \in \mathfrak{C}^n$, a morphism in M

$$\alpha : \mathcal{O}\binom{t}{\underline{c}} \otimes A_{\underline{c}} \to A_t \tag{4.2}$$

where $A_{\underline{c}} = \bigotimes_{i=1}^n A_{c_i}$. For $o \in \mathcal{O}\binom{t}{\underline{c}}$, we write $\alpha(o){:}A_{\underline{c}} \to A_t$ for the component of α at o

(3) For each $t \in \mathfrak{C}$, letting $\mathbb{1}_t \in \mathcal{O}\binom{t}{t}$ be its unit element in the operad $\mathcal{O}$, an identity map $\alpha_{\mathbb{1}_t} : A_t \to A_t$

We write $\mathbf{Alg}_{\mathcal{O}}(M)$ for the category of $\mathcal{O}$-algebras in M.

The maps α are also subject to similar equivariance and associativity axioms with respect to the permutation action and composition of $\mathcal{O}$. For a more complete definition including these axioms, see [6] Definition 13.2.3.

A morphism κ: $A \to B$ in $\mathbf{Alg}_{\mathcal{O}}(M)$ is a family of M-morphisms κ_t: $A_t \to B_t$ for each $t \in \mathfrak{C}$ such that, with κ standing in for the appropriate κ_t,

$$\kappa\alpha_A = \alpha_B(id \otimes \bigotimes_{i=1}^{n} \kappa). \tag{4.3}$$

Essentially, our morphism must be compatible with $\mathcal{O}$-action.

Example 4.4. Recall the associative operad $Assoc$ from Example 3.2. As an uncolored operad, an $Assoc$-algebra in M has only one object A_* in M, corresponding to the single color of the operad. Then an $Assoc$-algebra A is defined by a choice of such object and morphisms α: $Assoc(n) \otimes A_*^{\otimes n} \to A_*$. Since $Assoc(n)$ is S_n, these maps α correspond, for a given n, to different ways of multiplying n variables, each ordering of the variables in multiplication corresponding to a different permutation. Notice that a priori, these different way of multiplying the n variables are not necessarily the same. Since $Assoc$, as an operad, contains a unit, the category of $Assoc$-algebras in M, $\mathbf{Alg}_{Assoc}(M)$, is the category of associative and unital algebras in M, hence the name.

Notation 4.5. As this is an important category, we denote the category of associative and unital algebras in M by $\mathbf{Alg}(M) := \mathbf{Alg}_{Assoc}(M)$. Notice that this is compatible with Notation 2.4, when we used $\mathbf{Alg}(M)$ to define orthogonal functors.

Before moving on, it is worth making more explicit why this category is important to our quantum field theoretic motivations. Recall that algebraic quantum field theories assign to regions of spacetime associative algebras in a functorial, physically-motivated way. Up to a choice of category M, this category $\mathbf{Alg}(M)$ is precisely the target of these quantum field theory functors! Recalling Definition 2.5, which defines orthogonal functors, we can make the following definition.

Definition 4.6. Let $\bar{\mathcal{C}} = (\mathcal{C}, \perp)$ be an orthogonal category, and let M be any category. We define $\mathbf{Alg}(M)^{\bar{\mathcal{C}}}$ to be the functor category of $\perp$-commutative functors from $\bar{\mathcal{C}}$ to the category of associative and unital algebras in M.

Letting $\bar{\mathcal{C}}$ be a category of spacetimes of interest with an appropriate orthogonality relation, the category $\mathbf{Alg}(M)^{\bar{\mathcal{C}}}$ is then precisely the category of algebraic quantum field theories for this setting. We characterize this category using an operad which we develop in Section 5 and looking at algebras over that operad. Therefore, it is important to us that we further understand the relationships between functor categories and operads.

Now, we develop a diagram of adjunctions to help navigate between algebras over different operads, and eventually between different quantum field theories.

Consider $\mathfrak{C}$ as a discrete category. That is, as a category whose only morphisms are the identity morphisms on each object. Denote by $M^{\mathfrak{C}}$ the category of functors from $\mathfrak{C}$ to M. Then we can define the forgetful functor

$$U_{\mathcal{O}} : \mathbf{Alg}_{\mathcal{O}}(M) \to M^{\mathfrak{C}} \tag{4.7}$$

by mapping an $\mathcal{O}$-algebra A to the functor F_A, with $F_A(t) = A_t$ for t in $\mathfrak{C}$. By Theorem 2.9 in [1], this functor $U_{\mathcal{O}}$ is right adjoint to the free $\mathcal{O}$-algebra functor $F_{\mathcal{O}}\colon M^{\mathfrak{C}} \to \mathbf{Alg}_{\mathcal{O}}(M)$. We now give a construction of the free $\mathcal{O}$-algebra functor.

