DIFFERENTIAL GAMES AND VISCOSITY SOLUTIONS

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ABSTRACT. In this paper, we introduce differential games and the theory of viscosity solutions. We show how viscosity solutions can be used to study the Hamilton-Jacobi equation solved by the value function of a game. We also present explicit solutions and verification Theorems for some games.

CONTENTS

1. INTRODUCTION

This paper is largely based on Pierre Cardaliaguet's notes on Differential Games [1]. We work in $\mathbb{R}^N$, where the two-player game is defined by a *path* and a *payoff.*

The path is determined by an initial condition and an ordinary differential equation. This will be the same for all the games we deal with in this paper. The payoff varies, but it also depends on an initial condition, and on the players' strategies.

In this first Section, we introduce the pursuit-evasion and finite-horizon games, and prove some basic results. In Sections 2 and 3, we develop techniques which will be useful to prove more general results. We go back to the finite-horizon problem in Section 4 to prove those results.

Definition 1.1. (Pursuit-Evasion Differential Game) A pursuit-evasion differential game is characterized by the initial condition $x_0 \in \mathbb{R}^N$, and the following system:

$$\begin{cases} X_t' = f(X_t, u_t, v_t) \\ X_0 = x_0, \end{cases} \tag{1.2}$$

where $f : \mathbb{R}^N \times U \times V \to \mathbb{R}^N$ is smooth and bounded, and $X_t : [0, +\infty) \to \mathbb{R}^N$ is not a random variable, but a function which evolves according to f. The functions $u_t : [0, +\infty) \to U$ and $v_t : [0, +\infty) \to V$ are the controls for Players I and II respectively, and U, V are compact metric spaces. For a fixed target set $C \subset \mathbb{R}^N$, the payoff is

$$\mathcal{J}(x_0, u, v) = \inf \{t \geq 0 \mid X_t \in C\} .$$

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The notion of strategy formalizes the idea of a set of controls that each player is allowed to choose. A *feedback strategy* is one in which the player chooses the control according to the time and the current position of the system.

Definition 1.3. (Feedback Strategy) A feedback strategy for Player I is a map $\overline{u} : [0, +\infty) \times \mathbb{R}^N \to U$, and analogously for Player II.

To define a value function, we need a set of strategies for the players.

Definition 1.4. (Admissible Strategies) The sets $(\overline{U},\ \overline{V})$ are *admissible strategies* if

- All Lebesgue measurable maps $\overline{u} : [0, +\infty) \to U$ and $\overline{v} : [0, +\infty) \to V$ are in $\overline{U}$ and $\overline{V}$, respectively.
- For each $(\overline{u},\ \overline{v}) \in \overline{U} \times \overline{V}$ and $x_0 \in \mathbb{R}^N$, (1.2) has a unique solution.
- (shift) For any $u \in \overline{U}$, $\tau > 0$, the map u_τ defined by $u_\tau(t, x) := u(t + \tau, x)$ is also in $\overline{U}$, and analogously for $\overline{V}$.
- (concatenation) For any $u_1, u_2 \in \overline{U}$ and $s > 0$, the map u defined by
$$u(t) = \begin{cases} u_1(t), & \text{if } t \le s \\ u_2(t), & \text{otherwise} \end{cases}$$
is also in $\overline{U}$, and analogously for $\overline{V}$.

Now, we are ready to define our value functions.

Definition 1.5. (Upper Value function) The upper value function for a game with payoff $\mathcal{J}$ is
$$\mathbf{V}^+(x_0) = \inf_{\overline{u} \in \overline{U}} \sup_{\overline{v} \in \overline{V}} \mathcal{J}(x_0, \overline{u}, \overline{v})$$

Definition 1.6. (Lower Value function) The lower value function for a game with payoff $\mathcal{J}$ is
$$\mathbf{V}^-(x_0) = \sup_{\overline{v} \in \overline{V}} \inf_{\overline{u} \in \overline{U}} \mathcal{J}(x_0, \overline{u}, \overline{v})$$

Notice that the definitions of the value functions depend on a fixed pair of admissible strategies.

By properties of sup and inf, we have $\mathbf{V}^- \le \mathbf{V}^+$. The main goal of this paper is to prove the opposite inequality, $\mathbf{V}^+ \le \mathbf{V}^-$, so that $\mathbf{V}^- = \mathbf{V}^+ = \mathbf{V}$. In this case, we say the game has a *value*, which is $\mathbf{V}$. We will show this using the notion of viscosity solutions.

We associate two Hamiltonians to the pursuit-evasion game:
$$H^+(x, p) := \inf_{u \in U} \sup_{v \in V} \left\{ \langle p, f(x, u, v) \rangle \right\}, \text{ for all } (x, p) \in \mathbb{R}^N \times \mathbb{R}^N$$
$$H^-(x, p) := \sup_{v \in V} \inf_{u \in U} \left\{ \langle p, f(x, u, v) \rangle \right\}, \text{ for all } (x, p) \in \mathbb{R}^N \times \mathbb{R}^N$$

Similarly as before, $H^- \le H^+$, but in this case we take the sup and inf with respect to the metric spaces U and V, not the pair of admissible strategies. Let us denote by
$$\tilde{u}(x, p) \in \arg\min_{u \in U} \left\{ \max_{v \in V} \langle p, f(x, u, v) \rangle \right\} \text{ and } \tilde{v}(x, p) \in \arg\max_{v \in V} \left\{ \min_{u \in U} \langle p, f(x, u, v) \rangle \right\}.$$

The following Theorem enables us to check that a value function is indeed the value of the game, provided it satisfies some strong conditions. However, it is often not the case that $\mathbf{V}$ is so well-behaved.

Theorem 1.7. ***(Verification Theorem)*** *Suppose* $\mathbf{V}$ *is a nonnegative* C^1 *function satisfying*

$$H(x, DV(x)) + 1 = 0, \text{ for } x \notin C$$

and $\mathbf{V} = 0$ *on a closed target set* C. *Assume the maps* $x \mapsto \overline{u}^*(x) := \tilde{u}(x, DV(x))$ *and* $x \mapsto \overline{v}^*(x) := \tilde{v}(x, DV(x))$ *belong to* $\overline{U}$ *and* $\overline{V}$. *Then, the game has a value, which is* $\mathbf{V}$. *Moreover,* $\overline{u}^*$ *and* $\overline{v}^*$ *are optimal in the following sense:*

$$\mathbf{V}(x) = \sup_{\overline{v}\in\overline{V}} \mathcal{J}(x, \overline{u}^*, v) = \inf_{\overline{u}\in\overline{U}} \mathcal{J}(x, u, \overline{v}^*) = \mathcal{J}(x, \overline{u}^*, \overline{v}^*)$$

Proof. We will show $\mathbf{V}(x) \geq \sup_{\overline{v}\in\overline{V}} \mathcal{J}(x, \overline{u}^*, \overline{v})$. Fix $x_0 \in \mathbb{R}^N$, $\overline{v} \in \overline{V}$, and denote $X_t^{x_0, \overline{u}^*, \overline{v}}$ by X_t.

$$\begin{aligned}
\frac{d}{dt}\mathbf{V}(X_t) &= \langle D\mathbf{V}(X_t),\ f(X_t, \overline{u}^*, \overline{v})\rangle \\
&\leq \max_{\overline{v}\in\overline{V}} \langle D\mathbf{V}(X_t),\ f(X_t, \overline{u}^*, \overline{v})\rangle \\
&= H(X_t, D\mathbf{V}(X_t)) \\
&= -1
\end{aligned}$$

Let $T = \mathcal{J}(x_0, \overline{u}^*, \overline{v})$. Integrating the above inequality from 0 to $t < T$, we get

$$\mathbf{V}(X_t) - \mathbf{V}(x_0) \leq -t \implies \mathbf{V}(x_0) \ \geq \ t,$$

since $\mathbf{V}$ is nonnegative. So, if $\mathcal{J}(x_0, \overline{u}^*, \overline{v})$ were infinite, $\mathbf{V}(x_0)$ would be unbounded. By continuity of $\mathbf{V}$, we get $\mathbf{V}(x_0) \geq T$.

Hence, $\mathbf{V}(x) \geq \sup_{\overline{v}\in\overline{V}} \mathcal{J}(x, \overline{u}^*, \overline{v})$. The same idea applies in the proof of the opposite inequality $\mathbf{V}(x) \leq \inf_{\overline{u}\in\overline{U}} \mathcal{J}(x, \overline{u}, \overline{v}^*)$.

This completes the proof, since we've shown $\mathbf{V} \leq \mathbf{V}^- \leq \mathbf{V}^+ \leq \mathbf{V}$, and

$$\mathbf{V}(x) \leq \inf_{\overline{u}\in\overline{U}} \mathcal{J}(x, u, \overline{v}^*) \leq \sup_{\overline{v}\in\overline{V}} \mathcal{J}(x, \overline{u}^*, v) \leq \mathbf{V}(x), \text{ for all } x \in \mathbb{R}^N.$$

□

For the remainder of this paper, we will work with a different kind of game, which happens over a fixed time interval, and for which the payoff is a quantity to be maximized or minimized directly.

Definition 1.8. (Bolza problem) The Bolza problem is a finite-horizon game characterized by a terminal time T, an initial condition $(t_0, x_0) \in [0, T] \times \mathbb{R}^N$ and similar dynamics as the previous game:

$$\begin{cases} X_t' = f(t, X_t, u_t, v_t) \\ X_{t_0} = x_0. \end{cases} \tag{1.9}$$

For some fixed pair of admissible strategies $(\overline{U}, \overline{V})$ and $(u, v) \in (\overline{U}, \overline{V})$, we denote the unique solution to (1.9) by $X_t^{t_0, x_0, u, v}$. The payoff is

$$\mathcal{J}(t_0, x_0, u, v) = \int_{t_0}^{T} \ell(s, X_s^{t_0, x_0, u, v}, u_s, v_s)ds + g(X_T^{t_0, x_0, u, v}).$$

The function $\ell : [0,T] \times \mathbb{R}^N \times U \times V \to \mathbb{R}$ is called the *running cost* and the function $g : \mathbb{R}^N \to \mathbb{R}$ is called the *terminal cost*. The payoff for the game is the integral of the running cost over the runtime of the game plus the terminal cost. Furthermore, we normally assume the running and terminal costs are smooth and bounded.

