REPRESENTATION OF SEMISIMPLE LIE ALGEBRAS

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Abstract. In this paper, we summarize the classification of representations of semisimple lie algebras and look at properties of semisimple Lie algebras. We present the structure theory briefly and then state the theorem of highest weight for semisimple Lie algebras.

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1. Basic Preliminaries

Definition 1.1. A vector space $g$ over a field $F$ is called a Lie algebra with a map $g \times g \rightarrow g, (X,Y) \mapsto [X,Y]$ such that:

1. $(X,Y) \mapsto [X,Y]$ is bilinear.
2. $[X,Y] + [Y,X] = 0$.
3. $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ (Jacobi identity).

Definition 1.2. An ideal $I$ of a Lie algebra $g$ is a subalgebra of $g$ such that $[X,Y] \in I$ for all $X \in I$ and $Y \in g$.

Definition 1.3. A simple Lie algebra is a Lie algebra that is nonabelian and contains no nonzero proper ideals. A semisimple Lie algebra is a Lie algebra that is a direct sum of simple Lie algebras.

A real or complex Lie algebra is a Lie algebra over $\mathbb{R}$ or $\mathbb{C}$ respectively. We denote the $n^2$ dimensional vector space over $\mathbb{C}$ consisting of all $n \times n$ complex matrices $\mathfrak{gl}(n, \mathbb{C})$. It can be confirmed by simple calculation that $\mathfrak{gl}(n, \mathbb{C})$ is a Lie algebra with the bracket defined with normal matrix multiplication and addition:

$$\forall X, Y \in \mathfrak{gl}(n, \mathbb{C}), [X,Y] = XY - YX.$$  

Similarly, if $V$ is a vector space, we define $\mathfrak{gl}(V)$ to be a Lie algebra of all linear endomorphisms of $V$, where

$$\forall X, Y \in \mathfrak{gl}(V), [X,Y] = X \circ Y - Y \circ X.$$
Definition 1.4. The complexification of a real Lie algebra $\mathfrak{g}$ is a vector space $\mathfrak{g} \oplus i\mathfrak{g}$, denoted as $\mathfrak{g} + i\mathfrak{g}$. Its elements are of the form of $X + iY$, where $X, Y \in \mathfrak{g}$.

It is a Lie algebra with the bracket induced from that of $\mathfrak{g}$:

$$\forall X, Y, Z, W \in \mathfrak{g}, [X + iY, Z + iW] = ([X, Z] - [Y, W]) + i([Y, Z] + [X, W]).$$

That the induced Lie bracket satisfies 1.1 can be shown by simple calculation.

Definition 1.5. If $\mathfrak{g}$ is a Lie algebra, then a subalgebra of $\mathfrak{g}$ is a subspace of $\mathfrak{g}$ that is closed under the Lie bracket.

Definition 1.6. Given two Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$, a Lie algebra homomorphism is a linear map $\phi : \mathfrak{g} \to \mathfrak{g}'$ such that $\phi([X, Y]) = [\phi(X), \phi(Y)]$ for all $X, Y \in \mathfrak{g}$.

Definition 1.7. Let $\mathfrak{g}$ a Lie algebra and $V$ a vector space. A representation of $\mathfrak{g}$ on $V$ is a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ such that $\rho([X, Y]) = \rho(X) \circ \rho(Y) - \rho(Y) \circ \rho(X)$ for all $X, Y \in \mathfrak{g}$.

Definition 1.8. Given a representation $\rho$ of $\mathfrak{g}$ on $V$ as above, let a subspace $W \subset V$ be closed under the action of $\mathfrak{g}$, i.e.,

$$\rho(X)w \in W$$

for all $X \in \mathfrak{g}, w \in W$. Then the map $\rho|_W : \mathfrak{g} \to \mathfrak{gl}(W)$ such that $X \mapsto \rho(X)|_W$ is a subrepresentation of $\rho$. The subrepresentation is said to be proper if it is both nontrivial ($\rho(X) \neq id_V$ for some $X \in \mathfrak{g}$) and $W \subseteq V$.

Definition 1.9. An irreducible representation is a nonzero representation without any proper subrepresentation.

Definition 1.10. Let $\Pi$ be a representation of $\mathfrak{g}$ acting on the space $V$ and $\Sigma$ be a representation of $\mathfrak{g}$ on the space $W$. A linear map $\phi : V \to W$ is called an intertwining map of representations if $\phi(\Pi(A)v) = \Sigma(A)\phi(v)$ for all $A \in \mathfrak{g}$ and $v \in V$. If $\phi$ is invertible, $\phi$ is said to be an equivalence of representations. If $V$ and $W$ are isomorphic, then the representations are said to be equivalent.

