CLASSIFICATION OF THE 17 WALLPAPER GROUPS

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Abstract. Using basic group theory, we define and examine the various wallpaper groups. We then classify them based on their lattice types and prove there are exactly 17 different groups, up to an isomorphism. Additionally, we provide a short description and examples of each group.

Contents

1. Introduction

Looking at a piece of wallpaper, or any repeating pattern, one might be surprised that there is a lot of mathematics lurking behind that pattered paper. It turns out that any two-dimensional infinitely repeating pattern, called a wallpaper pattern, falls into one of 17 classes based on its symmetries. These symmetries include rotations, reflections, translations, and other transformations that preserve the pattern. These 17 classes, called the **wallpaper groups**, provide an interesting way to apply basic group theory to both geometry and art. Their three-dimensional analogues, the space groups, have scientific applications, as they are essential to crystallography. This paper seeks to enumerate and classify the 17 wallpaper groups.

To start, we will first introduce the group E_2, or the **Euclidean group**. It is the set of all **isometries** of the plane, or all functions that preserve distances in the plane. We will we not prove it here (see [1] for a full discussion and proof), but we can think of elements in this group as pairs $(\mathbf{v}, M)$, where $\mathbf{v}$ is a vector in $\mathbb{R}^2$ and M is a linear transformation of $\mathbb{R}^2$ that preserves distance. Here, M is the 2×2 matrix representation of the transformation in the standard basis. The group operation is defined as follows:

$$(\mathbf{w}, N)(\mathbf{v}, M) = (\mathbf{w} + N\mathbf{v}, NM).$$

The pair $(\mathbf{v}, M)$ is identified with the function $f(\mathbf{x}) = M\mathbf{x} + \mathbf{v}$. Now we see that the group operation represents function composition.

Additionally, it can be proved that for all $(\mathbf{v}, M) \in E_2$, M has a relatively simple form. Formally, M is a member of the orthogonal group $O_2(\mathbb{R})$, which is the group of all linear transformations that preserve distance. This group is quite interesting in itself, but all we care about is that it means M comes in one of two forms, A_θ or B_α, where

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{ and } B_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix}.$$

Geometrically, A_θ represents a rotation by θ radians about the origin, and B_α represents a reflection about a line that passes through the origin and makes a angle of $\frac{\alpha}{2}$ radians with the x-axis (again, see [1] for a full proof/discussion).

Obviously, E_2 contains translations, rotations, and reflections. E_2 also includes another type of transformation called a glide reflection, defined below:

Definition 1.1. A **glide reflection** is a reflection followed by a translation parallel to the reflection axis.

It turns out that all members of E_2 are one of these four transformation types, as summarized in the next theorem.

Theorem 1.2. *All members of E_2 are either a translation, a rotation, a reflection, or a glide reflection.*

Proof. (This proof is mostly taken from [1]). To prove the theorem, we will take a general transformation, and, based on its orthogonal matrix, classify it as one of the four types. Let our general transformation be $(\mathbf{v}, M)$. If $M = I$, where I is the identity matrix, then we have a translation. If we have $(\mathbf{v}, A_\theta)$ (where $\theta \neq 0$), then we claim we have a rotation. To see this, remember that a rotation keeps one point fixed, the center. Hence, we would like to solve the equation $f(\mathbf{c}) = A_\theta\mathbf{c} + \mathbf{v} = \mathbf{c}$ for $\mathbf{c}$ to find the center of this presumed rotation. Rewriting our equation, we have $\mathbf{v} = (I - A_\theta)\mathbf{c}$. Since

$$\det(I - A_\theta) = \begin{vmatrix} 1-\cos\theta & \sin\theta \\ -\sin\theta & 1-\cos\theta \end{vmatrix} = 2 - 2\cos\theta > 0,$$

$I - A_\theta$ is invertible and we have $\mathbf{c} = (I - A_\theta)^{-1}\mathbf{v}$. Thus, we can write the transformation as

$$f(\mathbf{x}) = A_\theta\mathbf{x} + \mathbf{v} = A_\theta\mathbf{x} + \mathbf{c} - A_\theta\mathbf{c} = A_\theta(\mathbf{x} - \mathbf{c}) + \mathbf{c},$$

so $(\mathbf{v}, A_\theta) = (\mathbf{c}, I)(\mathbf{0}, A_\theta)(-\mathbf{c}, I)$. This we can easily see is a rotation by θ radians about $\mathbf{c}$.

If $(\mathbf{v}, M) = (\mathbf{v}, B_\alpha)$, there are two cases. When $B_\alpha\mathbf{v} = -\mathbf{v}$, this means that $\mathbf{v}$ is perpendicular to the reflection line of B_α. Hence, $(\mathbf{v}, B_\alpha) = (\frac{1}{2}\mathbf{v}, I)(\mathbf{0}, B_\alpha)(-\frac{1}{2}\mathbf{v}, I)$. Because $\mathbf{v}$ is perpendicular, $(\mathbf{v}, B_\alpha)$ is a reflection about a line that makes an angle of $\frac{\alpha}{2}$ with the x-axis, and is shifted by $\frac{1}{2}\mathbf{v}$ from the origin.

When $B_\alpha\mathbf{v} \neq -\mathbf{v}$, define $\mathbf{w} = \mathbf{v} - B_\alpha\mathbf{v}$. Consequently,

$$B_\alpha\mathbf{w} = B_\alpha(\mathbf{v} - B_\alpha\mathbf{v}) = B_\alpha\mathbf{v} - B_\alpha^2\mathbf{v} = B_\alpha\mathbf{v} - \mathbf{v} = -\mathbf{w}.$$

Hence $\mathbf{w}$ is perpendicular to the reflection. Now we can project $\mathbf{v}$ onto $\mathbf{w}$ and define $2\mathbf{a} = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w}$, and $\mathbf{b} = \mathbf{v} - 2\mathbf{a}$. Thus, $\mathbf{a}$ is perpendicular to $\mathbf{b}$, so $\mathbf{b}$ is parallel to the reflection, and we can decompose the element as $(\mathbf{v}, B_\alpha) = (\mathbf{b}, I)(2\mathbf{a}, B_\alpha)$, which is a reflection followed by a parallel translation. Hence, $(2\mathbf{a}+\mathbf{b}, B_\alpha)$ is a glide reflection. See Figure 1 for a diagram.

