QUANTIZATION COMMUTES WITH REDUCTION

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Abstract. This paper explains the theorem that, under moderate hypotheses, Kähler quantization commutes with symplectic reduction.

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1. Introduction

In Hamilton's formulation of classical mechanics, the phase space of a mechanical system is given by a symplectic manifold (M, ω). A classical observable is a function on M. In quantum mechanics, on the other hand, the state of a particle is given by a unit vector in a separable Hilbert space $\mathbb{H}$ (over $\mathbb{C}$, always), up to a scalar multiplication in S^1. A quantum observable is an unbounded self-adjoint operator on $\mathbb{H}$. Many other concepts can also be defined in parallel between classical and quantum worlds.

In the twentieth century, many attempts had been made to provide a mathematical way enabling one to go from the classical theory to the quantum one. We refer

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to this process as *geometric quantization.* Roughly speaking, geometric quantization is a process that associates to each quantizable symplectic manifold M a separable Hilbert space $\mathbb{H}$, and to each quantizable function f an unbounded self-adjoint operator $Q(f)$ on $\mathbb{H}$, such that some additional conditions (ones that are dictated by physical experiments) are satisfied.

In a classical system, the Noether principle in physics says that each symmetry corresponds to an integral of motion. More precisely, suppose there is a G-action on a symplectic manifold M, where G is a Lie group of dimension k. Then there are k quantities that are conserved under the action. An accurate mathematical formulation of the Noether principle is given by the notion of *Marsden-Weinstein reduction*, or *symplectic reduction.* In a quantum system $\mathbb{H}$, a symmetry given by a G-representation on $\mathbb{H}$ restricts the possible states of a particle to the subspace $\mathbb{H}^G := \{\text{points fixed by } G\}$ of $\mathbb{H}$.

It is natural for us to ask whether our geometric quantization procedure commutes with imposing constraints by a Lie group symmetry on our system. In this expository paper, following [3], we formulate this question in the special case when G is compact, M is Kähler, and the geometric quantization scheme is taken to be the so-called *Kähler quantization.* We will also sketch a proof of it. In more general setups, this question is known as the Guillemin-Sternberg conjecture, and has been proved later in various settings. See for example [7][6].

In this paper, we begin with a brief review of classical mechanics and quantum mechanics in Section 2 in a mathematical point of view. In Section 3 and Section 4, we explain the process of geometric quantization and symplectic reduction, putting an emphasize on the Kähler case. Finally, in Section 5 we state and sketch the proof of the main theorem "quantization commutes with reduction." As for prerequisites, we assume the reader has basic acquaintance with symplectic manifolds and Hilbert spaces as well as some familiarity with complex geometry. Physics background is helpful but not necessary.

2. Classical Mechanics and Quantum Mechanics

2.1. **Hamiltonian Mechanics.** We begin with the simplest situation of a one-particle system in the state space $\mathbb{R}^n$. In this case, the phase space is given by $\mathbb{R}^{2n}$ whose points are of the form (q, p), where $q \in \mathbb{R}^n, p \in \mathbb{R}^n$ denote the position, momentum of the particle, respectively.

Suppose there is a conservative field on $\mathbb{R}^n$ given by a (smooth) potential function $U\colon \mathbb{R}^n \to \mathbb{R}$. Then the force to the particle at position q is given by $-\nabla U(q)$. Let m denote the mass of the particle. Then we have

$$\begin{cases} \dot{p} = -\nabla U(q), & \text{(Newton's second law)} \\ \dot{q} = p/m. & \text{(Definition of momentum)} \end{cases} \tag{2.1}$$

Define $H\colon \mathbb{R}^{2n} \to \mathbb{R}$ by $H(q,p) = \frac{|p|^2}{2m} + U(q)$. Then (2.1) can be rewritten as

$$\begin{cases} \dot{p} = -\frac{\partial H}{\partial q}, \\ \dot{q} = \frac{\partial H}{\partial p}. \end{cases} \tag{2.2}$$

These equations are known as **Hamilton's equations**. Given any initial data, they completely determine the state of the particle at any given time.

Now, suppose the particle is constrained to a submanifold N in $\mathbb{R}^n$, and $U \in C^\infty(N)$ is a potential function. Then the state space is just N, while the phase space can be identified with the cotangent bundle T^*N, which is equipped with a bundle metric induced by the submanifold metric on N. Let (q, p) be some local coordinates for T^*N, where $q \in N$ denotes the position and $p \in T_q^*N$ denotes the momentum. Then the kinetic energy of the particle is given by $T(q, p) = \frac{|p|^2}{2m}$. Let $H = T + U$ denote the total energy of the particle. The same Hamilton's equations (2.2) still holds in this case and determine the time evolution of the particle. For a proof of this using d'Alembert's principle and Legrendre transformation, one may consult [1]. Note that the special case $N = \mathbb{R}^n$ is exactly the unconstrained case discussed above.

More generally, let (M, ω) be a symplectic manifold. For a vector field X on M, let ι_X, $\mathcal{L}_X$ denote the contraction, Lie differentiation by X, respectively, as operators on $\Omega(M)$, the space of differential forms on M. Let d denote the exterior differentiation. Then for vector fields X, Y on M, we have the usual identities

$$\mathcal{L}_X = d\iota_X + \iota_X d, \tag{2.3}$$

$$\mathcal{L}_X \iota_Y - \iota_Y \mathcal{L}_X = \iota_{[X,Y]}. \tag{2.4}$$

For any smooth function f on M, let X_f denote the vector field defined by $\iota_{X_f}\omega = df$. Since $\omega \in \Omega^2(M)$ is nondegenerate at any point, X_f is unambiguously defined.

Definition 2.5. A **Hamiltonian system** is a symplectic manifold (M, ω) equipped with a function $H \in C^\infty(M)$, called the **Hamiltonian**. For such a system, the time evolution is given by the flow on M generated by the vector field $-X_H$, i.e. a point $x \in M$ evolves by

$$\dot{x} = (-X_H)_x. \tag{2.6}$$

Example 2.7. Let N denote any manifold and $M = T^*N$ denote its cotangent bundle. There is a canonical symplectic form ω on M, which is given as follows in local coordinates:

Let $(q_1, \cdots, q_n)$ be local coordinates on N. Then any $p \in T_q^*N$ for q in this coordinate neighborhood can be written as $p_1 dq_1|_q + \cdots p_n dq_n|_q$. Then $(q_1, \cdots, q_n, p_1, \cdots, p_n)$ are local coordinates on M. Let $\omega = -\sum_{j=1}^n dq_j \wedge dp_j$. One can check this definition is independent of the choice of coordinates, thus patches to a global form ω which is nondegenerate and closed (in fact exact), hence defines a symplectic structure on M.

Given a Hamiltonian $H \in C^\infty(M)$, the vector field X_H is determined by

$$\iota_{X_H}\left(-\sum_j dq_j \wedge dp_j\right) = dH = \sum_j \left(\frac{\partial H}{\partial q_j} dq_j + \frac{\partial H}{\partial p_j} dp_j\right).$$

Thus $-X_H$, in local coordinates, is given by

$$-X_H = \sum_i \left(\frac{\partial H}{\partial p_j} \frac{\partial}{\partial q_j} - \frac{\partial H}{\partial q_j} \frac{\partial}{\partial p_j} \right).$$

In particular, we see that (2.2) and (2.6) agree. Therefore equation (2.6) is a coordinate free generalization of the usual Hamilton's equations.

