BILLIARD DYNAMICS AND MASUR'S CRITERION

ANDREY SHAPIRO

ABSTRACT. This paper presents an accessible introduction to the mathematical theory of billiards. Specifically, it covers Masur's criterion, a cornerstone of many other crucial theorems in the field. This paper also covers two such theorems: the Veech dichotomy and McMullen's trichotomy, both of which classify motion and topology on an important subset of tables - lattice surfaces. The paper is self contained and only a basic understanding of linear algebra is assumed.

CONTENTS

1. INTRODUCTION

Despite being a relatively new field, the theory of billiard dynamics is very broad and reaches into many different fields of mathematics and physics. It stems from the game of billiards where a ball is hit on a table and bounces around. For our purposes, the ball is a point mass subject to no friction or other forces, and its motion follows the simple physical law of reflection: the angle of incidence is equal to the angle of reflection. The theory studies the patterns, distribution, and asymptotic dynamics of the ball's path. It was quickly discovered that this field has many applications in dynamical systems, group theory, and even in the physical modeling of particle collisions. Usually this field of mathematics is approached from the perspectives of dynamical systems and complex analysis, but in an effort to keep this paper accessible and self-contained, we will be using group theory, geometry, and a very limited amount of linear algebra.

We adapt the geometric definition of translation surface in terms of polygons and edge identifications as presented by Wright [10].

Definition 1.1. A *translation surface* is a finite, compact, path-connected, union of polygons in $\mathbb{C}$ where the edges of each polygon are identified in pairs such that

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paired edges are parallel and of equal length. A translation surface is an equivalence class where two such collections of polygons are said to define the same translation surface if one can be turned into the other using the following two transformations:

1) Cutting the polygon along a straight line, with those two sides being identified.
2) Translation and gluing of two identified edges.

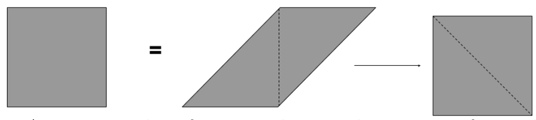


FIGURE 1. An example of 2 equal translation surfaces by cutting and gluing.

Definition 1.2. A *singularity* on a translation surface is a vertex of a polygon. (These will be the pockets of our billiard table).

Definition 1.3. A *saddle connection* is a line segment going between two singularities (which can be the same singularity) with no singularities on the interior of the line segment.

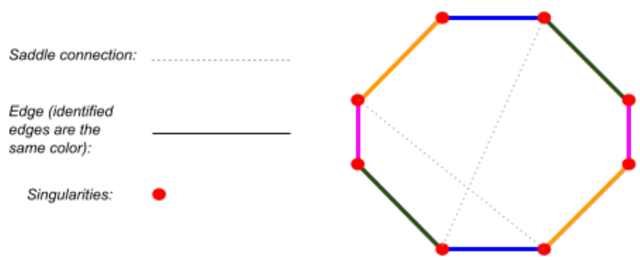
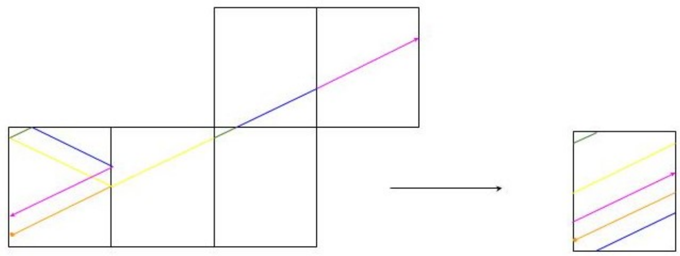


FIGURE 2. A translation surface and its vital components

Definition 1.4. A *billiard trajectory* in a translation surface S is defined by a point, or "billiard ball", in S and a direction. The ball travels in that direction until it reaches an edge, in which case, it continues at the identified point on the identified edge.

Definition 1.5. A billiard trajectory in a translation surface S is considered to be *periodic* if the ball returns to its original position and direction after some finite amount of time.