Hence we have proved all elements of E_2 are either a translation, a reflection, a rotation, or a glide reflection; and have also investigated the representation of the four isometries in pair form. $\square$

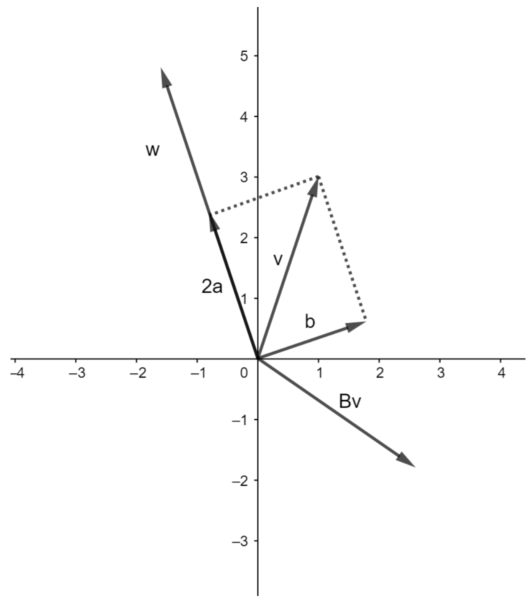


FIGURE 1. Decomposition of glide reflection

Definition 1.3. Let G be a subgroup of E_2. If I represents the identity matrix, then the **translation subgroup** T of G is defined as $T = \{(\mathbf{v}, I) \in G\}$.

As its name implies, T is the set of all translations in G. It is easy to check that T is a subgroup of G, as a translation composed with a translation gives another translation.

Definition 1.4. Let G be a subgroup of E_2. Then the **point group** H of G is defined as $H = \{M \mid (\mathbf{v}, M) \in G\}$.

The point group H encodes the types of reflections and rotations possible in G. Importantly, H is not necessarily a subgroup of G. For instance, if G had only glide reflections and no reflections or rotations, H would consist of reflections (matricies in the form B_α) and would therefore not be a subgroup.

Now we can finally formally define a wallpaper group:

Definition 1.5. A **wallpaper group** G is a subset of E_2 with a finite point group H and a translation subgroup T generated by two linearly independent translations.

The definition includes "two linearly independent translations" to make sure we have a pattern that repeats in the whole plane. Otherwise, we would include the frieze groups, which have translations in only one direction, and the rosette groups, which have no translations [3].

2. Lattices

Definition 2.1. The **lattice** L of a wallpaper group G it the set of all points in $\mathbb{R}^2$ that the origin gets mapped to by the functions in the translation subgroup T.

If we choose two translations in G, represented by their vectors $\mathbf{a}$ and $\mathbf{b}$, such that $\mathbf{a}$ is of minimum length and $\mathbf{b}$ is a vector of minimum length among those skew to $\mathbf{a}$, then $\mathbf{a}$ and $\mathbf{b}$ form a basis for the lattice. Hence, all points in the lattice are in the form $m\mathbf{a} + n\mathbf{b}$, where $m, n \in \mathbb{Z}$. (For a proof of this statement refer to [1])

Using inequalities, we can classify the types of lattices:

Theorem 2.2. *There are exactly 5 types of lattices.*

For the proof, see [1]

Here are their names and definitions:

(a) Oblique: $\|\mathbf{a}\| < \|\mathbf{b}\| < \|\mathbf{a} - \mathbf{b}\| < \|\mathbf{a} + \mathbf{b}\|$
(b) Rectangular: $\|\mathbf{a}\| < \|\mathbf{b}\| < \|\mathbf{a} - \mathbf{b}\| = \|\mathbf{a} + \mathbf{b}\|$
(c) Centered Rectangular: $\|\mathbf{a}\| < \|\mathbf{b}\| = \|\mathbf{a} - \mathbf{b}\| < \|\mathbf{a} + \mathbf{b}\|$
(d) Square: $\|\mathbf{a}\| = \|\mathbf{b}\| < \|\mathbf{a} - \mathbf{b}\| = \|\mathbf{a} + \mathbf{b}\|$
(e) Hexagonal: $\|\mathbf{a}\| = \|\mathbf{b}\| = \|\mathbf{a} - \mathbf{b}\| < \|\mathbf{a} + \mathbf{b}\|$

Figure 2 illustrates each case. Although it seems like we left out the case $\|\mathbf{a}\| = \|\mathbf{b}\| < \|\mathbf{a} - \mathbf{b}\| < \|\mathbf{a} + \mathbf{b}\|$, Figure 2F shows that if we choose a different basis, we get a centered rectangular lattice.

Another important property of the lattice is that the elements of the point group are symmetries of the lattice. This is summarized below:

Theorem 2.3. *The point group H of a wallpaper group G acts on the lattice L of the group.*

Proof. For proof see [1]. □

3. Preliminaries

To classify the wallpaper groups, we must define when they are "the same." For our purposes we will classify up to isomorphism, which means isomorphic groups are considered the same. Hence, the next theorem and its corollary will be useful (and presented without proof):

Theorem 3.1. *An isomorphism between wallpaper groups sends translations to translations, rotations to rotations, reflections to reflections, and glide reflections to glide reflections.*

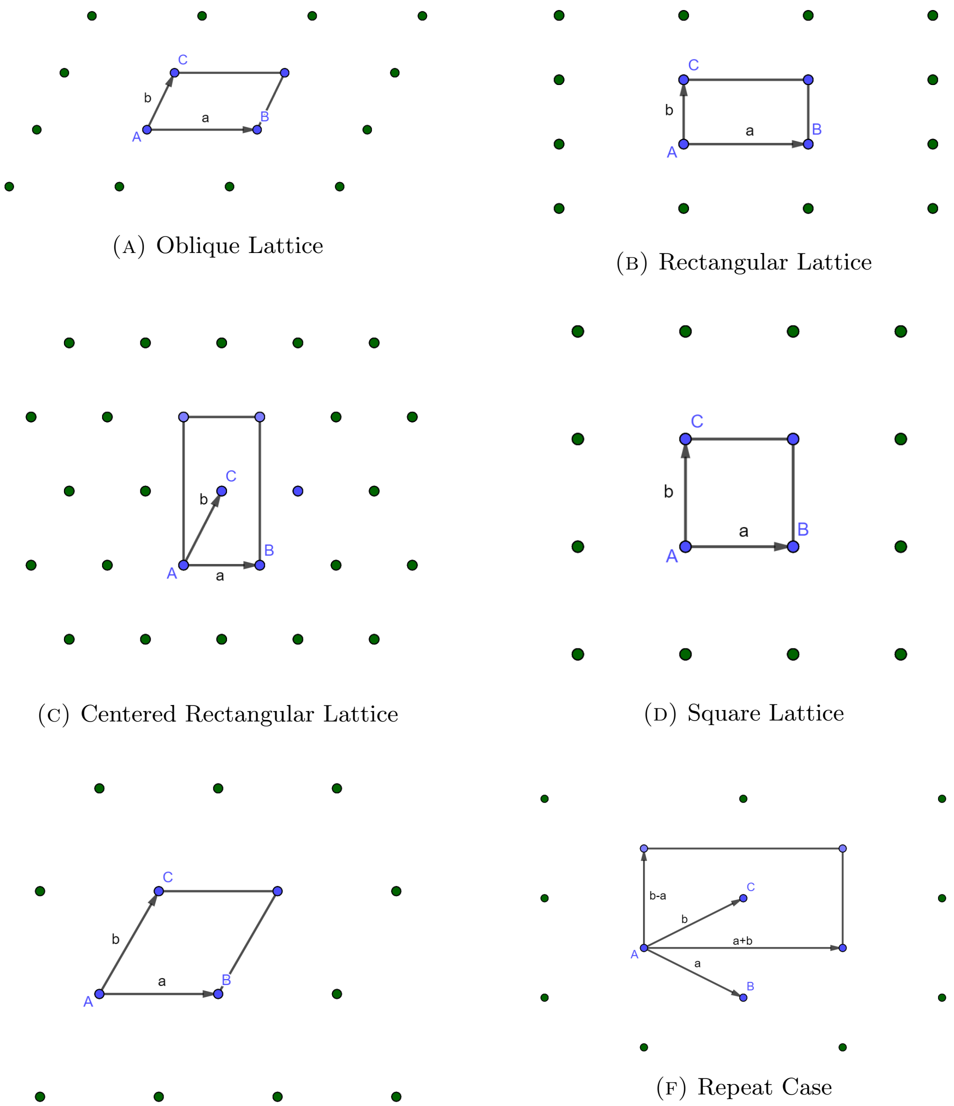
Corollary 3.2. *Two isomorphic wallpaper groups have isomorphic point groups.*

