REPRESENTATION OF SEMISIMPLE LIE ALGEBRAS

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Abstract. In this paper, we summarize the classification of representations of semisimple lie algebras and look at properties of semisimple Lie algebras. We present the structure theory briefly and then state the theorem of highest weight for semisimple Lie algebras.

Contents

1. Basic Preliminaries

Definition 1.1. A vector space $\mathfrak{g}$ over a field F is called a **Lie algebra** with a map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, (X, Y) \mapsto [X, Y]$ such that:

(1) $(X, Y) \mapsto [X, Y]$ is biliniear.
(2) $[X, Y] + [Y, X] = 0$.
(3) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (Jacobi identity).

Definition 1.2. An ideal I of a Lie algebra $\mathfrak{g}$ is a subalgebra of $\mathfrak{g}$ such that $[X, Y] \in I$ for all $X \in I$ and $Y \in \mathfrak{g}$.

Definition 1.3. A **simple Lie algebra** is a Lie algebra that is nonabelian and contains no nonzero proper ideals. A **semisimple Lie algebra** is a Lie algebra that is a direct sum of simple Lie algebras.

A real or complex Lie algebra is a Lie algebra over $\mathbb{R}$ or $\mathbb{C}$ respectively. We denote the n^2 dimensional vector space over $\mathbb{C}$ consisting of all $n \times n$ complex matrices $\mathfrak{gl}(n, \mathbb{C})$. It can be confirmed by simple calculation that $\mathfrak{gl}(n, \mathbb{C})$ is a Lie algebra with the bracket defined with normal matrix multiplication and addition:

$$\forall X, Y \in \mathfrak{gl}(n, \mathbb{C}), [X, Y] = XY - YX.$$

Similarly, if V is a vector space, we define $\mathfrak{gl}(V)$ to be a Lie algebra of all linear endomorphisms of V, where

$$\forall X, Y \in \mathfrak{gl}(V), [X, Y] = X \circ Y - Y \circ X.$$

Definition 1.4. The **complexification** of a real Lie algebra $\mathfrak{g}$ is a vector space $\mathfrak{g} \oplus \mathfrak{g}$, denoted as $\mathfrak{g} + i\mathfrak{g}$. Its elements are of the form of $X + iY$, where $X, Y \in \mathfrak{g}$. It is a Lie algebra with the bracket induced from that of $\mathfrak{g}$:

$$\forall X, Y, Z, W \in \mathfrak{g}, [X + iY, Z + iW] = ([X, Z] - [Y, W]) + i([Y, Z] + [X, W]).$$

That the induced Lie bracket satisfies 1.1 can be shown by simple calculation.

Definition 1.5. If $\mathfrak{g}$ is a Lie algebra, then a **subalgebra** of $\mathfrak{g}$ is a subspace of $\mathfrak{g}$ that is closed under the Lie bracket.

Definition 1.6. Given two Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$, a **Lie algebra homomorphism** is a linear map $\phi : \mathfrak{g} \to \mathfrak{g}'$ such that $\phi([X, Y]) = [\phi(X), \phi(Y)]$ for all $X, Y \in \mathfrak{g}$.

Definition 1.7. Let $\mathfrak{g}$ a Lie algebra and V a vector space. A **representation** of $\mathfrak{g}$ on V is a Lie algebra homomorphism

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V)$$

such that

$$\rho([X, Y]) = \rho(X) \circ \rho(Y) - \rho(Y) \circ \rho(X)$$

for all $X, Y \in \mathfrak{g}$.

Definition 1.8. Given a representation ρ of $\mathfrak{g}$ on V as above, let a subspace $W \subset V$ be closed under the action of $\mathfrak{g}$, i.e.,

$$\rho(X)w \in W$$

for all $X \in \mathfrak{g}, w \in W$. Then the map

$$\rho|_W : \mathfrak{g} \to \mathfrak{gl}(W)$$

such that $X \mapsto \rho(X)|_W$ is a **subrepresentation** of ρ. The subrepresentation is said to be **proper** if it is both nontrivial ($\rho(X) \neq id_V$ for some $X \in \mathfrak{g}$) and $W \subsetneq V$.

Definition 1.9. An **irreducible representation** is a nonzero representation without any proper subrepresentation.

