SYMMETRY GROUP OF A TESSELLATION IN THE HYPERBOLIC PLANE

QING QI

Abstract. The hyperbolic plane has lots of symmetrical properties, and so do the tessellations on it. This paper mainly explores the structure of symmetry groups of the hyperbolic plane and its tessellation, along with some of their geometric properties.

Contents

1. The hyperbolic plane 1
2. A tessellation on the hyperbolic plane 5
3. Symmetry group of the tessellation 8
Acknowledgments 9
4. Bibliography 9
References 9

1. The hyperbolic plane

The hyperbolic planes are models of Lobachevski’s geometry, in which the Euclidean fifth postulate doesn’t hold. There are multiple approaches to defining the hyperbolic plane, such as Lobachevski’s axiomatic approach, the Klein disk and the Poincare disk. We will define the hyperbolic plane using the Poincare upper half-plane model. The upper half-plane will be denoted as $H$.

Definition 1.1. A line in $H$ is either a semicircle meeting the real axis at right angles, or a vertical ray emanating from a point on the real axis.

From this definition, we know there is a unique line through any two points.

Date: August 31, 2020.
**Definition 1.2.** The angle between two lines is the angle between the two tangents to the lines at their intersection points (Fig. 1).

![Figure 1](image1.png)

**Definition 1.3.** Suppose $P$ and $Q$ are two points in the hyperbolic plane and $S$ and $T$ are the intersection points between the real axis and the line through $P$ and $Q$. Then we define the distance $d(P, Q)$ by $d(P, Q) = \left| \ln \frac{|OQ|}{|OP|} \right| \left| \ln \frac{|PR|}{|QR|} \right|$: see (Fig. 2).

![Figure 2](image2.png)

**Definition 1.4.** The set of points at infinity in $H$ is $\{i\infty\} \cup \mathbb{R}$.

Every line passes through exactly two points at infinity, and the two points at infinity determine the line.

**Definition 1.5.** A *symmetry* of the hyperbolic plane is a bijection $f : H \to H$ which preserves the distance. The set of all *symmetries* of $H$ is a group, called the *symmetry group* of $H$. We’ll denote this group by $G$.

By the definition, all symmetries take lines to lines, and preserve the angles.

**Theorem 1.6.** We define an action of $PSL_2(\mathbb{R})$ on the upper half plane as follows. For any matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL_2(\mathbb{R})$ and $z \in H$, we define $g(z) = \frac{az + b}{cz + d}$. This
rule satisfies \( g(z) = (-g)(z) \), and it defines an action of \( \text{PSL}_2(\mathbb{R}) \) on \( H \).

The group \( \text{PSL}_2(\mathbb{R}) \) acts via symmetries on \( H \). The full symmetry group \( G \) contains \( \text{PSL}_2(\mathbb{R}) \) as a index-2 subgroup, and is generated by \( \text{PSL}_2(\mathbb{R}) \) and the order-2 element \( f(z) = -\bar{z} \).

Proof. In [1] (Lemma, 3.1, 3.2, 3.5, 3.9), it is proved that all elements of \( \text{PSL}_2(\mathbb{R}) \) are symmetries of \( H \), as is \( f \).

Conversely, we will show any symmetry of \( H \) is either an element of \( \text{PSL}_2(\mathbb{R}) \), or a composition of \( f \) and an element in \( \text{PSL}_2(\mathbb{R}) \). Let \( s : H \to H \) be a symmetry mapping the line \( t_1 : x = 0 \) to the line \( t_2 \). First we will show that there exists \( \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{R}) \) which also maps \( t_1 \) to \( t_2 \). If \( t_2 \) is of the form \( x = a \), then we choose \( \alpha \) to be \( \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \). Otherwise suppose \( t_2 \) meets the real axis at points \((m,0)\) and \((n,0)\). And we can choose \( \alpha = \begin{pmatrix} n & m \\ 1 & n-m \end{pmatrix} \).

Now suppose \( \alpha^{-1}s \) maps \( i \) to \( ri \), so \( \begin{pmatrix} \frac{1}{r} & 0 \\ 0 & 1 \end{pmatrix} \alpha^{-1}s(i) = i \). Let \( \gamma = \begin{pmatrix} \frac{1}{r} & 0 \\ 0 & 1 \end{pmatrix} \alpha^{-1}s \), so \( \gamma(i) = i \) and \( \gamma(t_1) = t_1 \).