Again, Player I minimizes the payoff while Player II maximizes it. Analogously to the pursuit-evasion game, we define the following value functions.

Definition 1.10. (Upper Value function)

$$\mathbf{V}^+(t_0, x_0) = \inf_{\overline{u}\in\overline{U}} \sup_{\overline{v}\in\overline{V}} \mathcal{J}(t_0, x_0, \overline{u}, \overline{v})$$

Definition 1.11. (Lower Value function)

$$\mathbf{V}^-(t_0, x_0) = \sup_{\overline{v}\in\overline{V}} \inf_{\overline{u}\in\overline{U}} \mathcal{J}(t_0, x_0, \overline{u}, \overline{v})$$

The Hamiltonians associated with the finite-horizon game, for any $(t, x, p) \in [0,T] \times \mathbb{R}^N \times \mathbb{R}^N$, are:

$$H^+(t, x, p) := \inf_{u\in U} \sup_{v\in V} \{\langle p, f(t, x, u, v)\rangle + \ell(t, x, u, v)\} \tag{1.12}$$

$$H^-(t, x, p) := \sup_{v\in V} \inf_{u\in U} \{\langle p, f(t, x, u, v)\rangle + \ell(t, x, u, v)\} \tag{1.13}$$

Again, we work under Isaacs' condition $H^+ = H^- = H$ and assume we have minimizers $\tilde{u}(t, x, p)$ and maximizers $\tilde{v}(t, x, p)$:

$$\begin{aligned} H(t, x, p) &= \sup_{v\in V} \{\langle p, f(t, x, \tilde{u}(t, x, p), v)\rangle + \ell(t, x, \tilde{u}(t, x, p), v)\} \\ &= \inf_{u\in U} \{\langle p, f(t, x, u, \tilde{v}(t, x, p))\rangle + \ell(t, x, u, \tilde{v}(t, x, p))\} \end{aligned}$$

We prove the following verification Theorem for the Bolza problem, and afterwards give motivation for the Hamiltonians and the Theorems.

Theorem 1.14. ***(Verification Theorem)*** *Suppose* $\mathbf{V} : (0,T) \times \mathbb{R}^N \to \mathbb{R}$ *is a* C^1 *function, continuous on* $[0,T] \times \mathbb{R}^N$ *satisfying the following Hamilton-Jacobi equation:*

$$\partial_t \mathbf{V}(t, x) + H(t, x, D\mathbf{V}(t, x)) = 0, \text{ for all } (t, x) \in (0,T) \times \mathbb{R}^N,$$

and a terminal condition $\mathbf{V}(T, x) = g(x)$. *Assume the maps* $x \mapsto \overline{u}^*(t, x) := \tilde{u}(t, x, D\mathbf{V}(t, x))$ *and* $x \mapsto \overline{v}^*(t, x) := \tilde{v}(x, D\mathbf{V}(t, x))$ *belong to* $\overline{U}$ *and* $\overline{V}$. *Then, the game has a value, which is* $\mathbf{V}$. *Moreover,* $\overline{u}^*$ *and* $\overline{v}^*$ *are optimal in the following sense:*

$$\mathbf{V}(t, x) = \sup_{\overline{v}\in\overline{V}} \mathcal{J}(t, x, \overline{u}^*, v) = \inf_{\overline{u}\in\overline{U}} \mathcal{J}(t, x, u, \overline{v}^*) = \mathcal{J}(x, \overline{u}^*, \overline{v}^*)$$

Proof. We will show $\mathbf{V}(t, x) \geq \sup_{v\in\overline{V}} \mathcal{J}(t, x, \overline{u}^*, v)$ for any $(t, x) \in [0,T] \times \mathbb{R}^N$. Fix $\overline{v} \in \overline{V}$ and, for some initial condition $(t_0, x_0) \in [0,T] \times \mathbb{R}^N$, denote $X_t^{t_0, x_0, \overline{u}^*, \overline{v}}$ by X_t.

$$\begin{aligned}\frac{d}{dt}\left[\mathbf{V}(t,X_t)+\int_{t_0}^{t}\ell(t,X_t,\overline{u}^*,\overline{v})\right] &= \partial_t\mathbf{V}(t,X_t)+\langle D\mathbf{V}(t,X_t),\ f(X_t,\overline{u}^*,\overline{v})\rangle \\ &\qquad + \ell(t,X_t,\overline{u}^*,\overline{v}) \\ &\leq \partial_t\mathbf{V}(t,X_t)+\max_{v\in\overline{V}}\{\langle D\mathbf{V}(t,X_t),\ f(X_t,\overline{u}^*,v)\rangle \\ &\qquad\qquad + \ell(t,X_t,\overline{u}^*,v)\} \\ &= \partial_t\mathbf{V}(t,X_t)+H(t,X_t,D\mathbf{V}(t,X_t)) \\ &= 0\end{aligned}$$

Integrating the above inequality from t_1 to t_2, with $0 < t_1 < t_2 < T$, we get

$$\begin{aligned}&\mathbf{V}(t_2,X_{t_2})-\mathbf{V}(t_1,X_{t_1})+\int_{t_1}^{t_2}\ell(t,X_t,\overline{u}^*,\overline{v})\leq 0 \\ \implies\ &\mathbf{V}(t_1,X_{t_1})\geq\mathbf{V}(t_2,X_{t_2})+\int_{t_1}^{t_2}\ell(t,X_t,\overline{u}^*,\overline{v}).\end{aligned}$$

Therefore, if we let $t_1\to t_0$ and $t_2\to T$, by continuity of $\mathbf{V}$ we get

$$\begin{aligned}\mathbf{V}(t_0,x_0) &\geq \int_{t_0}^{T}\ell(t,X_t,\overline{u}^*,\overline{v})+\mathbf{V}(T,X_T) \\ &= \int_{t_0}^{T}\ell(t,X_t,\overline{u}^*,\overline{v})+g(X_T)\end{aligned}$$

Hence, $\mathbf{V}(t,x)\geq\sup_{\overline{v}\in\overline{V}}\mathcal{J}(t,x,\overline{u}^*,\overline{v})$ for all $(t_0,x_0)\in[0,T]\times\mathbb{R}^N$. The same idea applies in the proof of the opposite inequality $\mathbf{V}(t,x)\leq\inf_{\overline{u}\in\overline{U}}\mathcal{J}(t,x,\overline{u},\overline{v}^*)$.

This completes the proof, since we've shown $\mathbf{V}\leq\mathbf{V}^-\leq\mathbf{V}^+\leq\mathbf{V}$, and

$$\mathbf{V}(t,x)\leq\inf_{\overline{u}\in\overline{U}}\mathcal{J}(t,x,u,\overline{v}^*)\leq\sup_{\overline{v}\in\overline{V}}\mathcal{J}(t,x,\overline{u}^*,v)\leq\mathbf{V}(t,x),$$

for all $(t,x)\in[0,T]\times\mathbb{R}^N$.

□

Before examining the Hamiltonians and the Bolza problem closer, we need to introduce more general notions of strategies. These will be useful when we derive and motivate the Hamilton-Jacobi equation for the Bolza problem.

2. Strategies

The concept of a nonanticipative strategy broadens the scope of a differential game, since it allows each player to choose their control conditionally on the other's choice. As the name suggests, the players are not allowed to "look into the future", but they observe each other continuously.

Through the rest of the paper, we will denote by $\mathcal{U}(t_0,t_1)$ the set of bounded Lebesgue measurable maps $u:[t_0,t_1]\to U$. We set $\mathcal{U}(t_0):=\mathcal{U}(t_0,+\infty)$. Alternatively, if the game has a finite horizon T, we set $\mathcal{U}(t_0):=\mathcal{U}(t_0,T)$.

For arbitrary time intervals or those that follow from context, we use simply $\mathcal{U}$, and analogously for Player II. These are called the *open loop controls* for each player.

Definition 2.1. (Nonanticipative Strategy) A nonanticipative strategy for Player I is a measurable map $\alpha : \mathcal{V}(t_0) \to \mathcal{U}(t_0)$ such that for any $t \in [t_0, T]$ and $v_1, v_2 \in \mathcal{V}(t_0)$, if $v_1 = v_2$ almost everywhere on $[t_0, t]$, then $\alpha(v_1) = \alpha(v_2)$ a.e. on $[t_0, t]$.

We denote the set of nonanticipative strategies α for Player I by $\mathcal{A}(t_0)$, and the set of nonanticipative strategies $\beta : \mathcal{U}(t_0) \to \mathcal{V}(t_0)$ for Player II by $\mathcal{B}(t_0)$. Ideally, we would like to say that for each $(\alpha, \beta) \in \mathcal{A}(t_0) \times \mathcal{B}(t_0)$, there is a unique pair $(u, v) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$ such that

$$\alpha(v) = u, \text{ and } \beta(u) = v.$$

However, this is not generally true. This property does hold in the context of *delay strategies*. A delay strategy formalizes the idea that a player is reacting to information with some time delay.

Definition 2.2. (Delay Strategy) A delay strategy for Player I is a map $\alpha : \mathcal{V}(t_0) \to \mathcal{U}(t_0)$ with an associated delay $\tau > 0$, such that for any $t \geq t_0$ and $v_1, v_2 \in \mathcal{V}(t_0)$, if $v_1 = v_2$ almost everywhere on $[t_0, t]$, then $\alpha(v_1) = \alpha(v_2)$ a.e. on $[t_0, t + \tau]$.

We denote the set of delay strategies for Player I and Player II by $\mathcal{A}_d(t_0)$ and $\mathcal{B}_d(t_0)$, respectively. Note that every delay strategy is also a nonanticipative strategy, but the opposite does not necessarily hold.