Remark 2.8. The canonical symplectic form on T^*N differs from the usual definition by a sign. This is offsetted by the extra minus sign in (2.6). Our definition here makes the notations in Section 3.3 nicer. (Otherwise one needs the almost complex structure J to be compatible with $-\omega$ instead of ω.)

Define the **Poisson bracket** $\{\cdot,\cdot\}$ on the algebra $C^\infty(M)$ by

$$\{f, g\} = -\omega(X_f, X_g)(= \iota_{X_f}\iota_{X_g}\omega = X_f g),$$

which is anti-symmetric in f, g. Then the assignment

$$C^\infty(M) \to \mathfrak{X}(M),\ f \mapsto X_f \tag{2.9}$$

is linear. Moreover, applying (2.3)(2.4) and noticing that $d\omega = 0$, we obtain

$$\begin{aligned} \iota_{X_{\{f,g\}}}\omega &= d\omega(X_g, X_f) = d\iota_{X_f}\iota_{X_g}\omega = \mathcal{L}_{X_f}\iota_{X_g}\omega - \iota_{X_f} d\iota_{X_g}\omega \\ &= (\iota_{X_g}\mathcal{L}_{X_f} + \iota_{[X_f,X_g]})\omega - \iota_{X_f} d\iota_{X_g}\omega = \iota_{X_g}\iota_{X_f} d\omega + \iota_{[X_f,X_g]}\omega = \iota_{[X_f,X_g]}\omega, \end{aligned}$$

which implies that $X_{\{f,g\}} = [X_f, X_g]$. Next we show that $\{\cdot,\cdot\}$ satisfies the Jacobi identity. This follows from the computation that for any $f, g, h \in C^\infty(M)$, we have

$$\begin{aligned} 0 &= d\omega(X_f, X_g, X_h) = \sum_{cyc} X_f \omega(X_g, X_h) - \sum_{cyc} \omega([X_f, X_g], X_h) \\ &= \sum_{cyc} (-X_f\{g, h\} - \omega(X_{\{f,g\}}, X_h)) = \sum_{cyc} (-\{f, \{g, h\}\} + \{\{f, g\}, h\}) \\ &= 2\sum_{cyc} \{\{f, g\}, h\}. \end{aligned}$$

Hence, $(C^\infty(M), \{\cdot,\cdot\})$ is a Lie algebra and (2.9) is a Lie algebra homomorphism.

Example 2.10. Consider $T\mathbb{R}^n = \mathbb{R}^{2n}$ with the usual symplectic form $\omega = -\sum_j dq_j \wedge dp_j$, where q_j, p_j are the usual coordinate functions for $\mathbb{R}^{2n}$, $j = 1, \cdots, n$. We compute the pairwise Poisson brackets for these functions.

By definition, we first see that $X_{q_j} = \partial/\partial p_j$, $X_{p_j} = -\partial/\partial q_j$. Then it follows that

$$\{q_j, q_k\} = \{p_j, p_k\} = 0,\ \{q_j, p_k\} = \delta_{jk} \text{ for all } 1 \le j, k \le n. \tag{2.11}$$

Proposition 2.12. *Let $f \in C^\infty(M)$ be a classical observable for the Hamiltonian system (M, ω, H). Then the time evolution of f for a point is given by*

$$\dot{f} = \{f, H\}. \tag{2.13}$$

Proof. Since the time evolution of a point follows the flow generated by $-X_H$, we find that $\dot{f} = -X_H f = -\{H, f\} = \{f, H\}$, as desired. □

2.2. **Quantum Mechanics.** One of the most important features of quantum mechanics is that one cannot predict where a particle is (nor can one predict its momentum, acceleration, etc.) without observation. Mathematically, in a quantum system, quantities such as position and momentum are called quantum observables, and their values for a particle are nothing but probability distributions on corresponding codomains. This idea is most succinctly expressed in terms of Hilbert spaces. For basic definitions and properties of Hilbert spaces, such as unbounded operators, self-adjointness, spectral theorem for unbounded self-adjoint operators, one may consult [10][4].

As in the previous section, we begin by illustrating the situation of one particle in $\mathbb{R}^n$, with usual coordinates $x_1, \cdots, x_n$. The relevant Hilbert space in this case is $\mathbb{H} = L^2(\mathbb{R}^n)$ (coefficients are taken to be in $\mathbb{C}$). The state of the particle is by definition a unit vector in $\mathbb{H}$ up to a multiplicative scalar in S^1. The position operators and momentum operators are operators on $\mathbb{H}$ defined by

$$X_j = x_j \ (= \text{ multiplication by } x_j), \ P_j = -i\hbar\frac{\partial}{\partial x_j}, \ j = 1, \cdots, n,$$

respectively, on some suitable (dense) domains in $\mathbb{H}$. Here $i = \sqrt{-1}$ and $\hbar$ is Planck's constant. Then clearly the X_j are self-adjoint and the P_j are symmetric, but it is not so clear that the P_j are actually self-adjoint. We ignore this verification here (and later on) and refer readers to [4] for these domain technicalities. Given a potential function U on $\mathbb{R}^n$, the energy operator (or the quantum Hamiltonian) is given by

$$\hat{H} = \frac{P^2}{2m} + U(X) = -\frac{\hbar^2}{2m}\Delta + U(x),$$

where $\Delta = \sum_j \partial^2/\partial x_j^2$ is the Laplacian on $\mathbb{R}^n$. We compute that, on suitable dense domains,

$$[X_j, X_k] = [P_j, P_k] = 0, \ [X_j, P_k] = \delta_{jk} i\hbar. \tag{2.14}$$

A more general picture follows the next definitions.

Definition 2.15. (1) A **quantum system** is a separable Hilbert space $\mathbb{H}$ over $\mathbb{C}$ equipped with a quantum observable $\hat{H}$, called the **Hamiltonian**. Here, a **quantum observable** is an unbounded self-adjoint operator on $\mathbb{H}$. The state of the system is a unit vector $\psi \in \mathbb{H}$ up to a unit scalar, called a **wave function**.

(2) (Time evolution) The quantum system evolves by **Schödinger's equation**:

$$\dot{\psi} = \frac{\hat{H}\psi}{i\hbar}.$$

(3) (Observation) Given a quantum observable A on $\mathbb{H}$, one can **observe** the value of A. The result is a random real number λ subject to a specific probability distribution. After applying the observation, the wave function **collapses** to a unit eigenvector for A with eigenvalue λ. Moreover, the expected value of λ is given by

$$\langle A \rangle_\psi := \langle \psi, A\psi \rangle. \tag{2.16}$$

Here $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{H}$, which is (by physicist's convention) taken to be linear in the second variable and conjugate-linear in the first variable.

