SYMMETRY GROUP OF A TESSELLATION IN THE HYPERBOLIC PLANE

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Abstract. The hyperbolic plane has lots of symmetrical properties, and so do the tessellations on it. This paper mainly explores the structure of symmetry groups of the hyperbolic plane and its tessellation, along with some of their geometric properties.

Contents

1. The hyperbolic plane

The hyperbolic planes are models of Lobachevski's geometry, in which the Euclidean fifth postulate doesn't hold. There are multiple approaches to defining the hyperbolic plane, such as Lobachevski's axiomatic approach, the Klein disk and the Poincare disk. We will define the hyperbolic plane using the Poincare upper half-plane model. The upper half-plane will be denoted as H.

Definition 1.1. A *line* in H is either a semicircle meeting the real axis at right angles, or a vertical ray emanating from a point on the real axis.

From this definition, we know there is a unique line through any two points.

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Definition 1.2. The angle between two lines is the angle between the two tangents to the lines at their intersection points (Fig. 1).

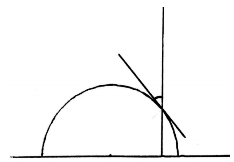

FIGURE 1

Definition 1.3. Suppose P and Q are two points in the hyperbolic plane and S and T are the intersection points between the real axis and the line through P and Q. Then we define the distance $d(P,Q)$ by $d(P,Q) = \left|\ln \frac{|OQ||PR|}{|OP||QR|}\right|$; see (Fig. 2).

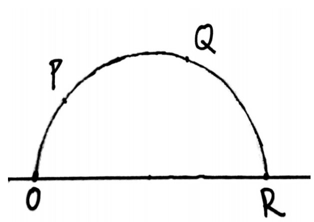

FIGURE 2

Definition 1.4. The set of points at infinity in H is $\{i\infty\} \cup \mathbb{R}$.

Every line passes through exactly two points at infinity, and the two points at infinity determine the line.

Definition 1.5. A *symmetry* of the hyperbolic plane is a bijection $f : H \to H$ which preserves the distance. The set of all *symmetries* of H is a group, called the *symmetry group* of H. We'll denote this group by G.

By the definition, all symmetries take lines to lines, and preserve the angles.

Theorem 1.6. *We define an action of $PSL_2(\mathbb{R})$ on the upper half plane as follows. For any matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL_2(\mathbb{R})$ and $z \in H$, we define $g(z) = \frac{az+b}{cz+d}$. This*

rule satisfies $g(z) = (-g)(z)$, and it defines an action of $PSL_2(\mathbb{R})$ on H.
The group $PSL_2(\mathbb{R})$ acts via symmetries on H. The full symmetry group G contains $PSL_2(\mathbb{R})$ as a index-2 subgroup, and is generated by $PSL_2(\mathbb{R})$ and the order-2 element $f(z) = -\bar{z}$.

Proof. In [1] (Lemma, 3.1, 3.2, 3.5, 3.9), it is proved that all elements of $PSL_2(\mathbb{R})$ are symmetries of H, as is f.
Conversely, we will show any symmetry of H is either an element of $PSL_2(\mathbb{R})$, or a composition of f and an element in $PSL_2(\mathbb{R})$. Let $s : H \to H$ be a symmetry mapping the line $t_1 : x = 0$ to the line t_2. First we will show that there exists $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{R})$ which also maps t_1 to t_2. If t_2 is of the form $x = a$, then we choose α to be $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$. Otherwise suppose t_2 meets the real axis at points $(m, 0)$ and $(n, 0)$. And we can choose $\alpha = \begin{pmatrix} n & \frac{m}{n-m} \\ 1 & \frac{1}{n-m} \end{pmatrix}$.

Now suppose $\alpha^{-1}s$ maps i to ri, so $\begin{pmatrix} \frac{1}{r} & 0 \\ 0 & 1 \end{pmatrix} \alpha^{-1}s(i) = i$. Let $\gamma = \begin{pmatrix} \frac{1}{r} & 0 \\ 0 & 1 \end{pmatrix} \alpha^{-1}s$, so $\gamma(i) = i$ and $\gamma(t_1) = t_1$.