Remark 1.6. As shown in the diagram below, the trajectory of a billiard ball on a billiard table with rational angles can be represented by keeping the path of the billiard ball straight, and reflecting the table over its edge (this process is called *unfolding*). Furthermore, we can identify opposite edges and map the billiard ball trajectory onto one table as several parallel line segments. From this arises the use of translation surfaces. Although a trajectory in translation surface may look different than in the billiard table, the trajectory's important attributes are preserved.



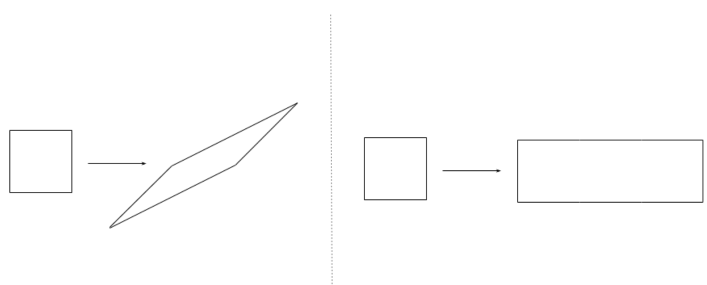
2. The g_t transformation and Masur's Criterion

Masur's criterion [4] is a crucial gateway to many theorems in billiards, such as the Veech dichotomy [7]. However, beforehand we must first introduce some key terms and a unique transformation (g_t) that can be applied to a translation surface which yields interesting results.

Definition 2.1. For a translation surface $S \in \mathbb{C}$, take arbitrary $x, y \in S$ and let P_x, P_y correspond to the path starting from x and y respectively, going in direction P. P is *uniquely ergodic* if for all such x, y, for any rectangle Q perpendicular to P, the average amount of time spent in Q by P_x is equal to the average amount of time spent in Q by P_y. This is written as $P_x(Q) = P_y(Q)$.

Definition 2.2. For our purposes, a *transformation* on a translation surface is an area-preserving 2-by-2 matrix by which every vector of the translation surface is multiplied. (Readers familiar with linear algebra will recognize that such transformations form the special linear group of 2-by-2 matrices with determinant ± 1 also known as SL(2,$\mathbb{R}$).)

As shown below, the matrix (left) $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ is a valid transformation while the matrix (right) $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ is not.



Definition 2.3. For all $t \in \mathbb{R}$, define the *geodesic flow* g_t as the transformation

$$\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}.$$

The matrices g_t for varying t comprise the diagonal subgroup of SL(2,$\mathbb{R}$) whose elements commute under matrix multiplication. We study the group SL(2,$\mathbb{R}$) and it main properties in depth later on in the paper.

We will now begin our approach to Masur's criterion [4], following the proof presented by Thierry Monteil [6].

Definition 2.4. A set of transformations (indexed by $t \in \mathbb{R}$) on a translation surface is *recurrent* if for some subsequence t_n, $\{S_{t_n}\}$ converges to some non-degenerate S_∞.

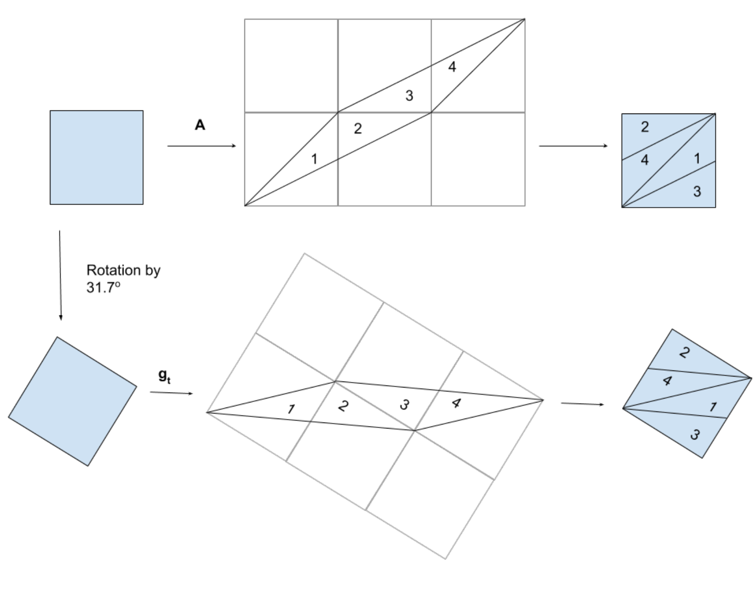
Remark 2.5. It may not be immediately clear how the set of transformations $\{g_t\}$ on a surface S could ever be recurrent. After all, intuition tells us that S would just be stretched out horizontally until it became a horizontal line. However, through the process of rearranging as defined in Definition 1.1, for some cases, we can show that $\{S_{g_t}\}$ returns periodically to its original form.