Remark 2.17. Although it is not needed for our purpose, it is worth mentioning that the probability distribution in (3) above can be explicitly written out as follows. Let $\sigma(A)$ denote the spectrum of A, which is a closed subset of $\mathbb{R}$. Then the spectral theorem for unbounded self-adjoint operators states that A can be rewritten as

$$A = \int_{\sigma(A)} \lambda d\mu^A(\lambda),$$

where μ^A is a projection-valued measure on $\sigma(A)$. Then the probability distribution is given by the probability measure μ^A_ψ on $\mathbb{R}$ supported in $\sigma(A)$, defined by

$$\mu^A_\psi(U) := \langle \psi, \mu^A(U)\psi \rangle$$

for any Borel set $U \subset \mathbb{R}$. One can check that its expectation is indeed given by (2.16):

$$\begin{aligned}\mathbb{E}(\mu^A_\psi) &= \int_{\mathbb{R}} \lambda d\mu^A_\psi(\lambda) = \int_{\sigma(A)} \lambda \langle \psi, d\mu^A(\lambda)\psi \rangle \\ &= \left\langle \psi, \left(\int_{\sigma(A)} \lambda d\mu^A(\lambda) \right) \psi \right\rangle = \langle A \rangle_\psi.\end{aligned}$$

See [4] for more details.

Proposition 2.18. *Let A be a quantum observable for the quantum system $(\mathbb{H}, \hat{H})$. Then the time evolution of the expected value of A is given by*

$$\frac{d\langle A \rangle_\psi}{dt} = \left\langle \frac{[A, \hat{H}]}{i\hbar} \right\rangle_\psi . \tag{2.19}$$

Proof. By Schödinger's equation and the self-adjointness of $\hat{H}$, we compute that

$$\begin{aligned}\frac{d\langle A \rangle_\psi}{dt} &= \langle \dot{\psi}, A\psi \rangle + \langle \psi, A\dot{\psi} \rangle = \frac{-\langle \hat{H}\psi, A\psi \rangle + \langle \psi, A\hat{H}\psi \rangle}{i\hbar} \\ &= \left\langle \psi, \frac{[A, \hat{H}]}{i\hbar}\psi \right\rangle = \left\langle \frac{[A, \hat{H}]}{i\hbar} \right\rangle_\psi ,\end{aligned}$$

as desired. □

Remark 2.20. For the theorem we are aiming for, introducing the concept of time evolution of a (classical or quantum) system is not really necessary. We include the discussions of time to make the physical picture complete and partly to motivate the Dirac axiom (3) below. In fact, if the time evolution is periodic, the extra information carried by time is nothing but a symmetry by the Lie group S^1 on our system. In the general situation, however, integrability problems could arise.

3. Geometric Quantization

This section constitutes a crash course in geometric quantization. Since this is not the main purpose for our paper, we will not go into too much detail for motivations. Interested readers may consult [2][4].

Although classical mechanics is elegant and well-understood, physicists eventually realized that it is inaccurate in extreme situations. Later, quantum mechanics was developed. It provides a better approximation to our real world in micro scales.

Attempts had been made to produce a quantum theory from the classical theory. Roughly speaking, we are seeking for a systematic way that associates to each classical space a quantum space, and to each classical observable a quantum observable. In other words, this means to assign to each symplectic manifold (M, ω) a Hilbert space $\mathbb{H}$ and to each (smooth, real valued) function f on M an unbounded self-adjoint operator $Q(f)$ on $\mathbb{H}$. In addition, this assignment should satisfy the following conditions, which are usually referred to as **Dirac axioms**:

(1) Q is $\mathbb{R}$-linear.
(2) $Q(1) = I$ is the identity operator.
Explanation: We expect that the classical observable 1 *corresponds to a quantum observable* $Q(1)$ *with* $\langle Q(1)\rangle_\psi = 1$ *for any wave function* ψ*.)*
(3) $Q(\{f, g\}) = [Q(f), Q(g)]/i\hbar$.
Explanation: From ((2.11) *v.s.* (2.14)*), (Proposition 2.12 v.s. Proposition 2.18)), we expect that* $\{\cdot,\cdot\}$ *on the classical side corresponds to* $[\cdot,\cdot]/i\hbar$ *on the quantum side.*
(4) Any complete set of functions on (M, ω) is mapped to a complete set of operators on $\mathbb{H}$.
Explanation: This condition is consistent with physical experiment and we shall take it for granted without further discussion. Imprecisely speaking, the condition says that the quantum space $\mathbb{H}$ *cannot be made any smaller.*

It turns out that a quantization scheme that satisfies (1)(2)(3) above exists for (M, ω) if a certain integral condition on (M, ω) holds. We will soon give an explicit construction for this so-called prequantization. However, unfortunately, it is proved that there is no quantization scheme that satisfies all the four conditions above (see Groenewold's "no-go" theorem [4]). We will instead weaken our requirements to only quantizing some but not all functions on M. We will impose an extra structure called a polarization on M and adapt the prequantization accordingly to get our final quantization. One can go further beyond this point, for example by introducing the half-forms that allows one to quantize more functions on M. However, for our purpose, we will not do so here.

3.1. **Prequantization.**

Definition 3.1. A symplectic manifold (M, ω) is **quantizable** if $[\omega/2\pi\hbar] \in H^2(M, \mathbb{Z})$.

Note that by scaling the symplectic form (change the speed of movement) if necessary we can make any symplectic manifold quantizable. From now on, we always assume (M, ω) to be quantizable.

Recall that elements in $H^2(M,\mathbb{Z})$ are in one-one correspondence with complex line bundles over M via the first Chern class. By the integral assumption we can choose a line bundle $L \to M$ with first Chern class $c_1(L) = [\omega/2\pi\hbar]$. Moreover, we can further choose a Hermitian metric $\langle\cdot,\cdot\rangle$ and a metric connection ∇ on L with curvature $F_\nabla = -2\pi i \cdot \omega/2\pi\hbar = \omega/i\hbar$. Such $(L, \nabla, \langle\cdot,\cdot\rangle)$ are called **prequantum data** for (M,ω).

Take $\mathbb{H}_{pre} = L^2(M,L)$, the L^2 completion of the space of square integrable sections of L with respect to $\langle\cdot,\cdot\rangle$. For any $f \in C^\infty(M)$, define $Q_{pre}(f) = i\hbar\nabla_{X_f} + f$ to be a (densely defined) unbounded operator on $\mathbb{H}_{pre}$. Then integration by parts shows that $Q_{pre}(f)$ is symmetric. In fact it is actually self-adjoint. Moreover, $Q_{pre}(1) = I$ is clearly satisfied.

Proposition 3.2. *For $f, g \in C^\infty(M)$ we have $[Q_{pre}(f), Q_{pre}(g)] = i\hbar Q_{pre}(\{f,g\})$ (on a suitable domain).*

Proof. We have arranged the curvature term such that

$$\begin{aligned}
[Q_{pre}(f), Q_{pre}(g)] &= i\hbar(i\hbar[\nabla_{X_f}, \nabla_{X_g}] + X_f g - X_g f)\\
&= i\hbar(i\hbar(\nabla_{[X_f,X_g]} + F_\nabla(X_f,X_g)) + \{f,g\} + \{f,g\})\\
&= i\hbar(i\hbar\nabla_{X_{\{f,g\}}} - \{f,g\} + 2\{f,g\})\\
&= i\hbar Q_{pre}(\{f,g\}).
\end{aligned}$$

□

3.2. **Quantization.** As commented earlier, prequantization is in some sense not small enough for the correct quantum theory. As an example, take $\mathbb{R}^n$ as our classical state space. Then the prequantization procedure for the phase space $\mathbb{R}^{2n}$ with the usual symplectic structure produces the Hilbert space $\mathbb{H}_{pre} = L^2(\mathbb{R}^{2n},\mathbb{C}) = L^2(\mathbb{R}^{2n})$ (here $\mathbb{C}$ denotes the trivial line bundle), which is "n-variables larger" than the model $L^2(\mathbb{R}^n)$ we usually use. To resolve this problem, we want to restrict our attention to elements in $\mathbb{H}_{pre}$ that are constant along n of the $2n$ directions. Below we shall make this heuristic idea precise.