Since γ preserves the distance, let $r = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} f \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, so r is the reflection across the upper half of the boundary of the disk $x^2 + y^2 = 1$, and then either $\gamma \circ r$ or γ preserves t_1. We might as well suppose it's γ that preserves t_1. Since the upper half-plane is oriented in the topological sense, every symmetry must either preserve or reverse the orientation. If γ preserves the orientation, then $\gamma = id$. Otherwise we choose $B \notin t_1$ and $\gamma(B) \neq B$. Let t_4 be the line that goes through B and meets t_1 vertically at point C. Since $d(B, C) = d(\gamma(B), C)$, $\gamma(B) = -\bar{B}$. So $\gamma(z) = f$ due to the continuity of γ and it's easy to check that f doesn't preserve the orientation, which is contradictory with our assumption. On the other hand, if γ doesn't preserve the orientation, then $\gamma = f$. Since f commutes with any $g \in PSL_2(\mathbb{R})$, we have the conclusion. □

Theorem 1.7. *The counter-clockwise rotation by the angle θ around the point i is* $r(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \in PSL_2(\mathbb{R})$.

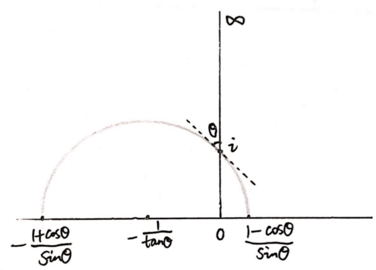

FIGURE 3

Proof. Let L_1 be the line $x = 0$, and L_2 be the result after L_1 is rotated counter-clockwise by the angle θ; see (Fig. 3). This symmetry of the hyperbolic plane preserves the orientation, so we can assume that the function of the rotation $r(\theta) : H \to H$ is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Also this function maps 0 and ∞ to the intersection points between L_2 and the real axis. A calculation in Euclidean geometry shows that the intersection points of L_2 and the real axis are $(\frac{1-\cos\theta}{\sin\theta}, 0)$ and $(-\frac{1+\cos\theta}{sin\theta}, 0)$. So we have

$$\begin{aligned} r(\theta)(i) &= i \\ r(\theta)(0) &= \frac{1-\cos\theta}{\sin\theta} \\ r(\theta)(\infty) &= -(\frac{1+\cos\theta}{sin\theta}) \\ ad - bc &= 1 \end{aligned}$$

There are two solutions of this system of equations, namely $\begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$ and $\begin{pmatrix} -\cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \end{pmatrix}$. They represent the same function in $PSL_2(\mathbb{R})$. □

Lemma 1.8. *The orientation-preserving elements of G are exactly the elements of the index-2 subgroup $PSL_2(\mathbb{R})$.*

Proof. We only need to check that $f_1(z) = az$, $f_2(z) = -\frac{1}{z}$ and $f_3(z) = z+b$ (a and b are real numbers) preserve the orientation, and $f_4(z) = \bar{z}$ reverses the orientation. For f_2, suppose L is the upper half of the boundary of the unit disk centered at 0. At point i, choose the tangent vector $(1, 0)$ and the normal vector $(0, 1)$. After being acted on by f_2, the tangent vector becomes $(-1, 0)$ and the normal vector is $(0, -1)$. So f_2 preserves the orientation. The other three functions can be examined in the same way. □

Theorem 1.9. *The counter-clockwise rotation by the angle θ around the point P is $\alpha r(\theta)\alpha^{-1}$, in which $\alpha : H \to H$ satisfies $\alpha(i) = P$ and $\alpha \in PSL_2(\mathbb{R})$.*

Proof. Assume $f = \alpha r(\theta)\alpha^{-1}$. Then $f(P) = \alpha r(\theta)(i) = \alpha(i) = P$. For any line t_1 through P, the line $r(\theta)\alpha^{-1}(t_1)$ can be obtained by rotating the line $\alpha^{-1}(t_1)$ counter-clockwise by the angle θ. Hence the angle between the line $f(t_1)$ and t_1 is θ because α is a conformal mapping and preserves the orientation. □

2. A tessellation on the hyperbolic plane

Just as the Euclidean plane can be tessellated by squares or triangles of the same size, the hyperbolic plane can also be tessellated by copies of a polygon. In the rest of this paper, we will study a special tessellation consisting of triangles whose three angles are all $\frac{2\pi}{7}$.