To demonstrate this, let us consider the unit square. If we apply the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},$$

the result can be cut and re-glued according to Definition 1.1 as shown. Thus, we could apply A to our translation surface S: $S_n = A^n(S)$ and for all $n \in \mathbb{N}$, $S_n = S$.

Now consider if we rotated the unit square so that the vectors $(1, \frac{\sqrt{5}-1}{2})$ and $(1, \frac{\sqrt{5}+1}{2})$ become horizontal and vertical respectively. Let us call this new translation surface U. Then, let us apply $g_{ln(\frac{\sqrt{5}+3}{2})}$ to U, and the result is identical to the rotation of S_1. Thus, we know that we can re-glue $g_{ln(\frac{\sqrt{5}+3}{2})}(U)$ so that it equals U. Hence, since S_n periodically returns to its original form, so does $g_t(U)$.



For readers with closer familiarity with linear algebra, the rotation was to make the eigenlines of A horizontal and vertical. We then let t be the natural log of the eigenvalue λ so that we obtain the matrix

$$g_t = \begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix}.$$

This is why $g_{ln(\lambda)}(U)$ is equivalent to the rotated form of $A(S)$.

Theorem 2.6 (Masur, [4]). *Suppose that $g_t(S)$ is recurrent. Then the vertical straight line flow is uniquely ergodic.*

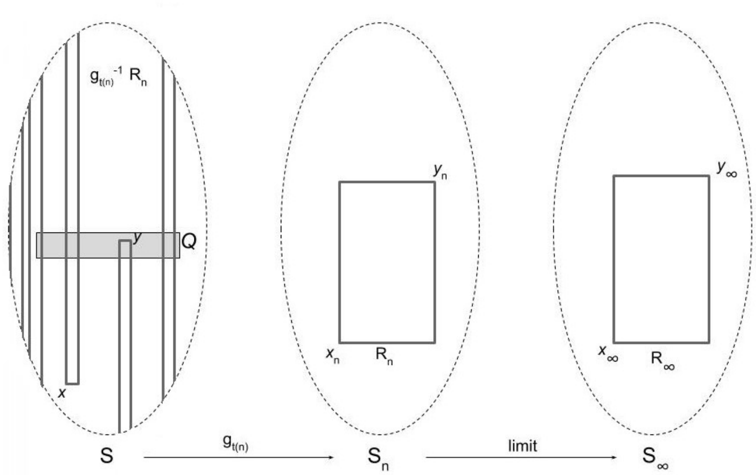
Proof. If $g_t(S)$ is recurrent, assume for contradiction that the vertical line flow is not uniquely ergodic.

Then for some $x, y \in S$ and some rectangle $Q \subset S$, the average amount of time spent in Q by the paths of x and y (x(Q) and y(Q) respectively), are such that $x(Q) \neq y(Q)$.

Since S is recurrent, we know that for some subsequence t_n, converges to some non-degenerate S_∞. We can then follow the paths of x and y as they converge to x_∞ and y_∞. Then, let us assume that there is an open subset of S that does not meet any singularities and which is big enough for us to draw a rectangle, $R_\infty \subset S_\infty$ with x_∞ and y_∞ as opposite vertices. Then, we know that for n large enough, we can still embed the rectangle $R_n \subset S_n$ with opposite vertices defined by x_n and y_n and whose dimensions (w_n, h_n) are very close to those of R_∞.