First, let $S_{\mathfrak{C}}$ be the groupoid with objects $\underline{c}$ of arbitrary finite length n and morphisms right permutations $\underline{c} \to \underline{c}\sigma$ for $\sigma \in S_n$.

Proposition 4.8. *Let X be an object in the functor category $M^{\mathfrak{C}}$. The free $\mathcal{O}$-algebra $F_{\mathcal{O}}(X)$ has objects given by the coend over the permutation groupoid,*

$$F_{\mathcal{O}}(X)_t := \int^{\underline{a}\in S_{\mathfrak{C}}} \mathcal{O}\binom{t}{\underline{a}} \otimes X(\underline{a}), \tag{4.9}$$

where $X(\underline{a})$ is the $|\underline{a}|$-fold tensor product of X applied componentwise. Notice that each $\mathcal{O}\binom{t}{\underline{a}} \otimes X(\underline{a})$ is an object in M by the Set-tensoring, and so $F_{\mathcal{O}}(X)_t$ is as well.

The $\mathcal{O}$-action is given by the composition maps γ in $\mathcal{O}$ and application of the coend universal property.

The coend $F_{\mathcal{O}}(X)_t$ can also be described as the coequalizer in the diagram

$$\amalg_{\underline{a}\to\underline{a}'}\mathcal{O}\binom{t}{\underline{a}'} \otimes X_{\underline{a}} \rightrightarrows \amalg_{\underline{a}\in S_{\mathfrak{C}}}\mathcal{O}\binom{t}{\underline{a}} \otimes X_{\underline{a}} \to F_{\mathcal{O}}(X)_t,$$

where the coproduct on the left should be taken for all pairs of objects which are related by a $S_{\mathfrak{C}}$-morphism. The appropriate parallel arrows are then those that replace a' with a or a with a', as written. As $F_{\mathcal{O}}(X)_t$ is the coequalizer, it then identifies those elements whose $\mathcal{O}\binom{t}{\underline{a}}$ components differ only by a permutation of the colors in $\underline{a}$.

Now, assume we have a morphism $\phi\colon \mathcal{O} \to \mathcal{P}$ of operads. From the earlier definition of $\mathbf{Op}$-morphisms, ϕ has an underlying map of colors, $\tilde{\phi}\colon \mathfrak{C} \to \mathfrak{D}$. Along this map of colors, there exists the pullback functor $\tilde{\phi}^*$. Concretely, for a functor F in $M^{\mathfrak{D}}$ and $c \in \mathfrak{C}$,

$$\tilde{\phi}^*(F)(c) = F\tilde{\phi}(c). \tag{4.10}$$

This functor has a left adjoint, $\tilde{\phi}_!(F)$, which we can construct in the following manner.

If F is a functor from $\mathfrak{C}$ to M, then let $\tilde{\phi}_!(F)$ be the left Kan extension of F along $\tilde{\phi}$. That is, $\tilde{\phi}_!(F)$ is the universal dotted arrow in the diagram

$$\begin{tikzcd} \mathfrak{D} \arrow[dr, dashed, "\tilde{\phi}_!(F)"] & \\ \mathfrak{C} \arrow[u, "\tilde{\phi}"] \arrow[r, "F"'] & M \end{tikzcd}$$

with a natural transformation ϵ from F to $\tilde{\phi}_!(F)\tilde{\phi}$. $\tilde{\phi}_!(F)$ is universal in the sense that for any other functor $G : \mathfrak{D} \to M$ with a natural transformation $\alpha : F \implies G\tilde{\phi}$, there exists a natural transformation $\sigma : \tilde{\phi}_!(F) \to G$ such that the following diagram commutes:

$$\begin{tikzcd} \tilde{\phi}_!(F)\tilde{\phi} \arrow[dr, dashed, "\sigma_{\tilde{\phi}}"] & \\ F \arrow[u, "\epsilon"] \arrow[r, "\alpha"'] & G\tilde{\phi} \end{tikzcd}$$

where for $c \in \mathbf{Ob}\,\mathfrak{C}$,

$$\sigma_{\tilde{\phi}}(c) = \sigma(\tilde{\phi}c) : \tilde{\phi}_!(F)\tilde{\phi}(c) \to G\tilde{\phi}(c) \tag{4.11}$$

Finally, again given a **Op**-morphism ϕ: $\mathcal{O} \to \mathcal{P}$, we can obtain a pullback functor of algebras, $\tilde{\phi}$: $\mathbf{Alg}_{\mathcal{P}}(M) \to \mathbf{Alg}_{\mathcal{O}}(M)$. For B a $\mathcal{P}$-algebra in M,

$$\tilde{\phi}(B)_t := B_{\tilde{\phi}(t)} \tag{4.12}$$

gives the objects in the $\mathcal{O}$-algebra. The $\mathcal{O}$-action is given by

$$\mathcal{O}\binom{t}{\underline{c}} \otimes \tilde{\phi}(B)_{\underline{c}} \xrightarrow{\phi \otimes id} \mathcal{P}\binom{\tilde{\phi}(t)}{\tilde{\phi}(\underline{c})} \otimes B_{\tilde{\phi}(\underline{c})} \xrightarrow{\alpha_B} B_{\tilde{\phi}(t)}$$

.