We construct the figure in this way: Let ℓ_0 be the upper half of the unit circle; this is a hyperbolic line through point i. Let ℓ_1 be line $\Re z = 0$. Choose a line ℓ_2, such that the angle between ℓ_2 and ℓ_1 and the angle between ℓ_2 and ℓ_0 is $2\pi 7$. Let C and A be the intersection points of ℓ_2 with ℓ_0 and ℓ_1 respectively. Let B be the reflection of C across the line ℓ_1; see (Fig. 4).

Lemma 2.1. *In the figure above, three sides of the triangle ABC are equal.*

Proof. Let l_2 go through C and divide $\angle ACB$ equally. Assume l_2 meets l_0 at D. Then the reflection of A across l_2 is B (otherwise there would be another point

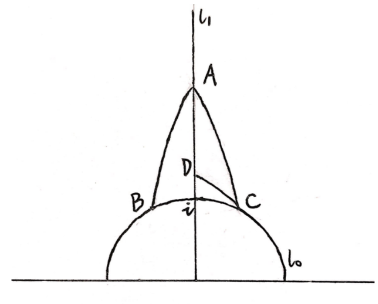

FIGURE 4

B_1 on the line l_1 that satisfies $\angle AB_1C = \frac{2\pi}{7}$). So $AC = BC$. And therefore $AB = AC = BC$.

□

Rotating $\triangle ABC$ around A in increment of $\frac{2\pi}{7}$ gives seven triangles around the point A. Repeating this process at various points will give the tessellation we are looking for.

Lemma 2.2. *$\triangle ABC$ and its copies can tessellate the whole plane.*

Proof. We construct the tessellation from the regular heptagon centered at A, which is formed after $\triangle ABC$ is rotated with respect to A by $\frac{2\pi}{7}$ for 6 times, and we call this original region R_0. The operation is the following: at each vertex E on the boundary of the region R_n, choose a triangle that includes E, and rotate it around E by the angle $\frac{2\pi}{7}$ for seven times. The new region is called R_{n+1}.

We will prove by induction that each R_n is convex. Suppose there are more than three triangles having the same vertex M on the boundary of R_n. Now every triangle involving M must have another vertex T on the boundary of R_{n-1}, and each pair (M, T) indicates there are two triangles that include M. Since the region R_{n-1} is convex, the number of this kind of T is at most two. So there are four triangles at M on the boundary of R_{n-1} respectively. But this can't be true because T_1 and T_2 must be in the same triangle.

Next we claim that there exists $\epsilon > 0$ with the following property: for any point P in R_n, and any point Q such that $d(P, Q) < \epsilon$, we have Q in R_{n+1}. Let M be the closest vertex to P, and R be the smallest regular heptagon centered at M. Let S be the set of points that are closer to M than to any other vertices in R. Then ∂S is in R and P must be contained in S. By compactness, there exists $\epsilon > 0$ such that for every T in S, we have $d(T, \partial R) > \epsilon$. So $d(P, \partial R_{n+1}) \geq d(P, \partial R) > \epsilon$. And this proves the claim.

Hence $\inf\{d(M, A) | M \in \partial R_n\} > n\epsilon$. So $\lim_{n\to\infty} R_n = H$. $\square$

We call this tessellation T.

In order to calculate the location of the vertices in T, the theorem below is needed.

Theorem 2.3. *Suppose $a, b, c, d \in R$, the angle θ between the two lines ac and bd satisfies $\sin\theta = \left|1 + \frac{3}{y+1}\right|$, in which y is the cross ratio $\frac{(b-a)(d-c)}{(b-c)(d-a)}$; see (Fig. 5).*

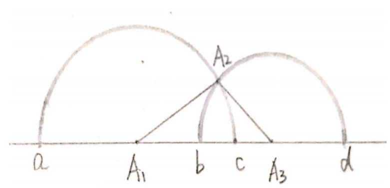

FIGURE 5

Proof. Let A_1 and A_3 be the centers of the two circles, and A_2 be the intersection point of the two lines.
Due to either $\theta + \angle A_1A_2A_3 = \frac{\pi}{2}$ or $|\theta - \angle A_1A_2A_3| = \frac{\pi}{2}$,
we have $\sin\theta = |\cos\angle A_1A_2A_3| = \left|\frac{(\frac{c-a}{2})^2+(\frac{d-b}{2})^2-(\frac{d-a}{2}+\frac{b-c}{2})^2}{2\frac{c-a}{2}\frac{d-b}{2}}\right| = \left|1 + \frac{3}{y+1}\right|$. $\square$

