OPTIMAL STOPPING TIME AND ITS APPLICATIONS TO ECONOMIC MODELS

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Abstract. This paper gives an introduction to an optimal stopping problem by using the idea of an Itô diffusion. We then prove the existence and uniqueness theorems for optimal stopping, which will help us to explicitly solve optimal stopping problems. We then apply our optimal stopping theorems to the economic model of stock prices introduced by Samuelson [5] and the economic model of capital goods. Both economic models show that those related agents are risk-loving.

Contents

1. Introduction

Since there are several outside factors in the real world, many variables in classical economic models are considered as stochastic processes. For example, the prices of stocks and oil do not just increase due to inflation over time but also fluctuate due to unpredictable situations.

Optimal stopping problems are questions that ask when to stop stochastic processes in order to maximize utility. For example, when should one sell an asset in order to maximize his profit? When should one stop searching for the keyword to obtain the best result? Therefore, many recent economic models often include optimal stopping problems. For example, by using optimal stopping, Choi and Smith [2] explored the effectiveness of the search engine, and Albrecht, Anderson, and Vroman [1] discovered how the search cost affects the search for job candidates.

In this paper, before introducing significant theorems in optimal stopping, we provide some background on Itô diffusions and boundary value problem. Most of this is from Øksendal's *Stochastic differential equations: an introduction with applications* [4]. A basic knowledge of stochastic calculus is assumed.

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Furthermore, we apply theorems in optimal stopping to two economic models: the model of stock prices by Samuelson [5] and a model of an employer who uses a capital to produce goods and sells the used the capital later. In both models, we also show whether those agents are risk-loving or risk-averse.

2. Preliminaries: Itô diffusion and boundary value problems

Most of the materials in this section are from Chapter 7 and Chapter 9 in Øksendal's *Stochastic differential equations: an introduction with applications* [4].

2.1. Itô diffusions and their properties.

Definition 2.1. An **Itô diffusion** is a stochastic process $(X_t)_{t\geq s}$ for some $s \geq 0$ satisfying a stochastic differential equation of the form

$$
\begin{aligned}
dX_t &= b(X_t)dt + \sigma(X_t)dB_t, \qquad t \geq s \\
X_s &= x,
\end{aligned}
\tag{2.2}
$$

where B_t is an m-dimensional Brownian motion and $b : \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are Lipschitz continuous functions.

The Lipschitz continuity condition assures us that the stochastic differential equation has a unique solution. We denote this unique solution by $X_t = X_t^{s,x}$.

The main difference between an Itô diffusion and a general continuous stochastic process is that the rate of change of an Itô diffusion only depends on the most current value of the diffusion. Therefore, we may expect that the process $(X_t)_{t\geq s}$ is time-homogeneous. Formally, it can be proved that the processes $\{X_{s+h}^{s,x}\}_{h\geq 0}$ and $\{X_h^{0,x}\}_{h\geq 0}$ have the same distributions.

From now on, we let $\mathcal{F}_t^{(m)}$ be the σ-algebra generated by $\{B_r; r \leq t\}$ for every $t \geq 0$. The existence and uniqueness theorem for differential equations implies that X_t is $\mathcal{F}_t^{(m)}$-measurable. Intuitively speaking, $\mathcal{F}_t^{(m)}$ contains the information of all fluctuations of the process before time t, so this information is sufficient to predict the state X_t.

For a function $g : \mathbb{R}^n \to \mathbb{R}$, we often use the notation $E^x[g(X_t)]$ as the expected value of $g(X_t)$ with respect to a given Itô diffusion $\{X_t\}_{t\geq 0}$ when its initial value is $X_0 = x \in \mathbb{R}^n$.

Example 2.3. One classical Itô diffusion is the geometric Brownian motion $(X_t)_{t\geq 0}$ defined by

$$
\begin{aligned}
dX_t &= \alpha X_t dt + \beta X_t dB_t, \\
X_0 &= x,
\end{aligned}
$$

where α and β are constant. In many economic models, changes in prices over time are modelled by geometric Brownian motions. The first term of the right-hand-side of the equation can be interpreted as the long-run inflation rate. The latter term of the right-hand-side of the equation can be interpreted as economic shocks, such as unpredictable inflation and depression. This stochastic differential equation can be solved explicitly as

$$X_t = x \exp\Big(\big(\alpha - \frac{1}{2}\beta^2\big)t + \beta B_t\Big).$$

An Itô diffusion also has the most important property that, if we know what has happened to the process before time t, then the process after time t and the process starting at X_t display the same behavior. This property is usually called Markov property for Itô diffusion, which is rigorously formulated as follows:

Theorem 2.4. *(Markov property for Itô diffusions) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded Borel measurable function. Then, for $t, h \geq 0$*

$$E^x[f(X_{t+h}) \mid \mathcal{F}_t^{(m)}] = E^{X_t}[f(X_h)]$$

The Markov property can be extended to a stronger version in which the fixed time t can be generalized to a stopping time τ with respect to $\mathcal{F}_t^{(m)}$, which is rigorously defined in Definition 2.5. Intuitively speaking, a stopping time is a random variable that decides which time to stop based on previous information.

Recall first that a collection of $\sigma-$algebras $\{\mathcal{N}_t\}_{t\geq 0}$ is called a filtration if $\mathcal{N}_k \subset \mathcal{N}_l$ for all $k \leq l$. Thus, we can define a stopping time as follows:

Definition 2.5. A $\mathbb{R}_{\geq 0} \cup \{\infty\}-$valued random variable τ is called a **stopping time** with respect to a filtration $\mathcal{N}_t$ if the set $\{\tau \leq t\}$ is $\mathcal{N}_t$-measurable for all $t \geq 0$.

Example 2.6. The stopping times we are going to consider in this paper are the first exit times of an Itô diffusion $(X_t)_{t\geq 0}$ from D, written as τ_D. This can be formally defined as

$$\tau_D = \inf\{t > 0 \mid X_t \notin D\}.$$

These stopping times are essential for many optimal stopping problems since it is intuitive to assume that the best time to stop the Itô diffusion $(X_t)_{t\geq 0}$ is when X_t reaches some threshold value. The economic models in Section 4 and Section 5 will exemplify such importance of the first exit time.

Theorem 2.7. *(Strong Markov property for Itô diffusions) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded Borel measurable function, and τ be a stopping time with respect to $\mathcal{F}_t^{(m)}$ such that $\tau < \infty$. Then, for $h \geq 0$*

$$E^x[f(X_{\tau+h}) \mid \mathcal{F}_t^{(m)}] = E^{X_\tau}[f(X_h)].$$

Note that a constant random variable τ is always a stopping time since $\{\tau \leq t\}$ is either the empty set or the whole space. By taking a constant stopping time $\tau = t$, we observe that the Markov property is a special case of the strong Markov property.

Next, we introduce the characteristic operator of an Itô diffusion, which will be useful in the future applications.

Definition 2.8. Let X_t be an Itô diffusion. The **characteristic operator** $\mathcal{A} = \mathcal{A}_X$ of X_t is defined by

$$\mathcal{A}f(x) = \lim_{U\downarrow x} \frac{E^x[f(X_{\tau_U})] - f(x)}{E^x[\tau_U]}, \tag{2.9}$$

for $f : \mathbb{R}^n \to \mathbb{R}$ and $x \in \mathbb{R}^n$ such that the limit exists. Here, $\lim_{U\downarrow x}$ means that we take the limit over a collection of open sets $\{U_k\}_{k\in\mathbb{N}}$ such that $U_{k+1} \subset U_k$ and $\bigcap_k U_k = \{x\}$. The set of functions f such that the limit (2.9) exists for all $x \in \mathbb{R}^n$ is denoted by $\mathcal{D}_\mathcal{A}$.