Altogether, this gives is the following diagram of adjunctions:

$$\begin{tikzcd} \mathbf{Alg}_{\mathcal{O}}(M) \arrow[d, bend left, "U_{\mathcal{O}}"] & & \mathbf{Alg}_{\mathcal{P}}(M) \arrow[ll, "\phi^*"'] \arrow[d, bend left, "U_{\mathcal{P}}"] \\ M^{\mathfrak{C}} \arrow[u, bend left, "F_{\mathcal{O}}"] \arrow[rr, bend left, "\tilde{\phi}_!"] & & M^{\mathfrak{D}} \arrow[u, bend left, "F_{\mathfrak{P}}"] \arrow[ll, bend left, "\tilde{\phi}^*"] \end{tikzcd}$$

By the Adjoint Lifting Theorem (see [5] Chapter 4.5) as applied in [1] Theorem 2.11, this is enough to show that ϕ^* has a left adjoint functor $\phi_!$, which is given as a reflexive coequalizer in Proposition 2.12 of [1]. Therefore, we have the following theorem.

Theorem 4.13. *Given an **Op**-morphism ϕ: $\mathcal{O} \to \mathcal{P}$, there exists an adjunction*

$$\phi_! : \boldsymbol{Alg}_{\mathcal{O}}(M) \rightleftarrows \boldsymbol{Alg}_{\mathcal{P}}(M) : \phi^*$$

.

So as long as we can find a morphism between two operads, we have a somewhat well-defined relationship between their categories of algebras.

5. Operads for Quantum Field Theory

In this section, we first define an auxiliary colored operad that is close to our quantum field theory operad, but without taking into consideration orthogonality. Then, we present the quantum field theory operad with built-in $\perp$-commutativity in three ways: as a coequalizer, via a quotient map, and finally through generators and relations.

First, we define the auxiliary colored operad. Recall the uncolored operad *Assoc* from Example 2.2.

Definition 5.1. Let $\mathcal{C}$ be a small category. The colored operad $\mathcal{O}_\mathcal{C}$ has $\mathcal{C}_0 := \mathbf{Ob}\,\mathcal{C}$ as its set of colors.

(1) For all $t \in \mathcal{C}_0$, $n \geq 0$, $\underline{c} \in \mathcal{C}_0^n$,

$$\mathcal{O}_\mathcal{C}\binom{t}{\underline{c}} := Assoc(n) \times \mathcal{C}(\underline{c}, t) \tag{5.2}$$

where $\mathcal{C}(\underline{c}, t) = \prod_{i=1}^n \mathcal{C}(c_i, t)$. We write elements of these sets as $(\sigma, \underline{f})$, where $\underline{f}$ is the n-fold product of $f_i{:}c_i \to t$.

(2) For $\sigma' \in S_n$, we have

$$\mathcal{O}_\mathcal{C}(\sigma')(\sigma, \underline{f}) = (\sigma\sigma', \underline{f}\sigma') \tag{5.3}$$

where σ' acts on σ by right multiplication in S_n and acts on $\underline{f}$ by permuting the elements of the n-tuple.

(3) The composition maps act as

$$\gamma((\sigma, \underline{f}); ((\sigma_1, \underline{g_1}), \dots, (\sigma_n, \underline{g_n})) = (\sigma(\sigma_1, \dots, \sigma_n), \underline{f}(\underline{g_1}, \dots, \underline{g_n})) \tag{5.4}$$

where σ acts by block permutations as in *Assoc*, and $\underline{f}(\underline{g_1}, \dots, \underline{g_n}) = (f_1 g_{11}, \dots, f_1 g_{1k_1}, f_2, g_{21}, \dots, f_n g_{nk_n})$.

Remark 5.5. One can understand these operations in the context of quantum field theory as a 3-step process. First, we apply the morphisms $\underline{f}$ to move $\underline{c}$ to t^n. Then, we permute the observables on t^n by right action of σ. Finally, we multiply the observables according to the order in which they appear. Notice that the multiplication is not necessarily commutative, but we will want to implement such commutativity if a pair of observables come from orthogonal morphisms.