Calculation 2.4. *The following is the calculation of the coordinates of a vertex in T. Assume the coordinates of A are $(0, x\tan(\frac{2\pi}{7}))$. Using the theorem above we have*

$$\sin\frac{2\pi}{7} = \left|1 + \frac{3}{y-1}\right|$$

$$y = \frac{((1 + \frac{1}{\cos\frac{2\pi}{7}})x - 1)((1 - \frac{1}{\cos\frac{2\pi}{7}})x + 1)}{((1 - \frac{1}{\cos\frac{2\pi}{7}})x - 1)((1 + \frac{1}{\cos\frac{2\pi}{7}})x + 1)}$$

The solution of this system of equations is
$x = \frac{-(8\cos\frac{2\pi}{7}+2\sin\frac{4\pi}{7})-\sqrt{(8\cos\frac{2\pi}{7}+2\sin\frac{4\pi}{7})^2-9(\sin\frac{4\pi}{7})^2}}{6(\sin\frac{2\pi}{7})^2}$
This gives us the location of A.

3. Symmetry group of the tessellation

Definition 3.1. Let T be the tessellation constructed in Lemma 2.2. A symmetry of the hyperbolic plane is said to be a *symmetry of T* if it preserves the set of vertices in T. The set of all symmetries of T is a group, called the *symmetry group* of T, and denoted G_1.

Theorem 3.2. *Let $\alpha : H \to H$ be the counter-clockwise rotation by $\frac{2\pi}{7}$ at the point A, let $\beta : H \to H$ be the rotation for angle π at point i, and let $\gamma : H \to H$ be the reflection with respect to the line AB. Then G_1 is generated by α, β and γ.*

Proof. Let G_2 is the group generated by α, β, and γ, and $D = \bigcup_{g\in G_2} \triangle ABC$. Let S be the set of vertices in D.
We claim that G_2 acts transitively on S.
Since every triangle in the district D is a copy of $\triangle ABC$ and $\alpha(B) = C$, there are at most two orbits of vertices in S. Suppose there are two different orbits of vertices. Denote every point in the same orbit with A as a red one and the others as a black one. Then every red point must be surrounded by black points(Fig. 6).

In this figure the location of $\triangle ABC$ is the same as that in Figure 4, and

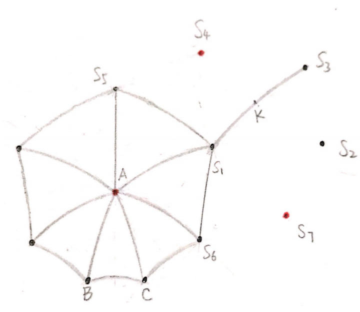

Figure 6

$S_2, S_3, S_4, S_5, A, S_6$, and S_7 are the closest vertices to S_1. We have $S_7 = \alpha^{-2}\gamma\alpha^2(A)$, $S_4 = \alpha^{-3}\gamma\alpha^3(A)$, so S_3, S_6 are red, and S_1, S_2, S_3 are black. Let K be the middle point of S_1 and S_3. Notice that K can be obtained by i after i rotates around A

for $\frac{6\pi}{7}$ and then rotate around S_4 for $\frac{2\pi}{7}$. So S_4 can rotate around K for π and we get S_2. Hence S_2 is both red and black. And this means there is only one orbit of vertices.

Therefore, for any vertex L in D, we have $w \in G_2$ such that $w(A) = L$. And in the notation of Thm 1.9, the rotation around L by the angle $\frac{2\pi}{7}$ is $w\alpha w^{-1}$. So with the construction in Lemma 2.2, we have $D = H$.

Since G_2 acts transitively on the set of vertices, for any $\triangle A_1A_2A_3$, there exists a element $g \in G_1$ that respectively maps A, B, C to A_1, A_2, A_3. Hence $G_2 = G_1$.

$\square$

Remark 3.3. With what we have calculated in 2.4, denote the coordinates of A as $(0, a)$. Then in the notation of Theorem 1.9, we have $\alpha = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} r_{\frac{2\pi}{7}} \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}$, $\beta(z) = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} r_{\frac{\pi}{7}} \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix} (\bar{z})$, and $\gamma(z) = -\frac{1}{z}$.

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4. Bibliography

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