Lemma 2.3. *For any pair $(\alpha, \beta) \in \mathcal{A}(t_0) \times \mathcal{B}(t_0)$, there is a unique pair of controls $(u, v) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$ such that*

$$\alpha(v) = u, \text{ and } \beta(u) = v \text{ a.e. on } [t_0, +\infty),$$

as long as either α or β is a delay strategy.

Proof. Without loss of generality, assume α is a delay strategy with delay $\tau > 0$. We prove the Lemma by induction, showing that for any $k \in \mathbb{N}$, there is a unique pair of controls $(u_k, v_k) : [t_0, t_0 + k\tau] \to U \times V]$ such that $\alpha(v_k) = u_k$ and $\beta = v_k$ on $[t_0, t_0 + k\tau]$.

For the base case, pick any $v_0 \in \mathcal{V}(t_0)$, set $u_1 := \alpha(v_0)$, and $v_1 := \beta(u_1)$. Note that since α is a delay strategy, $\alpha(v)$ is independent of v on $[t_0, t_0 + \tau]$. Hence, $\alpha(v_1) = \alpha(v_0) = u_1$, $\beta(u_1) = v_1$ on $[t_0, t_0 + \tau]$, and (u_1, v_1) is unique.

Now, assume the result is true for $k \in \mathbb{N}$, that is, there is a unique pair $(u_k, v_k) : [t_0, t_0 + k\tau] \to U \times V$ such that

$$\alpha(v_k) = u_k \text{ and } \beta(u_k) = v_k \text{ a.e. on } [t_0, t_0 + k\tau].$$

Extend v_k arbitrarily on $[t_0, t_0 + (k+1)\tau]$ to some v^*. Set $u_{k+1} = \alpha(v^*)$ and $v_{k+1} = \beta(u_{k+1})$. Since $v^* = v_k$ on $[t_0, t_0 + k\tau]$, we know $\alpha(v_k) = \alpha(v^*)$ on $[t_0, t_0 + (k+1)\tau]$.

Because β is nonanticipative, $u_k = \alpha(v_k)$, and $u_{k+1} = \alpha(v^*)$, this guarantees $\beta(u_k) = \beta(u_{k+1})$ on $[t_0, t_0 + k\tau]$. Hence, $v^* = v_k = v_{k+1}$ a.e. on $[t_0, t_0 + k\tau]$, so $u_{k+1} = \alpha(v^*) = \alpha(v_{k+1})$ a.e. on $[t_0, t_0 + (k+1)\tau]$ because α has delay τ.

So, for any $k \in \mathbb{N}$, we have a unique set of controls (u_k, v_k) such that $\alpha(v_k) = u_k$ and $\beta(u_k) = v_k$ a.e. on $[t_0, t_0 + k\tau]$. By construction, if $j \leq k$, then $(u_j, v_j) = (u_k, v_k)$ a.e. on $[t_0, t_0 + j\tau]$. Then, if we set

$$(u, v) = (u_k, v_k) \text{ on } [t_0, t_0 + k\tau]$$

for every $k \in \mathbb{N}$, we get the desired result.

□

The following Lemma allows us to prove more complex results for the value functions by introducing a different formulation, in terms of delay strategies.

Lemma 2.4. *For a game with payoff $\mathcal{J} : [0, +\infty) \times \mathbb{R}^N \times \mathcal{U}(t_0) \times \mathcal{V}(t_0) \to \mathbb{R}$, the upper and lower value functions*

$$\mathbf{V}^+(t_0, x_0) = \inf_{\alpha \in \mathcal{A}_d(t_0)} \sup_{\beta \in \mathcal{B}_d(t_0)} \mathcal{J}(t_0, x_0, \alpha, \beta)$$

and

$$\mathbf{V}^-(t_0, x_0) = \sup_{\beta \in \mathcal{B}_d(t_0)} \inf_{\alpha \in \mathcal{A}_d(t_0)} \mathcal{J}(t_0, x_0, \alpha, \beta)$$

can be equivalently expressed by

$$\mathbf{V}^+(t_0, x_0) = \inf_{\alpha \in \mathcal{A}_d(t_0)} \sup_{v \in \mathcal{V}(t_0)} \mathcal{J}(t_0, x_0, \alpha(v), v)$$

and

$$\mathbf{V}^-(t_0, x_0) = \sup_{\beta \in \mathcal{B}_d(t_0)} \inf_{u \in \mathcal{U}(t_0)} \mathcal{J}(t_0, x_0, u, \beta(u)).$$

Remark 2.5. When we write $\mathcal{J}(t, x, \alpha, \beta)$, we are implicitly using the result of the previous Lemma, which states that we can replace any $(\alpha, \beta) \in \mathcal{A}_d(t_0) \times \mathcal{B}_d(t_0)$ by some unique $(u, v) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$.

Proof. We will show the first equality. First, note that $\mathcal{V}(t_0) \subset \mathcal{B}_d(t_0)$, so for any $\alpha \in \mathcal{A}_d(t_0)$, we have

$$\sup_{\beta \in \mathcal{B}_d(t_0)} \mathcal{J}(t_0, x_0, \alpha, \beta) \geq \sup_{v \in \mathcal{V}(t_0)} \mathcal{J}(t_0, x_0, \alpha, v).$$

This gives $\mathbf{V}^+(t_0, x_0) \geq \inf_{\alpha \in \mathcal{A}_d(t_0)} \sup_{v \in \mathcal{V}(t_0)} \{\mathcal{J}(t_0, x_0, \alpha(v), v)\}$. Now, fix $\alpha \in \mathcal{A}_d(t_0)$. For any $\beta \in \mathcal{B}_d(t_0)$, we can find a pair $(u, v) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$ such that $\alpha(v) = u$ and $\beta(u) = v$. Hence, we can write $\mathcal{J}(t_0, x_0, \alpha, \beta) = \mathcal{J}(t_0, x_0, \alpha(v), v)$. Taking the sup over $v \in \mathcal{V}(t_0)$, we get

$$\mathcal{J}(t_0, x_0, \alpha, \beta) \leq \sup_{v \in \mathcal{V}(t_0)} \mathcal{J}(t_0, x_0, \alpha(v), v), \text{ for any } \beta \in \mathcal{B}_d(t_0).$$

Now, taking the supremum over $\beta \in \mathcal{B}_d(t_0)$ and then the infimum over $\alpha \in \mathcal{A}_d(t_0)$ on both sides, we get the opposite inequality

$$\mathbf{V}^+(t_0, x_0) \leq \inf_{\alpha \in \mathcal{A}_d(t_0)} \sup_{v \in \mathcal{V}(t_0)} \{\mathcal{J}(t_0, x_0, \alpha(v), v)\}.$$

□

3. Viscosity Solutions

The Hamiltonians and the associated Hamilton-Jacobi equations in the previous verification Theorems might have seemed unmotivated at first. For some games, however, it is possible to derive what their Hamilton-Jacobi equations look like using the dynamic programming property. In this section, we aim to explain where these equations come from and how they relate to viscosity solutions.

Suppose we have the following maximization problem in discrete-time optimal control:

$$\mathbf{V}(x_0) := \max_{\{\alpha_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \mathcal{J}(x_t, \alpha_t), \quad \{\alpha_t\}_{t=0}^{\infty} \subset A,$$

where A is the control set, $\mathcal{J}$ is a given payoff function, β is the discount factor, and x_t obeys an evolution equation such as

$$x_{t+1} = h(x_t, \alpha_t).$$

In this case, the value function $\mathbf{V}$ will generally need to satisfy a dynamic programming property such as the following:

$$\mathbf{V}(x_t) = \max_{\alpha_t \in A} \{\mathcal{J}(x_t, \alpha_t) + \beta \mathbf{V}(x_{t+1})\},$$

Hence, the problem can be simplified into maximizing the sum of only two quantities. Intuitively, at each step, the value function captures all the “value” left in the future.

In some differential games, there is an equivalent dynamic programming property, in which the time increments need not be integers. In the case of the Bolza problem, it has the following form:

$$\mathbf{V}^+(t_0, x_0) = \inf_{\alpha \in \mathcal{A}_d(t_0)} \sup_{v \in \mathcal{V}(t_0)} \left\{ \int_{t_0}^{t_0+h} \ell(s, X_s^{\alpha,v}, \alpha(v)_s, v_s) ds + \mathbf{V}^+(t_0 + h, X_{t_0+h}^{\alpha,v}) \right\}$$

After taking the limit of the above with the time increment $h > 0$ going to zero, we get a *Hamilton-Jacobi equation* for the value function. The details can be found in Cardaliaguet’s notes [1]. The idea is that dynamic programming can be rewritten as

$$\inf_{\alpha} \sup_{v} \left\{ \frac{1}{h} \int_{t_0}^{t_0+h} \ell(s, X_s, \alpha_s, v_s) ds + \frac{\mathbf{V}^+(t_0 + h, X_{t_0+h}^{\alpha,v}) - \mathbf{V}^+(t_0, x_0)}{h} \right\} = 0,$$

for $h \neq 0$. Taking $h \to 0$, the above looks like

$$\partial_t \mathbf{V}^+(t_0, x_0) + \inf_{\alpha} \sup_{v} \{\ell(t_0, x_0, \alpha_0, v_0) + \langle D\mathbf{V}(t_0, x_0), f(t_0, x_0, \alpha_0, v_0)\rangle\} = 0.$$

Recalling the definition of the Hamiltonian for the Bolza problem, this has the form of the verification Theorem proved in Section 1:

$$\partial_t V(t, x) + H(t, x, DV(t, x)) = 0.$$

In general terms, we denote the Hamilton-Jacobi equation for a game by

$$H(x, V(x), DV(x), D^2V(x)) = 0, \tag{3.1}$$

where $DV(x)$ and $D^2V(x)$ are, respectively, the spatial gradient and the Hessian matrix of V at x.