We may interpret the characteristic operator as a generalization of differentiation. If X_t is a deterministic Itô diffusion on $\mathbb{R}$ such that $X_t = t$ for $t \geq 0$, the corresponding characteristic operator is the right-hand derivative. In general, even though the definition of the characteristic operator seems complicated, the computation is much easier if the function f is twice-differentiable, as shown in the following theorem.

Theorem 2.10. *Let $f \in C^2$. Then $f \in \mathcal{D}_{\mathcal{A}}$ and*

$$\mathcal{A}f = \sum_i b_i \frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}.$$

According to this theorem, if the Itô diffusion is deterministic, the characteristic operator is indeed a directional derivative along vector field b.

Theorem 2.11. *(Dynkin's formula) Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice-differentiable and compactly supported. Suppose τ is a stopping time satisfying $E^x[\tau] < \infty$. Then*

$$E^x[f(X_\tau)] = f(x) + E^x\left[\int_0^\tau \mathcal{A}f(X_s)ds\right].$$

Dynkin's formula is analogous to the first fundamental theorem of calculus. Dynkin (1965) provided a more general version of Theorem 2.11. Yet, this simplified version is sufficient to study the relationship between the expected value at the stopping time and the characteristic operator.

2.2. **Harmonic function and the stochastic Dirichlet problem.** Since our interest in this paper is to find a stopping time that induces the optimal expected reward, to understand how to calculate the expected reward given a simple form of stopping time is crucial. In the following, we let $(X_t)_{t\geq 0}$ be an Itô diffusion satisfying (2.2).

Definition 2.12. Let f be a locally bounded and Borel measurable real-valued function on D. f is called X**-harmonic** in D if

$$f(x) = E^x[f(X_{\tau_U})]$$

for all $x \in D$ and all bounded open sets U with $\bar{U} \subset D$.

One crucial example of harmonic functions is the expected value of a function on the boundary with respect to the first exit times of an Itô diffusion from a measurable set.

Theorem 2.13. *Let ϕ be a bounded measurable function on ∂D. Then, the function $u : D \to \mathbb{R}$ defined by*

$$u(x) = E^x[\phi(X_{\tau_D})]$$

is X-harmonic and $\lim_{t\uparrow\tau_D} u(X_t) = \phi(X_{\tau_D})$ a.s. for all $x \in D$.

Lemma 2.14. *Let f be X-harmonic in D. Then $\mathcal{A}f = 0$ in D.*

Theorem 2.13 and Lemma 2.14 together directly imply the following corollary.

Corollary 2.15. *Let ϕ be a bounded measurable function on ∂D. Then, the function $u : D \to \mathbb{R}$ defined by*

$$u(x) = E^x[\phi(X_{\tau_D})]$$

satisfies that $\mathcal{A}f = 0$ in D and $\lim_{t\uparrow\tau_D} u(X_t) = \phi(X_{\tau_D})$ a.s. for all $x \in D$.

This corollary is crucial to compute the expected reward given a simple form of stopping time since we can obtain the expected reward by solving the partial differential equation $\mathcal{A}f = 0$ with known boundary conditions. More details are discussed in Section 4 and Section 5.

3. Existence and uniqueness theorems for optimal stopping

Most of the materials in this section are from Chapter 10.1 in Øksendal's *Stochastic differential equations: an introduction with applications* [4]. First, let us formally define optimal stopping problems as follows.

Definition 3.1. Let X_t be an Itô diffusion on $\mathbb{R}^n$ and let g be a given continuous function on $\mathbb{R}^n$ such that $g(x) \geq 0$ for all $x \in \mathbb{R}^n$. Then, we call g a **reward function**. An **optimal stopping time** is a stopping time $\tau^* = \tau^*(x, \omega)$ for $\{X_t\}$ such that

$$E^x[g(X_{\tau^*})] = \sup_\tau E^x[g(X_\tau)]$$

for all $x \in \mathbb{R}^n$, where the sup is taken over all stopping times τ for $\{X_t\}$. If $\omega \in \Omega$ is a point such that $\tau(\omega) = \infty$, then we interpret $g(X_\tau(\omega))$ as 0. Also, we define the corresponding **optimal expected reward** g^* as

$$g^*(x) = \sup_\tau E^x[g(X_\tau)].$$

Note that the reward function here depends on only the stopping state of the diffusion. However, some reward functions in the real world also depend on the stopping time and all states of the diffusion before it stops. These optimal problems can be solved by introducing a new Itô diffusion associated to the reward function. We will discuss more about this in Section 4 and Section 5.

The condition that g is a nonnegative function makes the notation $E^x[g(X_\tau)]$ well-defined. This condition can be relaxed in the future as will be explained later.

The reason why we do not define the optimal expected reward g^* as the expected reward of an optimal stopping time is that a stopping time may not exist in general. An example of this is shown as follows:

Example 3.2. Let X_t be a deterministic Itô diffusion on $\mathbb{R}$ such that $X_t = t$ for $t \geq 0$ and let g be a reward function such that $g(x) = \frac{x^2}{1+x^2}$. Then, $\sup_\tau E^x[g(X_\tau)] \leq 1$ since the function g is bounded above by 1. For any $a > x$, we obtain

$$1 \geq \sup_\tau E^x[g(X_\tau)] \geq E^x[g(X_{\tau_{(-\infty,a)}})] = g(a) = \frac{a^2}{1+a^2}.$$

By taking sup over (x, ∞) on the right hand side, we obtain $\sup_\tau E^x[g(X_\tau)] = 1$. However, there is no stopping time τ such that $E^x[g(X_\tau)] = 1$ since $g(x) < 1$ for all $x \in \mathbb{R}$.

Next, we introduce a supermeanvalued function with respect to an Itô diffusion.

Definition 3.3. A measurable function $f : \mathbb{R}^n \to [0, \infty]$ is called **supermeanvalued** if

$$f(x) \geq E^x[f(X_\tau)] \tag{3.4}$$

for all stopping times τ and all $x \in \mathbb{R}^n$. If f is also lower semicontinuous, then f is called **superharmonic**.

Intuitively speaking, being a supermeanvalued function means that stopping the process suddenly will give more reward than stopping the process later no matter what. Therefore, it makes sense that, if the reward function is already a supermeanvalued function, the optimal stopping is just stopping at the initial state.

Definition 3.5. Let $x \in \mathbb{R}^n$. A measurable function $f : \mathbb{R}^n \to [0, \infty]$ is called **supermeanvalued** at x if

$$f(x) \geq E^x[f(X_\tau)]$$

for all stopping times $\tau \leq \tau_{B_1(x)}$, where $\tau_{B_1(x)}$ is the first exit time of the ball $B_1(x) := \{y \in \mathbb{R}^n \mid |x - y| < 1\}$. If f is also lower semicontinuous, then f is called **superharmonic**.

It can be proved that f is a supermeanvalued function if and only if f is supermeanvalued at x for all $x \in \mathbb{R}^n$. This means that the property of being a supermeanvalued function is a local property.

We will provide some important properties of supermeanvalued and superharmonic functions as follows:

Lemma 3.6. *(1) If f is superharmonic (supermeanvalued) and $\alpha > 0$, then αf is superharmonic (supermeanvalued).*
(2) If f_1, f_2 are superharmonic (supermeanvalued), then $f_1 + f_2$ is superharmonic (supermeanvalued).
(3) $f_1, f_2, \ldots$ are superharmonic (supermeanvalued) functions and $f_k \uparrow f$ pointwise, then f is superharmonic (supermeanvalued).
(4) If f is supermeanvalued and $\sigma \leq \tau$ are stopping times, then $E^x[f(X_\sigma)] \geq E^x[f(X_\tau)]$.
(5) If f is supermeanvalued and $H \subset \mathbb{R}^n$ is a Borel set, then $\tilde{f}(x) := E^x[f(X_{\tau_H})]$ is supermeanvalued.