Definition 3.3. A **polarization** of a symplectic manifold (M,ω) is a (smooth) complex distribution $P \subset TM \otimes \mathbb{C}$ that is

1. Lagrangian: For each $x \in M$, P_x is a Lagrangian subspace of $(T_xP \otimes \mathbb{C}, \omega_x)$.
2. Integrable (involutive): For any complexified vector fields X, Y on M lying in P, the vector field $[X,Y]$ also lies in P.
3. Stable in the real part: fibers of $P \cap \bar{P}$ have the same dimension.

Fix a polarization P of (M,ω). A section $s \in \Gamma(M,L)$ is **polarized** if $\nabla_X s = 0$ for all $X \in \bar{P}$. It makes no difference to use P instead of $\bar{P}$ here, but our definition will be notationally more desirable for later developments in Section 3.3.

Take $\mathbb{H}$ to be the closure, in $\mathbb{H}_{pre}$, of the space of square integrable polarized sections of L. A function $f \in C^\infty(M)$ is said to be **quantizable** if $Q_{pre}(f)$ preserves the space of polarized sections, in which case it restricts to an unbounded self-adjoint operator $Q(f)$ on $\mathbb{H}$.

In conclusion, we have obtained a quantization scheme that associates to each quantizable polarized symplectic manifold a Hilbert space and to each quantizable function an unbounded self-adjoint operator. This construction satisfies the conditions (1) to (4) we require for a quantization scheme, except that not all functions are quantizable. We did not actually make sense of (4). Roughly speaking, polarized sections are constant in half of the directions, and this is what makes (4) true (see [2][4] for more discussion).

At this point one may ask which functions are actually quantizable. Here is a sufficient condition.

Proposition 3.4. *If X_f preserves $\bar{P}$, then f is quantizable.*

Proof. Let s be a polarized section. For any $X \in \bar{P}$, by assumption, $[X, X_f] \in \bar{P}$ and $\nabla_X s = 0$. Thus we have

$$\begin{aligned}\nabla_X Q_{pre}(f)s &= i\hbar\nabla_X\nabla_{X_f}s + (Xf)s\\ &= i\hbar(\nabla_{X_f}\nabla_X s + \nabla_{[X,X_f]}s + F_\nabla(X, X_f)s) + \iota_X(df)s\\ &= \omega(X, X_f)s + \omega(X_f, X)s\\ &= 0.\end{aligned}$$

Therefore $Q_{pre}(f)$ is also a polarized section. □

As a special case,

Proposition 3.5. *If f is constant along (integral curves of) $\bar{P}$, then f is quantizable and $Q(f) = f$ is multiplication by f.*

Proof. The assumption says that $\omega(X_f, X) = 0$ for all $X \in \bar{P}$. Pointwise, this says that $(X_f)_x$ lies in the symplectic complement of $\bar{P}_x$ in (T_xM, ω_x). Noting that $\bar{P}$, like P, is Lagrangian, it follows that $X_f \in \bar{P}$. Therefore $\nabla_{X_f}s = 0$ for any polarized section and the result follows. □

3.3. **Kähler quantization.** As special cases of Definition 3.3, we say a polarization P is **purely real** if $P = \bar{P}$; we say P is **purely imaginary** if $P \cap \bar{P} = M \subset TM$, i.e. if $P \cap \bar{P}$ is fiberwise trivial.

An example of purely real polarization is given by the vertical polarization for $M = T^*N$ equipped with its canonical symplectic form. In this case one will usually need to introduce the method of half-form quantization such that the set of quantizable functions is nonempty. See [4] for examples and discussion.

For the purpose of this paper, we will be concerned with purely complex polarization. (From now on we will use some standard concepts in complex geometry which can be found in [5].) In this case, $TM \otimes \mathbb{C} = P \oplus \bar{P}$. Define an almost complex structure $J \in \Gamma(End(TM \otimes \mathbb{C}))$ on M to be multiplication by i on P and $-i$ on $\bar{P}$. Then integrability of P implies that J integrates to a complex structure on M. If in addition, J is compatible with ω, then (M, ω, J) is Kähler. In this case we call P a **Kähler polarization**. It is easily verified that the Lagrangian property of P implies that $\omega(\cdot, J\cdot)$ is symmetric. Therefore the only requirement for (M, ω, J) to be Kähler is the taming condition $\omega(v, Jv) > 0$ for all $v \neq 0$.

Conversely, if (M, ω, J) is a Kähler manifold, then ω is a real $(1,1)$-form. Therefore the holomorphic tangent bundle $P = T^{1,0}M$ is Lagrangian. It defines a Kähler polarization of M.

We summarize the discussion above with the following proposition.

Proposition 3.6. *Kähler polarizations of (M, ω) are naturally in one-one correspondence with Kähler structures (M, ω, J). The explicit correspondence is given as above.*

We say a Kähler manifold is **quantizable** if its underlying symplectic manifold is quantizable. Therefore, the constructions in Section 3.1 and Section 3.2 give a quantization scheme for quantizable Kähler manifolds. This special kind of geometric quantization is often referred to as **Kähler quantization**. We will now examine this case more carefully and give a more explicit description of the quantum space $\mathbb{H}$.

Let $\nabla^{0,1}\colon \Omega^{p,q}(M, L) \to \Omega^{p,q+1}(M, L)$ be the antiholomorphic part of ∇. Since $\nabla \circ \nabla\colon \Omega^0(M, L) \to \Omega^2(M, L)$ defines the curvature $F_\nabla = \omega/i\hbar$, which is a $(1,1)$-form, we see that its image is contained in $\Omega^{1,1}(M, L)$. In particular, $\nabla^{0,1} \circ \nabla^{0,1} = 0$. This shows that there is a unique holomorphic structure on L such that $\nabla^{0,1} = \bar{\partial}_L$ (see [5, Theorem 2.6.26]). Since ∇ is also compatible with the Hermitian metric, it is the Chern connection on the Hermitian holomorphic line bundle $(L, \langle\cdot,\cdot\rangle)$.

Recall that the curvature of a Hermitian holomorphic line bundle is defined to be the curvature of its Chern connection.

Definition 3.7. A **quantum datum** for a quantizable Kähler manifold M is a Hermitian holomorphic line bundle over M with curvature $\omega/i\hbar$.

Let L be a quantum datum for M. Under the correspondence in Proposition 3.6, polarized sections on L are exactly sections s with $\bar{\partial}_L s = 0$, i.e. the holomorphic ones. By usual regularity theory we see that elements in the quantum space $\mathbb{H}$ are holomorphic (in particular smooth). Hence $\mathbb{H}$ is exactly the space of square integrable holomorphic sections of L. In particular, if M is compact, then $\mathbb{H} = H^0(M, L)$ is the space of all holomorphic sections of L (which is finite-dimensional by standard Hodge theory [5, Theorem 4.1.13]).