Since \( \gamma \) preserves the distance, let \( r = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} f \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \), so \( r \) is the reflection across the upper half of the boundary of the disk \( x^2 + y^2 = 1 \), and then either \( \gamma \circ r \) or \( \gamma \) preserves \( t_1 \). We might as well suppose it’s \( \gamma \) that preserves \( t_1 \). Since the upper half-plane is oriented in the topological sense, every symmetry must either preserve or reverse the orientation. If \( \gamma \) preserves the orientation, then \( \gamma = id \). Otherwise we choose \( B \notin t_1 \) and \( \gamma(B) \neq B \). Let \( t_4 \) be the line that goes through \( B \) and meets \( t_1 \) vertically at point \( C \). Since \( d(B,C) = d(\gamma(B),C) \), \( \gamma(B) = -\bar{B} \). So \( \gamma(z) = f \) due to the continuity of \( \gamma \) and it’s easy to check that \( f \) doesn’t preserve the orientation, which is contradictory with our assumption. On the other hand, if \( \gamma \) doesn’t preserve the orientation, then \( \gamma = f \). Since \( f \) commutes with any \( g \in \text{PSL}_2(\mathbb{R}) \), we have the conclusion. \( \square \)
Theorem 1.7. The counter-clockwise rotation by the angle $\theta$ around the point $i$ is
\[
r(\theta) = \begin{pmatrix} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\ -\sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix} \in PSL_2(\mathbb{R}).
\]

Proof. Let $L_1$ be the line $x = 0$, and $L_2$ be the result after $L_1$ is rotated counter-clockwise by the angle $\theta$; see (Fig. 3). This symmetry of the hyperbolic plane preserves the orientation, so we can assume that the function of the rotation $r(\theta) : H \to H$ is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Also this function maps 0 and $\infty$ to the intersection points between $L_2$ and the real axis. A calculation in Euclidean geometry shows that the intersection points of $L_2$ and the real axis are $(\frac{1 - \cos \theta}{\sin \theta}, 0)$ and $(\frac{-1 + \cos \theta}{\sin \theta}, 0)$. So we have
\[
\begin{align*}
    r(\theta)(i) &= i \\
    r(\theta)(0) &= \frac{1 - \cos \theta}{\sin \theta} \\
    r(\theta)(\infty) &= -\left(\frac{1 + \cos \theta}{\sin \theta}\right) \\
    \det(ad - bc) &= 1
\end{align*}
\]

There are two solutions of this system of equations, namely $\begin{pmatrix} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\ -\sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix}$ and $\begin{pmatrix} -\cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & -\cos \frac{\theta}{2} \end{pmatrix}$. They represent the same function in $PSL_2(\mathbb{R})$. \qed
Lemma 1.8. The orientation-preserving elements of $G$ are exactly the elements of the index-2 subgroup $\text{PSL}_2(\mathbb{R})$.

Proof. We only need to check that $f_1(z) = az$, $f_2(z) = -\frac{1}{z}$ and $f_3(z) = z + b$ ($a$ and $b$ are real numbers) preserve the orientation, and $f_4(z) = \bar{z}$ reverses the orientation. For $f_2$, suppose $L$ is the upper half of the boundary of the unit disk centered at 0. At point $i$, choose the tangent vector $(1, 0)$ and the normal vector $(0, 1)$. After being acted on by $f_2$, the tangent vector becomes $(-1, 0)$ and the normal vector is $(0, -1)$. So $f_2$ preserves the orientation. The other three functions can be examined in the same way. □

Theorem 1.9. The counter-clockwise rotation by the angle $\theta$ around the point $P$ is $\alpha r(\theta) \alpha^{-1}$, in which $\alpha : H \to H$ satisfies $\alpha(i) = P$ and $\alpha \in \text{PSL}_2(\mathbb{R})$.

Proof. Assume $f = \alpha r(\theta) \alpha^{-1}$. Then $f(P) = \alpha r(\theta)(i) = \alpha(i) = P$. For any line $t_1$ through $P$, the line $r(\theta) \alpha^{-1}(t_1)$ can be obtained by rotating the line $\alpha^{-1}(t_1)$ counter-clockwise by the angle $\theta$. Hence the angle between the line $f(t_1)$ and $t_1$ is $\theta$ because $\alpha$ is a conformal mapping and preserves the orientation. □

2. A tessellation on the hyperbolic plane

Just as the Euclidean plane can be tessellated by squares or triangles of the same size, the hyperbolic plane can also be tessellated by copies of a polygon. In the rest of this paper, we will study a special tessellation consisting of triangles whose three angles are all $\frac{2\pi}{7}$.