Proof. See [4]. □

To check whether a function is supermeanvalued/superharmonic or not seems complicated since we have to check (3.4) for all stopping times τ. However, if we know that the function is twice-differentiable, we can easily check this by using its characteristic function and the following corollary.

Corollary 3.7. *If $f \in C^2(\mathbb{R}^n)$, then f is superharmonic if and only if $\mathcal{A}f \leq 0$ where $\mathcal{A}$ is the characteristic operator of X_t.*

Proof. If f is superharmonic, then for every $x \in D$, $E^x[f(X_{\tau_U})] \leq f(x)$ for every bounded open set U with $\bar{U} \subset \mathbb{R}^n$ such that $x \in D$. Therefore, by the definition of the characteristic operator

$$\mathcal{A}f(x) = \lim_{U \downarrow x} \frac{E^x[f(X_{\tau_U})] - f(x)}{E^x[\tau_U]} \leq 0.$$

If $\mathcal{A}f \leq 0$, we want to show that f is supermeanvalued at every point $x \in \mathbb{R}^n$. Consider a stopping time τ that is less than the exit time of the unit ball $B_1(x)$ centered at x. By the C^∞ Urysohn's lemma, there exists $\tilde{f} \in C_c^\infty(\mathbb{R}^n)$ such that $f = \tilde{f}$ on $B_2(x)$. Thus, if $X_0 = x$, then $X_\tau \in \bar{B}_1(x) \subset B_2(x)$, which means that

$f(X_\tau) = \tilde{f}(X_\tau)$. Therefore, Theorem 2.11 implies that

$$E^x[f(X_\tau)] = E^x[\tilde{f}(X_\tau)] = \tilde{f}(x) + E^x\left[\int_0^\tau \mathcal{A}\tilde{f}(X_s)ds\right]$$
$$= f(x) + E^x\left[\int_0^\tau \mathcal{A}f(X_s)ds\right] \le f(x),$$

which means that f is supermeanvalued at every point $x \in \mathbb{R}^n$. □

Definition 3.8. Let h be a real measurable function on $\mathbb{R}^n$. If f is a superharmonic (supermeanvalued) function and $f \le h$, we say that f is a superharmonic (supermeanvalued) majorant of h w.r.t. X_t. Suppose there exists a function $\widehat{h}$ such that

(1) $\widehat{h}$ is a superharmonic majorant of h and
(2) if f is another superharmonic majorant of h, then $\widehat{h} \le f$.

Then, we call $\widehat{h}$ the least superharmonic majorant of h.

In general, the definition does not guarantee the existence of the least superharmonic majorant. However, in our cases, the reward functions we are going to study mostly are continuous functions, and it can be proved that nonnegative and continuous functions have the least superharmonic majorant [4]. Therefore, mostly, the existence of the least superharmonic majorant is not our concern.

We will build a relationship between the least superharmonic majorant and the optimal reward function and finally show that these functions are the same.

Lemma 3.9. *Suppose that a continuous reward function $g \ge 0$ has the optimal reward function g^* and the least superharmonic majorant $\widehat{g}$. Then, $g^* \le \widehat{g}$.*

Proof. By the definition of a supermeanvalued function, we obtain

$$\widehat{g}(x) \ge E^x[\widehat{g}(X_\tau)] \ge E^x[g(X_\tau)].$$

Thus, by taking the supremum over a stopping time τ on both sides, we have that $\widehat{g}(x) \ge \sup_\tau E^x[g(X_\tau)] = g^*(x)$, as desired. □

Our next question is whether the converse inequality also holds or not. To do this, we will first consider the case that g is a bounded function. We will find a stopping time that yields the reward arbitrarily close to the least superharmonic majorant as follows:

Lemma 3.10. *Let g be a bounded continuous reward function $g \ge 0$. For $\epsilon > 0$, we define*

$$D_\epsilon = \{x \in \mathbb{R}^n \mid g(x) < \widehat{g}(x) - \epsilon\}.$$

Then,

$$\widehat{g}(x) - \epsilon \le E^x[g(X_{\tau_\epsilon})]$$

for all $x \in \mathbb{R}^n$.

Proof. Define $\tilde{g}_\epsilon(x) = E^x[\widehat{g}(X_{\tau_\epsilon})]$. Lemma 3.6(5) implies that $\tilde{g}_\epsilon$ is a supermeanvalued function. Let

$$\sup_{x\in\mathbb{R}^n} \{g(x) - \tilde{g}_\epsilon(x)\} = a.$$

Note that $a < \infty$ since g is bounded above. Therefore, $(\tilde{g}_\epsilon + a)(x) \geq g(x)$ and $\tilde{g}_\epsilon + a$ is a supermeanvalued function. Then $\tilde{g}_\epsilon + a$ is a supermeanvalued majorant of g. Thus,

$$\tilde{g}_\epsilon(x) + a \geq \widehat{g}(x),$$

for all $x \in \mathbb{R}^n$. If $a > 0$, then there exists $x_0 \in \mathbb{R}^n$ such that $g(x_0) = \tilde{g}_\epsilon(x_0) + a - \theta$ for some $\theta \in (0, \min\{\epsilon, a\})$. Thus,

$$g(x_0) = \tilde{g}_\epsilon(x_0) + a - \theta \geq \widehat{g}(x_0) - \theta \geq \widehat{g}(x_0) - \epsilon,$$

which means that $x_0 \notin D_\epsilon$. Then,

$$\tilde{g}_\epsilon(x_0) = E^x[\widehat{g}(X_{\tau_\epsilon})] = \widehat{g}(x_0).$$

This implies that

$$g(x_0) = \tilde{g}_\epsilon(x_0) + a - \theta = \widehat{g}(x_0) + a - \theta > \widehat{g}(x_0),$$

which is impossible since $\widehat{g} \geq g$. Thus, $a \leq 0$, implying that $g(x) \leq \tilde{g}_\epsilon(x)$ for all $x \in \mathbb{R}^n$. Therefore, $\tilde{g}_\epsilon$ is a superharmonic majorant of g, so

$$\widehat{g}(x) \leq \tilde{g}_\epsilon(x) = E^x[\widehat{g}(X_{\tau_\epsilon})] \leq E^x[(g + \epsilon)(X_{\tau_\epsilon})] = E^x[g(X_{\tau_\epsilon})] + \epsilon,$$

which means that $\widehat{g}(x) - \epsilon \leq E^x[g(X_{\tau_\epsilon})]$ for all $x \in \mathbb{R}^n$. □

Based on Lemma 3.9 and Lemma 3.10, we have sufficient information to conclude that the least superharmonic majorant and the optimal reward are exactly the same in the following theorem.

Theorem 3.11. *(Existence theorem for optimal stopping time I) Suppose that a continuous reward function $g \geq 0$ has the optimal reward function g^* and the least superharmonic majorant $\widehat{g}$. Then, $g^*(x) = \widehat{g}(x)$ for all $x \in \mathbb{R}^n$.*

Proof. We will first consider the case that g is bounded. By Lemma 3.10 and the definition of an optimal stopping time,

$$\widehat{g}(x) - \epsilon \leq E^x[g(X_{\tau_\epsilon})] \leq g(x).$$

Since ϵ can be an arbitrarily small positive number, we obtain $\widehat{g}(x) \leq g(x)$. Thus, by Lemma 3.9, we can conclude that $\widehat{g}(x) = g(x)$ for all $x \in \mathbb{R}^n$.