Given a Lie algebra $\mathfrak{g}$, we can define a map $ad_X : \mathfrak{g} \to \mathfrak{g}, X \in \mathfrak{g}$ such that $Y \mapsto [X, Y]$. Thus, $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), X \mapsto ad_X$. We can see that $ad$ is a representation of $\mathfrak{g}$ on $\mathfrak{gl}(\mathfrak{g})$. We call this the adjoint representation of $\mathfrak{g}$.

Definition 1.11. Given a Lie algebra $\mathfrak{g}$, the Killing Form $B$ on $\mathfrak{g}$ is a symmetric bilinear form given by

$$B(X, Y) = \text{tr}(ad_X \circ ad_Y), \forall X, Y \in \mathfrak{g}.$$
Let \( \{e_i\} \) be a basis of \( \mathfrak{g} \). Then, we can see that
\[
ad_{e_i} \circ ad_{e_j}(e_k) = [e_i, [e_j, e_k]] = \sum_m \sum_l f^m_k f^l_j e_m
\]
holds. Thus, by the definition of \( B \),
\[
B(e_i, e_j) = \sum_k \sum_l f^k_i f^l_j
\]
holds. Since \( B \) is a bilinear form, we can compute \( B(X, Y) \) for all \( X, Y \in \mathfrak{g} \) from this. One important property of the adjoint representation is that \( ad_X \) is skew hermitian with respect of \( B \), i.e.,
\[
\forall X, Y, Z \in \mathfrak{g}, B(ad_X(Y), Z) = -B(Y, ad_X(Z)).
\]
This can be verified by simple computation.

**Definition 1.12.** A **compact real Lie algebra** \( \mathfrak{p} \) is a real Lie algebra such that its Killing form is negative definite. If \( \mathfrak{g} \) is a complex semisimple Lie algebra, then a **compact real form** of \( \mathfrak{g} \) is a real subalgebra \( \mathfrak{p} \) of \( \mathfrak{g} \) that is a compact real Lie algebra, and such that the complexification of \( \mathfrak{p} \) is \( \mathfrak{g} \).

It turns out that every complex semisimple Lie algebra \( \mathfrak{g} \) has a compact real form. In this paper, we will only consider Lie algebra that is the complexification of some simply-connected compact real subalgebra of \( \mathfrak{gl}(n, \mathbb{C}) \). In particular, when \( \mathfrak{g} \) is used without context in this paper, it is a Lie algebra that can be written as \( \mathfrak{p} + i\mathfrak{p} \), where \( \mathfrak{p} \) is a simply-connected compact real subalgebra of \( \mathfrak{gl}(n, \mathbb{C}) \).

## 2. Cartan subalgebra

**Definition 2.1.** A Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \) is a complex subspace of \( \mathfrak{g} \) with following properties.

1. For all \( H_1, H_2 \in \mathfrak{h} \), \([H_1, H_2] = 0\).
2. For all \( X \in \mathfrak{g} \), if \([H, X] = 0\) for all \( H \in \mathfrak{h} \), then \( X \in \mathfrak{h} \).
3. For all \( H \in \mathfrak{h} \), \( ad_H \) is diagonalizable.

In other words, a Cartan subalgebra of \( \mathfrak{g} \) is maximal commutative in \( \mathfrak{g} \), and all \( ad_H \)'s are simultaneously diagonalizable. The following theorem provides us the existence of a Cartan subalgebra for any \( \mathfrak{g} \) and a way to compute it. The proof is nontrivial and is omitted.

**Theorem 2.2.** Let \( \mathfrak{p} \) be a compact real form of \( \mathfrak{g} \), and let \( \mathfrak{t} \) be any maximal commutative subalgebra of \( \mathfrak{p} \). Define \( \mathfrak{h} \subset \mathfrak{g} \) to be \( \mathfrak{h} = \mathfrak{t} + i\mathfrak{t} \). Then, \( \mathfrak{h} \) is a Cartan subalgebra of \( \mathfrak{g} \).