Proof. For the proof of these statements, see [1] □

With just these basic definitions, we can quickly prove this meaningful result:

Theorem 3.3 (Crystallographic Restriction)**.** *The point group H of a wallpaper group G can only contain rotations of order 1,2,3,4, or 6.*

Proof. Since H is finite, all elements (including rotations) must be of finite order. Now consider a line of points in the lattice $A - O - B$, each separated by distance



(A) Oblique Lattice

(B) Rectangular Lattice

(C) Centered Rectangular Lattice

(D) Square Lattice

(E) Hexagonal Lattice

(F) Repeat Case

FIGURE 2. Lattice Types

d. Consider a rotation of order q, which is a rotation by $\frac{2\pi}{q}$ radians. Rotate point A by $\frac{2\pi}{q}$ and B by $-\frac{2\pi}{q}$ about O. The resulting points A', B' lie on a line parallel to the original line, as demonstrated in Figure 3. Since this rotation preserves the lattice, the distance between A' and B' must equal $d' = md$ where m is an integer. By trigonometry, $d' = 2d\cos\frac{2\pi}{q}$. Hence, we have $2\cos\frac{2\pi}{q} = m$, which only happens when $\cos\frac{2\pi}{q} = 0, \pm\frac{1}{2}, \pm 1$. This corresponds to $q = 1, 2, 3, 4$ or 6. $\square$

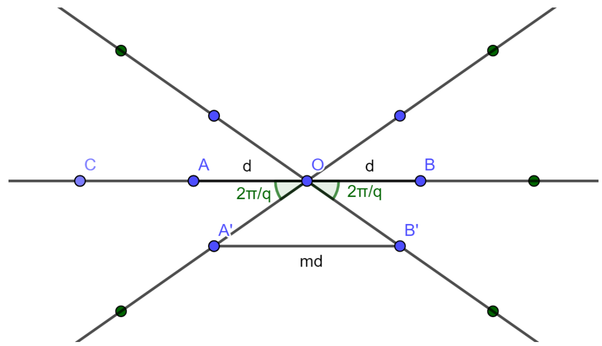


FIGURE 3. Illustration of the proof of Theorem 3.3

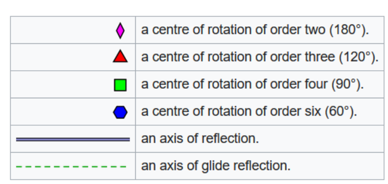
A potentially confusing aspect of wallpaper groups is their names. The naming system may seem convoluted at first, but it makes more sense considering that it is inherited from classifying the 230 space groups, the three dimensional analogs of wallpaper groups. The standard notation contains up to 4 letters and numbers, like **p4mm**. The first letter is either **p** or **c**, where **p** means a primitive cell and **c** means a centered cell. The distinction between the types of cells is mostly conventional, introduced to make descriptions of the groups easier. A primitive cell contains lattice points only on its vertices, while a centered cell has lattice points on the vertices and one in the center. We also choose a "main" translation axis, which is usually taken to be horizontal.

The second symbol is a number **n**, which indicates the highest order rotation. The third symbol represents a reflection or glide reflection perpendicular to the main axis, where **m** means a reflection, **g** indicates a glide reflection axis, and **1** means neither. The fourth symbol represents a symmetry axis at angle α to the main axis, where $\alpha = \pi$ for $n = 1, 2$, $\alpha = \frac{\pi}{4}$ for $n = 4$, and $\alpha = \frac{\pi}{3}$ for $n = 3, 6$. The type of transformation in fourth slot is represented by **m**,**1**, or **g**, which are interpreted the same way as the third symbol. If the third and fourth symbols are omitted, it means there are no reflections or glide reflections. Additionally, the notation sometimes drops numbers or letters if they can be inferred and do not cause conflicts. For example, the full name of **cm** is **c1m1**.

4. Classification

We will classify the 17 groups by examining each lattice type as a case, closely following the method laid out in [1].

Some notational notes: G is the wallpaper group being considered, and A_θ and B_α are matrices defined the same as above. The symbol **a** is the shortest translation of the group, oriented horizontally, and **b** is the other translation and basis vector of the lattice. The point group is represented by H. Additionally, the following section has many lattice diagrams, all of which use the legend to the right.



Case 1: Oblique lattice. The only symmetries in $O_2(\mathbb{R})$ that preserve an oblique lattice (parallelogram) are the identity I or a rotation by π radians, $-I$. Hence, the possible point groups are $\{I\}$ or $\{I, -I\}$.

Case 1.1: The point group is $\{I\}$. This gives us the group **p1**. This is the simplest group, containing only translations. Hence, there are almost no restrictions, the only one being that the lattice forms a parallelogram. The lengths of the translations and the angle between them can have any value.