If g is unbounded, we define $g_N = \min(N, g)$ for all $N \in \mathbb{N}$. Thus, g_N is bounded, which means that $g_N^* = \widehat{g_N}$. The sequence of functions $\{\widehat{g_N}\}_N = \{g_N^*\}_N$ is pointwise increasing since $g_N \leq g_{N+1}$, so $g_N^* \leq g_{N+1}^*$. Thus, the sequence $\{\widehat{g_N}\}_N$ converges to some measurable function $h : \mathbb{R}^n \to [0, \infty]$. Since $\widehat{g_N}$ is a superharmonic function for all $N \in \mathbb{N}$ and $\widehat{g_N} \uparrow h$ pointwise, then, by Lemma 3.6(3), h is supermeanvalued.

We will show that $h \geq \widehat{g}$. Consider that $g_N \leq \widehat{g_N}$. Then, for all $x \in \mathbb{R}^n$,

$$g(x) = \lim_{N \to \infty} \min(N, g(x)) = \lim_{N \to \infty} g_N(x) \leq \lim_{N \to \infty} \widehat{g_N}(x) = h(x).$$

Since h is a supermeanvalued function, h is a superharmonic majorant of g, so $h \geq \widehat{g}$. Also, $g^* \geq g_N^*$ since $g \geq g_N$ for all $N \in \mathbb{N}$. Thus,

$$g^*(x) \geq \lim_{N \to \infty} g_N^*(x) = \lim_{N \to \infty} \widehat{g_N}(x) = h(x) \geq \widehat{g}(x)$$

for all $x \in \mathbb{R}^n$. By Lemma 3.9, we conclude that $h = \widehat{g} = g^*$, as desired. □

As we discussed before, even though the optimal reward function exists, it is not necessarily true that the optimal stopping time always exists. Lemma 3.10 tells us that stopping at the first exit time from D_ϵ is close to being optimal, so we may expect the first exit time from $D = D_0 = \{x \in \mathbb{R}^n \mid g(x) < \widehat{g}(x)\}$ may yield the optimal reward function. However, this is generally not true.

Example 3.12. Recall Example 3.2. We can see that

$$D = \{x \in \mathbb{R} \mid g(x) < 1\} = \mathbb{R}.$$

Thus, the first exit time τ_D from D is always infinite, so $E^x[g(X_{\tau_D})] = 0$, which is not optimal.

The next theorem gives us the sufficient condition of the existence for the optimal stopping time.

Theorem 3.13. *(Existence theorem for optimal stopping time II) For arbitrary continuous $g \geq 0$ let*

$$D = \{x \mid g(x) < g^*(x)\}.$$

For $N \in \mathbb{N}$, we define $g_N = g \wedge N, D_N = \{x \mid g_N(x) < \widehat{g_N}(x)\}$ and $\sigma_N = \tau_{D_N}$. If $\tau_D < \infty$ a.s. and the family $\{g(X_{\sigma_N})\}_N$ is uniformly integrable, then

$$g^*(x) = E^x[g(X_{\tau_D})]$$

and $\tau^* = \tau_D$ is an optimal stopping time.

Proof. See [4]. □

Even though Theorem 3.11 and Theorem 3.13 explicitly give us how to find the optimal reward function and the optimal stopping time, they are not practical in general since the least superharmonic majorant is not easy to be computed. The following corollary will help us to find the optimal reward function and the optimal stopping time.

Corollary 3.14. *Let $g \geq 0$ be a continuous function. Suppose there exists a Borel set H such that $\tilde{g}_H(x) := E^x[g(X_{\tau_H})]$ is a supermeanvalued majorant of g. Then $g^*(x) = \tilde{g}_H(x)$, so $\tau^* = \tau_H$ is optimal.*

Proof. Since $\tilde{g}_H$ is a supermeanvalued majorant of g, $\tilde{g}_H \geq \widehat{g}$. Also, by the definition of the optimal reward function and Theorem 3.11,

$$\tilde{g}_H(x) = E^x[g(X_{\tau_H})] \leq \sup_\tau E^x[g(X_\tau)] = g^*(x) = \widehat{g}(x).$$

Thus, $g^* = \tilde{g}_H$ with the optimal stopping time τ_H. □

This is the most crucial result in this paper to calculate the optimal reward function. In many optimal stopping problems in the real world, it is generally not too difficult to find a Borel set H. To find this, we may expect that H needs to have some patterns depending on the corresponding situation. For example, one will sell an asset if its price reaches a particular value, so we may expect H to be $(-\infty, a)$ for some $a \in \mathbb{R}$. Then, we need to find an appropriate value of a that makes $\tilde{g}_H$ a supermeanvalued function by using Corollary 3.7 to calculate a. More details are explained in Section 4 and Section 5.

Corollary 3.15. *Assume that*

$$D = \{x \mid g(x) < \widehat{g}(x)\}$$

and define $\tilde{g}(x) = E^x[g(X_{\tau_D})]$. *If* $\tilde{g} \geq g$, *then* $\tilde{g} = g^*$.

Proof. Since $X_{\tau_D} \notin D$, $g(X_{\tau_D}) \geq \widehat{g}(X_{\tau_D})$. Thus, $g(X_{\tau_D}) = \widehat{g}(X_{\tau_D})$, which means that

$$\tilde{g}(x) = E^x[g(X_{\tau_D})] = E^x[\widehat{g}(X_{\tau_D})].$$

Since $\widehat{g}$ is a supermeanvalued function, Lemma 3.6(5) implies that $\tilde{g}$ is a supermeanvalued function. Then, $\tilde{g}$ is asupermeanvalued majorant of g. We can conclude by Corollary 3.14 that $\tilde{g} = g^*$. □

In the next theorem, we will show that if an optimal stopping time exists, then τ_D is also an optimal stopping time.

Theorem 3.16. *(Uniqueness theorem for optimal stopping) Define as before*

$$D = \{x \mid g(x) < g^*(x)\} \subset \mathbb{R}^n.$$

Suppose there exists an optimal stopping time τ^* *for the problem.. Then* $\tau^* \geq \tau_D$ *for all* $x \in D$ *and* $g^*(x) = E^x[g(X_{\tau_D})]$ *for all* $x \in \mathbb{R}^n$.

Proof. For the sake of contradiction, assume that $\tau^* \ngeq \tau_D$ for some $x_0 \in D$. This means that $P^x[\tau^* < \tau_D] > 0$. Then

$$\begin{aligned} g^*(x) = E^x[g(X_{\tau^*})] &= E^x[g(X_{\tau^*})1_{\tau^*<\tau_D}] + E^x[g(X_{\tau^*})1_{\tau^*\geq\tau_D}] \\ &= E^x[g(X_{\tau^*}) \mid \tau^* < \tau_D]P^x[\tau^* < \tau_D] + E^x[g(X_\tau)1_{\tau^*\geq\tau_D}] \\ &< E^x[g^*(X_{\tau^*}) \mid \tau^* < \tau_D]P^x[\tau^* < \tau_D] + E^x[g^*(X_{\tau^*})1_{\tau\geq\tau_D}] \\ &= E^x[g^*(X_{\tau^*})1_{\tau^*<\tau_D}] + E^x[g^*(X_{\tau^*})1_{\tau^*\geq\tau_D}] = E^x[g^*(X_{\tau^*})] \leq g^*(x), \end{aligned}$$

which is a contradiction. Thus, $\tau^* \geq \tau_D$ for all $x \in D$.

Consider the case that $x \in D$. Since $\tau^* \geq \tau_D$, Lemma 3.6(4) and the fact that $\widehat{g}$ is superharmonic imply that

$$g^*(x) = E^x[g(X_{\tau^*})] \leq E^x[\widehat{g}(X_{\tau^*})] \leq E^x[\widehat{g}(X_{\tau_D})] \leq \widehat{g}(x).$$

Since $\widehat{g} = g^*$, we conclude $g^*(x) = E^x[g(X_{\tau_D})]$ for all $x \in D$.

If $x \notin \bar{D}$, we have $\tau_D = 0$ a.s., so $g^*(x) = g(x) = E^x[g(X_{\tau_D})]$. For the case that x is a boundary point of D, see [4]. □

One consequence of this theorem is a way to show the nonexistence of an optimal stopping time by assuming its existence and showing that τ_D is not an optimal stopping time. See more details in Chapter 4.