Definition 1.10. Let Π be a representation of $\mathfrak{g}$ acting on the space V and Σ be a representation of $\mathfrak{g}$ on the space W. A linear map $\phi : V \to W$ is called an **intertwining map** of representations if $\phi(\Pi(A)v) = \Sigma(A)\phi(v)$ for all $A \in \mathfrak{g}$ and $v \in V$. If ϕ is invertible, ϕ is said to be an equivalence of representations. If V and W are isomorphic, then the representations are said to be **equivalent**.

Given a Lie algebra $\mathfrak{g}$, we can define a map $ad_X : \mathfrak{g} \to \mathfrak{g}, X \in \mathfrak{g}$ such that $Y \mapsto [X, Y]$. Thus, $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), X \mapsto ad_X$. We can see that ad is a representation of $\mathfrak{g}$ on $\mathfrak{gl}(\mathfrak{g})$. We call this the **adjoint representation** of $\mathfrak{g}$.

Definition 1.11. Given a Lie algebra $\mathfrak{g}$, the **Killing Form** B on $\mathfrak{g}$ is a symmetric bilinear form given by

$$B(X, Y) = \operatorname{tr}(ad_X \circ ad_Y), \forall X, Y \in \mathfrak{g}.$$

The computation of B might be unclear from its definition. But it can be calculated explicitly using the structure constants. We choose a basis $\{e_i\}$ of $\mathfrak{g}$. The **structure constants** of $\mathfrak{g}$, f_{ij}^k, are defined by

$$[e_i, e_j] = \sum_k f_{ij}^k e_k.$$

Let $\{e_i\}$ be a basis of $\mathfrak{g}$. Then, we can see that

$$ad_{e_i} \circ ad_{e_j}(e_k) = [e_i, [e_j, e_k]] = \sum_m \sum_l f_{il}^m f_{jk}^l e_m$$

holds. Thus, by the definition of B,

$$B(e_i, e_j) = \sum_k \sum_l f_{il}^k f_{jk}^l$$

holds. Since B is a bilinear form, we can compute $B(X,Y)$ for all $X, Y \in \mathfrak{g}$ from this. One important property of the adjoint representation is that ad_X is skew hermitian with respect of B, i.e.,

$$\forall X, Y, Z \in \mathfrak{g}, B(ad_X(Y), Z) = -B(Y, ad_X(Z)).$$

This can be verified by simple computation.

Definition 1.12. A **compact real Lie algebra** $\mathfrak{p}$ is a real Lie algebra such that its Killing form is negative definite. If $\mathfrak{g}$ is a complex semisimple Lie algebra, then a **compact real form** of $\mathfrak{g}$ is a real subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ that is a compact real Lie algebra, and such that the complexification of $\mathfrak{p}$ is $\mathfrak{g}$.

It turns out that every complex semisimple Lie algebra $\mathfrak{g}$ has a compact real form. In this paper, we will only consider Lie algebra that is the complexification of some simply-connected compact real subalgebra of $\mathfrak{gl}(n, \mathbb{C})$. In particular, when $\mathfrak{g}$ is used without context in this paper, it is a Lie algebra that can be written as $\mathfrak{p} + i\mathfrak{p}$, where $\mathfrak{p}$ is a simply-connected compact real subalgebra of $\mathfrak{gl}(n, \mathbb{C})$.

2. Cartan subalgebra

Definition 2.1. A Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a complex subspace of $\mathfrak{g}$ with following properties.

(1) For all $H_1, H_2 \in \mathfrak{h}$, $[H_1, H_2] = 0$.
(2) For all $X \in \mathfrak{g}$, if $[H, X] = 0$ for all $H \in \mathfrak{h}$, then $X \in \mathfrak{h}$.
(3) For all $H \in \mathfrak{h}$, ad_H is diagonalizable.

In other words, a Cartan subalgebra of $\mathfrak{g}$ is maximal commutative in $\mathfrak{g}$, and all ad_H's are simultaneously diagonalizable. The following theorem provides us the existence of a Cartan subalgebra for any $\mathfrak{g}$ and a way to compute it. The proof is nontrivial and is omitted.