Following this approach, we define a colored sequence containing those pairs of morphisms whose observables should commute.

Definition 5.6. $P_\perp$ is the $\mathcal{C}_0$-colored sequence such that for $n \neq 2$, $t \in \mathcal{C}_0$, and $\underline{c} \in \mathcal{C}_0^n$, $P_\perp\binom{t}{\underline{c}} = \emptyset$, and for $n = 2$, $\underline{c} \in \mathcal{C}_0^2$, $P_\perp\binom{t}{\underline{c}} = \perp \cap \mathcal{C}(\underline{c}, t)$. That is, for each set of three objects in $\mathcal{C}$, labeled c_1, c_2, and t, the associated set in $P_\perp$ is the set of pairs of $\perp$-commutative functors, one from $c_1 \to t$ and the other $c_2 \to t$.

Recall that we have the forgetful functor U: $\mathbf{Op}_\mathcal{C} \to \mathbf{Seq}_\mathcal{C}$ taking a colored operad to a colored sequence. We can define parallel morphisms in $\mathbf{Seq}_\mathcal{C}$ from $P_\perp$ to a colored sequence closer to our quantum field theory operad as follows.

Definition 5.7. The sets in $P_\perp$ contain pairs of morphisms, so we can write an element of a set in $P_\perp$ as (f_1, f_2), where f_1 and f_2 are orthogonal morphisms with a shared target. On the other hand, the sets in $\mathcal{O}_\mathcal{C}$ with color vector length $n = 2$ are $Assoc(2) \times (f_1, f_2)$, where f_1, f_2 are some morphisms with a common target. We then have two morphisms of sequences, $s_{\perp,1}$ and $s_{\perp,2}$, that correspond to the two choices of an element in $Assoc(2)$. $s_{\perp,1}, s_{\perp,2}\colon P_\perp \rightrightarrows U\mathcal{O}_\mathcal{C}$ are given on elements by:

(1) $s_{\perp,1}(f_1, f_2) := (e, (f_1, f_2))$
(2) $s_{\perp,2}(f_1, f_2) := (\tau, (f_1, f_2))$

where e and τ are the two elements of S_2.

Recall our earlier quantum field theoretic interpretation of these sets of operations in Remark 5.5. Considering these operations sets as parts of operads, the images of (f_1, f_2) under $s_{\perp,1}$ and $s_{\perp,2}$ correspond to applying the morphisms f_1 and f_2, and then multiplying the resulting observables in either the same or opposite order given, respectively. By assumption, $(f_1, f_2) \in \perp$, and so we want these two operations to be the same. This leads to the following definition.

Definition 5.8. For the orthogonal category $\bar{\mathcal{C}}$, the associated quantum field theory operad $\mathcal{O}_{\bar{\mathcal{C}}} \in \mathbf{Op}_{\mathcal{C}_0}$ is the coequalizer

$$\mathcal{O}_{\bar{\mathcal{C}}} := \mathrm{colim}\left(\begin{tikzcd} F(P_\perp) \arrow[r, shift left, "s_{\perp,1}"] \arrow[r, shift right, "s_{\perp,2}"'] & \mathcal{O}_\mathcal{C} \end{tikzcd} \right) \tag{5.9}$$

where $F(P_\perp)$ is the free $\mathcal{C}_0$-colored operad generated by $P_\perp$.

We can equivalently define $\mathcal{O}_{\bar{\mathcal{C}}}$ as a levelwise quotient of $\mathcal{O}_\mathcal{C}$. The relevant equivalence relation is as follows.

Definition 5.10. Within each set $\mathcal{O}_\mathcal{C}\binom{t}{\underline{c}}$, we say that $(\sigma, \underline{f}) \sim_\perp (\sigma', \underline{f'})$ if and only if

(1) $\underline{f} = \underline{f'}$
(2) The map $\sigma\sigma'^{-1}\colon \underline{f}\sigma^{-1} \mapsto \underline{f}\sigma'^{-1}$ is a sequence of transpositions of adjacent pairs of orthogonal morphisms. That is, $\sigma\sigma'^{-1} = \tau_1 \dots \tau_N$, where N is a positive integer and $\tau_1, \dots, \tau_N \in S_n$ are transpositions such that for all $k = 1, \dots, N$,

$$\tau_k : \underline{f}\sigma^{-1}\tau_1 \dots \tau_{k-1} \mapsto \underline{f}\sigma^{-1}\tau_1 \dots \tau_k \tag{5.11}$$

is a transposition of a pair of $\mathcal{C}$-morphisms that are orthogonal (in $\perp$ as a pair) and adjacent in the sequence of morphisms $\underline{f}\sigma^{-1}\tau_1 \dots \tau_{k-1}$.