Now, let us apply the transformation $g_{t_n}^{-1}$ on S_n and R_n inside it. The result is S with a long rectangle R inside it. The height of the rectangle is $e^{t_n} h_n$.

Because R is just a transformation of R_n under g_t, we can look at R_n as a translation surface and then we know that its boundaries are preserved and so thus, R can't overlap itself.



The left side of the rectangle is the vertical path defined by x from time 0 to $e^{t_n} h_n$. Likewise, the right side is the vertical path defined by y from time 0 to $-e^{t_n} h_n$. Here, we can see that the difference between the time that the paths of x and y spend in Q is at most the height of Q, h_Q.

Then, if we let $T = e^{t_n} h_n$, $|x(Q) - y(Q)| \leq \lim_{T\to\infty} \frac{h_Q}{T}$. Thus, as n approaches infinity, $\lim_{T\to\infty} \frac{h_Q}{T} = 0$ and so, $|x(Q) - y(Q)| = 0$ which is a contradiction.

Now let us consider the case where we can not find an open set into which we can embed R_∞. We know that S_∞ is path-connected and non-degenerate. So, we know that there exists a path from x_∞ to y_∞ surrounded by an open set not meeting any singularity. Thus, there exists a finite sequence $x^1_\infty, x^2_\infty, ..., x^k_\infty = y_\infty$ such that the rectangles formed by any two adjacent elements of the sequence all lie within the open set. Then, we can apply our previous reasoning to determine that $x^1(Q) = x^2(Q) = ... = x^k(Q) = y(Q)$.

Thus, the straight line flow is uniquely ergodic.

□

3. Regarding group theory and the upper half plane

Before we can continue on to some important theorems that come as a result of Masur's criterion, we must first recall some definitions from group theory [1].

Definition 3.1. A *group* G is a set with some operation * such that:
1) If $a, b \in G$ then $a * b \in G$.
2) If $a, b, c \in G$ then $(a * b) * c = a * (b * c)$
3) There exists some identity e, such that for all $a \in G$, $a * e = e * a = a$
4) For all $a \in G$, there is some $b \in G$ such that $a * b = b * a = e$.

A *subgroup* of a group G is a group that is a subset of G closed under the same operation.

Often, a group will be presented in the form (X,*), where X is the set and * is the operation.

Definition 3.2. A *group action* * by some group G on a set S is a map $G \times S \to S$ such that:
1) for all $s \in S$, $e * s = s$, and
2) for all $g, g' \in G$, $g * (g' * s) = (gg') * s$.

Definition 3.3. A *stabilizer* of an element of some set is the group of actions on that element which return the same element. Thus, the stabilizer of a translation surface S is the set of 2-by-2 matrices which when applied to S yields an equivalent translation surface under Definition 1.1.

Definition 3.4. For some element s in a set S that is acted on by a group G, the *orbit* $O_s = \{g * s | g \in G\}$

Theorem 3.5. *The set of all transformations $SL(2, \mathbb{R})$ is a group under matrix multiplication.*

Proof. Let M and N be transformations. Writing them as matrices

$$M = \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix}, N = \begin{bmatrix} n_1 & n_2 \\ n_3 & n_4 \end{bmatrix},$$

their respective determinants must take the forms $m_1m_4 - m_2m_3 = 1$ and $n_1n_4 - n_2n_3 = 1$ to satisfy the unit determinant condition.