4. Symplectic Reduction

From now on symmetry enters our picture. In this section we sketch a reduction procedure for classical systems with additional symmetry, called symplectic reduction, which is fairly standard and can be found in any introductory textbook in symplectic geometry, for example [9]. This section is independent of quantum aspects discussed previously in Section 2.2 and Section 3.

4.1. Hamiltonian Action and Moment Map.

Let (M, ω) be a symplectic manifold equipped with an action by some Lie group G, which is, for simplicity, assumed to be compact and connected. Assume, by replacing the G-action by its conjugate action if necessary, that G acts on M on the right. Let $\mathfrak{g}$ denote the Lie algebra of G. For any $\xi \in \mathfrak{g}$, let $\xi^\#$ denote the associated vector field of ξ on M, i.e.

$$\xi^\#_x = \frac{d}{dt}\bigg|_{t=0} (x \cdot \exp(t\xi)),\ x \in M.$$

Then, by standard computation one verifies that $\xi \mapsto \xi^\#$ is a Lie algebra homomorphism.

We call the G-action on M a **symplectic action** if each $g \in G$ acts on M by a symplectomorphism. Alternatively, since G is connected, by (2.3) and closeness of ω,

$$\begin{aligned} \text{The } G\text{-action is symplectic } &\iff \mathcal{L}_{\xi^\#}\omega = 0 \text{ for all } \xi \in \mathfrak{g} \\ &\iff \iota_{\xi^\#}\omega \text{ is closed for all } \xi \in \mathfrak{g}. \end{aligned}$$

A vector field X on M is said to be **symplectic** if $\iota_X\omega$ is closed; it is said to be **Hamiltonian** if $\iota_X\omega$ is exact. The space of symplectic, Hamiltonian vector field on M are denoted $\mathfrak{X}^{symp}(M), \mathfrak{X}^{ham}(M)$, respectively. Then the map $f \mapsto X_f$ as defined in Section 2.1 fits into the exact sequence

$$0 \to \mathbb{R} \to C^\infty(M) \to \mathfrak{X}^{ham}(M) \to 0.$$

The condition for the G-action to be symplectic is precisely that the image of the map $\xi \mapsto \iota_{\xi^\#}\omega$ is contained in $\mathfrak{X}^{symp}(M)$. When we are in the lucky situation where the image in actually in $\mathfrak{X}^{ham}(M)$, we can further lift this map $\mathfrak{g} \to \mathfrak{X}^{ham}(M)$ to a map $\mathfrak{g} \to C^\infty(M)$.

Definition 4.1. The G-action on M is **Hamiltonian** if $\xi \mapsto \iota_{\xi^\#}\omega$ defines a map $\mathfrak{g} \to \mathfrak{X}^{ham}(M)$ which lifts to a Lie algebra homomorphism $\mu^* \colon \mathfrak{g} \to C^\infty(M)$. Here the Lie algebra structure on $C^\infty(M)$ is given by the Poisson bracket.

The map μ^* is called a **comoment map** for the Hamiltonian G-action. Its dual map $\mu \colon M \to \mathfrak{g}^*$ defined by

$$\mu^\xi(x) = \langle \mu(x), \xi \rangle := \mu^*(\xi)(x),\ \xi \in \mathfrak{g},\ x \in M \tag{4.2}$$

is called the **moment map** for the Hamiltonian G-action.

Under our assumption that G is connected, an alternative definition of Hamiltonian G-action is given by

Proposition 4.3. *The G-action on M is Hamiltonian if and only if there is a map $\mu \colon M \to \mathfrak{g}^*$, called the moment map, satisfying:*

(i) $\iota_{\xi^\#}\omega = d\mu^\xi$ for all $\xi \in \mathfrak{g}$;
(ii) μ is equivariant with respect to the G-action on M and the G-coadjoint action on $\mathfrak{g}^*$.

In application, the moment map μ appears more often than the comoment map μ^*. We could have simply take the less intuitive Proposition 4.3 as our definition for Hamiltonian G-actions and for moment maps (and this is the correct definition when G is not connected). Anyway, we sketch a proof for equivalence of the two definitions in our case.

Proof. Equation (4.2) gives a one-one correspondence between maps $\mu^*\colon \mathfrak{g} \to C^\infty(M)$ and $\mu\colon M \to \mathfrak{g}^*$. Under this correspondence, condition (i) is a restatement of the condition that $\mu^*\colon \mathfrak{g} \to C^\infty(M)$ lifts $\mathfrak{g} \to \mathfrak{X}^{ham}(M)$. It remains to check that condition (ii) is equivalent to μ^* being a Lie algebra homomorphism. This is a straightforward calculation which we leave to readers. □

Succinctly, one calls (M, ω, G, μ) as above a **Hamiltonian G-space**.

4.2. **The Marsden-Weinstein Reduction.** Let (M, ω, G, μ) be a Hamiltonian G-space. Assume $M_0 = \mu^{-1}(0)$ is nonempty. By equivariance of μ, we see that M_0 is G-invariant.

Lemma 4.4. 0 *is a regular value for μ if and only if G acts on M_0 locally freely.*

Proof.

$$\begin{aligned} & 0 \text{ is a regular value for } \mu \\ \iff & d\mu_x \text{ is surjective for all } x \in M_0 \\ \iff & \iota_{\xi^\#}\omega \neq 0 \text{ on } M_0, \text{ for all } \xi \in \mathfrak{g}\backslash\{0\} \text{ (by Proposition 4.3(i))} \\ \iff & \xi^\# \text{ is nonvanishing on } M_0 \text{ for all } \xi \in \mathfrak{g}\backslash\{0\} \\ \iff & G \text{ acts on } M_0 \text{ locally freely.} \end{aligned}$$
□

Theorem 4.5 (Marsden-Weinstein-Mayer)**.** *Suppose (M, ω, G, μ) is a Hamiltonian G-space and G acts freely on $M_0 = \mu^{-1}(0) \neq \emptyset$. Let $\pi\colon M_0 \to M_G$ denote the orbit projection and $i\colon M_0 \to M$ denote the inclusion. Then π is a principal G-bundle and $i^*\omega$ descends to a symplectic form ω_G on M_G.*

Proof. By Lemma 4.4, 0 is a regular value of μ, thus M_0 is a smooth manifold. Since the G-action on M_0 is free and G is compact by our assumption, the first statement is immediate.

Fix $x \in M$, and let $\mathcal{O}_x = x \cdot G$ denote the orbit containing x. Then $T_x\mathcal{O}_x \subset T_xM_0$ consists of all $\xi_x^\#$, $\xi \in \mathfrak{g}$. For any $u = \xi_x^\# \in T_x\mathcal{O}_x$ and $v \in T_xM_0$, we have $\omega_x(u, v) = d\mu_x^\xi(v) = 0$. Therefore, by dimension counting, T_xM_0 and $T_x\mathcal{O}_x$ are symplectic complements of each other as subspaces of (T_xM, ω_x). Consequently, ω_x descends to a symplectic form $(\omega_G)_{\pi(x)}$ on $T_xM_0/T_x\mathcal{O}_x = T_{\pi(x)}M_G$. One can check that for different x, these patch together to a nondegenerate 2-form on M_G which is closed since π^* is injective and

$$\pi^* d\omega_G = d\pi^*\omega_G = di^*\omega = i^*d\omega = 0.$$
□

The symplectic manifold (M_G, ω_G) is called the **Mayer-Weinstein reduction**, or **symplectic reduction**, of the Hamiltonian G-space (M, ω, G, μ).