Proposition 5.12. *The quotient map $p_{\bar{\mathcal{C}}}\colon \mathcal{O}_\mathcal{C} \to \mathcal{O}_\mathcal{C}/\sim_\perp$ gives an operad equivalent to that given in Definition 5.8. That is, $\mathcal{O}_\mathcal{C}/\sim_\perp = \mathcal{O}_{\bar{\mathcal{C}}}$.*

We leave the proof of this to Proposition 3.9 of [1], but provide an example.

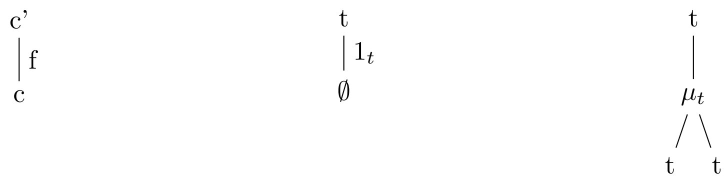
Example 5.13. Let $(f_1, f_2) \in \perp$ for $\perp$ an orthogonality relation on $\bar{\mathcal{C}}$. Then $(e, (f_1, f_2)) \sim_\perp (\tau, (f_1, f_2))$. We have that $(f_1, f_2) = (f_1, f_2)$ as elements in $\mathbf{Mor}\,\bar{\mathcal{C}}$. To see that the two operations are congruent under $\sim_\perp$, we want to examine the map $e\tau^{-1} = \tau$: $(f_1, f_2)e^{-1} = (f_1, f_2) \mapsto (f_1, f_2)\tau^{-1} = (f_2, f_1)$. This permutation is a single transposition, exchanging f_1 and f_2. Since f_1 and f_2 are adjacent in (f_1, f_2) and orthogonal, $(e, (f_1, f_2)) \sim_\perp (\tau, (f_1, f_2))$.

Before presenting $\mathcal{O}_{\bar{\mathcal{C}}}$ from generators and relations, we note that these first two constructions are functorial by Proposition 3.11 of [1].

Proposition 5.14. *The assignment of an orthogonal category $\bar{\mathcal{C}} \mapsto \mathcal{O}_{\bar{\mathcal{C}}}$ is a functor from the category **OrthCat** to the category **Op**. The assignment of objects is as given in Definition 5.8 and Proposition 5.12. To an orthogonal functor $F : \bar{\mathcal{C}} \to \bar{\mathcal{D}}$, a morphism in **OrthCat**, we assign the **Op**-morphism with the underlying map of colors $F : \mathcal{C}_0 \to \mathcal{D}_0$ and mapping of sets $\mathcal{O}_{\bar{\mathcal{C}}}\binom{t}{\underline{c}} \to \mathcal{O}_{\bar{\mathcal{D}}}\binom{F(t)}{F(\underline{c})}$ given by $(\sigma, \underline{f}) \mapsto (\sigma, F(\underline{f}))$, where $F(\underline{f})$ is to be applied componentwise.*

We now give a presentation of our operad by generators and relations. First, we will construct an auxiliary operad that does not depend on a choice of $\perp$ Then, by adding in appropriate relations, we convert this auxiliary operad into our quantum field theory operad. We will present the operations graphically using trees meant to be read from the bottom, upwards.

First, consider the generators $G_{\mathcal{C}}$: for each $t \in \mathcal{C}_0$ and $(f : c \to c') \in \mathbf{Mor}\,\mathcal{C}$, we have



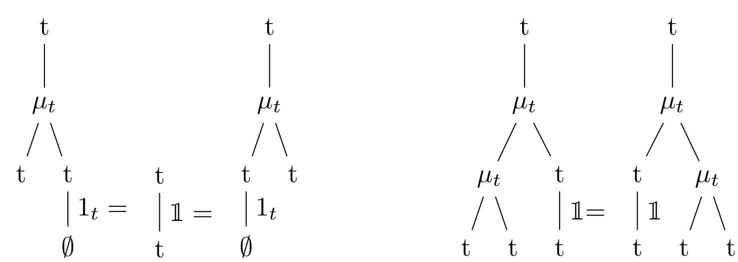
These give a sequence with, for example, $G_{\mathcal{C}}\binom{c'}{c}$ the set of morphisms from c to c'.

Similarly, we define three more sequences, representing the relations.

(1) The functoriality relations R_{Fun}

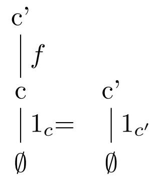
for every $t \in \mathcal{C}_0$ and pairs of composable morphisms f and g

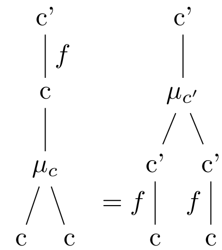
(2) The algebra relations R_{Alg}



for every $t \in \mathcal{C}_0$

(3) The compatibility relations R_{FA}





Notice that for each equality of trees, the two sides of the equation are not a priori the same in $F(G_{\mathcal{C}})$.