All of the results on optimal stopping we have right now require the assumption that g is a nonnegative continuous function. This assumption can be relaxed to the condition that g is bounded below since we can add some constant to g so that the new function is nonnegative, and we can apply the results to the new function. Furthermore, this condition can even be relaxed more to the condition that the family $\{g^-(X_\tau)\}$, where τ is a stopping time and $g^-(x, 0) = \min(g(x), 0)$, is uniformly integrable, whereby all of the results still hold. See [4] and [6] for more details.

4. Applications to an economic model: Stock prices

The application of optimal stopping times to stock prices has been extensively studied. One example is the work by Samuelson [5], who studied the optimal time for selling an asset. Suppose that one initially has an asset whose price follows a geometric Brownian motion X_t given by

$$\begin{aligned} dX_t &= rX_t dt + \alpha X_t dB_t \\ X_0 &= x, \end{aligned}$$

where r and α are constants. Also, selling an asset requires a transaction cost $a > 0$, which is constant over time. Intuitively speaking, the transaction cost is resistant to economic changes.

If one decides to sell one's asset at time t, one will obtain the net profit $(X_t - a)^+ = \max\{0, X_t - a\}$ since one should not sell the asset at the price below a. Assume that ρ is a discount rate, so that the discounted profit to sell the asset at time t measured in time 0 dollars is

$$g(t,\omega) = e^{-\rho t}(X_t(\omega) - a)^+.$$

Therefore, the optimal stopping problem of the seller is

$$\Phi(x) = \sup_\tau E^x[e^{-\rho t}(X_t - a)^+].$$

Observe that the reward function has nonnegative values, so we can apply the optimal stopping theorems we have to this problem. Note first that our stopping problem is not time-homogeneous yet, so we introduce the new Itô diffusion Y_t in order to obtain a time-homogeneous optinal stopping problem as follows:

$$dY_t = \begin{bmatrix} dt \\ dX_t \end{bmatrix} = \begin{bmatrix} 1 \\ rX_t \end{bmatrix} dt + \begin{bmatrix} 0 \\ \alpha X_t \end{bmatrix} dB_t; \quad Z_0 = (s,x).$$

Therefore, we can define a new reward function

$$\bar{g}(s,x) = e^{-\rho t}(x-a)^+.$$

So we can rewrite our optimal stopping problem as

$$\Phi(x) = \sup_\tau \mathbb{E}^{(0,x)}[\tilde{g}(\tau, X_\tau)],$$

which is time-homogeneous with the new Itô diffusion Y_t. Note that the characteristic operator $\mathcal{A}$ of Y_t is

$$\mathcal{A}\phi = \frac{\partial \phi}{\partial s} + rx\frac{\partial \phi}{\partial x} + \frac{1}{2}\alpha^2 x^2 \frac{\partial^2 \phi}{\partial x^2},$$

for all $\phi \in C^2(\mathbb{R}_{\geq 0} \times \mathbb{R})$. In view of Corollary 3.14, we will find a Borel set H such that $\tilde{g}_H$ is a supermeanvalued majorant of g. To do this, we expect that H may be written as $H_{x_0} = \mathbb{R}_{\geq 0} \times (0, x_0)$ since it is intuitive to assume that the asset seller is indifferent to selling the asset with some price x_0, and the price cannot be below 0 due to the geometric Brownian motion.

Let $\tau(x_0)$ be the first exit time from H_{x_0}. To calculate

$$h(s,x) = \tilde{g}_{\tau(x_0)}(s,x) = E^{(s,x)}[\bar{g}(Y_{\tau(x_0)})],$$

we need to solve the following boundary value problem

$$\begin{aligned} \frac{\partial h}{\partial s} + rx\frac{\partial h}{\partial x} + \frac{1}{2}\alpha^2 x^2 \frac{\partial^2 h}{\partial x^2} &= 0, \text{ for } 0 < x < x_0 \\ h(s,0) &= 0 \\ h(s,x_0) &= e^{-\rho s}(x_0-1)^+. \end{aligned} \tag{4.1}$$

We guess a solution of the form $h(s,x) = e^{-\rho s}\phi(x)$, so we can rewrite the equations as

$$\begin{aligned} -\rho\phi(x) + rx\phi'(x) + \frac{1}{2}\alpha^2 x^2(x)\phi'' &= 0, \text{ for } 0 < x < x_0 \\ \phi(0) &= 0 \\ \phi(x_0) &= (x_0-1)^+. \end{aligned} \tag{4.2}$$

Therefore, the general solution ϕ of the ordinary differential equation (4.2) is

$$\phi(x) = C_1 x^{\gamma_1} + C_2 x^{\gamma_2},$$

where C_1, C_2 are arbitrary constant and

$$\gamma_i = \alpha^{-2}\left[\left(\frac{1}{2}\alpha^2 - r\right) \pm \sqrt{\left(\frac{1}{2}\alpha^2 - r\right)^2 + 2\rho\alpha^2}\right],$$

for $i \in \{1,2\}$, where $\gamma_2 < 0 < \gamma_1$.

Since $\phi(0) = 0$, we must have $C_2 = 0$; otherwise, ϕ would not be well-defined at 0 since $\gamma_2 < 0$. Moreover, the boundary condition tells us that $\phi(x_0) = (x_0 - 1)^+$, so a simple calculation shows us that $C_1 = x_0^{-\gamma_1}(x_0 - a)^+$. Then, we can derive the formula of h

$$\tilde{g}_{\tau(x_0)} = h(s,x) = e^{-\rho s}(x_0 - a)^+\left(\frac{x}{x_0}\right)^{\gamma_1}.$$

Note that

$$\gamma_1 - 1 = \alpha^{-2}\left[\sqrt{\left(\frac{1}{2}\alpha^2 + r\right)^2 + 2(\rho - r)\alpha^2} - \left(\frac{1}{2}\alpha^2 + r\right)\right] \geq 0$$

if and only if $\rho \geq r$, and the equality holds when $\rho = r$. We will consider three cases.

Case 1: $\rho < r$. Then, $\gamma_1 < 1$, so for fixed x,

$$\lim_{x_0\to\infty} \tilde{g}_{\tau(x_0)} = \lim_{x_0\to\infty} e^{-\rho s}x^{\gamma_1}(x_0^{1-\gamma_1} - a x_0^{-\gamma_1})^+ = \infty,$$

which means that we can choose a stopping time that yields an arbitrarily large reward. Therefore, τ^* does not exist, and $g^* = \infty$. Intuitively speaking, when the average growth of the price of the asset is higher than the discount rate, the seller does not have an incentive to sell anytime since he can just keep the asset a bit longer to gain more discounted profit.

Case 2: $\rho = \gamma$. Then, $\gamma_1 = 1$, so for fixed x,

$$g^*(x) \geq \lim_{x_0\to\infty} \tilde{g}_{\tau(x_0)}(x) = \lim_{x_0\to\infty} e^{-\rho s}x(1 - a x_0^{-1})^+ = e^{-\rho s}x. \tag{4.3}$$

Define $\tilde{g}(x) = e^{-\rho s}x$ for all $x \in \mathbb{R}^+$. Thus, (4.3) implies that $\tilde{g} \leq g^*$. Note that $\tilde{g}$ is a superharmonic function since

$$\mathcal{A}\tilde{g}(s,x) = e^{-\rho s}x(-\rho + r) = 0$$

for all s and x. Also, $\tilde{g}(x) = e^{-\rho s}x \geq e^{-\rho s}(x-a) = g(x)$, so $\tilde{g}$ is a supemeanvalued majorant of g, which means that $\tilde{g} \geq \widehat{g}$. By Theorem 3.10, $g^*(x) = \tilde{g}(x) = e^{-\rho s}x$ for all $x \in \mathbb{R}$.