Case 1.2: The point group is $\{\pm I\}$. This gives us the group **p2**, which contains translations and 180° rotations, but no reflections or glide reflections. Like **p1**, the lengths of the parallelogram lattice can be any length and meet at any angle. But because of the rotations, the pattern inside the cell has some restrictions (see Figure 4)

The point groups $\{I\}$ and $\{\pm I\}$ will appear multiple times in later cases, but we will ignore them, knowing they do not give us any new groups.

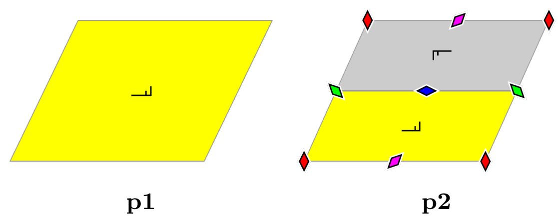


FIGURE 4. Lattice diagrams for Case 1 [6, 7]

Case 2: Rectangular lattice. We can easily see that in addition to $\{\pm I\}$, the horizontal reflection B_0 and vertical reflection B_π preserve the lattice. Hence, all point groups in this case must be subgroups of $\{I, -I, B_0, B_\pi\}$. As discussed before, we will skip point the groups $\{I\}$ and $\{\pm I\}$.

Case 2.1: The point group is $\{I, B_0\}$. This point group gives us two different groups, depending on how B_0 is realized in the group. If the group contains reflections (of the form $(\mathbf{0}, B_0)$), then we have the group **pm**. The group **pm** has no rotations, but has reflections and translations. The reflection axes are all parallel, and parallel to one side of the lattice cell.

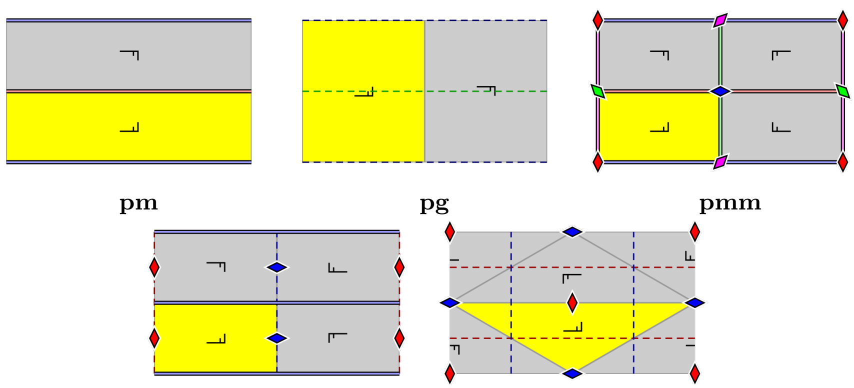
If the wallpaper group contains no reflections, but has $\{I, B_0\}$ as its point group, then B_0 must be realized as a glide reflection. This gives us the group **pg**. This group has only glide reflections and translations. Similar to **pm**, all glide reflection axes are parallel and are parallel to one side of the lattice cell.

Case 2.2: The point group is $\{I, B_\pi\}$. This is equivalent to Case 2.1, as we are just switching horizontal with vertical.

Case 2.3: The point group is $\{\pm I, B_0, B_\pi\}$. This case gives three different groups, again based on how B_0 and B_π are realized. If both B_0 and B_π are realized as reflections, we get **pmm**. This group contains reflections in two different directions and 180° rotations.

If B_0 is realized by a reflection and B_π by a glide reflection, we get **pmg**. And if the roles of B_π and B_0 are reversed, we still have **pmg**, as we are just exchanging horizontal and vertical like Case 2.2. This group contains all four types of transformations. It has reflection axes in one direction and glide reflection axes in a perpendicular direction.

If there are no reflections, then B_0 and B_π are realized as glide reflections and we have **pgg**. This group has no reflections, but has glide reflections in two perpendicular directions.



pm **pg** **pmm**

pmg **pgg**

FIGURE 5. Lattice diagrams for Case 2 [8, 9, 10, 11, 12]

Case 3: Centered Rectangular lattice. Like in the case of the rectangular lattice, the transformations that preserve the lattice are $\{I, -I, B_0, B_\pi\}$, so all point groups must be subgroups of that. Again, we will ignore $H = \{I\}$ and $H = \{\pm I\}$.

Case 3.1: The point group is $\{I, B_0\}$. This case gives only one group, **cm**, which we will investigate further than most other groups. The reflection B_0 must be realized as $(\mathbf{v}, B_0) \in G$ for some $\mathbf{v} \in \mathbb{R}^2$. Choose the origin to be a point along the reflection/glide axis so that $2\mathbf{v}$ is a multiple of $\mathbf{a}$. Now we have two cases:
If $2\mathbf{v} = k\mathbf{a}$, where k is an even integer, then $(-\mathbf{v}, I) = (-\frac{1}{2}k\mathbf{a}, I)$ is a translation in the group. Hence, also using that $\mathbf{a}$ and $\mathbf{v}$ lie along the reflection axis of B_0, $(-\mathbf{v}, I)(\mathbf{v}, B_0) = (\mathbf{0}, B_0)$ is a reflection in the group.

Elements that are not translations in the group will have the form $(m\mathbf{a}+n\mathbf{b}, B_0)$. Using that $(m\mathbf{a} + n\mathbf{b}, B_0)(\mathbf{0}, B_0) = (m\mathbf{a} + n\mathbf{b}, I)$, thus $m, n \in \mathbb{Z}$. Hence, we have

$$(m\mathbf{a} + n\mathbf{b}, B_0) = ((m + \tfrac{1}{2}n)\mathbf{a} + \tfrac{1}{2}n(2\mathbf{b} - \mathbf{a}), B_0)$$

where $m, n \in \mathbb{Z}$. If n is even and $m = -\frac{1}{2}$, then we have horizontal reflections that are parallel. If n is odd, we have glide reflection axes that are not reflection axes ($m + \frac{1}{2}n$ is not an integer). Note, all these axes (glide or standard reflection) are parallel.

If k is odd, then we have that

$$(-\tfrac{1}{2}(k+1)\mathbf{a}+\mathbf{b}, I)(\tfrac{1}{2}k\mathbf{a}, B_0) = (\tfrac{1}{2}(2\mathbf{b}-\mathbf{a}), B_0)$$

is element of G. This is a reflection as $2\mathbf{b}-\mathbf{a}$ is perpendicular to the horizontal reflection axis. We can move our origin so $(\mathbf{0}, B_0)$ is an element of the group, and then we get the case when k is even.
As we have seen, **cm** has no rotations, but has glide reflections and normal reflections, which are all parallel, alternate, and are separated by multiples of $\frac{1}{2}(2\mathbf{b}-\mathbf{a})$.