Theorem 4.6. *In the same setup as Theorem 4.5, if in addition, there is a G-invariant almost complex structure J on M making (M, ω, J) a Kähler manifold, then J descends to an almost complex structure J_G on M_G making (M_G, ω_G, J_G) a Kähler manifold.*

Proof. By assumption, $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$ is a G-invariant metric on M. For $x \in M$ and $v \in T_{\pi(x)}M_G$, define $J_G v = \pi_*((J\tilde{v})^T)$, where $\tilde{v} \in T_x M_0$ is any lift of v and $(\cdot)^T$ denotes the orthogonal projection to $T_x M_0$ with respect to g. We check that J_G satisfies our requirement.

First we claim that J restricts to a bijection $T_x\mathcal{O}_x \to (T_x M_0)^\perp$ (hence also to a bijection in the backward direction). Here $(\cdot)^\perp$ denotes the orthogonal complement with respect to g. Fix $\tilde{u} \in T_x M$. For any $\tilde{w} \in T_x M_0$, we have

$$\omega(\tilde{w}, \tilde{u}) = g(\tilde{w}, -J\tilde{u}). \tag{4.7}$$

Now the claim follows since $\tilde{u} \in T_x\mathcal{O}_x$ and $J\tilde{u} \in (T_x M_0)^\perp$ are both equivalent to (4.7) being zero.

We show that $J_G v$ is well-defined. First, suppose $\tilde{v}' \in T_x M_0$ also lifts v. Then $\tilde{v}' - \tilde{v} \in T_x\mathcal{O}_x$. Hence $J(\tilde{v}' - \tilde{v}) \in (T_x M_0)^\perp$, from which we see that $J_G v$ does not depend on the choice of $\tilde{v}$. Second, suppose $x' \in M_0$ with $\pi(x') = \pi(x)$. Write $x' = R_g x$, where R denotes the G-action. Then $\tilde{v}' := (R_g)_*(\tilde{v}) \in T_{x'}M_0$ also lifts v. Since J and g are G-invariant, we see that $(J\tilde{v}')^T = (R_g)_*((J\tilde{v})^T)$. It follows that $J_G v$ does not depend on the choice of x either.

We show that (M_G, ω_G, J_G) is Kähler. First we show $J_G^2 v = -v$. Since $(J\tilde{v})^T$ lifts $J_G v$, we see

$$J_G^2 v = \pi_*((J(J\tilde{v})^T)^T) = \pi_*((J^2\tilde{v} - J(J\tilde{v})^\perp)^T) = \pi_*(-\tilde{v}) - \pi_*(J((J\tilde{v})^\perp)) = -v.$$

Here we have noted the claim before that J restricts to $(T_x M_0)^\perp \to T_x\mathcal{O}_x$.

Next we show $g_G(\cdot,\cdot) = \omega_G(\cdot, J_G\cdot)$ defines an inner product on M_G. Take any $u, v \in T_{\pi(x)}M_G$ with lifts $\tilde{u}, \tilde{v} \in T_x M_0$. Then

$$\begin{aligned} g_G(u,v) &= \omega_G(u, J_G v) = \pi^*\omega_G(\tilde{u}, (J\tilde{v})^T) = i^*\omega(\tilde{u}, (J\tilde{v})^T) \\ &= \omega(\tilde{u}, (J\tilde{v})^T) = g(J\tilde{u}, (J\tilde{v})^T) = g((J\tilde{u})^T, (J\tilde{v})^T). \end{aligned}$$

Hence g_G is nonnegative and symmetric. It is nondegenerate since

$$(J\tilde{u})^T = 0 \iff \tilde{u} \in T_x\mathcal{O}_x \iff u = 0.$$

Lastly, J_G is integrable since the vanishing of the Nijenhuis tensor for J implies the same for J_G. Therefore (M_G, ω_G, J_G) is Kähler. □

5. Quantization Commutes with Reduction

5.1. **Statement of the Theorem.** Suppose (M, ω, J) is a quantizable Kähler manifold and L is a quantum datum for M. Assume that there is a Hamiltonian action by some compact connected Lie group G on M compatible with all these structures. Explicitly, this means

- The G-action on (M, ω) is Hamiltonian with some moment map $\mu\colon M \to \mathfrak{g}^*$. Moreover, G acts on M by biholomorphisms.
- There is a natural $\mathfrak{g}$-action on smooth sections of L defined by $\xi \cdot s := (Q_{pre}(\mu^\xi)/i\hbar)s = (\nabla_{\xi^\#} + \mu^\xi/i\hbar)s$. By property of Q_{pre} we see $[\xi, \eta]\cdot s = \xi\cdot(\eta\cdot s) - \eta\cdot(\xi\cdot s)$, thus this indeed defines a $\mathfrak{g}$-action. The action gives rises to a Lie algebra morphism $\mathfrak{g} \to \mathfrak{X}(L)$ lifting the map $(\mathfrak{g} \to \mathfrak{X}(M),\ \xi \to \xi^\#)$,

which in turn integrates to a G-action on the line bundle L (some global topological condition is assumed) by biholomorphisms, lifting the G-action on M.

- The Hermitian metric $\langle \cdot, \cdot \rangle$ on L is G-invariant. Thus L is a G-equivariant Hermitian holomorphic line bundle.
- (As a consequence) the Chern connection ∇ and its curvature are both G-invariant.

Assume $M_0 = \mu^{-1}(0) \neq \emptyset$ and G acts freely on M_0. Then Theorem 4.6 says that the symplectic reduction $M_G = M_0/G$ is Kähler. Let $i\colon M_0 \to M$, $\pi\colon M_0 \to M_G$, $\omega_G \in \Omega^{1,1}(M_G)$ be as in the theorem.

Lemma 5.1. *The quantum datum L for M naturally descends to a quantum datum L_G for M_G.*

Proof. Since the G-action on L lifts that on M and G acts freely on M_0, the restricted line bundle $L_0 := L|_{M_0} \to M_0$ descends to a complex line bundle $L_G := L_0/G \to M_0/G = M_G$. Since $\langle \cdot, \cdot \rangle$ is G-invariant, it descends to a Hermitian metric $\langle \cdot, \cdot \rangle_G$ on L_G.

Next we show that ∇ descends to a connection ∇_G on L_G. Since $L_0 = \pi^* L_G$, the pullback by π defines a map

$$\pi^*\colon \Omega^*(M_G, L_G) \to \Omega^*(M_0, L_0). \tag{5.2}$$

An element $\alpha \in \Omega^*(M, L)$ is said to be invariant (resp. horizontal) if $\mathcal{L}_{\xi^\#}\alpha = 0$ (resp. $\iota_{\xi^\#}\alpha = 0$) for all $\xi \in \mathfrak{g}$. It is said to be basic if it is both invariant and horizontal. Then, by usual argument one checks that the map (5.2) defines an isomorphism onto $\Omega^*_{bas}(M_0, L_0)$, the set of basic elements in $\Omega^*(M_0, L_0)$.