The next question is whether there exists an optimal stopping time for this problem. Assume that there exists an optimal stopping time τ^*. The uniqueness theorem for optimal stopping implies that $\tau^* \geq \tau_D$ for all $x \in D$, where

$$D = \{x \mid g(x) < g^*(x)\} = \mathbb{R}^2.$$

Thus, $\tau^* \geq \tau_D = \tau_{\mathbb{R}^2} = \infty$, so $g^* = 0$, which contradicts what we had before. We conclude that there is no optimal stopping time for this problem.

Intuitively speaking, when the average growth of the price and the discount rate are the same, the average discounted outcome before deducted by the transaction fee is still the same over time. Therefore, the seller will just wait to sell until the discounted cost of the transaction fee becomes negligible.

Case 3: $\rho > \gamma$. Then, $\gamma_1 > 1$. This is the most interesting case because the seller will not keep the asset forever since the discount rate overcomes the growth rate of the price. Our goal is to determine which x_0 maximizes $\tilde{g}_{\tau(x_0)}(s,x)$ for all x. Note that this maximization makes sense since the function $\tilde{g}_{\tau(x_0)}(s,x)$ does not depend on s and depends linearly on y. For $x_0 > a$, we observe that

$$\frac{\partial \tilde{g}_{\tau(x_0)}(s,x)}{\partial x_0} = e^{-\rho s}x^{\gamma_1}\big((1-\gamma_1)x_0^{-\gamma_1} + a\gamma_1 x_0^{-\gamma_1-1}\big).$$

By setting the partial derivative equal to 0 and checking the second partial derivative, the value of a_0 that maximizes $\tilde{g}_{\tau(x_0)}(s,x)$ is

$$x_0 = x_{max} = \frac{a\gamma_1}{\gamma_1 - 1} > a$$

since $\gamma_1 > 1$. Therefore, our best candidate for a supermeanvalued majorant of g is $\tilde{g}_{a_{max}}$. By Corollary 3.14, it is sufficient to verify that $\tilde{g}_{x_{max}}$ is a supermeanvalued function with respect to Y_t and is greater than or equal g.

To show that $\tilde{g}_{x_{max}}$ is a supermeanvalued function, by Corollary 3.6, we need to prove $\mathcal{A}\tilde{g}_{x_{max}} < 0$. The boundary value problem (4.1) shows that $\mathcal{A}\tilde{g}_{x_{max}}(s,x) = 0$ for all $x < x_{max}$. For $x > x_{max}$, $\tilde{g}_{x_{max}}(s,x) = e^{-\rho s}(x-a)$ since $x > a$. Thus,

$$\begin{aligned}\mathcal{A}\tilde{g}_{a_{max}}(s,x) &= e^{-\rho s}\big(-\rho(x-a) + rx\big) = e^{-\rho s}\big((r-\rho)x + \rho a\big)\\ &\leq ae^{-\rho s}\left((r-\rho)\frac{\gamma_1}{\gamma_1-1} + \rho\right) = ae^{-\rho s}\left(\frac{r\gamma_1 - \rho}{\gamma_1 - 1}\right).\end{aligned}$$

Also, we can rewrite γ_1 as

$$\gamma_1 = \frac{2\rho}{\sqrt{\big(\frac{1}{2}\alpha^2 - r\big)^2 + 2\rho\alpha^2} - \big(\frac{1}{2}\alpha^2 - r\big)} \frac{2\rho}{\big(\frac{1}{2}\alpha^2 + r\big) - \big(\frac{1}{2}\alpha^2 - r\big)} = \frac{\rho}{r}$$

because $\big(\frac{1}{2}\alpha^2 + r\big)^2 - \big(\frac{1}{2}\alpha^2 - r\big)^2 = 2r\alpha^2 < 2\rho\alpha^2$. At $x = x_{max}$, we do the same by using the left-side derivative and the right-side derivative. Thus, we obtain $\mathcal{A}\tilde{g}_{x_{max}}(s,a,y) \leq 0$, as desired.

To prove that $\tilde{g}_{x_{max}} \geq g$, for fixed s and x, we have shown that the value of x_0 that maximizes $\tilde{g}_{x_0}$ is $x_0 = x_{max}$. Therefore,

$$\tilde{g}_{x_{max}}(s,x) \geq \tilde{g}_a(s,x) = g(s,x).$$

In conclusion, $\tilde{g}_{x_{max}}$ is a supermeanvalued majorant of g. Therefore, by Corollary 3.14, the optimal time for the seller to sell the asset is when the price of the asset reaches $x_{max} = \frac{a\gamma_1}{\gamma_1 - 1}$. The corresponding expected discount profit is

$$\tilde{g}_{x_{max}}(0, x) = \left(\frac{\gamma_1 - 1}{a}\right)^{\gamma_1 - 1}\left(\frac{x}{\gamma_1}\right)^{\gamma_1}.$$

The optimal stopping time we discovered shows that one should act as a risk lover in order to maximize the profit. To see this, if the change of the price becomes more unpredictable. this means that α is getting bigger. We may rewrite γ_1 as

$$\gamma_1 = \left(\frac{1}{2} - s\right) + \sqrt{\left(\frac{1}{2} - s\right)^2 + \frac{2\rho s}{r}} =: \phi(s),$$

where $s = \frac{r}{\alpha^2}$. Consider that

$$\phi'(s) = -1 + \left(\left(s - \frac{1}{2} + \frac{\rho}{r}\right)^2 + \frac{\rho}{r} - \frac{\rho^2}{r^2}\right)^{-\frac{1}{2}}\left(s - \frac{1}{2} + \frac{\rho}{r}\right) > 0$$

since $\rho > r$, which means that ϕ is an increasing function. As α increases, $s = \frac{r}{\alpha^2}$ decreases, which means that $\gamma_1 = \phi(s)$ decreases. Thus, $x_{max} = \frac{a\gamma_1}{\gamma_1 - 1}$ increases as α increases. This means that, by keeping the average performance the same, the larger variance of the change of price induces the seller to set a higher price threshold.

5. Applications to an economic model: Capital goods

Assume that a firm owns capital goods, such as machines and tools. In order to produce goods, the firm needs to hire a skilled worker to use those capital goods and pays him a constant wage over time. Meanwhile, while the worker is using the capital to produce goods, the capital depreciates over time. At any time, the firm may choose to stop hiring the worker and then sell the capital immediately at the price which linearly correlates to its latest performance. The question here is when the firm should stop hiring the worker and sell the capital in order to maximize the net outcome.

Let W_t be the accumulated wage until time t. Since we assume that the wage is constant, we can derive the formula of W_t as follows:

$$W_t = \bar{w}t,$$

for some constant $\bar{w} > 0$.

On the other hand, let A_t be the current productivity rate of the capital goods at time t, where $t \geq 0$. We assume that the change of A_t over time t consists of two main components: a deterministic depreciation rate dA_t^c and a stochastic change dA_t^u. The deterministic depreciation rate reflects the rate of the deterioration of the capital goods in the normal state. It is intuitive to assume that, the closer the performance of the capital is to the level that the capital can produce nothing, the lower the average rate of the depreciation is. Therefore, let

$$dA_t^c = \alpha A_t dt,$$

for some constant $\alpha \leq 0$.

A stochastic deterioration reflects possible shocks during the work. For example, the worker might have an innovative idea to develop the capital, which boosts the productivity. On the other hand, the worker might accidentally damage the machine the day after, which exacerbates the depreciation of the capital. These shocks can be modeled by a Brownian motion amplified by the current productivity level of the capital goods. Therefore, we derive

$$dA_t^u = \beta A_t dB_t,$$

for some constant $\beta \geq 0$

Thus, by combining dA_t^c and dA_t^u, we obtain a stochastic differential equation for A_t as follows:

$$dA_t = \alpha A_t dt + \beta A_t dB_t \text{ and } A_0 = a,$$

where a is the initial level of the productivity.