Case 3.2: The point group is $\{I, B_\pi\}$. As previous cases, this the same as Case 3.1, except we exchange vertical and horizontal.

Case 3.3: The point group is $\{\pm I, B_0, B_\pi\}$. This case gives a group called **cmm**. Using the same logic as Case 3.1, we can prove that B_0 and B_π are realized as both reflections and glide reflections. Along with 180° rotations, this group has both reflection and glide reflection axes, in both horizontal and vertical directions.

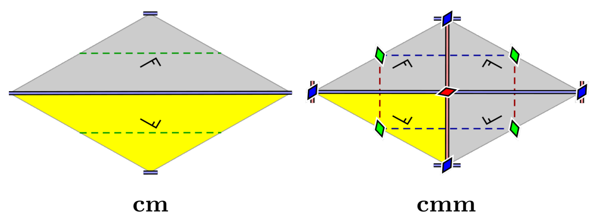


FIGURE 6. Lattice diagrams for Case 3 [13, 14]

Case 4: Square lattice. The maximal point group for this case is
$H = \{I, A_{\frac{\pi}{2}}, -I, A_{\frac{3\pi}{2}}, B_0, B_{\frac{\pi}{2}}, B_\pi, B_{\frac{3\pi}{2}}\}$. The subgroups of $\{I\}$, $\{\pm I\}$, $\{I, B_0\}$, $\{I, B_\pi\}$, $\{\pm I, B_0, B_\pi\}$ have already been investigated before, so we will skip them.

Case 4.1: The point group is $\{I, B_{\frac{\pi}{2}}\}$ *or* $\{I, B_{\frac{3\pi}{2}}\}$. When the point group is $\{I, B_{\frac{\pi}{2}}\}$, if we choose $\mathbf{a}+\mathbf{b}$ and $\mathbf{b}$ as a new lattice basis, we see this case is isomorphic to **cm**. A similar change of basis for the case where $H = \{I, B_{\frac{3\pi}{2}}\}$ gives us **cm** again.

Case 4.2: The point group is $\{\pm I, B_{\frac{\pi}{2}}, B_{\frac{3\pi}{2}}\}$. Again, a change of basis like Case 4.1 gives us the group **cmm**.

Case 4.3: The point group is $\{I, A_{\frac{\pi}{2}}, -I, A_{\frac{3\pi}{2}}\}$. This case gives the group **p4**, which has no reflections or glide reflections, but has rotations of order 4 and order 2.

Case 4.4: The point group is $\{I, A_{\frac{\pi}{2}}, -I, A_{\frac{3\pi}{2}}, B_0, B_{\frac{\pi}{2}}, B_\pi, B_{\frac{3\pi}{2}}\}$. As we will see, this case gives rise to two different groups. The point group is D_4 (the symmetry group of the square) and is generated by elements $A_{\frac{\pi}{2}}$ and B_0, so all that matters is how these elements are realized.

The first situation is when B_0 can be realized as a reflection. Then we get the group **p4m**. This actually forces all reflections in the point group to be realized as reflections in the wallpaper group, which we will now prove. Choose an origin so that $(\mathbf{0}, A_{\frac{\pi}{2}})$ is in G. Since B_0 is realized as a reflection, then it is realized in the form $(\mu\mathbf{b}, B_0)$, since $\mathbf{b}$ is perpendicular to the reflection axis. Now, we have that

$$(\mathbf{0}, A_{\frac{\pi}{2}})(\mu\mathbf{b}, B_0) = (-\mu\mathbf{a}, B_{\frac{\pi}{2}})$$
$$(-\mu\mathbf{a}, B_{\frac{\pi}{2}})^2 = (-\mu\mathbf{a} - \mu\mathbf{b}, I)$$

are elements of **p4m**, so $\mu \in \mathbb{Z}$. Thus,

$$(-\mu\mathbf{b}, I)(\mu\mathbf{b}, B_0) = (\mathbf{0}, B_0)$$

is in G, and by composing this with $(\mathbf{0}, A_{\frac{\pi}{2}})$ multiple times, we see that $B_0, B_{\frac{\pi}{2}}, B_\pi, B_{\frac{3\pi}{2}}$ are all realized by reflections.

The group **p4m** has reflections in four different directions, all separated by 45°. It also contains glide reflection axes in two directions, and 90° and 180° rotations.

The second situation is when B_0 cannot be realized as a reflection. Choose an origin so that $(\mathbf{0}, A_{\frac{\pi}{2}}) \in G$ and consider the realization of B_0, $(\mathbf{v}, B_0)$. Since $\mathbf{a}$ and $\mathbf{b}$ are linearly independent, thus $(\mathbf{v}, B_0) = (\lambda\mathbf{a} + \mu\mathbf{b}, B_0)$ for some $\lambda, \mu \in \mathbb{R}$. Hence,

$$(\lambda\mathbf{a} + \mu\mathbf{b}, B_0)(\lambda\mathbf{a} + \mu\mathbf{b}, B_0) = (2\lambda\mathbf{a}, I).$$

So $(2\lambda\mathbf{a}, I) \in G$ and 2λ must be an integer. If 2λ is even, then

$$(-\lambda\mathbf{a}, I)(\lambda\mathbf{a} + \mu\mathbf{b}, B_0) = (\mu\mathbf{b}, B_0)$$

is a reflection in G ($\mathbf{b}$ is perpendicular to the horizontal reflection axis of B_0). This is a contradiction, so 2λ must be odd and

$$((\tfrac{1}{2} - \lambda)\mathbf{a}, I)(\lambda\mathbf{a} + \mu\mathbf{b}, B_0) = (\tfrac{1}{2}\mathbf{a} + \mu\mathbf{b}, B_0)$$

is a glide reflection in G. Additionally, we have that

$$(\mathbf{0}, A_{\frac{\pi}{2}})(\tfrac{1}{2}\mathbf{a} + \mu\mathbf{b}, B_0) = (\tfrac{1}{2}\mathbf{b} - \mu\mathbf{a}, B_{\frac{\pi}{2}})$$

and

$$(\tfrac{1}{2}\mathbf{b} - \mu\mathbf{a}, B_{\frac{\pi}{2}})^2 = ((\tfrac{1}{2} - \mu)(\mathbf{a} + \mathbf{b}), I)$$

are elements in G, so $\frac{1}{2} - \mu$ is an integer. Hence

$$((\mu - \tfrac{1}{2})\mathbf{a}, I)(\tfrac{1}{2}\mathbf{b} - \mu\mathbf{a}, B_{\frac{\pi}{2}}) = (\tfrac{1}{2}\mathbf{a} + \tfrac{1}{2}\mathbf{b}, B_{\frac{\pi}{2}})$$

is an element of G. Since $\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$ is perpendicular to the reflection axis of $B_{\frac{\pi}{2}}$, thus $B_{\frac{\pi}{2}}$ is realized by a reflection (not glide) in G. Similarly, B_π is only a glide reflection in G, whereas $B_{\frac{3\pi}{2}}$ is a normal reflection axis in G. The realizations of B_π and $B_{\frac{3\pi}{2}}$ can be proved by composing the previous results with the appropriate rotations.