Definition 5.15. Consider R_{Fun}, R_{Alg}, and R_{FA} as colored sequences. Let

$$R_{\mathcal{C}} := R_{Fun} \sqcup R_{Alg} \sqcup R_{FA} \tag{5.16}$$

be all of these relations taken together.

We construct two **Seq**-morphisms $r_{\mathcal{C},i}\colon R_{\mathcal{C}} \to G_{\mathcal{C}}$ for $i = 1, 2$. Let $r_{\mathcal{C},1}$ act like an embedding functor, taking trees in $R_{\mathcal{C}}$ to the tree in $G_{\mathcal{C}}$ that is the same as the left side of that tree's equality in $R_{\mathcal{C}}$. Let $r_{\mathcal{C},2}$ act in the same way, but taking trees in $R_{\mathcal{C}}$ to trees in $G_{\mathcal{C}}$ that are the same as the right side of the equality. These morphisms behave the same way under application of the free operad functor, respecting the usual axioms.

We also consider the **Op**-morphism $q_{\mathcal{C}}$, which we will use to quotient our freely generated operad down to the operad $\mathcal{O}_{\mathcal{C}}$ in the following proposition. We define the morphism by giving its action on the generators of $F(G_{\mathcal{C}})$, which defines its actions across the operad by its properties as an **Op**-morphism.

$$q_{\mathcal{C}}(1_t) = (e, *) \tag{5.17}$$

$$q_{\mathcal{C}}(f) = (e, f) \tag{5.18}$$

$$q_{\mathcal{C}}(\mu_t) = (e, (id_t, id_t)) \tag{5.19}$$

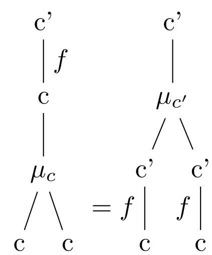
Here, $*$ is the tuple of length 0.

Proposition 5.20. *Let $\mathcal{C}$ be a small category. Then with F the free operad functor, $\mathcal{O}_{\mathcal{C}}$ is the coequalizer in $\boldsymbol{Op}_{\mathcal{C}_0}$ in the diagram*

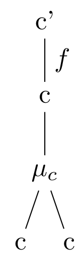
$$\begin{tikzcd} F(R_{\mathcal{C}}) \arrow[r, shift left, "r_{\mathcal{C},1}"] \arrow[r, shift right, "r_{\mathcal{C},2}"'] & F(G_{\mathcal{C}}) \arrow[r, "q_{\mathcal{C}}"] & \mathcal{O}_{\mathcal{C}} \end{tikzcd}$$

As a bit of insight into how this coequalizer applies the relations to the generators, we give a calculation showing that $q_{\mathcal{C}} r_{\mathcal{C},1} = q_{\mathcal{C}} r_{\mathcal{C},2}$.

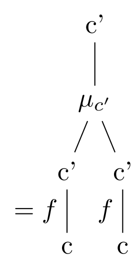
Example 5.21. Consider the equality on the right in the compatibility relations. The operation



under the morphism $r_{\mathcal{C},1}$ is



Under the morphism $r_{\mathcal{C},2}$, it is



The image under $r_{\mathcal{C},1}$, when thought of as an operation in an operad by applying F, is $\gamma_G(f;\mu_c)$. The image under $r_{\mathcal{C},2}$ is $\gamma_G(\mu_{c'};(f,f)$. Since $q_{\mathcal{C}}$ is an **Op**-morphism, we have that

(5.22)

$$q_{\mathcal{C}}(\gamma_G(f,\mu_c)) = \gamma_{\mathcal{O}}(q_{\mathcal{C}}(f), q_{\mathcal{C}}(\mu_c)) = \gamma_{\mathcal{O}}((e,f),(e,(id_c,id_c))) = (e,(fid_c,fid_c)) = (e,(f,f))$$

Similarly, we have

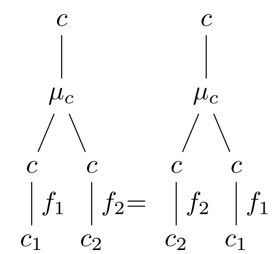
(5.23)

$$q_{\mathcal{C}}(\gamma_G(\mu_{c'};(f,f))) = \gamma_{\mathcal{O}}(q_{\mathcal{C}}(\mu_{c'});(q_{\mathcal{C}}(f),q_{\mathcal{C}}(f))) = \gamma_{\mathcal{O}}((e,(id_{c'},id_{c'}));((e,f),(e,f))) = (e,(f,f))$$

and so the compositions of **Op**-morphisms are equal on this relation.