Taking their product of these transformations, we get

$$M * N = \begin{bmatrix} m_1n_1 + m_2n_3 & m_1n_2 + m_2n_4 \\ m_3n_1 + m_4n_3 & m_3n_2 + m_4n_4 \end{bmatrix}.$$

Then the determinant of the product yields

$$\begin{aligned} &(m_1n_1 + m_2n_3)(m_3n_2 + m_4n_4) - (m_1n_2 + m_2n_4)(m_3n_1 + m_4n_3) \\ &= m_1m_3n_1n_2 - m_1m_3n_1n_2 + m_2m_4n_3n_4 - m_2m_4n_3n_4 \\ &+(m_1m_4 - m_2m_3) \cdot n_1n_4 - (m_1m_4 - m_2m_3) \cdot n_2n_3 \end{aligned}$$

Simplifying, we get $M * N = 0 + 0 + 1(n_1n_4 - n_2n_3) = 1$ and thus their product $M * N$ always belongs to $SL(2, \mathbb{R}$.

We know that matrix multiplication is associative and thus, $SL(2, \mathbb{R})$ is associative. We also know that the matrix $e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity matrix and that since its determinant is equal to 1, $SL(2, \mathbb{R})$ has an identity element.

From linear algebra, we know that any matrix with non-zero determinant, M has some N such that $M * N = N * M = e$.

We then know the determinant of $M * N$ takes the form

$$\begin{aligned} &(m_1n_1 + m_2n_3)(m_3n_2 + m_4n_4) - (m_1n_2 + m_2n_4)(m_3n_1 + m_4n_3) \\ &= m_1m_3n_1n_2 - m_1m_3n_1n_2 + m_2m_4n_3n_4 - m_2m_4n_3n_4 \\ &+(m_1m_4 - m_2m_3) \cdot n_1n_4 - (m_1m_4 - m_2m_3) \cdot n_2n_3 = 1. \end{aligned}$$

Simplifying, we get: $1 = 0 + 0 + n_1n_4 - n_2n_3$ and thus, $N \in SL(2, \mathbb{R})$.

Thus, $SL(2, \mathbb{R})$ is a group.

□

We will now define the upper half plane and its properties, using as reference Garrett's paper [3].

Definition 3.6. The *upper half plane* $\mathcal{H}$ is a subset of the complex plane $\mathbb{C}$ such that for all $x + iy \in \mathcal{H}$, $y > 0$.

Definition 3.7. The *Möbius transformation* is the group action on $z \in \mathcal{H}$ by $SL(2, \mathbb{R})$ such that for $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{R})$,

$$M(h) = \frac{az + b}{cz + d}.$$

Theorem 3.8. *The Möbius transformation is a group action.*

Proof. 1)

$$e(z) = \frac{z + 0}{0 \cdot z + 1} = z$$

2) Let $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $g' = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$ where $g, g' \in SL(2, \mathbb{R})$. Then we have that

$$gg' = \begin{bmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{bmatrix}$$

and

$$gg'(z) = \frac{(aa' + bc')z + ab' + bd'}{(ca' + dc')z + cb' + dd'}.$$

Meanwhile, we have

$$g(g'(z)) = g(\frac{a'z + b'}{c'z + d'}) = \frac{a\frac{a'z+b'}{c'z+d'} + b}{c\frac{a'z+b'}{c'z+d'} + d} = \frac{(aa' + bc')z + ab' + bd'}{(ca' + dc')z + cb' + dd'}$$

. So all that is left is to show that the transformation is indeed a map of the form $G \times S \to S$ defining a group action. □

Proposition 3.9. *For any $z \in \mathcal{H}$, and for any $M \in SL(2, \mathbb{R})$,*

$$Im(M(z)) = \frac{Im(z)}{|cz + d|^2}$$

where Im(z) denotes the complex component of Z.

Proof. We know:

$$Im(M(z)) = Im(\frac{ax + ayi + b}{cx + cyi + d}) = Im(\frac{(ax + b) + ayi}{(cx + d) + cyi} \cdot \frac{(cx + d) - cyi}{(cx + d) - cyi})$$

$$= \frac{(ax + b)(-cy) + (cx + d)(ay)}{(cx + d)^2 + (cy)^2} = \frac{(ad - bc)(y)}{|cz + d|^2}.$$

Then, because $ad - bc = 1$, we get that

$$Im(M(z)) = \frac{y}{|cz + d|^2} = \frac{Im(z)}{|cz + d|^2}.$$

This also shows that the Möbius transformation is indeed a group action. □

4. Regarding the hyperbolic plane and lattice surfaces

We will now define the properties of lattice surfaces, which are the focus of the Veech dichotomy and McMullen's trichotomy.