We construct the figure in this way: Let $\ell_0$ be the upper half of the unit circle; this is a hyperbolic line through point $i$. Let $\ell_1$ be line $\Re z = 0$. Choose a line $\ell_2$, such that the angle between $\ell_2$ and $\ell_1$ and the angle between $\ell_2$ and $\ell_0$ is $2\pi/7$. Let $C$ and $A$ be the intersection points of $\ell_2$ with $\ell_0$ and $\ell_1$ respectively. Let $B$ be the reflection of $C$ across the line $\ell_1$; see ( Fig. 4 ).

Lemma 2.1. In the figure above, three sides of the triangle $ABC$ are equal.

Proof. Let $l_2$ go through $C$ and divide $\angle ACB$ equally. Assume $l_2$ meets $l_0$ at $D$. Then the reflection of $A$ across $l_2$ is $B$ (otherwise there would be another point
B₁ on the line l₁ that satisfies \( \angle AB₁C = \frac{2\pi}{7} \). So \( AC = BC \). And therefore \( AB = AC = BC \). 

\[ \square \]

Rotating \( \triangle ABC \) around \( A \) in increment of \( \frac{2\pi}{7} \) gives seven triangles around the point \( A \). Repeating this process at various points will give the tessellation we are looking for.

**Lemma 2.2.** \( \triangle ABC \) and its copies can tessellate the whole plane.

**Proof.** We construct the tessellation from the regular heptagon centered at \( A \), which is formed after \( \triangle ABC \) is rotated with respect to \( A \) by \( \frac{2\pi}{7} \) for 6 times, and we call this original region \( R₀ \). The operation is the following: at each vertex \( E \) on the boundary of the region \( Rₙ \), choose a triangle that includes \( E \), and rotate it around \( E \) by the angle \( \frac{2\pi}{7} \) for seven times. The new region is called \( Rₙ₊₁ \).

We will prove by induction that each \( Rₙ \) is convex. Suppose there are more than three triangles having the same vertex \( M \) on the boundary of \( Rₙ \). Now every triangle involving \( M \) must have another vertex \( T \) on the boundary of \( Rₙ₋₁ \), and each pair \((M, T)\) indicates there are two triangles that include \( M \). Since the region \( Rₙ₋₁ \) is convex, the number of this kind of \( T \) is at most two. So there are four triangles at \( M \) on the boundary of \( Rₙ₋₁ \) respectively. But this can’t be true because \( T₁ \) and \( T₂ \) must be in the same triangle.

Next we claim that there exists \( \epsilon > 0 \) with the following property: for any point \( P \) in \( Rₙ \), and any point \( Q \) such that \( d(P, Q) < \epsilon \), we have \( Q \) in \( Rₙ₊₁ \). Let \( M \) be the
closest vertex to $P$, and $R$ be the smallest regular heptagon centered at $M$. Let $S$ be the set of points that are closer to $M$ than to any other vertices in $R$. Then $\partial S$ is in $R$ and $P$ must be contained in $S$. By compactness, there exists $\varepsilon > 0$ such that for every $T$ in $S$, we have $d(T, \partial R) > \varepsilon$. So $d(P, \partial R_{n+1}) \geq d(P, \partial R) > \varepsilon$. And this proves the claim.

Hence $\inf\{d(M, A)\mid M \in \partial R_n\} > n\varepsilon$. So $\lim_{n \to \infty} R_n = H$. □

We call this tessellation $T$.

In order to calculate the location of the vertices in $T$, the theorem below is needed.

**Theorem 2.3.** Suppose $a, b, c, d \in \mathbb{R}$, the angle $\theta$ between the two lines $ac$ and $bd$ satisfies $\sin \theta = \left| 1 + \frac{3}{y+1} \right|$, in which $y$ is the cross ratio $\frac{(b-a)(d-c)}{(b-c)(d-a)}$, see (Fig. 5).