Theorem 2.2. *Let $\mathfrak{p}$ be a compact real form of $\mathfrak{g}$, and let $\mathfrak{t}$ be any maximal commutative subalgebra of $\mathfrak{p}$. Define $\mathfrak{h} \subset \mathfrak{g}$ to be $\mathfrak{h} = \mathfrak{t} + i\mathfrak{t}$. Then, $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$.*

It is also possible to prove that every Cartan subalgebra of $\mathfrak{g}$ is in this form, and all of them have the same dimension. Thus we can just choose any Cartan subalgebra of $\mathfrak{g}$ to represent it. From now on, if $\mathfrak{h}$ is used without context, it is the fixed Cartan subalgebra of $\mathfrak{g}$.

3. Structure Theory

Definition 3.1. A **root** of $\mathfrak{g}$ is a nonzero linear functional $\alpha \in \mathfrak{h}^*$ such that

$$\forall H \in \mathfrak{h}, [H, X] = \alpha(H)X$$

for some eigenvector $X \in \mathfrak{g}$. We call X a **root vector**. The **root space** $\mathfrak{g}_\alpha$ is the set of all root vectors of α.

We want to examine the structure of $\mathfrak{g}$ more throughly. We can decompose $\mathfrak{g}$ into $\mathfrak{h} \oplus \mathfrak{h}^\perp$. Since ad_H's for all $H \in \mathfrak{h}$ are simultaneously diagonalizable, its eigenvectors form a basis of $\mathfrak{g}$. Since $\mathfrak{h}$ is maximal commutative, the eigenvectors with eigenvalue of 0 span $\mathfrak{h}$. Other eigenvectors span the root space. Therefore, the following theorem holds.

Theorem 3.2. ***(Cartan Decomposition)*** *$\mathfrak{g}$ can be decomposed as a direct sum as follows:*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha,$$

where R is the set of all roots.

Now we can prove that the Killing form B on $\mathfrak{g}$ is an inner product when restricted to $\mathfrak{h}$. The only thing that needs to be shown is that B is nondegenerate on $\mathfrak{h}$. This can be done by observing that $\alpha(T)$ is pure imaginary for all root α and $T \in \mathfrak{t}$. This comes from the nature of construction of $\mathfrak{h}$ (Theorem 2.2).
We can also show that if α is a root, then $-\alpha$ is also a root. Let $X = X_1 + iX_2 \in \mathfrak{g}$ be a root vector of α. Using that $\alpha(H)$ is a pure imaginary eigenvalue of ad_H if $H \in \mathfrak{p} \subset \mathfrak{h}$, one can see that $-X_1 + iX_2$ is a root vector of $-\alpha$. Thus $-\alpha$ is a root.

Proposition 3.3. *If α is a root, we can find $X_\alpha \in \mathfrak{g}_\alpha, Y_\alpha \in \mathfrak{g}_{-\alpha}, H_\alpha \in \mathfrak{h}$ such that*

$$[H_\alpha, X_\alpha] = 2X_\alpha$$
$$[H_\alpha, Y_\alpha] = -2Y_\alpha$$
$$[X_\alpha, Y_\alpha] = H_\alpha$$

Proof. We take $X = X_1 + iX_2$ to be any nonzero element of $\mathfrak{g}_\alpha$ ($X_1, X_2 \in \mathfrak{t}$). Let $Y = -X_1 + iX_2$. Then $Y \in \mathfrak{g}_{-\alpha}$. Let $H = [X, Y]$. Using the Jacobi identity, it can be shown that $ad_{H'}([X, Y]) = 0$ for all $H' \in \mathfrak{h}$. Thus $H \in \mathfrak{h}$. Now, normalize X, Y, and H:

$$H_\alpha := \frac{2}{\alpha(H)} H, X_\alpha := \sqrt{\frac{2}{\alpha(H)}} X, Y_\alpha := \sqrt{\frac{2}{\alpha(H)}} Y$$

Direct calculation shows that the commutation relations hold. □

Definition 3.4. A **base** is a set of roots Δ such that

(1) Δ is a basis of $\mathfrak{h}$.
(2) Any root α can be expressed as

$$\alpha = \sum_i n_i \alpha_i, \alpha_i \in \Delta$$

where n_i's are integers either all ≥ 0 or all ≤ 0.