The group given from this second case is called **p4g**. This group contains similar transformations to **p4m**, but oriented differently. There are glide axes in perpendicular directions, and reflection axes oriented 45° to the glides. It also has 90° and 180° rotations. Looking at Figure 7, we can see that the order 4 rotation centers lie at the intersection of reflection axes for **p4m**, but not for **p4g**. This will be important in showing these two groups are not isomorphic.

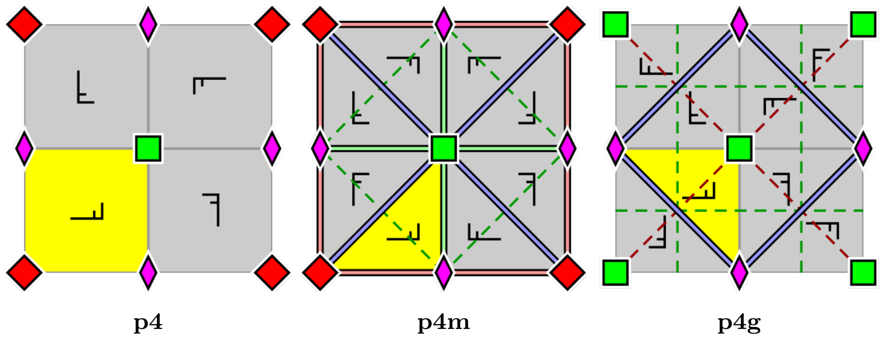


FIGURE 7. Lattice diagrams for Case 4 [15, 16, 17]

Case 5: Hexagonal lattice. The maximal point group is $\{I, A_{\frac{\pi}{3}}, A_{\frac{2\pi}{3}}, -I, A_{\frac{4\pi}{3}}, A_{\frac{5\pi}{3}}, B_0, B_{\frac{\pi}{3}}, B_{\frac{2\pi}{3}}, B_\pi, B_{\frac{4\pi}{3}}, B_{5\frac{\pi}{3}}\}$, which is just D_6, the symmetry group of a regular hexagon. All point groups in this case must be subgroups of D_6. The subgroups $\{I\}, \{\pm I\}, \{\pm I, B_0, B_\pi\}$ will not be considered as they have already been dealt with.

Case 5.1: The point group is $\{I, B_{\frac{k\pi}{3}}\}$ for $0 \le k \le 5, k \in \mathbb{Z}$. For $k = 0$, we have $\{I, B_0\}$, which is case 3.2, so we get **cm**. When $k = 1$, if we replace $\mathbf{a}, \mathbf{b}$ with $\mathbf{a} + \mathbf{b}, \mathbf{b} - \mathbf{a}$, this becomes a centered rectangular lattice with point group $\{I, B_0\}$ (effectively rotating by $\frac{\pi}{6}$ radians), so we get **cm** again. For the rest of the k values, we can do a similar change of basis to arrive at group the **cm**.

Case 5.2: The point group is $\{\pm I, B_{\frac{\pi}{3}}, B_{\frac{4\pi}{3}}\}$ or $\{\pm I, B_{\frac{2\pi}{3}}, B_{\frac{5\pi}{3}}\}$. With a change of a basis similar to Case 5.1, we get centered rectangular lattices with point group $\{\pm I, B_0, B_\pi\}$, and get **cmm**.

Case 5.3: The point group is $\{I, A_{\frac{2\pi}{3}}, A_{\frac{4\pi}{3}}\}$. This gives us the group **p3**, which has no reflections or glide reflections, but has rotation centers of order 3.

Case 5.4: The point group is $\{I, A_{\frac{2\pi}{3}}, A_{\frac{4\pi}{3}}, B_0, B_{\frac{2\pi}{3}}, B_{\frac{4\pi}{3}}\}$. Even though there are 3 different reflections in the point group, this gives us only one group, **p31m**. Choose an origin so that $(\mathbf{0}, A_{\frac{2\pi}{3}})$ is in the group. Consider a realization of B_0, $(\lambda\mathbf{a} + \mu\mathbf{b}, B_0)$. We have that

$$(\lambda\mathbf{a} + \mu\mathbf{b}, B_0)^2 = (2\lambda\mathbf{a}, I)$$
$$((\lambda\mathbf{a} + \mu\mathbf{b}, B_0)(\mathbf{0}, A_{\frac{4\pi}{3}}))^2 = (\lambda\mathbf{a} + \mu\mathbf{b}, B_{\frac{2\pi}{3}})^2 = ((\lambda + 2\mu)\mathbf{b}, I)$$
$$((\lambda\mathbf{a} + \mu\mathbf{b}, B_0)(\mathbf{0}, A_{\frac{2\pi}{3}}))^2 = (\lambda\mathbf{a} + \mu\mathbf{b}, B_{\frac{4\pi}{3}})^2 = ((\lambda - \mu)(\mathbf{a} - \mathbf{b}), I)$$

are all elements of G, so $2\lambda, \lambda + 2\mu$ and $\lambda - \mu$ are all integers. Consequently, $\lambda + 2\mu + 2(\lambda - \mu) - 2\lambda = \lambda$ is an integer, and it follows that μ is also an integer. Hence,

$$(-\lambda\mathbf{a} - \mu\mathbf{b}, I)(\lambda\mathbf{a} + \mu\mathbf{b}, B_0) = (\mathbf{0}, B_0)$$

is an element of G. Applying rotations, we see that $B_0, B_{\frac{2\pi}{3}}$, and $B_{\frac{4\pi}{3}}$ are all realized by reflections, and there are no possibilities of other groups.

The group **p31m** has reflection and glide axes in 3 different axes, separated by 120°. It also has 120° rotations.

When describing **p31m**, we should be careful as it often gets confused with the next group **p3m1**. In fact, many textbooks (including references [1] and [2]) incorrectly switch this label. As discussed in [4], it is believed this interchange is a notational error that has been copied into many reference textbooks.