Now, let $s \in \Gamma(M_G, L_G)$ be arbitrary. We check that $(i^*\nabla)(\pi^* s)$ is basic. It is invariant since both $\pi^* s$ and $i^*\nabla$ are G-invariant. It is horizontal because, for any $\xi \in \mathfrak{g}$, $\mu^\xi = 0$ on M_0 implies that $\nabla_{\xi^\#}(\pi^* s) = \xi \cdot s = 0$.

Combining the two arguments above we see that

$$\nabla_G s := (\pi^*)^{-1}((i^*\nabla)(\pi^* s)) \in \Omega^1(M_G, L_G)$$

is well-defined, making ∇_G a connection on L_G satisfying $\pi^*\nabla_G = i^*\nabla$. It follows that

$$\pi^* F_{\nabla_G} = i^* F_\nabla = i^*(\omega/i\hbar) = \pi^*(\omega_G/i\hbar),$$

hence $F_{\nabla_G} = \omega_G/i\hbar$ is a $(1,1)$-form on M_G. Therefore $\bar\partial_{L_G} := \nabla_G^{0,1}$ defines on L_G a holomorphic structure. Clearly ∇_G is metric compatible, thus it is the Chern connection for the Hermitian holomorphic bundle L_G. We conclude that L_G is a quantum datum for M_G. □

Let $\mathbb{H}, \mathbb{H}_G$ denote the quantum spaces for M, M_G obtained by Kähler quantization using quantum data L, L_G, respectively. Assume M, thus M_G, is compact, then by the discussion in Section 3.3, $\mathbb{H} = H^0(M, L)$, $\mathbb{H} = H^0(M_G, L_G)$ are spaces of holomorphic sections. Since G acts on L by holomorphic maps, there is a natural G-action on $\mathbb{H}$. Let $\mathbb{H}^G$ denote the fixed point set of this action, which is a closed subspace of $\mathbb{H}$, hence also Hilbert.

Let $s \in \mathbb{H}^G = H^0(M, L)^G$ be arbitrary. Then $s|_{M_0} \in \Gamma(M_0, L_0)^G$ descends to a section $\hat{s} \in \Gamma(M_G, L_G)$.

We are now set up to formulate the main theorem.

Theorem 5.3 (Quantization commutes with reduction)**.** *Assume M is a compact quantizable Kähler manifold with a quantum datum L and there is a Hamiltonian G-action with moment map μ that is compatible L in the sense explained above. Assume also that G acts freely on $\mu^{-1}(0) \neq \emptyset$. Then, with the notations above, the map $s \mapsto \hat{s}$ defines a linear bijection $\mathbb{H}^G \to \mathbb{H}_G$.*

In the last two sections we will sketch a proof for this theorem following [3].

5.2. **Complexification of a Compact Lie Group.** In this section we state a few properties about complex Lie groups, the complex analog of a real Lie group. Readers may consult [8][3] for missing proofs and more developments.

A (real) Lie group is called a **complex Lie group** if it is a complex manifold and its group operations are holomorphic. Morphisms between complex Lie groups are (real) Lie group homomorphisms that are holomorphic.

For a Lie group G, its **complexification**, denoted $G_{\mathbb{C}}$, is a complex Lie group equipped with a Lie group homomorphism $i_G\colon G \to G_{\mathbb{C}}$ such that for every Lie group homomorphism $\alpha\colon G \to H$ to some complex Lie group H, there is a unique complex Lie group homomorphism $\phi\colon G_{\mathbb{C}} \to H$ such that $\alpha = \phi \circ i_G$. Clearly, if it exists, $G_{\mathbb{C}}$ is unique up to isomorphism.

Every compact Lie group G with Lie algebra $\mathfrak{g}$ admits a complexification $G_{\mathbb{C}}$, whose Lie algebra is given by $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$. This can be explicitly shown as follows. Without loss of generality assuming G is connected, by standard Lie group theory we may write G as a direct product of a torus $(S^1)^m$ and a semisimple Lie group G_1. Then $\Gamma = \pi_1(G_1)$ is finite. Let $(\tilde{G}_1)_{\mathbb{C}}$ denote the simply connected Lie group with Lie algebra $(\mathfrak{g}_1)_{\mathbb{C}} = \mathfrak{g}_1 \otimes \mathbb{C}$, with a complex structure induced by the exponential map $(\mathfrak{g}_1)_{\mathbb{C}} \xrightarrow{\cong} (\tilde{G}_1)_{\mathbb{C}}$. Then the complex Lie group $G_{\mathbb{C}} := (\mathbb{C}^*)^m (\tilde{G}_1)_{\mathbb{C}}/\Gamma$ is the complexification of G (equipped with the natural embedding $G \hookrightarrow G_{\mathbb{C}}$). It can be shown that the group $G_{\mathbb{C}}$ admits a direct decomposition $G_{\mathbb{C}} = GP$ where $P = \exp(i\mathfrak{g})$.

Suppose G acts on some compact complex manifold X by biholomorphisms. Then it extends to a $G_{\mathbb{C}}$-holomorphic action on X. Explicitly, let J denote the almost complex structure on X, then for $\xi \in \mathfrak{g}$, define $(i\xi)^\# = J\xi^\#$. Since G acts by holomorphic maps, we have $[\xi^\#, X] = \mathcal{L}_{\xi^\#}(JX) = J\mathcal{L}_{\xi^\#}X = J[\xi^\#, X]$ for any $\xi \in \mathfrak{g}$ and $X \in \mathfrak{X}(X)$. Hence, for any $\xi, \eta \in \mathfrak{g}$, we have

$$[\xi^\#, (i\eta)^\#] = J[\xi^\#, \eta^\#] = [\xi, i\eta]^\#.$$

Moreover, since the Nijenhuis tensor vanishes, we have

$$\begin{aligned}[(i\xi)^\#, (i\eta)^\#] &= J[J\xi^\#, \eta^\#] + J[\xi^\#, J\eta^\#] + [\xi^\#, \eta^\#] \\ &= J[J\xi^\#, \eta^\#] = -[\xi^\#, \eta^\#] = [i\xi, i\eta]^\#.\end{aligned}$$

Therefore $(\cdot)^\#$ indeed defines a Lie algebra homomorphism $\mathfrak{g}_{\mathbb{C}} \to \mathfrak{X}(X)$. By the explicit description of $G_{\mathbb{C}}$ above, one sees that this integrates to a $G_{\mathbb{C}}$-holomorphic action on X. Here, the compactness assumption on X can be replaced by a certain integrability assumption.

5.3. **Proof of the Theorem.**

Lemma 5.4. *For $\xi \in \mathfrak{g}$, the gradient of μ^ξ is $(i\xi)^\#$.*

Proof. We compute that

$$\langle \nabla \mu^\xi, \cdot \rangle = d\mu^\xi(\cdot) = \omega(\xi^\#, \cdot) = \langle J\xi^\#, \cdot \rangle = \langle (i\xi)^\#, \cdot \rangle.$$

Thus $\nabla \mu^\xi = (i\xi)^\#$, as desired. □

Write $G_\mathbb{C} = GP$ where $P = \exp(i\mathfrak{g})$. Let $M_s := M_0 G_\mathbb{C} = M_0 GP = M_0 P$.