The elements of Δ are called the **positive simple roots**.

To continue our discussion, we admit the following proposition without proof.

Proposition 3.5. *For any set of roots, a base Δ exists.*

Definition 3.6. $\mu \in \mathfrak{h}$ is an **integral element** if $\mu(H_\alpha)$ is an integer for all root α. An integral element μ is a **dominant integral element** if $\mu(H_\alpha) \geq 0$ for all positive simple root α's.

Definition 3.7. Let π be a finite-dimensional representation of $\mathfrak{g}$ on a vector space V. We call $\mu \in \mathfrak{h}^*$ a **weight** if there exists a nonzero $v \in V$ such that

$$\pi(H)v = \mu(H)v$$

for all $H \in \mathfrak{h}$. Here, v is called a **weight vector** for the weight μ, and the set of all vectors satisfying the equation is called the **weight space** with weight μ. The dimension of the weight space is called the **multiplicity** of the weight.

Let v be a weight vector with weight μ and X_α is an element of the root space $\mathfrak{g}_\alpha$. Since $[H, X_\alpha] = \alpha(H)X_\alpha$,

$$\pi(H)\pi(X_\alpha)v = [\pi(X_\alpha)\pi(H) + \pi([H, X_\alpha])]v = (\mu + \alpha)(H)\pi(X_\alpha)v.$$

Thus, either $\pi(X_\alpha)v$ is zero or a weight vector with weight $\mu + \alpha$. From this, the following theorem holds.

Proposition 3.8. *Every finite-dimensional representation (π, V) is the direct sum of its weight spaces.*

Proof. The complete reducibility of $\mathfrak{g}$ allows π to decompose as a direct sum of irreducible representations, thus we can only consider the case when π is irreducible. Let U be a subspace of V spanned by all weight vectors. U is invariant under $\pi(H)$ for all $H \in \mathfrak{h}$ by the definition of weight, and we have just shown that it is also invariant under $\pi(X_\alpha)$ for all $X \in \mathfrak{g}_\alpha$. By the Cartan decomposition of $\mathfrak{g}$, U is invariant under the action of $\mathfrak{g}$. Since π is irreducible and $U \neq \{0\}$, $U = V$ holds. □

Definition 3.9. Let $\mu_1, \mu_2 \in \mathfrak{h}^*$. We say that μ_1 is **higher** than μ_2 if there exist non-negative real numbers $a_1, \ldots, a_r$ such that

$$\mu_1 - \mu_2 = \sum_i a_i \alpha_i$$

where $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ is the set of positive simple roots. If π is a representation of $\mathfrak{g}$, then a weight μ_0 for π is said to be a **highest weight** if μ_0 is higher than any other weights.

4. Theorem of Highest Weight

The ultimate goal of this paper is to prove the following statements for semisimple Lie algebras.

(1) Every Irreducible representation has a highest weight.
(2) Two irreducible representations with the same weight are equivalent.
(3) The highest weight of every irreducible representation is a dominant integral element.
(4) Every dominant integral element occurs as the highest weight of an irreducible representation.

Definition 4.1. (π, V) is a highest weight cyclic representation with weight μ_0 if there exists $v \neq 0$ in V such that

(1) v is a weight vector with weight μ_0.
(2) $\pi(X_\alpha)v = 0$ for all positive roots α.
(3) The smallest invariant subspace of V containing v is V.

From the definition, it follows that π has a highest weight μ_0, and the weight space of μ_0 is one-dimensional. To prove this, consider the subspace U of V spanned by the elements of the form $\pi(Y_{\alpha_1})\dots\pi(Y_{\alpha_l})v$, where $Y_{\alpha_i} \in \mathfrak{g}_{-\alpha_i}$ and α_i's are positive roots. Using commutation relations, it is easy to prove that U is invariant. Then, by definition, $V = U$. It follows that V is spanned by the weight vectors of the above form, which have weights lower than μ_0, μ_0 is the highest weight for V. Also, the only weight vectors with weight μ_0 are multiples of v because all other weight vectors have weights lower than μ_0. Thus the weight space of μ_0 is one-dimensional.