Case 5.5: The point group is $\{I, A_{\frac{2\pi}{3}}, A_{\frac{4\pi}{3}}, B_{\frac{\pi}{3}}, B_\pi, B_{\frac{5\pi}{3}}\}$. Similar to Case 5.4, we will show this only gives one group, **p3m1**. Choose an origin so that $(\mathbf{0}, A_{\frac{2\pi}{3}})$ is a member of G. Consider the realization of $B_{\frac{\pi}{3}}$, $(\lambda\mathbf{a} + \mu\mathbf{b}, B_{\frac{\pi}{3}})$. Squaring this, we have that

$$(\lambda\mathbf{a} + \mu\mathbf{b}, B_{\frac{\pi}{3}})^2 = ((\lambda + \mu)(\mathbf{a} + \mathbf{b}), I),$$

is a member of G, so $\mu + \lambda$ is an integer. Now, with more calculations we have that

$$(\mathbf{0}, A_{\frac{2\pi}{3}})(\lambda\mathbf{a} + \mu\mathbf{b}, B_{\frac{\pi}{3}}) = (\lambda(\mathbf{b} - \mathbf{a}) - \mu\mathbf{a}, B_\pi)$$
$$(\lambda(\mathbf{b} - \mathbf{a}) - \mu\mathbf{a}, B_\pi)^2 = (\lambda(2\mathbf{b} - \mathbf{a}), I)$$

are members of the group, so λ is an integer. Consequently, μ is also an integer, and we have that

$$(-\lambda\mathbf{a} - \mu\mathbf{b}, I)(\lambda\mathbf{a} + \mu\mathbf{b}, B_{\frac{\pi}{3}}) = (\mathbf{0}, B_{\frac{\pi}{3}})$$

is a member of G. Applying rotations, we have that $B_{\frac{\pi}{3}}, B_\pi$ and $B_{\frac{5\pi}{3}}$ are realized as $(\mathbf{0}, B_{\frac{\pi}{3}}), (\mathbf{0}, B_\pi)$ and $(\mathbf{0}, B_{\frac{5\pi}{3}})$, which are all reflections. Hence, since the realizations of the B elements in the point group are all characterized, and these elements generate the point group, we only get one wallpaper group from this case.

Similar to **p31m**, **p3m1** has reflection and glide reflection axes in three different directions, along with 120° rotations. Looking at Figure 8, notice that for group **p3m1** all rotation centers lie at the intersection of reflection axes, while this is not the case for **p31m**. We will use this fact latter to show these groups are not isomorphic.

Case 5.6: The point group is $\{I, A_{\frac{\pi}{3}}, A_{\frac{2\pi}{3}}, -I, A_{\frac{4\pi}{3}}, A_{\frac{5\pi}{3}}\}$. This gives us the group **p6**. This group has no reflections or glide reflections, but has rotations of multiple orders. As we can see from the point group, there are rotations of order 2,3, and 6.

Case 5.7: The point group is $\{I, A_{\frac{\pi}{3}}, A_{\frac{2\pi}{3}}, -I, A_{\frac{4\pi}{3}}, A_{\frac{5\pi}{3}}, B_0, B_{\frac{\pi}{3}}, B_{\frac{2\pi}{3}}, B_\pi, B_{\frac{4\pi}{3}}, B_{5\frac{\pi}{3}}\}$. This gives us group **p6m**. Using the same computations as in cases 5.4 and 5.5, we can show that $B_0, \ldots, B_{\frac{5\pi}{3}}$ must all be realized as reflections.

This group contains reflection and glide axes in six different directions, along with three different types of rotations: 180° rotations, 120° rotations, and 60° rotations.

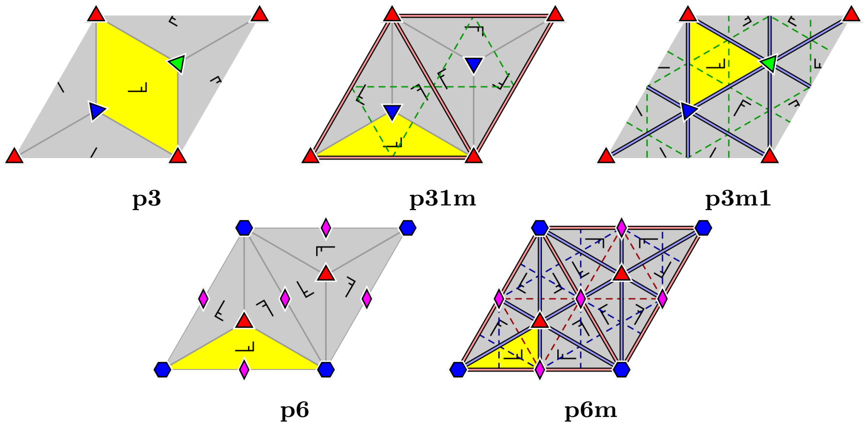


FIGURE 8. Lattice diagrams for Case 5 [18, 19, 20, 21, 22]

5. UNIQUENESS

Now that we have seen 17 different groups, we must prove they are all are distinct isomorphism classes. From Corollary 3.2, we know that if two groups are isomorphic, they must have isomorphic point groups. Hence, we only have to check that wallpaper groups that have point groups of the same order are not isomorphic, as if two point groups have different order, then their associated wallpaper groups are not isomorphic. We include the following helpful table of wallpaper groups and their respective point groups, taken from [1].

G	H	G	H
p1	trivial	**p4**	$\mathbb{Z}_4$
p2	$\mathbb{Z}_2$	**p4m**	D_4
pm	$\mathbb{Z}_2$	**p4g**	D_4
pg	$\mathbb{Z}_2$	**p3**	Z_3
pmm	$\mathbb{Z}_2 \times \mathbb{Z}_2$	**p3m1**	D_3
pmg	$\mathbb{Z}_2 \times \mathbb{Z}_2$	**p31m**	D_3
pgg	$\mathbb{Z}_2 \times \mathbb{Z}_2$	**p6**	$\mathbb{Z}_6$
cm	$\mathbb{Z}_2$	**p6m**	D_6
cmm	$\mathbb{Z}_2 \times \mathbb{Z}_2$		

In the following theorems, we will prove uniqueness by collecting the wallpaper groups by the order of their point group, and then proving all groups in a collection are distinct from each other.

Theorem 5.1. *No two of* **p2**, **pm**, **pg**, *and* **cm** *are isomorphic.*

Proof. The group **p2** contains rotations, while none of the others do, so it is not isomorphic to **pm**, **pg** or **cm**. Groups **pm** and **cm** contain reflections, while **pg** does not, so **pg** is not isomorphic to **cm** or **pg**.