Definition 4.1. A *lattice* Λ in $SL(2, \mathbb{R})$ is a subgroup of the group $SL(2, \mathbb{R})$ such that for any $z \in \mathcal{H}$ there exists $g \in \Lambda$ such that under the Möbius transformation, $g(z) \in F$ where F is a subset of $\mathcal{H}$ with finite area in the hyperbolic plane.

Definition 4.2. A *lattice surface* is a translation surface with a stabilizer Λ which forms a lattice in $SL(2, \mathbb{R})$.

Before we can use the above definitions, we must first discuss how geometry on $\mathcal{H}$ (the hyperbolic plane) is defined.

Definition 4.3. An *H-point* is any point in $\mathcal{H}$. A *H-line* is either a perpendicular to the x-axis or a semicircle with its center on the x-axis.

The H-line is well-defined since any 2 H-points have one and only one H-line passing through them.

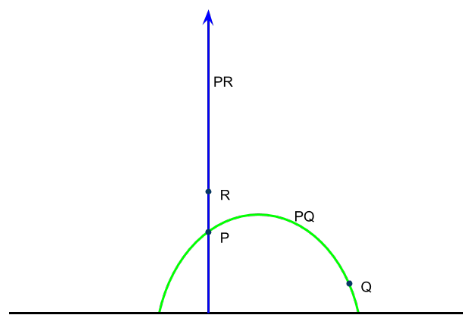


FIGURE 3. The line PR is a ray in H while the line PQ is a semicircle

Definition 4.4. Consider two H-points a,b. The always-positive *H-distance* d(ab) is calculated as follows:

1) If the H-line passing through a,b is a line then $d(ab) = ln(\frac{|ax|}{|bx|})$ where $|ax|$ and $|bx|$ are euclidean distances to the x-axis.

2) If the H-line is a semicircle then $d(ab) = ln(\frac{|ax_1|\cdot|bx_2|}{|ax_2|\cdot|bx_1|})$ where $|ax_1|, |ax_2|, |bx_1|, |bx_2|$ are the euclidean distances to the points x_1, x_2 where the semicircle intersects the x-axis.

Remark 4.5. It is often helpful to imagine $\mathcal{H}$ as an infinitely large disc where the x-axis is its circumference and going off into infinity in the positive y direction leads you to the center of the disc.

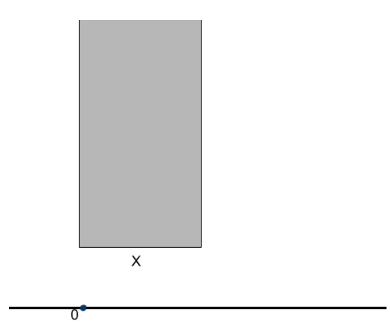
These definitions are taken from the online resource [2].

Example 4.6. Here is an example of how area on $\mathcal{H}$ may be deceiving. As shown below, let us take an area bounded by 2 vertical rays a distance x away and the line y=1.

At first, the area seems to be infinite, but if we consider that as a and b go off to infinity in the y-direction, so does the semicircle connecting them, and we get: $\lim_{y\to\infty} d(ab) = \lim_{y\to\infty} ln(\frac{|ax_1|\cdot|bx_2|}{|ax_2|\cdot|bx_1|})$. We know that as y approaches infinity, the ratios $\frac{ax_1}{ax_2}$ and $\frac{bx_2}{bx_1}$ approach 1, so we then know: $\lim_{y\to\infty} d(ab) = ln(1) = 0$.

Thus, the area is actually pinched at the top and is bounded.