Suppose that τ is the time that the firm stop hiring the worker. This is indeed a stopping time since a decision made by the firm is based on a profile of the previous productivity development. Therefore, the net outcome of the firm all through the entire time is

$$g(\tau) = \int_0^\tau (A_t - \bar{w})e^{-\rho t}dt + \gamma A_t e^{-\rho\tau},$$

where $\rho \in (0,1)$ is the discount rate and γ is the price of the capital per one unit of the productivity. The first term on the right-hand-side is the net discounted outcomes of the firm before cancelling the contract. The latter term on the right-hand-side is the discounted profit after selling the used capital.

To make a decision when the firm should stop hiring the worker, the firm has to find a stopping time τ that maximizes $g(\tau)$. Therefore, the firm's problem now is to find τ^* and to solve the following optimal stopping problem:

$$\Phi(a) = \sup_\tau E^a\left[\int_0^\tau (A_t - \bar{w})e^{-\rho t}dt + \gamma A_\tau e^{-\rho\tau}\right] = g(\tau^*).$$

For every $s > 0$,

$$g(s) = \int_0^s (A_t - \bar{w})e^{-\rho t}dt + \gamma A_s e^{-\rho s} \geq -\int_0^s \bar{w}e^{-\rho t}dt \geq -\int_0^\infty \bar{w}e^{-\rho t}dt = -\frac{\bar{w}}{\rho}.$$

Thus, the reward function g is bounded below, so we can apply the optimal stopping theorems we have to this problem.

Note first that our optimal stopping problem is not time-homogeneous yet, so we introduce a new Itô diffusion Z_t in order to obtain a time-homogeneous optimal stopping problem as follows

$$dZ_t = \begin{bmatrix} dt \\ dA_t \\ dY_t \end{bmatrix} = \begin{bmatrix} 1 \\ \alpha A_t \\ (A_t - \bar{w})e^{-\rho t} \end{bmatrix} dt + \begin{bmatrix} 0 \\ \beta A_t \\ 0 \end{bmatrix} dB_t; \quad Z_0 = (s, a, y).$$

Then, we can define a new reward function

$$\bar{g}(s, a, y) = \gamma a e^{-\rho s} + y,$$

and we can rewrite our optimal stopping problem as

$$\Phi(a) = \sup_\tau E^{(0,a,0)}[\bar{g}(\tau, A_\tau, Y_\tau)],$$

which is time-homogeneous with the new Itô diffusion Z_t. Observe that the characteristic operator $\mathcal{A}$ of Z_t is given by

$$\mathcal{A}\phi = \frac{\partial \phi}{\partial s} + \alpha a \frac{\partial \phi}{\partial a} + (a - \bar{w})e^{-\rho s}\frac{\partial \phi}{\partial y} + \frac{1}{2}\beta^2 a^2 \frac{\partial^2 \phi}{\partial a^2},$$

for all $\phi \in C^2(\mathbb{R}_{\geq 0} \times \mathbb{R}^2)$. In view of Corollary 3.14, we will find a Borel set H such that $\tilde{g}_H$ is a supermeanvalued majorant of g. To do this, we expect that H may be written as $H_{a_0} = \mathbb{R}_{\geq 0} \times (a_0, \infty) \times \mathbb{R}$ since it is intuitive to assume that the firm should sell the capital when its productivity falls to some threshold a_0.

Let $\tau(a_0)$ be the first exit time from H_{a_0}. Also, for $N \in \mathbb{N}$, define $\tau^N(a_0)$ to be the first exit time from $H_{a_0}^N = \mathbb{R}_{\geq 0} \times (a_0, N) \times \mathbb{R}$. To calculate

$$h(s, a, y) = \tilde{g}_{\tau(a_0)}(s, a, y) = E^{(s,a,y)}[\bar{g}(Z_{\tau(a_0)})],$$

we will first calculate

$$h_N(s, a, y) = \tilde{g}_{\tau^N(a_0)}(s, a, y) = E^{(s,a,y)}[\bar{g}(Z_{\tau^N(a_0)})].$$

Then, we can take the limit of h_N as $N \to \infty$ to obtain h since it can be proved that the sequence $\{\bar{g}(Z_{\tau^N(a_0)})\}_N$ is uniformly integrable for all (s, a, y) by using the test function $\psi(x) = x^2$, and the sequence converges to $\bar{g}(Z_{\tau(a_0)})$ a.s.

To obtain h_N, we need to solve the following boundary value problem

$$\frac{\partial h_N}{\partial s} + \alpha a \frac{\partial h_N}{\partial a} + (a - \bar{w})e^{-\rho s}\frac{\partial h_N}{\partial y} + \frac{1}{2}\beta^2 a^2 \frac{\partial^2 h_N}{\partial a^2} = 0 \text{ for } a_0 < a < N \tag{5.1}$$
$$h_N(s, a_0, y) = \gamma a_0 e^{-\rho s} + y$$
$$h_N(s, N, y) = \gamma N e^{-\rho s} + y.$$

We guess a solution of the form $h_N(s, a, y) = y + e^{-\rho s} f_N(a)$, so we can rewrite the equations as

$$(a - \bar{w} - \rho f_N(a)) + \alpha a \frac{\partial f_N}{\partial a} + \frac{1}{2}\beta^2 a^2 \frac{\partial^2 f_N}{\partial a^2} = 0 \text{ for } a_0 < a < N \tag{5.2}$$
$$f_N(a_0) = \gamma a_0$$
$$f_N(N) = \gamma N.$$

Furthermore, we assume that

$$f_N(a) = \phi_N(a) + \frac{a}{\rho - \alpha} - \frac{\bar{w}}{\rho},$$

for some $\phi_N : \mathbb{R} \to \mathbb{R}$, so the equation (5.2) can be written as

$$-\rho \phi_N(a) + \alpha a \frac{\partial \phi_N}{\partial a} + \frac{1}{2}\beta^2 a^2 \frac{\partial^2 \phi_N}{\partial a^2} = 0. \tag{5.3}$$

The general solution ϕ_N of the ordinary differential equation (5.3) is

$$\phi_N(a) = C_1^{(N)} a^{\gamma_1} + C_2^{(N)} a^{\gamma_2},$$

where $C_1^{(N)}, C_2^{(N)}$ are arbitrary constants and

$$\gamma_i = \beta^{-2}\left[\left(\frac{1}{2}\beta^2 - \alpha\right) \pm \sqrt{\left(\frac{1}{2}\beta^2 - \alpha\right)^2 + 2\rho\beta^2}\right],$$

for $i \in \{1, 2\}$ where $\gamma_2 < 0 < \gamma_1$. Therefore,

$$f_N(a) = C_1^{(N)} a^{\gamma_1} + C_2^{(N)} a^{\gamma_1} + ma - b,$$

where $m = \frac{1}{\rho - a}$ and $b = \frac{\bar{w}}{\rho}$. By the boundary conditions that $f_N(a_0) = \gamma a_0$ and $\gamma_N(N) = \gamma N$, we obtain

$$C_1^{(N)} = \frac{[b + (\gamma - m)a_0]N^{\gamma_2 - 1} - a_0^{\gamma_2}[bN^{-1} - (m - \gamma)N]}{a_0^{\gamma_1}N^{\gamma_2 - 1} - a_0^{\gamma_2}N^{\gamma_1 - 1}} \longrightarrow 0$$

$$C_2^{(N)} = \frac{b + (\gamma - m)a_0 - a_0^{\gamma_1}[bN^{-\gamma_1} - (m - \gamma)N^{1-\gamma_1}]}{a_0^{\gamma_2} - a_0^{\gamma_1}N^{\gamma_2 - \gamma_1}} \longrightarrow \frac{b + (\gamma - m)a_0}{a_0^{\gamma_2}},$$

as $N \to \infty$. Thus, as $N \to \infty$, we obtain

$$f_N(a) \longrightarrow f(a) = \left(\frac{a}{a_0}\right)^{\gamma_2}\left(\frac{\bar{w}}{\rho} + \Big(\gamma - \frac{1}{\rho - \alpha}\Big)a_0\right) + \frac{a}{\rho - \alpha} - \frac{\bar{w}}{\rho}.$$

Therefore,

$$\tilde{g}_{\tau(a_0)}(s, a, y) = h(s, a, y) = e^{-\rho s}\left[\left(\frac{a}{a_0}\right)^{\gamma_2}\left(\frac{\bar{w}}{\rho} + \Big(\gamma - \frac{1}{\rho - \alpha}\Big)a_0\right) + \frac{a}{\rho - \alpha} - \frac{\bar{w}}{\rho}\right] + y.$$

We will consider the following two cases.