We also assume that $H : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}_N$ is continuous and *elliptic*:

$$H(x, r, p, X) \geq H(X, r, p, Y), \text{ if } X,\ Y \in \mathcal{S}_N, \text{ and } X \leq Y, \tag{3.2}$$

where $\mathcal{S}_N$ denotes the set of symmetric $N \times N$ matrices. Recall that for $P \in \mathcal{S}_N$, we say $P \geq 0$ if and only if

$$\langle Pv, v\rangle \geq 0 \text{ for all } v \in \mathbb{R}^N.$$

The Hamilton-Jacobi equation might have no smooth solutions, which is the case even for relatively simple games. For instance, the evolution equation

$$
\begin{cases} \partial_t V(t,x) + H(x, DV(t,x)) = 0, \text{ for } (t,x) \in (0,\infty) \times \mathbb{R}^N \\ V(0,x) = g(x) \end{cases} \tag{3.3}
$$

in general has no smooth solutions, as argued by L. C. Evans in [2]. However, if we modify the equation to be

$$
\begin{cases} V_t(t,x) + H(x, DV(t,x)) - \epsilon \Delta V = 0, \text{ for } (t,x) \in (0,\infty) \times \mathbb{R}^N \\ V(0,x) = g(x), \end{cases}
$$

then a smooth solution exists. Hopefully, taking $\epsilon \to 0$ gives us some kind of solution to (3.3). However, such a solution is continuous but not necessarily differentiable (see Evans [2] for a detailed proof), so it might not make sense to talk about its gradient.

The following formulation, which is due to Michael Crandall and Pierre-Louis Lions, expands the possible set of solutions to a Hamilton-Jacobi equation. It allows for continuous functions that are not differentiable. These are called *viscosity solutions.*

This extension to C^0 functions is consistent, as one can check that a (smooth) classical solution is also a solution in the viscosity sense. We begin by defining sub- and supersolutions.

Definition 3.4. (Viscosity subsolution) A function u is called a viscosity subsolution of equation (3.1) if it is upper semicontinuous (u.s.c.), and if, for every C^2 function ϕ such that $u - \phi$ has a local maximum at x, we have

$$H(x, u(x), D\phi(x), D^2\phi(x)) \geq 0$$

Definition 3.5. (Viscosity supersolution) A function u is a viscosity supersolution of (3.1) if it is lower semicontinuous (l.s.c.) and if, for every C^2 function ϕ such that $u - \phi$ has a local minimum at x, we have

$$H(x, u(x), D\phi(x), D^2\phi(x)) \leq 0$$

Definition 3.6. (Viscosity solution) A function u is a viscosity solution of (3.1) if it is both a subsolution and a supersolution.

We have the following Lemma for viscosity solutions:

Lemma 3.7. *In the definition of viscosity subsolution, we can replace "local maximum" by "strict local maximum".*

We omit the proof, which goes by choosing an appropriate modification to each test function (see Cardaliaguet [1] for details).

Theorem 3.8. *A function u is a viscosity subsolution to* (3.1) *if and only if u is upper semicontinuous, and for every C^2 function ϕ such that $u(x_0) = \phi(x_0)$ and $u \leq \phi$ in a neighborhood of x_0, we have*

$$H(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \geq 0,$$

and symmetrically for supersolutions.

Proof. First, assume u is a viscosity subsolution to (3.1). Then, u is u.s.c., and if $\phi \in C^2$ is such that $u \leq \phi$ on a neighborhood of x_0, and $u(x_0) = \phi(x_0)$, then $u - \phi$ achieves a maximum at x_0, and so we get the result by definition of subsolution.

Now, assume the latter part of the statement. For every $\phi \in C^2$ such that $u - \phi$ achieves a local maximum at x_0, let $\tilde{\phi} = \phi + [u(x_0) - \phi(x_0)]$. We have $u(x_0) = \tilde{\phi}(x_0)$, so the result holds for $\tilde{\phi}$, and because $D\tilde{\phi} = D\phi$, we get

$$H(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \geq 0,$$

hence ϕ is a viscosity subsolution to (3.1).

□

Viscosity subsolutions and supersolutions are also stable under half-relaxed limits, which are an analogous version of Gamma-convergence in variational calculus.

Lemma 3.9. *If f is an upper semicontinuous function which achieves a strict maximum at x_0, and $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of u.s.c. functions such that, for every x,*

$$f(x) = \limsup_{z_n \to x,\ n \to \infty} f_n(z_n),$$

then there is a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ and a sequence $\{x_k\}_{k\in\mathbb{N}}$ such that x_k is a maximum for f_{n_k}, $x_k \to x_0$, and $f_{n_k}(x_k) \to f(x_0)$ as $k \to +\infty$.

Proof. Since x_0 is a strict maximum for f, which is u.s.c., we know there is some $r > 0$ such that the following hold:

$$f(x_0) > \max_{\overline{B_r(x_0)} \setminus \{x_0\}} f(x),$$

and, in particular,

$$f(x_0) = \max_{\overline{B_r(x_0)}} f(x) > \max_{\partial B_r(x_0)} f(x).$$

Consider the sequence $\{x_n\}_{n\in\mathbb{N}}$, where we define

$$x_n := \max_{\overline{B_r(x_0)}} f_n(x).$$

By definition of x_n, we have

$$\limsup_{n\to\infty} f_n(x_n) \geq \limsup_{n\to\infty} f_n(z_n),$$

for any sequence $\{z_n\}_{n\in\mathbb{N}}$. By compactness of the ball and general facts about sequences, we also know there is a subsequence $\{x_{n_k}\}$ satisfying

$$\limsup_{n\to\infty} f_n(x_n) = \lim_{n\to\infty} f_{n_k}(x_{n_k}), \text{ and } x_{n_k} \to \overline{x},$$

Hence, by definition of f, we get

$$f(\overline{x}) \geq \limsup_{n\to\infty} f_n(x_n) \geq \limsup_{z_n \to x_0, n\to\infty} f_n(z_n) = f(x_0),$$

Therefore, $\overline{x} = x_0$, since x_0 is a strict local maximum for f. So, x_0 is a cluster point of $\{x_n\}$, and thus there is a subsequence $\{x_{n_k}\}$ for which f_{n_k} attains a maximum at x_{n_k}, $x_{n_k} \to x_0$, and $f_{n_k}(x_{n_k}) \to f(x_0)$. □

Theorem 3.10. *Let H and $\{H_n\}_{n\in\mathbb{N}}$ be continuous Hamiltonians. Suppose $\{\mathbf{V}_n\}$ is a uniformly bounded sequence such that each V_n is a viscosity subsolution to*

$$H_n(x, W(x), DW(x), D^2W(x)) = 0,$$

with $H_n \to H$ locally uniformly as $n \to \infty$ Then, the ***upper half-relaxed limit*** *$\mathbf{V}^*$ of $\{\mathbf{V}_n\}$, defined by*

$$\mathbf{V}^*(x) = \limsup_{x_n \to x, n\to\infty} \mathbf{V}_n(x_n),$$

is also a subsolution to (3.1).

Proof. Let ϕ be a C^2 function such that $\mathbf{V} - \phi$ has a maximum at x_0. We want to show

$$H(x, \mathbf{V}(x), D\phi(x), D^2\phi(x)) \geq 0.$$

By the previous Lemma, we know there is a sequence $\{x_k\}$ such that each x_k is a maximum for $\mathbf{V}_{n_k} - \phi$ and $x_k \to x_0$ as $k \to \infty$. So, we have

$$H_{n_k}(x_k, \mathbf{V}_{n_k}(x_k), D\phi(x_k), D^2\phi(x_k)) \geq 0.$$

Because $H_{n_k} \to H$ locally uniformly, $x_k \to x_0$, $\mathbf{V}_{n_k}(x_k) \to \mathbf{V}^*(x_0)$, and $\phi \in C^2$, we get

$$H(x_0, \mathbf{V}^*(x_0), D\phi(x_0), D^2\phi(x_0)) = \lim_{k\to\infty} H_{n_k}(x_k, \mathbf{V}_{n_k}(x_k), D\phi(x_k), D^2\phi(x_k)) \geq 0$$

So, $\mathbf{V}^*$ is a subsolution to (3.1). □

4. The finite-horizon problem

In this section, we will prove further results about the finite-horizon game. We recall the definition of the Bolza problem.

Definition 4.1. (Bolza Problem) The Bolza problem is a finite-horizon differential game where the dynamics are, for some fixed initial condition $(t_0, x_0) \in [0, T] \times \mathbb{R}^N$,

$$\begin{cases} X_t' = f(t, X_t, u_t, v_t) \\ X_{t_0} = x_0, \end{cases} \tag{4.2}$$

where $(u, v) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$ and U, V are compact metric spaces.

We assume f is bounded and Lipschitz with respect to the space and time variables. This guarantees that for every $(u, v) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$, (4.2) has a unique solution, which we denote by $X_t^{t_o, x_0, u, v}$. The payoff $\mathcal{J}$ is defined by

$$\mathcal{J}(t_0, x_0, u, v) = \int_{t_0}^{T} \ell(s, X_s^{t_0, x_0, u, v}, u_s, v_s) ds + g(X_T^{t_0, x_0, u, v}).$$

We assume that ℓ is continuous, bounded, and uniformly Lipschitz with respect to the space variable. We also assume g is a bounded Lipschitz continuous function.

In this setup, Player I *minimizes* the payoff, while Player II *maximizes* it. We formalize this by defining the upper and lower value functions.

Definition 4.3. (Upper Value Function) The upper value function for the Bolza problem is

$$\mathbf{V}^+(t_0, x_0) := \inf_{\alpha \in \mathcal{A}_d(t_0)} \sup_{\beta \in \mathcal{B}_d(t_0)} \mathcal{J}(t_0, x_0, \alpha, \beta) \tag{4.4}$$

Definition 4.5. (Lower Value Function) The lower value function for the Bolza problem is

$$\mathbf{V}^-(t_0, x_0) := \sup_{\beta \in \mathcal{B}_d(t_0)} \inf_{\alpha \in \mathcal{A}_d(t_0)} \mathcal{J}(t_0, x_0, \alpha, \beta) \tag{4.6}$$

In the interest of brevity, we work with the upper value function for the most part, since its properties can be extended to the lower value function due to the following Lemma.