The proof that the area converges and is therefore finite requires a detour into calculus and will therefore be omitted.



Now that we have shown what it means for an area to be finite in $\mathcal{H}$, we can explore the simplest example of a lattice surface: the flat torus. The first part of this next proof will follow Garrett's exposition [3].

Definition 4.7. A *flat torus* is a translation surface whose underlying polygon is a parallelogram.

The unit square defines a torus under the edge identifications $(x, y) \sim (x+1, y) \sim (x, y+1)$ in $\mathbb{R}^2$, and under similar identifications, any parallelogram defines a torus. From this arises the term *flat torus.*

Theorem 4.8. *The flat torus is a lattice surface.*

Proof. Let us consider the set formed by the composition C of the following matrices : $\{A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\}$.

As shown in Prop. 3.10,

$$Im(M(z)) = \frac{Im(z)}{|cz + d|^2}.$$

We also know that because $ad - bc = 1$ either c or d are non-zero. Then we know that $|cz + d| > 0$ because $|z| > 0$.

Then, for any given $z \in \mathcal{H}$ and $M \in S$,

$$Im(M(z)) < \frac{Im(z)}{0} = \infty.$$

Thus, because $c, d \in \mathbb{Z}$ there exists $sup(Im(M(z))$ and $sup(Im(M(z)) = max(Im(M(z)))$.

Let z_m be an element of the orbit of z such that $Im(z_m) = max(Im(M(z)))$ (there will be an infinite amount of such elements, so we will choose an arbitrary z_m).

Returning to our original matrices, we see that $A^n(z) = z + n$ because

$$A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}.$$

We also see that $B(x) = -1/z$.

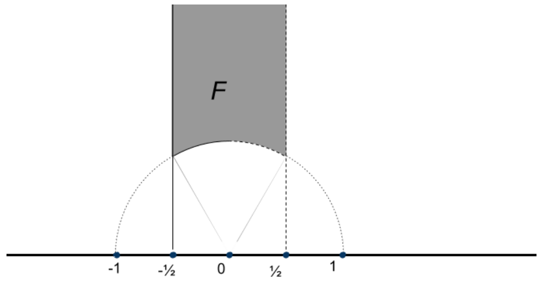
Using matrix A, we can always make the real component of z_m obey the inequality $\frac{-1}{2} \leq Re(A^n(z_m)) < \frac{1}{2}$. It is important to note that this leaves $Im(z_m)$ unchanged.

Then, let us assume that $|z_m| < 1$. It follows that

$$Im(B(z_m)) = \frac{Im(z_m)}{|z|^2} > Im(z_m)$$

which is a contradiction, and thus, $|z_m| \geq 1$.

Thus, we get that for all $z \in \mathcal{H}$ there is some $z_m \in O_z$ such that $z_m \in F = \{x \in \mathcal{H} | \frac{-1}{2} \leq Re(x) < \frac{1}{2}\ and\ |x| \geq 1\}$.



We know that if we take our flat torus T, and we apply A(M) (left) and B(M) (right) as shown below, the result is equal to T. Thus, A and B are elements of the stabilizer of T. Furthermore, since A(T)=B(T)=T we know that $B(A(T)) = A(T) = T$ and thus, $s(T) = T$ for all $s \in C$. Thus, any $s \in C$ is also an element of the stabilizer of T.

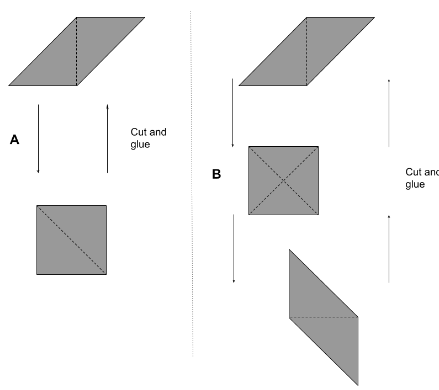


Figure 4. Here we see how A (left) and B (right) stabilize any parallelogram.