Proposition 4.2. *Every irreducible representation of $\mathfrak{g}$ is a highest weight cyclic representation, with a unique highest weight μ_0.*

Proof. Every irreducible representation is a direct sum of its weight spaces. Finite dimensional representations have only finitely many weights. Thus we can find μ_0 such that no weight $\mu \neq \mu_0$ is higher than it. There exists a corresponding weight vector v. Then, it follows that $\pi(X_\alpha)v = 0$ for all positive roots α. Since the representation is irreducible, the only invariant subspace containing v is V. Thus, the representation is a highest weight cyclic representation. □

Conversely, every highest weight cyclic representation of $\mathfrak{g}$ is irreducible, because the smallest invariant subspace of V containing v is V itself.

Proposition 4.3. *Two irreducible representations of $\mathfrak{g}$ with the same highest weight are equivalent.*

Proof. Suppose (π, V) and (σ, W) are two representations of $\mathfrak{g}$ with the same highest weight μ_0. Let $v \in V$ and $w \in W$ be weight vectors of μ_0 in each representation. Consider the representation $V \oplus W$. It is easy to show that μ_0 is also the highest weight in $V \oplus W$. Let U be the smallest invariant subspace of $V \oplus W$ that contains (v, w). U is a highest weight cyclic representation, and therefore irreducible. Define two projection maps $P_1 : V \oplus W \to V, (v, w) \mapsto v$ and $P_2 : V \oplus W \to W, (v, w) \mapsto w$. Since $P_1|_U$ and $P_2|_U$ are intertwining maps and U, V, W are irreducible, the intertwining maps are invertible by Schur's lemma. Thus U, V, W are all isomorphic to each other. □

Proposition 4.4. *If π is an irreducible representation of $\mathfrak{g}$, then the highest weight μ_0 of π is a dominant integral element.*

Proof. Given the highest weight μ_0, its weight vector v_0, and a positive root α, either $\mu_0 - \alpha$ is a root or $\pi(Y_\alpha)v_0 = 0$. Inductively, there exists an integer $m \geq 0$ such that $\mu_0 - n\alpha$ is a root for all $n \leq m$ and $\pi(Y_\alpha)v_m = 0$ where v_k is a weight vector of $\mu_0 - k\alpha$. Using commutation relations and induction, it can be proven that $\pi(X_\alpha)v_k = [k\mu_0(H_\alpha) - k(k-1)]v_{k-1}$ holds for all $k > 0$. Thus, when $k = m + 1$, since $v_{m+1} = 0$, $(m + 1)(\mu_0(H_\alpha) - m) = 0$. It follows that $\mu_0(H_\alpha)$ is a nonnegative integer. □

The only thing left to be shown now is that every dominant integral element is the highest weight of an irreducible representation. This is a nontrivial task. Before I finish the paper, I will briefly present Verma modules to construct such representation. A Verma module is an infinite-dimensional representation on a vector space V_μ that has a highest weight μ. It can be defined for every element μ of $\mathfrak{h}^*$, not necessarily an integral element, but it is not finite-dimensional. Thus, it doesn't match our requirements by itself. However, if μ is a dominant integral weight, then V_μ contains a maximal invariant subspace U_μ such that V_μ/U_μ is finite dimensional, irreducible, and has highest weight μ.

We can construct V_μ by looking at its properties. Since μ is a highest weight, the representation π of $\mathfrak{g}$ on V_μ should satisfy $\pi(X_\alpha) = 0$ for all positive roots α. Also, since μ is a weight, $\pi(H - \mu(H)1) = 0$ for all $H \in \mathfrak{h}$. It turns out that we can always find such a representation, using the concept of universal enveloping algebra. The universal enveloping algebra of $\mathfrak{g}$ always have an ideal generated from X_α's and $(H - \mu(H)1$'s, so we can take the quotient of the universal enveloping algebra by the ideal, which is isomorphic to V_μ.

Now we have the theorem of highest weight, and we can classify all finite-dimensional representations of $\mathfrak{g}$ according to it.

Acknowledgments

I thank all organizers and participants of the University of Chicago Mathematics REU 2020 for the opportunity to present this paper. Especially, it is a pleasure to thank my mentor, Pranjal Warade, for resolving my questions about the topic, giving comments, and helping me with the writing. I also thank Professor Peter May for reviewing and proofreading.

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