Lemma 4.7. *If* $\mathbf{V}^-$ *is the lower value function of a Bolza problem with running cost* ℓ *and terminal cost* g*, then* $\mathbf{V}^- = -\tilde{\mathbf{V}}^+$*. Here,* $\tilde{\mathbf{V}}^+$ *is the upper value function for a Bolza problem with running cost* $-\ell$ *and terminal cost* $-g$*, where Player I is in the role of Player II in the previous game, and conversely.*

One can check that this follows from properties of supremum and infimum. Our goal is to prove the dynamic programming property for this game. First, we show that $\mathbf{V}^+$ is uniformly Lipschitz in the space variable with respect to time by using the following Lemma:

Lemma 4.8. *If* $f, g : A \times B \to \mathbb{R}$ *are functions, where* A *and* B *are arbitrary sets, and we know*

$$\sup_{a\in A,\ b\in B} |f(a,b) - g(a,b)| \leq K,$$

for some constant $K \geq 0$*, then*

$$|\inf_{a\in A} \sup_{b\in B} f(a,b) - \inf_{a\in A} \sup_{b\in B} g(a,b)| \leq K,$$

as long as one of the $\inf\sup$ *is finite.*

Proof. By assumption, we know

$$f(a,b) \leq K + g(a,b), \text{ for all } (a,b) \in A \times B$$

So, taking the supremum over $b \in B$, and then the infimum over $a \in A$ on both sides, we get

$$\inf_{a\in A} \sup_{b\in B} f(a,b) - \inf_{a\in A} \sup_{b\in B} g(a,b) \leq K,$$

and the same trick works for the other inequality, which shows both $\inf\sup$ are finite. □

Theorem 4.9. *There exists a constant* $C > 0$ *such that, for all* $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$ *and any* $t \in [0,T]$*, we have*

$$|\mathbf{V}^+(t,x) - \mathbf{V}^+(t,y)| \leq C\|x-y\|$$

Proof. Fix $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, $t_0 \in [0,T]$, and $(u,v) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$. Denoting $X_s^{t_0,z_0,u,v}$ by $X_s^{z_0}$ for any z_0, our goal is to estimate

$$|\mathcal{J}(t_0,x_0,u,v) - \mathcal{J}(t_0,y_0,u,v)| =$$

$$= \left| \int_{t_0}^T \ell(s, X_s^{x_0}, u_s, v_s) - \ell(s, X_s^{y_0}, u_s, v_s) ds + g(X_T^{x_0}) - g(X_T^{y_0}) \right| \tag{4.10}$$

$$\leq \int_{t_0}^T |\ell(s, X_s^{x_0}, u_s, v_s) - \ell(s, X_s^{y_0}, u_s, v_s)| ds + |g(X_T^{x_0}) - g(X_T^{y_0})| \tag{4.11}$$

Because f is uniformly Lipschitz with respect to the space variable, we have the following:

$$|X_t^{x_0} - X_t^{y_0}| = |x_0 - y_0 + \int_0^t f(s, X_s^{x_0}, u_s, v_s) - f(s, X_s^{y_0}, u_s, v_s) ds|$$

$$\leq |x_0 - y_0| + \int_{t_0}^t C_f |X_s^{x_0} - X_s^{y_0}| ds$$

So, by Gronwall's Inequality, we get

$$|X_t^{x_0} - X_t^{y_0}| \leq |x_0 - y_0| e^{C_f t}$$

Using in (4.11) the Lipschitz estimates on ℓ and g, we get

$$\begin{aligned}|\mathcal{J}(t_0,x_0,u,v)-\mathcal{J}(t_0,y_0,u,v)| &\leq \int_{t_0}^{T} C_\ell |X_s^{x_0}-X_s^{y_0}|ds + C_g|X_T^{x_0}-X_T^{y_0}| \\ &\leq C_\ell|x_0-y_0|\int_{t_0}^{T} e^{C_f s}ds + C_g|x_0-y_0|e^{C_f T} \\ &= K|x_0-y_0|,\end{aligned}$$

for some constant K. Since this is true for every $(u,v)\in\mathcal{U}(t_0)\times\mathcal{V}(t_0)$, we have

$$\sup_{(u,v)\in\mathcal{U}(t_0)\times\mathcal{V}(t_0)} |\mathcal{J}(t_0,x_0,u,v)-\mathcal{J}(t_0,y_0,u,v)| \leq K|x_0-y_0|,$$

and so by Lemma (4.8), we get the result. □

Lipschitz continuity of $\mathbf{V}^+$ allows us to prove the following result, which states that near-optimal strategies at a point remain near-optimal in a neighborhood of that point.

Lemma 4.12. *Fix $(t_0,x_0)\in[0,T]\times\mathbb{R}^N$. Suppose $\alpha\in\mathcal{A}_d$ is an ϵ-optimal strategy for $\mathbf{V}^+$, that is,*

$$\sup_{v\in\mathcal{V}(t_0)} \mathcal{J}(t_0,x_0,\alpha(v),v) \leq \mathbf{V}^+(t_0,x_0)+\epsilon$$

Then, there is some $\delta>0$ such that if $y_0\in B(x_0,\delta)$, α is a 2ϵ-optimal strategy for $V(t_0,y_0)$.

Proof. We want to show

$$\sup_{v\in\mathcal{V}(t_0)} \mathcal{J}(t_0,y_0,\alpha(v),v) \leq \mathbf{V}^+(t_0,y_0)+2\epsilon$$

In the previous Lemma, we showed that $\mathcal{J}$ is also uniformly Lipschitz in the space variable. Hence, fixing $v\in\mathcal{V}(t_0)$, we have

$$\begin{aligned}|\mathcal{J}(t_0,y_0,\alpha(v),v)-\mathcal{J}(t_0,x_0,\alpha(v),v)| &\leq K|x_0-y_0| \\ \implies \mathcal{J}(t_0,y_0,\alpha(v),v) &\leq \mathcal{J}(t_0,x_0,\alpha(v),v)+K|x_0-y_0|\end{aligned}$$

for some constant K. So, using the ϵ-optimality of α, we get

$$\begin{aligned}\mathcal{J}(t_0,y_0,\alpha(v),v) &\leq \mathbf{V}^+(t_0,x_0)+\epsilon+K|x_0-y_0| \\ \mathcal{J}(t_0,y_0,\alpha(v),v) &\leq \mathbf{V}^+(t_0,y_0)+\epsilon+2K|x_0-y_0|,\end{aligned}$$

using Lipschitz continuity of $\mathbf{V}^+$. We get the result by choosing $\delta=\frac{\epsilon}{4K}$. □

Now, we are finally able to prove the dynamic programming property.

Theorem 4.13. ***(Dynamic Programming Property)***
Suppose we have the previous assumptions on f, ℓ, and g. Fix $(t_0,x_0)\in[0,T]\times\mathbb{R}^N$, and let $0<h<T$. Then, the following holds:

(4.14)
$$\mathbf{V}^+(t_0,x_0)=\inf_{\alpha\in\mathcal{A}_d}\sup_{v\in\mathcal{V}}\left\{\int_{t_0}^{t_0+h}\ell(s,X_s^{\alpha,v},\alpha(v)_s,v_s)ds+\mathbf{V}^+(t_0+h,X_{t_0+h}^{\alpha,v})\right\},$$

where we denote $X_t^{t_0,x_0,\alpha(v),v}$ by $X_t^{\alpha,v}$ for any $(\alpha,v)\in\mathcal{A}_d(t_0)\times\mathcal{V}(t_0)$.

Proof. Let us denote the right-hand side of (4.14) by $W(t_0, x_0)$. First, we show $\mathbf{V}^+ \leq W + \epsilon$ for all $\epsilon > 0$, and so $\mathbf{V}^+ \leq W$. Fix $\epsilon > 0$, and let α be an ϵ-optimal strategy for W, that is,

$$\sup_{v \in \mathcal{V}(t_0)} \left\{ \int_{t_0}^{t_0+h} \ell(s, X_s^{\alpha,v}, \alpha(v)_s, v_s) ds + \mathbf{V}^+(t_0 + h, X_{t_0+h}^{\alpha,v}) \right\} \leq W(t_0, x_0) + \epsilon$$

Since f is bounded, X_t remains inside a compact region K for any $(t, \alpha, v) \in [0, T] \times \mathcal{A}_d \times \mathcal{V}$. Therefore, choosing δ as in the previous Lemma, we know $\{B(x, \frac{\delta}{2})\}_{x \in \mathbb{R}^N}$ is an open cover of K, and so we can find a finite subcover $\{B(x_j, \frac{\delta}{2})\}_{j=1}^n$.

Let α_j be an ϵ-optimal strategy for $\mathbf{V}^+(t_0 + h, x_j)$. For any $y \in K$, we know $y \in B(x_j, \frac{\delta}{2})$ for some j, and so α_j is a 2ϵ-optimal strategy for $\mathbf{V}^+(t_0 + h, y)$. Let $\mathcal{O}_j := B(x_j, \frac{\delta}{2}) \setminus \bigcup_{i<j} \mathcal{O}_i$. Now, define a new strategy $\overline{\alpha}$ by

$$\overline{\alpha}(v)_s = \begin{cases} \alpha(v)_s, \text{ if } s \in [0, h] \\ \alpha_j(v)_s, \text{ if } s \in (h, T] \text{ and } X_{h-\tau}^{\alpha,v} \in \mathcal{O}_j, \end{cases}$$

where we choose $\tau > 0$ in a way that guarantees us that $X_h^{\alpha,v}$ is also in $B(x_j, \delta)$: $\tau < \min\{h, \frac{\delta}{2\|f\|_\infty}\}$. For τ small enough, we can assume it is a delay common to α and $\{\alpha_j\}_{j=1}^n$.

To show that $\overline{\alpha}$ is a delay strategy, suppose $v_1 = v_2$ a.e. on $[t_0, s]$. If $s \leq t_0 + h - \tau$, we know $\overline{\alpha}(v_1) = \alpha(v_1) = \alpha(v_2) = \overline{\alpha}(v_2)$ a.e. on $[t_0, s + \tau]$ because α is a delay strategy.