Case 1: $\gamma \geq \frac{1}{\rho - \alpha}$. This means that $\tilde{g}_{\tau(a_0)}$ is increasing in a_0. Thus, for $a_0 \leq a$, $\tilde{g}_{\tau(a_0)}(s, a, y) \leq \tilde{g}_{\tau(a)}(s, a, y)$. Moreover, for $a_0 > a$, $\tilde{g}_{\tau(a_0)}(s, a, y) = \tilde{g}_{\tau(a)}(s, a, y) = \bar{g}(s, a, y)$. We will show that $\bar{g}$ is a supermeanvalued function. Consider that for all $a \in \mathbb{R}^+$

$$\mathcal{A}\bar{g}(a) = e^{-\rho s}\big(a(1 - \gamma(\rho - \alpha)) - \bar{w}\big) \leq -\bar{w}e^{-\rho s} \leq 0.$$

Thus, $\bar{g}$ itself is a supermeanvalued function, so $g^* = \bar{g}$ and $\tau = 0$. Therefore, $\Psi(a) = \bar{g}(a) = \gamma a$.

We may interpret $\rho - \alpha$ as the total depreciation of the capital, which consists of the actual depreciation and the discount rate. Therefore, the term $\frac{1}{\rho - \alpha}$ intuitively means the multiplier of the opportunity cost if the firm decides to sell the capital. This means that the condition $\gamma \geq \frac{1}{\rho - \alpha}$ means that the price of the capital is at least the multiplier of the opportunity cost if the firm decides to sell the capital. Thus, it is better off for firm to sell the capital at the very beginning state instead of to keep the capital and sell later. Therefore, the result that $\tau = 0$ is an optimal stopping time intuitively makes sense.

Case 2: $\gamma < \frac{1}{\rho - \alpha}$. Our goal is to determine which a_0 maximizes $\tilde{g}_{\tau(a_0)}(s, a, y)$ for all s and y. Note that this maximization makes sense since the function $\tilde{g}_{\tau(a_0)}(s, a, y)$ does not depend on s and depends linearly on y. We observe that

$$\frac{\partial \tilde{g}_{\tau(a_0)}(s, a, y)}{\partial a_0} = e^{-\rho s}a^{\gamma_2}\Big[-\frac{\bar{w}}{\rho}\gamma_2 a_0^{-\gamma_2 - 1} + \left(\gamma - \frac{1}{\rho - \alpha}\right)(1 - \gamma_2)a_0^{-\gamma_2}\Big].$$

By setting the partial derivative equal to 0 and checking the second partial derivative, the value of a_0 that maximizes $\tilde{g}_{H_{a_0}}(s, a, y)$ is

$$a_0 = a_{max} = \left(\frac{1}{\rho - \alpha} - \gamma\right)^{-1}\frac{\bar{w}\gamma_2}{\rho(\gamma_2 - 1)}.$$

Therefore, our best candidate for a supermeanvalued majorant of g is $\tilde{g}_{a_{max}}$. By Corollary 3.13, it is sufficient to verify that $\tilde{g}_{a_{max}}$ is a supermeanvalued function with respect to Z_t and is greater than or equal g.

To show that $\tilde{g}_{a_{max}}$ is a supermeanvalued function, by Corollary 3.6, we need to prove $\mathcal{A}\tilde{g}_{a_{max}} \leq 0$. The boundary value problem (5.1) shows that $\mathcal{A}\tilde{g}_{a_{max}}(s, a, y) =$

0 for all $a < a_{max}$. For $a > a_{max}$, $\tilde{g}_{a_{max}}(s,a,y) = \gamma a e^{-\rho s} + y$, so

$$\begin{aligned}\mathcal{A}\tilde{g}_{a_{max}}(a) &= e^{-\rho s}\left(a(\rho-\alpha)(\frac{1}{\rho-\alpha}-\gamma)-\bar{w}\right) \le e^{-\rho s}\left(\frac{\gamma_2(\rho-\alpha)}{\rho(\gamma_2-1)}\bar{w}-\bar{w}\right)\\ &= e^{-\rho s}\left(\frac{-\gamma_2\alpha+\rho}{(\gamma_2-1)\rho}\right)\bar{w}.\end{aligned}$$

Since we can rewrite γ_2 as

$$\gamma_2 = -\frac{2\rho}{\left(\frac{1}{2}\beta^2-\alpha\right)+\sqrt{\left(\frac{1}{2}\beta^2-\alpha\right)^2+2\rho\beta^2}} > \frac{\rho}{\alpha}$$

and $\gamma_2 - 1 < 0$. At $x = x_{max}$, we do the same by using the left-side derivative and the right-side derivative. We can conclude that $\mathcal{A}\tilde{g}_{a_{max}}(s,a,y) \le 0$, as desired.

To prove that $\tilde{g}_{a_{max}} \ge g$, for fixed s, a, and y, we have shown that the value of a_0 that maximizes $\tilde{g}_{a_0}$ is $a_0 = a_{max}$. Therefore,

$$\tilde{g}_{a_{max}}(s,a,y) \ge \tilde{g}_a(s,a,y) = g(s,a,y).$$

To sum up, $\tilde{g}_{a_{max}}$ is a supermeanvalued majorant of g. Therefore, by Corollary 3.14, the firm should stop hiring the worker if the productivity level of the technology reaches the value $a_{max} = \left(\frac{1}{\rho-\alpha}-\gamma\right)^{-1}\frac{\bar{w}\gamma_2}{\rho(\gamma_2-1)}$. The corresponding expected discounted outcome is

$$\tilde{g}_{a_{max}}(0,a,0) = \left(-\frac{a}{\gamma_2}\left(\frac{1}{\rho-\alpha}-\gamma\right)\right)^{\gamma_2}\left(\frac{\rho(1-\gamma_2)}{\bar{w}}\right)^{\gamma_2-1}+\frac{a}{\rho-\alpha}-\frac{\bar{w}}{\rho}.$$

The optimal stopping time we discovered shows that the firm here acts as a risk lover. To see this, if the change of the productivity becomes more unpredictable, this can be interpreted as β is getting larger. We can rewrite γ_2 as

$$\gamma_2 = -\frac{2\rho}{\left(\frac{1}{2}\beta^2-\alpha\right)+\sqrt{\left(\frac{1}{2}\beta^2-\alpha\right)^2+2\rho\beta^2}} < 0,$$

which is increasing in β. Therefore, $a_{max} = \left(\frac{1}{\rho-\alpha}-\gamma\right)^{-1}\frac{\bar{w}\gamma_2}{\rho(\gamma_2-1)}$ is decreasing in β. This means that, by keeping the average performance the same, the larger fluctuation of technology deterioration induces the firm's incentive to set a lower critical level of the productivity.

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