Hence, the stabilizer of T forms a lattice in $SL(2, \mathbb{R})$ and thus, the flat torus is a lattice surface.

□

5. The Veech Dichotomy and McMullen's Trichotomy

Now that we have seen an example of a lattice surface, we can move on to discuss two theorems that arise as a result of Masur's criterion.

Theorem 5.1 (Veech, [7]). *For any lattice surface, in the direction of any saddle connection, the straight line flow (or billiard ball path) is periodic. In all other directions, the straight line flow is uniquely ergodic.*

Proof. The proof essentially follows from Masur's criterion, but it uses advanced concepts from hyperbolic geometry including the relationship between lattice surfaces and tangent bundles over the moduli space of genus g compact Riemann surfaces. We refer the reader to Wright's doctoral topic proposal [9].

□

Example 5.2. As proven previously, we know that the flat torus is a lattice surface. It then follows by the Veech dichotomy that all trajectories in the direction of saddle connections are periodic and all other trajectories are uniquely ergodic. This result has been known for over 100 years and was one of the first theorems in billiard dynamics (as a result of Weyl's theorem [8]), however, it is still useful for understanding the much broader Veech dichotomy, which covers all lattice surfaces (such as the octagon, decagon, and many more).

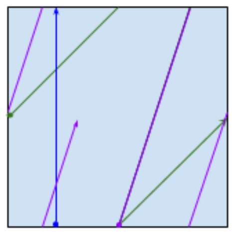


FIGURE 5. The green and blue trajectories are periodic orbits in directions of saddle connections. The purple trajectory is an example of a uniquely ergodic path

Finally, we present a crucial theorem classifying lattice surfaces of genus 2.

Definition 5.3. An *L-shaped table L(a,b)* is a billiard table with side lengths as pictured bellow.

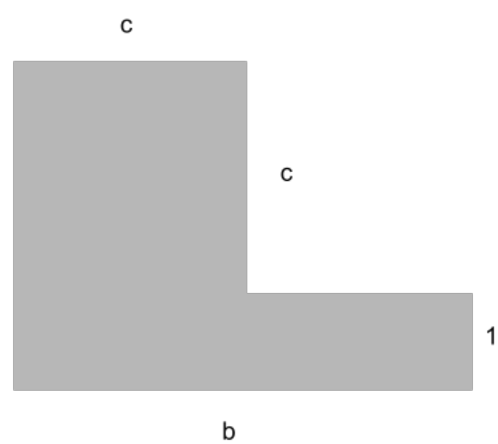


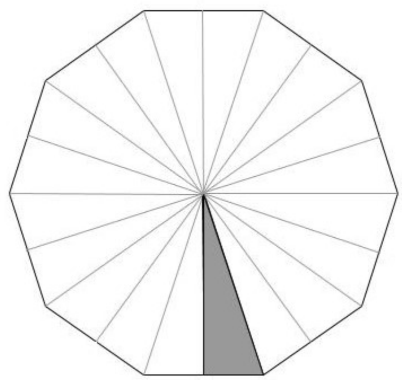
FIGURE 6. $c = (a + \sqrt{a^2 - 4b})/2$

Theorem 5.4 (McMullen, [5]). *Let T be a billiard table which unfolds into a translation surface X of genus 2 (a lattice surface which, when its identified edges are glued, forms a surface with two holes). Then X is a lattice surface if and only if T is equivalent to*

1) a table tiled by congruent triangles of angles $(\pi/2, \pi/3, \pi/6)$ and $(\pi/2, \pi/4, \pi/4)$;
2) the triangle with angles $(\pi/2, 2\pi/5, \pi/10)$; or
3) the L-shaped table L(a,b) for some $a, b \in \mathbb{Z}$

Remark 5.5. Two billiard tables are equivalent if they unfold to the same translation surface.

To demonstrate this we will look at a canonical example.



Example 5.6. As shown, the regular decagon falls into the second category, and since it is a genus 2 surface after gluing identified edges, we know that it is a lattice surface.

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