If $s \geq t_0 + h - \tau$, we know $X_{t_0+h}^{\alpha(v_1),v_1} = X_{t_0+h}^{\alpha(v_2),v_2} \in \mathcal{O}_j$, for some j. Thus, $\overline{\alpha}(v_1) = \alpha(v_1) = \alpha(v_2) = \overline{\alpha}(v_2)$ a.e. on $[t_0, t_0 + h]$, and $\overline{\alpha}(v_1) = \alpha_j(v_1) = \alpha_j(v_2) = \overline{\alpha}(v_2)$ a.e. on $[t_0 + h, s + \tau]$ since α_j is a delay strategy.

With this choice of $\overline{\alpha}$, we get, for any fixed $v \in \mathcal{V}(t_0)$,

$$\mathcal{J}(t_0, x_0, \overline{\alpha}(v), v) =$$

$$\int_{t_0}^{t_0+h} \ell(s, X_s^{\alpha,v}, \alpha(v)_s, v_s) ds + \int_{t_0+h}^{T} \ell(s, X_s^{\alpha_j,v}, \alpha_j(v)_s, v_s) ds + g(X_T^{\overline{\alpha},v}),$$

for some j depending on v.

Because α_j is 2ϵ-optimal for $\mathbf{V}^+(t_0 + h, X_{t_0+h}^{\alpha,v})$ and α is ϵ-optimal for $W(t_0, x_0)$, we have

$$\int_{t_0}^{t_0+h} \ell(s, X_s^{\alpha,v}, \alpha(v)_s, v_s) ds + \int_{t_0+h}^{T} \ell(s, X_s^{\alpha_j,v}, \alpha_j(v)_s, v_s) ds + g(X_T^{\overline{\alpha},v})$$

$$\begin{aligned} &\leq \int_{t_0}^{t_0+h} \ell(s, X_s^{\alpha,v}, \alpha(v)_s, v_s) ds + \mathbf{V}^+(t_0 + h, X_{t_0+h}^{\alpha,v}) + 2\epsilon \\ &\leq \sup_{v \in \mathcal{V}} \left\{ \int_{t_0}^{t_0+h} \ell(s, X_s^{\alpha,v}, \alpha(v)_s, v_s) ds + \mathbf{V}^+(t_0 + h, X_{t_0+h}^{\alpha,v}) \right\} + 2\epsilon \\ &\leq W(t_0, x_0) + 3\epsilon \end{aligned}$$

Therefore, for any $v \in \mathcal{V}(t_0)$, the following holds:

$$\mathcal{J}(t_0, x_0, \overline{\alpha}(v), v) \leq W(t_0, x_0) + 3\epsilon,$$

so taking the supremum over $v \in \mathcal{V}(t_0)$, we get

$$\begin{aligned}\mathbf{V}^+(t_0, x_0) = \inf_{\alpha \in \mathcal{A}_d} \sup_{v \in \mathcal{V}} \mathcal{J}(t_0, x_0, \alpha(v), v) &\le \sup_{v \in \mathcal{V}} \mathcal{J}(t_0, x_0, \overline{\alpha}(v), v) \\ &\le W(t_0, x_0) + 3\epsilon\end{aligned}$$

This proves the first inequality. Now, we show $W \le \mathbf{V}^+$. Fix $\epsilon > 0$ and let α be ϵ-optimal for $\mathbf{V}^+(t_0, x_0)$, that is,

$$\sup_{v \in \mathcal{V}(t_0)} \mathcal{J}(t_0, x_0, \alpha(v), v) \le \mathbf{V}^+(t_0, x_0) + \epsilon$$

Fix $v_0 \in \mathcal{V}(t_0)$. Define a new strategy $\alpha^{v_0} \in \mathcal{A}_d(t_0 + h)$ by

$$\alpha^{v_0}(v)_t = \alpha(\overline{v})_t, \text{ where } v \in \mathcal{V}(t_0 + h) \text{ and } \overline{v}_s = \begin{cases} v_{0_s}, & \text{if } s \in [t_0, t_0 + h] \\ v_s, & \text{otherwise} \end{cases}$$

Notice that the subscript t in the above ranges from $t_0 + h$ to T, since α^{v_0} is in $\mathcal{A}_d(t_0 + h)$. Even though the players are not allowed to look into the future in the context of delay strategies, we do not have a *Markovian* property.

That is, if two controls v_1 and v_2 agree on an interval $[t,\ \tau]$ with $t > t_0$, it is not necessarily true that $\alpha(v_1) = \alpha(v_2)$ almost everyhere on $[t,\ \tau]$, if $\alpha \in \mathcal{A}_d(t_0)$. Because we do not have this Markovian property, we need the above definition of a new strategy.

With this strategy, we have

$$\mathbf{V}^+(t_0 + h, X^{\alpha, v_0}_{t_0+h}) \le \sup_{v \in \mathcal{V}(t_0+h)} \mathcal{J}(t_0 + h, X^{\alpha, v_0}_{t_0+h}, \alpha^{v_0}(v), v) \tag{4.15}$$

Denote the set of controls $v \in \mathcal{V}(t_0)$ such that $v = v_0$ a.e. on $[t_0, t_0 + h]$ by $\mathcal{V}(t_0, t_0 + h, v_0)$. For any $v \in \mathcal{V}(t_0, t_0 + h, v_0)$, we get

$$\mathcal{J}(t_0, x_0, \alpha(v), v) =$$

$$= \int_{t_0}^{t_0+h} \ell(s, X^{\alpha, v_0}_s, \alpha(v_0)_s, v_{0_s}) ds + \int_{t_0+h}^{T} \ell(s, X^{\alpha, v}_s, \alpha^{v_0}(v^h)_s, v^h_s) ds + g(X^{\alpha, \overline{v}}_T)$$

$$= \int_{t_0}^{t_0+h} \ell(s, X^{\alpha, v_0}_s, \alpha(v_0)_s, v_{0_s}) ds + \mathcal{J}(t_0 + h, X^{\alpha, v_0}_{t_0+h}, \alpha^{v_0}(v^h), v^h),$$

where $v^h = v|_{[t_0+h, T]}$.

Taking the sup on the above yields

$$\begin{aligned}\sup_{v \in \mathcal{V}(t_0, t_0+h, v_0)} \mathcal{J}(t_0, x_0, \alpha(v), v) = &\int_{t_0}^{t_0+h} \ell(s, X^{\alpha, v_0}_s, \alpha(v_0)_s, v_{0_s}) ds \\ &+ \sup_{v \in \mathcal{V}(t_0+h)} \mathcal{J}(t_0 + h, X^{\alpha, v_0}_{t_0+h}, \alpha^{v_0}(v), v)\end{aligned}$$

Combining this equality with (4.15), we get

$$\int_{t_0}^{t_0+h} \ell(s, X^{\alpha, v_0}_s, \alpha(v_0)_s, v_{0_s}) ds + \mathbf{V}^+(t_0 + h, X^{\alpha, v_0}_{t_0+h})$$

$$\leq \sup_{v\in\mathcal{V}(t_0,t_0+h,v_0)} \mathcal{J}(t_0,x_0,\alpha(v),v)$$
$$\leq \sup_{v\in\mathcal{V}(t_0)} \mathcal{J}(t_0,x_0,\alpha(v),v)$$
$$\leq \mathbf{V}^+(t_0,x_0)+\epsilon$$

Since this is true for any $v_0 \in \mathcal{V}(t_0)$, the following holds:

$$\sup_{v\in\mathcal{V}} \left\{ \int_{t_0}^{t_0+h} \ell(s, X_s^{\alpha,v_0}, \alpha(v_0)_s, v_{0_s})ds + \mathbf{V}^+(t_0+h, X_{t_0+h}^{\alpha,v_0}) \right\} \leq \mathbf{V}^+(t_0,x_0)+\epsilon$$

This gives the result since ϵ was arbitrary and $W(t_0,x_0)$ is the inf of the left-hand side with respect to α in $\mathcal{A}_d(t_0)$. □

The next Theorem is the reason we introduced the theory of viscosity solutions in Section 3. Comparison principles are crucial for proving the existence of the value function for some games.

Theorem 4.16. ***(Comparison Principle for Bounded Domains)*** *Let $H : \mathcal{O}\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ be continuous, with $\mathcal{O}\subset\mathbb{R}^N$ a bounded open set, $\mathbf{V}_1 : \overline{\mathcal{O}} \to \mathbb{R}$ be a subsolution, and $\mathbf{V}_2 : \overline{\mathcal{O}} \to \mathbb{R}$ be a supersolution of*

$$H(x, V(x), DV(x)) = 0, \text{ for all } x \in \mathcal{O}. \tag{4.17}$$

Suppose that for any $(r,s)\in\mathbb{R}\times\mathbb{R}$ and $(x,p)\in\mathcal{O}\times\mathbb{R}$, we have

$$H(x,r,p) - H(x,s,p) \leq -\gamma(r-s), \text{ if } r \geq s,$$

for some constant $\gamma > 0$. Assume H also satisfies the following:

$$|H(x,r,p) - H(y,r,p)| \leq C(1+\|p\|)\|x-y\|.$$

With these assumptions, if $\mathbf{V}_1 \leq \mathbf{V}_2$ on $\partial\mathcal{O}$, then $\mathbf{V}_1 \leq \mathbf{V}_2$ in $\mathcal{O}$.

Proof. We want to show $\sup_{x\in\overline{\mathcal{O}}}\{(\mathbf{V}_1-\mathbf{V}_2)(x)\} \leq 0$. Because both $\mathbf{V}_1$ and $-\mathbf{V}_2$ are upper semicontinuous, we know $\mathbf{V}_1-\mathbf{V}_2$ achieves a maximum on the compact set $\overline{\mathcal{O}}$, so we can replace the sup by max.