Notice that this construction gives us the auxiliary operad, rather than the quantum field theory operad that takes orthogonality into consideration. However, we can recover that operad in the same manner by augmenting our relations.

Given $\bar{\mathcal{C}} = (\mathcal{C},\perp)$ an orthogonal category, let $R_{\bar{\mathcal{C}}} = R_{\mathcal{C}} \sqcup R_O$, where R_O are the orthogonality relations given by



for all $(f_1 : c_1 \to c, f_2 : c_2 \to c) \in \perp$. Then let $r_{\bar{\mathcal{C}},i}$ be the same as the morphisms $r_{\mathcal{C},i}$, extended to R_O in the same manner.

Theorem 5.24. *Let $\bar{\mathcal{C}} = (\mathcal{C}, \perp)$ be an orthogonal category. Then with F the free operad functor, $\mathcal{O}_{\bar{\mathcal{C}}}$ is the coequalizer in $\boldsymbol{Op}_{\mathcal{C}_0}$ in the diagram*

$$\begin{tikzcd} F(R_{\bar{\mathcal{C}}}) \arrow[r, shift left, "r_{\bar{\mathcal{C}},1}"] \arrow[r, shift right, "r_{\bar{\mathcal{C}},2}"'] & F(G_{\mathcal{C}}) \arrow[r, "p_{\bar{\mathcal{C}}} q_{\mathcal{C}}"] & \mathcal{O}_{\mathcal{C}} \end{tikzcd}$$

where $p_{\bar{\mathcal{C}}}$ is the quotient map from Proposition 5.12.

The proof of Theorem 5.22 can be found under Theorem 3.14 of [1].

6. Algebras over Quantum Field Theory Operads

In this section, we illustrate the connection between our operads $\mathcal{O}_{\bar{\mathcal{C}}}$ and algebraic quantum field theories over $\bar{\mathcal{C}}$.

Remark 6.1. A quantum field theory, specifically from the approach of algebraic quantum field theory, is typically defined by the Haag-Kastler axioms, originally given in [2]. For the purposes of this section, it is enough to consider a quantum field theory as a $\perp$-commutative functor from a category of spacetimes to a category of associative and unital algebras, where the orthogonality relation is generally defined in a way such that observables from causally disjoint regions of spacetime commute. Further, one can think of the morphisms in the spacetime category as inclusions of one spacetime into an encompassing one. Then two orthogonal morphisms can be thought of as embeddings from two smaller, causally disjoint regions into a single larger spacetime that contains both smaller spacetimes.

We use our presentation of the operad by generators and relations to show that the category of algebras over our quantum field theory operad is precisely the category of quantum field theories corresponding to our choice of small category $\mathcal{C}$ and orthogonality relation $\perp$. In doing so, we identify an operad which parametrizes the algebraic actions of a given algebraic quantum field theory, providing a different perspective from which one can view the relationship between quantum field theories over the same spacetimes.

Theorem 6.2. *Let $\bar{\mathcal{C}} = (\mathcal{C}, \perp)$ be an orthogonal category. The category of algebras in M over the colored operad $\mathcal{O}_{\bar{\mathcal{C}}}$, $\boldsymbol{Alg}_{\mathcal{O}_{\bar{\mathcal{C}}}}(M)$, is equivalent to $\boldsymbol{Alg}(M)^{\bar{\mathcal{C}}}$,the category of $\perp$-commutative functors from $\bar{\mathcal{C}}$ to the category of associative and unital algebras in M.*

Remark 6.3. Recall that the category of algebras in M over the associative operad, $Assoc$, is the same as the category of associative and unital algebras in M. Therefore, $\mathbf{Alg}(M) = \mathbf{Alg}_{Assoc}(M)$.

In light of Remark 6.1, Theorem 6.2 can be considered as stating that $\mathbf{Alg}_{\mathcal{O}_{\bar{\mathcal{C}}}}(M)$ is the category of valid quantum field theories on $\bar{\mathcal{C}}$ taking values in M.

Before proving Theorem 6.2, we recall what it means for two categories to be equivalent. For the following definitions, consider a functor F from a category $\mathcal{C}$ to a category $\mathcal{D}$.

Definition 6.4. F is essentially surjective if for every $Y \in \mathbf{Ob}\,\mathcal{D}$, there exists $X \in \mathbf{Ob}\,\mathcal{C}$ and an isomorphism in $\mathcal{D}$ from FX to Y.

Definition 6.5. F is full if for every $X, Y \in \mathbf{Ob}\,\mathcal{C}$, $F\colon \mathcal{C}(X,Y) \to \mathcal{D}(FX, FY)$ is surjective.