Lemma 5.5. *M_s is an open subset of M on which $G_\mathbb{C}$ acts freely.*

Proof. By Lemma 5.4, for any $\xi \in \mathfrak{g}$, we have $(i\xi)^\# = \nabla \mu^\xi$ which is orthogonal to M_0 and nonvanishing on M_0 for any $\xi \in \mathfrak{g} \backslash \{0\}$. By a dimension count, we conclude that M_s contains a neighborhood of M_0 in M. Since every $G_\mathbb{C}$-orbit of M_s intersects M_0, this implies that M_s contains a neighborhood of any of its point. Therefore M_s is open. Moreover, since any $\xi^\#|_{M_0}$, $\xi \in \mathfrak{g}$, is contained in TM_0 and is nonvanishing if $\xi \neq 0$, we know $\xi^\#|_{M_0}$ is nonvanishing for any $\xi \in \mathfrak{g}_\mathbb{C} \backslash \{0\}$. It follows that the stabilizer of every point in M_0, thus of every point in M_s, is discrete. Hence the $G_\mathbb{C}$-action on M_s is locally free.

To show the action is free, let $x \in M_0$, $g \in G$, $p \in P$ be such that $x = xgp$. It suffices to prove that $g = p = e$ is the identity. Write $p = \exp(i\xi)$, $\xi \in \mathfrak{g}$. Let $f(t) = \mu^\xi(xg \exp(it\xi))$. Then

$$f(0) = f(1) = 0, \; f'(t) = (i\xi)^\#_{xg\exp(it\xi)} \mu^\xi = |\nabla \mu^\xi|^2 (xg \exp(it\xi)) \geq 0.$$

Hence $f'(t) = 0$ for all $t \in [0, 1]$, in particular for $t = 0$. It follows that $\xi = 0$, $p = e$. Since G acts on M_0 freely, we also have $g = e$. □

Since the action is holomorphic, the quotient $M_G = M_s/G_\mathbb{C}$ is a complex manifold. Explicitly, the associated almost complex structure J'_G on M_G is given by $J'_G(v) = (\pi_G)_*(J\tilde{v})$ for $v \in T_{\pi_G(x)} M_G$, $x \in M_s$, and $\tilde{v} \in T_x M_s$ a lift of v. Here $\pi_G \colon M_s \to M_G$ denotes the quotient map. In particular, noting that $(T_x M_0)^\perp$ is contained in the tangent space of the orbit $xG_\mathbb{C}$ at x, we may take $x \in M_0$, $\tilde{v} \in T_x M_0$. Then we see that J'_G agrees with J_G defined as in the proof of Theorem 4.6. Therefore, the two complex structures on M_G, either obtained from symplectic reduction or from the quotient $M_s/G_\mathbb{C}$, coincide.

The G-action on the line bundle L extends to a $G_\mathbb{C}$-action by holomorphic bundle morphisms. Explicitly, the infinitesimal $\mathfrak{g}_\mathbb{C}$-action on the smooth sections of L given by

$$\xi \cdot s := (\nabla_{\xi^\#} + \mu^\xi / i\hbar) s, \; \xi \in \mathfrak{g}_\mathbb{C} \tag{5.6}$$

integrates to a global $G_\mathbb{C}$-action since M is compact.

Now, quotienting by $G_{\mathbb{C}}$ gives a map between holomorphic line bundles

$$\begin{array}{ccc} L_s & \longrightarrow & L_G \\ \downarrow & & \downarrow \\ M_s & \longrightarrow & M_G, \end{array}$$

where $L_s = L|_{M_s}$. It follows that the pullback map defines an isomorphism

$$\pi_G^*\colon H^0(M_G, L_G) \xrightarrow{\cong} H^0(M_s, L_s)^{G_{\mathbb{C}}}.$$

Lemma 5.7. *Any G-invariant holomorphic section s of L (resp. L_s) is $G_{\mathbb{C}}$-invariant. Moreover, $|s|$ attains maximum on M_0.*

Proof. Let $\xi \in \mathfrak{g}$ be arbitrary. Since s is holomorphic and ∇ is compatible with the holomorphic structure, we have $\nabla_{\xi^\# + iJ\xi^\#} s = 0$. Hence $\nabla_{(i\xi)^\#} s = i\nabla_{\xi^\#} s$, which implies that $(i\xi)\cdot s = (\nabla_{(i\xi)^\#} + \mu^\xi/\hbar)s = i(\xi\cdot s) = 0$. It follows that s is $G_{\mathbb{C}}$-invariant. Moreover, we have

$$(i\xi)^\# |s|^2 = \langle \nabla_{(i\xi)^\#} s, s\rangle + \langle s, \nabla_{(i\xi)^\#} s\rangle = -2\mu^\xi |s|^2/\hbar. \tag{5.8}$$

If s is defined on L and $s \neq 0$, then (5.8) implies that $|s|$ attains maximum only on M_0. Maximum is attained by compactness of M.

If s is defined on L_s, then for any $y = x\exp(i\xi) \in M_s$, where $x \in M_0$, $\xi \in \mathfrak{g}$, we see by Lemma 5.4 that $\mu^\xi(x\exp(it\xi)) \geq 0$ for $t \in [0,1]$. Therefore $|s(y)|^2 \leq |s(x)|^2$ by integrating (5.8). Hence $|s|$ attains maximum on M_0 by compactness of M_0. □

Remark 5.9. With a little more care one can show that any G-invariant holomorphic section of L is zero above any point in $M\backslash M_s$. More generally, this result still holds with L replaced by any $L^{\otimes k}$, $k > 0$.

Now, we see that the map $s \mapsto \hat{s}$ in the statement of Theorem 5.3 factors as

$$H^0(M,L)^G = H^0(M,L)^{G_{\mathbb{C}}} \xrightarrow{r} H^0(M_s, L_s)^{G_{\mathbb{C}}} \xrightarrow[\cong]{(\pi_G^*)^{-1}} H^0(M_G, L_G).$$

It remains to show that the restriction map r is bijective. If $r(s) = 0$ for some s, then the second part of Lemma 5.7 implies that $s = 0$. Hence r is injective. To prove surjectivity, we resort to the following proposition for which we will not give the full proof.

Proposition 5.10. *The complement of M_s in M is contained in a complex subvariety of M of codimension at least one.*

Proof. Since $iF_\nabla = \omega/\hbar$ is a positive real $(1,1)$-form, L is a positive line bundle. For each $k > 0$, $L^{\otimes k}$ is a G-equivariant line bundle, and by the Kodaira embedding theorem we know that $H^0(M, L^{\otimes k}) \neq 0$ for some k. However, it is less clear why $H^0(M, L^{\otimes k})^G$ should be nontrivial for some $k > 0$. We refer readers to [3] for a direct proof of this result and [11] for a proof using some geometric invariant theory. The proposition follows from this existence result and Remark 5.9. □

By Lemma 5.7, any $s \in H^0(M_s, L_s)^{G_\mathbb{C}}$ is bounded. Therefore, by Proposition 5.10 and the Riemann extension theorem, it extends to a holomorphic section of L, which is also $G_\mathbb{C}$-invariant by continuity. This proves the surjectivity of r and finishes the proof of Theorem 5.3.

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