Assume for contradiction that

$$0 < M := \max_{x\in\overline{\mathcal{O}}}\{\mathbf{V}_1(x)-\mathbf{V}_2(x)\} := \mathbf{V}_1(x_0)-\mathbf{V}_2(x_0).$$

Note that this also implies $x_0 \in \mathcal{O}$. Now, we introduce the *doubling variable technique.* For any $\epsilon > 0$, define a new function $\psi_\epsilon : \overline{\mathcal{O}}\times\overline{\mathcal{O}} \to \mathbb{R}$ by setting

$$\psi_\epsilon(x,y) := \mathbf{V}_1(x) - \mathbf{V}_2(y) - \frac{1}{2\epsilon}\|x-y\|^2.$$

The function ψ_ϵ is also u.s.c., so let

$$M_\epsilon := \max_{(x,y)\in\overline{\mathcal{O}}\times\overline{\mathcal{O}}}\{\psi_\epsilon(x,y)\} := \psi_\epsilon(x_\epsilon,y_\epsilon).$$

Because $\mathbf{V}_1$ and $-\mathbf{V}_2$ are upper semicontinuous functions defined on a compact set, they are bounded above by some constant K, and so, for any $\epsilon > 0$, M_ϵ is bounded above by $2K$. Since M_ϵ is nondecreasing with ϵ, we know $\lim_{\epsilon\to 0^+} M_\epsilon$ exists.

$$0 < M \leq M_\epsilon = \mathbf{V}_1(x_\epsilon) - \mathbf{V}_2(y_\epsilon) - \frac{1}{2\epsilon}\|x_\epsilon - y_\epsilon\|^2 \leq 2K$$

This shows $\lim_{\epsilon\to 0^+}\|x_\epsilon - y_\epsilon\| = 0$. Now, let $(\overline{x},\ \overline{x})$ be a cluster point of $\{(x_{\epsilon_n}, y_{\epsilon_n})\}_{n\in\mathbb{N}}$ for some sequence with $\epsilon_n \to 0^+$ as $n\to\infty$. using upper semi-continuity, we get

$$\begin{aligned} 0 < M \leq \lim_{n\to\infty} M_{\epsilon_n} &\leq \liminf_{n\to\infty} \mathbf{V}_1(x_{\epsilon_n}) - \mathbf{V}_2(y_{\epsilon_n}) \\ &\leq \limsup_{n\to\infty} \mathbf{V}_1(x_{\epsilon_n}) - \mathbf{V}_2(y_{\epsilon_n}) \\ &= \mathbf{V}_1(\overline{x}) - \mathbf{V}_2(\overline{x}) \leq M \end{aligned}$$

So, any limit point of $\{(x_{\epsilon_n}, y_{\epsilon_n})\}_{n\in\mathbb{N}}$ must be a maximum of $\mathbf{V}_1 - \mathbf{V}_2$. This guarantees we have no cluster points on $\partial\mathcal{O}$, since no maximum can occur on $\partial\mathcal{O}$.

Moreover, $\lim_{\epsilon\to 0^+} M_\epsilon = \lim_{\epsilon\to 0^+} \mathbf{V}_1(x_\epsilon) - \mathbf{V}_2(y_\epsilon) = M$, and so

$$\lim_{\epsilon\to 0^+} \frac{\|x - y_\epsilon\|^2}{\epsilon} = \lim_{\epsilon\to 0^+} \mathbf{V}_1(x_\epsilon) - \mathbf{V}_2(y_\epsilon) - M = 0$$

For any $\epsilon > 0$, notice that the map $x \mapsto \mathbf{V}_1(x) - \mathbf{V}_2(y_\epsilon) - \frac{1}{2\epsilon}\|x - y_\epsilon\|^2$ has a maximum at x_ϵ. So, because $\mathbf{V}_1$ is a subsolution of (4.17), and letting $\phi(x) = \mathbf{V}_2(y_\epsilon) + \frac{1}{2\epsilon}\|x - y_\epsilon\|^2$, we have

$$H(x_\epsilon, \mathbf{V}_1(x_\epsilon), D\phi(x_\epsilon)) = H(x_\epsilon, \mathbf{V}_1(x_\epsilon), \frac{x_\epsilon - y_\epsilon}{\epsilon}) \geq 0.$$

Similarly for $\mathbf{V}_2$, the map $y \mapsto \mathbf{V}_2(y) - \mathbf{V}_1(x_\epsilon) + \frac{1}{2\epsilon}\|x_\epsilon - y\|^2$ achieves a minimum at y_ϵ. Letting $\phi(y) = \mathbf{V}_1(x_\epsilon) - \frac{1}{2\epsilon}\|x_\epsilon - y\|^2$ in the definition of supersolution, we get

$$H(y_\epsilon, \mathbf{V}_2(y_\epsilon), D\phi(y_\epsilon)) = H(y_\epsilon, \mathbf{V}_2(y_\epsilon), \frac{x_\epsilon - y_\epsilon}{\epsilon}) \leq 0.$$

So, using our assumptions on H,

$$\begin{aligned} 0 &\leq H(x_\epsilon, \mathbf{V}_1(x_\epsilon), \frac{x_\epsilon - y_\epsilon}{\epsilon}) - H(y_\epsilon, \mathbf{V}_2(y_\epsilon), \frac{x_\epsilon - y_\epsilon}{\epsilon}) \\ &= H(x_\epsilon, \mathbf{V}_1(x_\epsilon), \frac{x_\epsilon - y_\epsilon}{\epsilon}) - H(x_\epsilon, \mathbf{V}_2(y_\epsilon), \frac{x_\epsilon - y_\epsilon}{\epsilon}) + \\ &\quad + H(x_\epsilon, \mathbf{V}_2(y_\epsilon), \frac{x_\epsilon - y_\epsilon}{\epsilon}) - H(y_\epsilon, \mathbf{V}_2(y_\epsilon), \frac{x_\epsilon - y_\epsilon}{\epsilon}) \\ &\leq -\gamma M_\epsilon + C\left(1 + \frac{\|x_\epsilon - y_\epsilon\|}{\epsilon}\right)\|x_\epsilon - y_\epsilon\| \\ &= -\gamma M, \quad \text{as } \epsilon \to 0^+. \end{aligned}$$

This is a contradiction since $\gamma,\ M > 0$, so this completes the proof. □

There is a similar comparison principle for a Hamilton-Jacobi equation such as the one we derived for the Bolza Problem.

Theorem 4.18. ***(Comparison Principle for evolution equations in unbounded domains)***

Let $H : [0,T]\times\mathbb{R}^N\times\mathbb{R}^N \to \mathbb{R}$ be continuous, $\mathbf{V}_1 : [0,T]\times\mathbb{R}^N \to \mathbb{R}$ be a subsolution, and $\mathbf{V}_2 : [0,T]\times\mathbb{R}^N \to \mathbb{R}$ be a supersolution of

(4.19) $$\partial_t V(t,x) + H(t,x,DV(x)) = 0, \text{ for } (t,x) \in [0,T)\times\mathbb{R}^N.$$

Suppose that for any $p_1,\ p_2$ in $\mathbb{R}^N$ and (t,x) in $[0,T]\times\mathbb{R}^N$, we have

$$|H(t,x,p_1) - H(t,x,p_2)| \leq C\|p_1 - p_2\|,$$

and also that for any $(t_1, x_1),\ (t_2, x_2) \in [0,T] \times \mathbb{R}^N$ *and* $p \in \mathbb{R}^N$

$$|H(t_1, x_1, p) - H(t_2, x_2, p)| \leq C(1 + \|p\|)\|(t_1, x_1) - (t_2, x_2)\|, \quad p \in \mathbb{R}^N$$

for some constant C.

With these assumptions, if $\mathbf{V}_1(T, x) \leq \mathbf{V}_2(T, x)$ *for all* $x \in \mathbb{R}^N$*, then* $\mathbf{V}_1 \leq \mathbf{V}_2$ *in* $[0,T] \times \mathbb{R}^N$.

The idea of the proof is the same as in the previous Comparison Principle, with a few modifications needed. One can find the details in Cardaliaguet's notes [1].

The next Lemma is the last result needed to prove the existence of the value for the Bolza problem.

Lemma 4.20. $\mathbf{V}^+$ *is a subsolution to*

$$\begin{cases} \partial_t W(t,x) + H^+(t, x, DW(t,x)) = 0 \\ W(T, x) = g(x) \end{cases} \tag{4.21}$$

Proof. Suppose ϕ is a C^1 function such that $\mathbf{V}^+ - \phi$ has a local maximum at $(t_0, x_0) \in [0,T] \times \mathbb{R}^N$. We need to show

$$\partial_t \phi(t_0, x_0) + H^+(t_0, x_0, D\phi(t_0, x_0)) \geq 0.$$

$$\partial_t \phi(t_0, x_0) + \inf_{u \in U} \sup_{v \in V} \{\langle D\phi(t_0, x_0), f(t_0, x_0, u, v)\rangle + \ell(t_0, x_0, u, v)\} \geq 0$$

We know there exist k, $r > 0$ such that for any $t \in (t_0 - k, t_0 + k)$ and $x \in B_r(x_0)$, the following holds:

$$\mathbf{V}^+(t,x) \leq \phi(t,x) + \mathbf{V}^+(t_0, x_0) - \phi(t_0, x_0).$$

With h small enough, $X_{t_0+h}^{t_0, x_0 \alpha, v}$ is in $B_r(x_0)$, since f is bounded. With h as such, we have

$$\begin{aligned} 0 &\leq \inf_{\alpha \in \mathcal{A}_d} \sup_{v \in \mathcal{V}} \left\{ \phi(t_0 + h, X_{t_0+h}^{\alpha, v}) - \phi(t_0, x_0) + \int_{t_0}^{t_0+h} \ell(s, X_s^{\alpha, v}, \alpha(v)_s, v_s) ds \right\} \\ &\leq \sup_{v \in \mathcal{V}} \left\{ \phi(t_0 + h, X_{t_0+h}^{u,v}) - \phi(t_0, x_0) + \int_{t_0}^{t_0+h} \ell(s, X_s^{u,v}, u, v_s) ds \right\}, \end{aligned}$$

for any constant strategy u. Dividing both sides by $h > 0$, we get

$$0 \leq \sup_{v \in \mathcal{V}} \left\{ \frac{\phi(t_0 + h, X_{t_0+h}^{u,v}) - \phi(t_0, x_0)}{h} + \frac{1}{h} \int_{t_0}^{t_0+h} \ell(s, X_s^{u,v}, u, v_s) ds. \right\}$$