Definition 6.6. F is faithful if for every $X, Y \in \mathbf{Ob}\,\mathcal{C}$, $F\colon \mathcal{C}(X,Y) \to \mathcal{D}(FX, FY)$ is injective.

Definition 6.7. F is an equivalence of categories if it is essentially surjective, full, and faithful.

Let $F\colon \mathbf{Alg}_{\mathcal{O}_{\bar{\mathcal{C}}}}(M) \to \mathbf{Alg}(M)^{\bar{\mathcal{C}}}$ be the functor defined on objects in the following way. Let $A \in \mathbf{Alg}_{\mathcal{O}_{\bar{\mathcal{C}}}}(M)$. From Definition 4.1, A assigns to each object $c \in \mathcal{C}_0$ an object $A_c \in \mathbf{Ob}\,M$. Let the functor $F(A)$ act on objects by sending $F(A)(c)$ to A_c, equipped with an algebra structure. Since $\mathbf{Alg}(M) = \mathbf{Alg}_{Assoc}(M)$, giving the object an algebra structure is the same as defining maps

$$Assoc(n) \otimes A_c^{\otimes n} \to A_c. \tag{6.8}$$

This is accomplished by using the $\mathcal{O}_{\bar{\mathcal{C}}}$-action. Concretely, for $\sigma \in Assoc(n)$, $n \geq 0$,

$$\alpha_{Assoc}(\sigma) = \alpha_{\mathcal{O}_{\bar{\mathcal{C}}}}(\sigma, id^{\otimes n}). \tag{6.9}$$

Morphisms in $\mathbf{Alg}_{\mathcal{O}_{\bar{\mathcal{C}}}}(M)$ between objects A and B are maps κ in M from A_t to B_t for each $c \in \mathcal{C}_0$ such that, using (3.3), $\kappa\alpha_A = \alpha_B(id \otimes \bigotimes_{i=1}^{n} \kappa)$. With a slight abuse of notation, writing $F(A)(c) = A_c$, this corresponds to maps $F(A)(c) \to F(B)(c)$ that respect $\mathcal{O}_{\bar{\mathcal{C}}}$-action. By considering $Assoc$-action as $\mathcal{O}_{\bar{\mathcal{C}}}$-action with every morphism the identity morphism, we see that this implies the maps $F(A)(c) \to F(B)(c)$ respect $Assoc$-action.

Therefore, for F, G functors in $\mathbf{Alg}(M)^{\bar{\mathcal{C}}}$, and morphisms $f : c \to d$, the following diagram commuting on objects implies that it also commutes on the algebra structure, and so morphisms in $\mathbf{Alg}_{\mathcal{O}_{\bar{\mathcal{C}}}}(M)$ are natural transformations between the functors $\mathbf{Alg}(M)^{\bar{\mathcal{C}}}$, which are morphisms in that category.

$$\begin{tikzcd} A_c \arrow[r] \arrow[d, "Ff"] & B_c \arrow[d, "Gf"] \\ A_d \arrow[r] & B_d \end{tikzcd} \tag{6.10}$$

Proof. Now, we check that F is essentially surjective. Given $Y \in \mathbf{Ob}\,\mathbf{Alg}(M)^{\bar{\mathcal{C}}}$, let the mapping of objects in $\mathcal{C}_0$ to objects in M by Y give the assignment of objects for an algebra $A \in \mathbf{Alg}_{\mathcal{O}_{\bar{\mathcal{C}}}}(M)$. In addition, since our functor F forgets the action of the morphisms in the operad's operation sets, we may define the $\mathcal{O}_{\bar{\mathcal{C}}}$-action such that its restriction to the identity morphisms agrees with the *Assoc*-action.

To check that F is full, consider a morphism in $\mathbf{Alg}(M)^{\bar{\mathcal{C}}}$, which is a natural transformation in the diagram (6.10). Since this is given for every morphism f, including the identity morphism on each element of $\mathcal{C}_0$, this natural transformation gives maps $A_t \to B_t$ for each object t of $\mathcal{C}_0$. This gives a morphism κ such that F applied to this morphism is the desired natural transformation.

Finally, we check that F is faithful. Because the arrows in $\mathbf{Alg}(M)$ determine the morphism in $\mathbf{Alg}(M)^{\bar{\mathcal{C}}}$, as well as an $\mathbf{Op}$-morphism in $\mathbf{Alg}_{\mathcal{O}_{\bar{\mathcal{C}}}}(M)$, F, the maps F: $\mathbf{Alg}_{\mathcal{O}_{\bar{\mathcal{C}}}}(M)(X,Y) \to \mathbf{Alg}(M)^{\bar{\mathcal{C}}}(FX,FY)$ are injective. Therefore, F is faithful. $\square$

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