![Figure 5](image)

**Proof.** Let $A_1$ and $A_3$ be the centers of the two circles, and $A_2$ be the intersection point of the two lines.

Due to either $\theta + \angle A_1A_2A_3 = \frac{\pi}{2}$ or $|\theta - \angle A_1A_2A_3| = \frac{\pi}{2}$,

we have $\sin \theta = |\cos \angle A_1A_2A_3| = \left| \frac{(c-a)^2 + (d-b)^2 - (d-c + b-a)^2}{2(c-a)(d-b)} \right| = \left| 1 + \frac{3}{y+1} \right|$. □

**Calculation 2.4.** The following is the calculation of the coordinates of a vertex in $T$. Assume the coordinates of $A$ are $(0, x \tan(\frac{2\pi}{7}))$. Using the theorem above we have

$$\sin \frac{2\pi}{7} = \left| 1 + \frac{3}{y-1} \right|$$

$$y = \frac{((1 + \frac{1}{\cos \frac{2\pi}{7}})x - 1)((1 - \frac{1}{\cos \frac{2\pi}{7}})x + 1)}{((1 - \frac{1}{\cos \frac{2\pi}{7}})x - 1)((1 + \frac{1}{\cos \frac{2\pi}{7}})x + 1)}$$
The solution of this system of equations is
\[
x = \frac{-(8 \cos \frac{2\pi}{7} + 2 \sin \frac{4\pi}{7}) - \sqrt{(8 \cos \frac{2\pi}{7} + 2 \sin \frac{4\pi}{7})^2 - 9(\sin \frac{4\pi}{7})^2}}{6(\sin \frac{4\pi}{7})^2}
\]
This gives us the location of \( A \).

3. Symmetry group of the tessellation

**Definition 3.1.** Let \( T \) be the tessellation constructed in Lemma 2.2. A symmetry of the hyperbolic plane is said to be a *symmetry of \( T \)* if it preserves the set of vertices in \( T \). The set of all symmetries of \( T \) is a group, called the *symmetry group of \( T \)*, and denoted \( G_1 \).

**Theorem 3.2.** Let \( \alpha : H \to H \) be the counter-clockwise rotation by \( \frac{2\pi}{7} \) at the point \( A \), let \( \beta : H \to H \) be the rotation for angle \( \pi \) at point \( i \), and let \( \gamma : H \to H \) be the reflection with respect to the line \( AB \). Then \( G_1 \) is generated by \( \alpha, \beta \) and \( \gamma \).

**Proof.** Let \( G_2 \) be the group generated by \( \alpha, \beta \), and \( \gamma \), and \( D = \bigcup_{g \in G_2} \triangle ABC \). Let \( S \) be the set of vertices in \( D \).
We claim that \( G_2 \) acts transitively on \( S \).
Since every triangle in the district \( D \) is a copy of \( \triangle ABC \) and \( \alpha(B) = C \), there are at most two orbits of vertices in \( S \). Suppose there are two different orbits of vertices. Denote every point in the same orbit with \( A \) as a red one and the others as a black one. Then every red point must be surrounded by black points (Fig. 6).

In this figure the location of \( \triangle ABC \) is the same as that in Figure 4, and

![Figure 6](image-url)

\( S_2, S_3, S_4, S_5, A, S_6, \) and \( S_7 \) are the closest vertices to \( S_1 \). We have \( S_7 = \alpha^{-2} \gamma \alpha^2(A) \), \( S_4 = \alpha^{-3} \gamma \alpha^3(A) \), so \( S_3, S_6 \) are red, and \( S_1, S_2, S_3 \) are black. Let \( K \) be the middle point of \( S_1 \) and \( S_3 \). Notice that \( K \) can be obtained by \( i \) after \( i \) rotates around \( A \).
for $\frac{6\pi}{7}$ and then rotate around $S_4$ for $\frac{2\pi}{7}$. So $S_4$ can rotate around $K$ for $\pi$ and we get $S_2$. Hence $S_2$ is both red and black. And this means there is only one orbit of vertices.

Therefore, for any vertex $L$ in $D$, we have $w \in G_2$ such that $w(A) = L$. And in the notation of Thm 1.9, the rotation around $L$ by the angle $\frac{2\pi}{7}$ is $w\alpha w^{-1}$. So with the construction in Lemma 2.2, we have $D = H$.

Since $G_2$ acts transitively on the set of vertices, for any $\triangle A_1A_2A_3$, there exists an element $g \in G_1$ that respectively maps $A, B, C$ to $A_1, A_2, A_3$. Hence $G_2 = G_1$.

\[\Box\]

Remark 3.3. With what we have calculated in 2.4, denote the coordinates of $A$ as $(0, a)$. Then in the notation of Theorem 1.9, we have $\alpha = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}$, $\beta(z) = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} r_{\frac{\pi}{7}} \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}(z)$, and $\gamma(z) = -\frac{1}{z}$.

Acknowledgments

It is a pleasure to thank my mentor, Brian Lawrence for he inspired me with the possible paths when I was confused and generously offered me many materials to learn. He also provided me with plenty of helpful comments when I wrote this paper.

4. Bibliography

References