Fix $\varepsilon > 0$. We know there exists $v \in \mathcal{V}(t_0)$ such that the following holds:

$$-\varepsilon \leq \frac{\phi(t_0 + h, X_{t_0+h}^{u,v}) - \phi(t_0, x_0)}{h} + \frac{1}{h} \int_{t_0}^{t_0+h} \ell(s, X_s^{u,v}, u, v_s) ds. \tag{4.22}$$

However, we still cannot take $h \to 0^+$, since we do not know how v_t behaves near 0. Note that for the second term in (4.22), we have this estimate:

$$\left| \int_{t_0}^{t_0+h} \ell(s, X_s^{u,v}, u, v_s) ds - \int_{t_0}^{t_0+h} \ell(t_0, X_0, u, v_s) ds \right|$$

$$
\begin{aligned}
&\le \int_{t_0}^{t_0+h} |\ell(s, X_s^{u,v}, u, v_s) - \ell(t_0, x_0, u, v_s)|\, ds \\
&\le \int_{t_0}^{t_0+h} C_\ell(\|X_s^{u,v} - x_0\| + |s - t_0|) ds \\
&\le \; C_\ell \int_{t_0}^{t_0+h} (1 + \|f\|_\infty)|s - t_0| ds \\
&\le o(h).
\end{aligned}
$$

For the first term, we have

$$
\phi(t_0 + h, X_{t_0+h}) - \phi(t_0, x_0) = \int_{t_0}^{t_0+h} \partial_t \phi(s, X_s) + \langle D\phi(X_s), f(s, X_s, u, v_s)\rangle ds
$$

Because ϕ is C^1 and f is bounded, we know

$$
\frac{\int_{t_0}^{t_0+h} \partial_t \phi(s, X_s) ds}{h} \to \partial_t \phi(t_0, x_0), \text{ as } h \to 0.
$$

and using the Cauchy-Schwarz inequality with M being the bound on f, we get

$$
\begin{aligned}
&\left| \int_{t_0}^{t_0+h} \langle D\phi(s, X_s), f(s, X_s, u, v_s)\rangle ds - \int_{t_0}^{t_0+h} \langle D\phi(t_0, x_0), f(t_0, x_0, u, v_s)\rangle ds \right| \\
\le\; & \int_{t_0}^{t_0+h} |\langle D\phi(s, X_s), f(s, X_s, u, v_s)\rangle - \langle D\phi(t_0, x_0), f(s, X_s, u, v_s)\rangle \\
&\qquad + \langle D\phi(t_0, x_0), f(s, X_s, u, v_s)\rangle - \langle D\phi(x_0), f(t_0, x_0, u, v_s)\rangle| ds \\
=\; & \int_{t_0}^{t_0+h} |\langle D\phi(s, X_s) - D\phi(x_0), f(s, X_s, u, v_s)\rangle \\
&\qquad + \langle D\phi(x_0), f(s, X_s, u, v_s) - f(t_0, x_0, u, v_s)\rangle| ds \\
\le\; & M \int_{t_0}^{t_0+h} \|D\phi(s, X_s) - D\phi(t_0, x_0)\| ds \\
&\; + \|D\phi(t_0, x_0)\| \int_{t_0}^{t_0+h} \|f(s, X_s, u, v_s) - f(t_0, x_0, u, v_s)\| ds \\
\le\; & o(h)
\end{aligned}
$$

So, recalling that V is a compact metric space, we have:

$$
\begin{aligned}
&\frac{1}{h}\int_{t_0}^{t_0+h} \langle D\phi(s, X_s), f(s, X_s, u, v_s)\rangle ds + \frac{1}{h}\int_{t_0}^{t_0+h} \ell(s, X_s, u, v_s) ds \\
\le\; & \frac{1}{h}\int_{t_0}^{t_0+h} \langle D\phi(t_0, x_0), f(t_0, x_0, u, v_s)\rangle ds + \frac{1}{h}\int_{t_0}^{t_0+h} \ell(t_0, x_0, u, v_s) ds + o(h) \\
\le\; & \frac{1}{h}\int \max_{v\in V} \{\langle D\phi(t_0, x_0), f(t_0, x_0, u, v)\rangle\}\, ds + \frac{1}{h}\int \max_{v\in V} \{\ell(t_0, x_0, u, v)\}\, ds + o(h)
\end{aligned}
$$

Plugging these estimates in (4.22) and letting $h \to 0$ yields the following:

$$\begin{aligned}
-\varepsilon &\leq \frac{\phi(t_0+h, X^{u,v}_{t_0+h}) - \phi(t_0,x_0)}{h} + \frac{1}{h}\int_{t_0}^{t_0+h} \ell(s, X^{u,v}_s, u, v_s)ds \\
-\varepsilon &\leq \partial_t\phi(t_0,x_0) + \max_{v\in V}\{\langle D\phi(t_0,x_0), f(t_0,x_0,u,v)\rangle\} + \max_{v\in V}\{\ell(t_0,x_0,u,v)\} \\
-\varepsilon &\leq \partial_t\phi(t_0,x_0) + \max_{v\in V}\{\langle D\phi(t_0,x_0), f(t_0,x_0,u,v)\rangle + \ell(t_0,x_0,u,v)\}
\end{aligned}$$

Since this is true for any $u \in U$ and $\varepsilon > 0$, we get

$$0 \leq \partial_t\phi(t_0,x_0) + H^+(t_0,x_0,D\phi(x_0)).$$

□

Similarly, $\mathbf{V}^-$ is a supersolution of

$$\begin{cases} \partial_t W(t,x) + H^-(t,x,DW(t,x)) = 0 \\ W(T,x) = g(x) \end{cases}$$

Theorem 4.23. ***(Existence of the Value for the Bolza Problem)***
Under Isaacs' condition, with our usual definitions of the value functions—(4.4) *and* (4.6)*—, and the Hamiltonians—*(1.12) *and* (1.13)*—, the game has a value, that is,* $\mathbf{V}^- = \mathbf{V}^+ = \mathbf{V}$.

Proof. First, we check that the Hamiltonian satisfies the assumptions in (4.18) by recalling our assumptions on f and ℓ. We know

$$|\langle p_1, f(t,x,u,v)\rangle + \ell(t,x,u,v) - \langle p_2, f(t,x,u,v)\rangle - \ell(t,x,u,v)| \leq M\|p_1 - p_2\|,$$

for any $(u,v) \in U \times V$, so by Lemma (4.8) the first assumption is satisfied:

$$|H(t,x,p_1) - H(t,x,p_2)| \leq C\|p_1 - p_2\| \text{ for all } p_1, p_2 \in \mathbb{R}^N$$

For the second assumption, we estimate the following:

$$\begin{aligned}
&|\langle p, f(t_1,x_1,u,v)\rangle + \ell(t_1,x_1,u,v) - \langle p, f(t_2,x_2,u,v)\rangle - \ell(t_2,x_2,u,v)| \\
\leq\ & |\langle p, f(t_1,x_1,u,v) - f(t_2,x_2,u,v)\rangle| + |\ell(t_1,x_1,u,v) - \ell(t_2,x_2,u,v)| \\
\leq\ & C_f\|(t_1,x_1) - (t_2,x_2)\|\,\|p\| \ + \ C_\ell\|(t_1,x_1) - (t_2,x_2)\|,
\end{aligned}$$

so that again, by Lemma (4.8), we get the result:

$$|H(t_1,x_1,p) - H(t_2,x_2,p)| \leq C(1+\|p\|)\|(t_1,x_1) - (t_2,x_2)\|, \quad p \in \mathbb{R}^N.$$

Under Isaacs' condition, the same Hamilton-Jacobi equation has $\mathbf{V}^-$ as a supersolution and $\mathbf{V}^+$ as a subsolution. Moreover, $\mathbf{V}^+(T,x) = \mathbf{V}^-(T,x) = g(x)$ for all $x \in \mathbb{R}^N$. Using the comparison principle (4.18) we know $\mathbf{V}^+ \leq \mathbf{V}^-$ on $[0,T] \times \mathbb{R}^N$ and, by definition, $\mathbf{V}^- \leq \mathbf{V}^+$. So $\mathbf{V}^- = \mathbf{V}^+ = \mathbf{V}$ on $[0,T] \times \mathbb{R}^N$, and the game has a value.

□

The following Theorem gives an explicit formula for the value of a finite-horizon game. The proof is rather lengthy, and the interested reader can find it in the notes by Cardaliaguet [1].

Theorem 4.24. ***(Hopf-Lax Representation Formula)*** *Consider the following evolution equation for* $W : [0,T] \times \mathbb{R}^N \to \mathbb{R}$*:*

$$\begin{cases} \partial_t W(t,x) + H(DW(t,x)) = 0 \text{ on } (0,T) \times \mathbb{R}^N \\ W(T,x) = g(x) \end{cases} \tag{4.25}$$

If g *is super-linear, that is,*

$$\lim_{\|x\|\to\infty} \frac{g(x)}{\|x\|} = \infty,$$

and H *is Lipschitz continuous:*

$$|H(p) - H(q)| \leq C\|p - q\|, \text{ for all } p, q \in \mathbb{R}^N$$

then the unique viscosity solution to (4.25) *is given by*

$$\mathbf{V}(t,x) = (g^*(q) - (T-t)H(q))^*(x),$$

where for a given function $\phi : E \to (-\infty, +\infty]$ *with nonempty domain from a normed vector space* E*,* $\phi^* : E^* \to (-\infty, +\infty]$ *is the convex conjugate of* ϕ*:*

$$\phi^*(f) = \sup_{x\in E} \langle f, x\rangle - \phi(x).$$

If $E = \mathbb{R}^N$*, we can equivalently write*

$$\phi^*(x) = \sup_{q\in\mathbb{R}^N} \langle q, x\rangle - \phi(q).$$

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