Finally, **cm** has glide reflections whose glide part does not lie in the group. Remembering our investigations from Case 3.2, we can plug $m = 0$ and $n = 1$ into the formula to get that

$$(\tfrac{1}{2}\mathbf{a} + \tfrac{1}{2}(2\mathbf{b} - \mathbf{a}), B_0) = (\tfrac{1}{2}\mathbf{a}, I)(\tfrac{1}{2}(2\mathbf{b} - \mathbf{a}), B_0)$$

is a member of **cm**. The glide part $(\frac{1}{2}\mathbf{a}, I)$ is not a translation in **cm**, but when composed with the reflection it is. Compare this with **pm**. Choose origin so $(\mathbf{0}, B_0) \in$ **pm**. Hence, all non-translations have the form

$$(m\mathbf{a} + n\mathbf{b}, B_0) = (m\mathbf{a} + n\mathbf{b}, I)(\mathbf{0}, B_0) = (m\mathbf{a}, I)(n\mathbf{b}, B_0)$$

where $m, n \in \mathbb{Z}$. For glide reflections ($m \neq 0$), we see the glide is a multiple of **a** and is therefore in the group. Hence, **cm** and **pm** are not isomorphic.

□

Theorem 5.2. *No two of* **pmm**, **pmg**, **pgg**, **cmm**, *and* **p4** *are isomorphic*

Proof. Since **p4** contains no reflections or glide reflections, but the other four have reflections or glide reflections, **p4** is not isomorphic to **pmm**, **pmg**, **pgg**, or **cmm**. Groups **cmm**, **pmg**, and **pmm** all contain reflections, while **pgg** does not, so **pgg** is not isomorphic to the other groups. Similar to Theorem 5.1, the glide part of any glide reflection in **pmm** belongs to the group, unlike groups **pmg** and **cmm**, which have axes for exclusively glide reflections. Hence, **pmm** is not isomorphic to **cmm** or **pmg**.

Lastly, in **pmg**, since B_π is realized only as a glide reflection, all reflections are horizontal. Therefore, the composition of two reflections gives a translation. Compare this with **cmm**, where there are vertical and horizontal reflections, which when composed give a rotation by π radians. Hence, **cmm** and **pmg** are not isomorphic.

□

Theorem 5.3. **p4m** *and* **p4g** *are not isomorphic.*

Proof. Consider the element $(\mathbf{0}, A_{\frac{\pi}{2}})$, which is a member of both groups with proper choice of origin. We will see that $(\mathbf{0}, A_{\frac{\pi}{2}})$ can be written as the composition of two reflections in **p4m**, but not **p4g**.

From Case 4.5, we know that for **p4m**, $(\mathbf{0}, B_0)$ and $(\mathbf{0}, B_{\frac{\pi}{2}})$ are members of **p4m**. Also, we have $(\mathbf{0}, B_{\frac{\pi}{2}})(\mathbf{0}, B_0) = (\mathbf{0}, A_{\frac{\pi}{2}})$.

Now we will prove that it is impossible to factorize $(\mathbf{0}, A_{\frac{\pi}{2}})$ as two reflections in **p4g**. Any factorization has the form $(\mathbf{0}, A_{\frac{\pi}{2}}) = (\mathbf{v}, B_\alpha)(\mathbf{w}, B_\beta)$. Since $B_\alpha \cdot B_\beta = A_{\alpha-\beta}$ (left to the reader as an exercise), our only possible options are $\alpha = \frac{\pi}{2}, \beta = 0$; $\alpha = \pi, \beta = \frac{\pi}{2}$; $\alpha = \frac{3\pi}{2}, \beta = \pi$; or $\alpha = 0, \beta = \frac{3\pi}{2}$. In all four cases, one of the factors contains either B_0 or B_π. Since B_0 or B_π are realized only as glide reflections, factors containing them will never be reflections. Hence, it is impossible to factorize $(\mathbf{0}, A_{\frac{\pi}{2}})$ as two reflections in **p4g**.

Hence, we have that **p4g** and **p4m** are not isomorphic. □

Theorem 5.4. **p31m** *and* **p3m1** *are not isomorphic.*

Proof. Similar to the previous theorem, we will prove that $(\mathbf{a}, A_{\frac{2\pi}{3}})$ can be factorized as the product of two reflections in **p3m1**, but not in **p31m**. From Case 5.5, we know that we can factorize $(\mathbf{a}, A_{\frac{2\pi}{3}})$ as

$$(\mathbf{a}, I)(\mathbf{0}, B_\pi)(\mathbf{0}, B_{\frac{\pi}{3}}) = (\mathbf{a}, B_\pi)(\mathbf{0}, B_{\frac{\pi}{3}}) = (\mathbf{a}, A_{\frac{2\pi}{3}}).$$

Now, consider the hypothetical factorization in **p31m**, $(\mathbf{a}, A_{\frac{2\pi}{3}}) = (\mathbf{v}, B_\alpha)(\mathbf{w}, B_\beta)$. Since $B_\alpha \cdot B_\beta = A_{\alpha-\beta}$, we have three cases: $\alpha = \frac{2\pi}{3}, \beta = 0$; $\alpha = \frac{4\pi}{3}, \beta = \frac{2\pi}{3}$; or $\alpha = 0, \beta = \frac{4\pi}{3}$.

Consider the case $\alpha = \frac{2\pi}{3}, \beta = 0$. Since $(\mathbf{v}, B_{\frac{2\pi}{3}})$ and $(\mathbf{w}, B_0)$ are both reflections, $\mathbf{v} = \lambda(\mathbf{b} - 2\mathbf{a})$ and $\mathbf{w} = \mu(2\mathbf{b} - \mathbf{a})$. Hence, we have

$$(\lambda(\mathbf{b} - 2\mathbf{a}), B_{\frac{2\pi}{3}})(\mu(2\mathbf{b} - \mathbf{a}), B_0) = ((\mu - 2\lambda)\mathbf{a} + (\lambda + \mu)\mathbf{b}, A_{\frac{2\pi}{3}}) = (\mathbf{a}, A_{\frac{2\pi}{3}}),$$

so $\mu - 2\lambda = 1$ and $\lambda + \mu = 0$. Solving gives us $\lambda = -\frac{1}{3}$ and $\mu = \frac{1}{3}$. But this means that

$$(\tfrac{1}{3}(2\mathbf{b} - \mathbf{a}), B_0)(\mathbf{0}, B_0) = (\tfrac{1}{3}(2\mathbf{b} - \mathbf{a}), I)$$

is a translation in **p31m**. But this is a contradiction, as $\frac{2}{3}, -\frac{1}{3} \notin \mathbb{Z}$. The other two cases are the same as this, but rotated by $\frac{2\pi}{3}$ radians, so they are also impossible. Hence, $(\mathbf{a}, A_{\frac{2\pi}{3}})$ cannot be factorized in **p31m** as the composition of two reflections.

Hence, **p3m1** and **p31m** are not isomorphic.

□

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