It is also possible to prove that every Cartan subalgebra of \( \mathfrak{g} \) is in this form, and all of them have the same dimension. Thus we can just choose any Cartan subalgebra of \( \mathfrak{g} \) to represent it. From now on, if \( \mathfrak{h} \) is used without context, it is the fixed Cartan subalgebra of \( \mathfrak{g} \).
3. Structure Theory

Definition 3.1. A root of \( g \) is a nonzero linear functional \( \alpha \in \mathfrak{h}^* \) such that
\[
\forall H \in \mathfrak{h}, \ [H, X] = \alpha(H) X
\]
for some eigenvector \( X \in g \). We call \( X \) a root vector. The root space \( g_{\alpha} \) is the set of all root vectors of \( \alpha \).

We want to examine the structure of \( g \) more thoroughly. We can decompose \( g \) into \( \mathfrak{h} \oplus \mathfrak{h}^\perp \). Since \( \text{ad}_H \)'s for all \( H \in \mathfrak{h} \) are simultaneously diagonalizable, its eigenvectors form a basis of \( g \). Since \( \mathfrak{h} \) is maximal commutative, the eigenvectors with eigenvalue of 0 span \( \mathfrak{h} \). Other eigenvectors span the root space. Therefore, the following theorem holds.

Theorem 3.2. (Cartan Decomposition) \( g \) can be decomposed as a direct sum as follows:
\[
g = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} g_{\alpha},
\]
where \( R \) is the set of all roots.

Now we can prove that the Killing form \( B \) on \( g \) is an inner product when restricted to \( \mathfrak{h} \). The only thing that needs to be shown is that \( B \) is nondegenerate on \( \mathfrak{h} \). This can be done by observing that \( \alpha(T) \) is pure imaginary for all root \( \alpha \) and \( T \in \mathfrak{t} \). This comes from the nature of construction of \( \mathfrak{h} \) (Theorem 2.2).

We can also show that if \( \alpha \) is a root, then \( -\alpha \) is also a root. Let \( X = X_1 + iX_2 \in g \) be a root vector of \( \alpha \). Using that \( \alpha(H) \) is a pure imaginary eigenvalue of \( \text{ad}_H \) if \( H \in \mathfrak{p} \subset \mathfrak{h} \), one can see that \( -X_1 + iX_2 \) is a root vector of \( -\alpha \). Thus \( -\alpha \) is a root.

Proposition 3.3. If \( \alpha \) is a root, we can find \( X_{\alpha} \in g_{\alpha}, Y_{\alpha} \in g_{-\alpha}, H_{\alpha} \in \mathfrak{h} \) such that
\[
[H_{\alpha}, X_{\alpha}] = 2X_{\alpha},
[H_{\alpha}, Y_{\alpha}] = -2Y_{\alpha},
[X_{\alpha}, Y_{\alpha}] = H_{\alpha}.
\]

Proof. We take \( X = X_1 + iX_2 \) to be any nonzero element of \( g_{\alpha} \) (\( X_1, X_2 \in \mathfrak{t} \)). Let \( Y = -X_1 + iX_2 \). Then \( Y \in g_{-\alpha} \). Let \( H = [X, Y] \). Using the Jacobi identity, it can be shown that \( \text{ad}_{H'}([X, Y]) = 0 \) for all \( H' \in \mathfrak{h} \). Thus \( H \in \mathfrak{h} \). Now, normalize \( X, Y \), and \( H \):
\[
H_{\alpha} := \frac{2}{\alpha(H)}, X_{\alpha} := \sqrt{\frac{2}{\alpha(H)}}, Y_{\alpha} := \sqrt{\frac{2}{\alpha(H)}} Y.
\]
Direct calculation shows that the commutation relations hold. \( \square \)

Definition 3.4. A base is a set of roots \( \Delta \) such that
1. \( \Delta \) is a basis of \( \mathfrak{h} \).
2. Any root \( \alpha \) can be expressed as
\[
\alpha = \sum_i n_i \alpha_i, \ \alpha_i \in \Delta
\]
where \( n_i \)'s are integers either all \( \geq 0 \) or all \( \leq 0 \).
The elements of \( \Delta \) are called the positive simple roots.

To continue our discussion, we admit the following proposition without proof.
Proposition 3.5. For any set of roots, a base \( \Delta \) exists.

Definition 3.6. \( \mu \in h \) is an integral element if \( \mu(H_\alpha) \) is an integer for all root \( \alpha \). An integral element \( \mu \) is a dominant integral element if \( \mu(H_\alpha) \geq 0 \) for all positive simple root \( \alpha \)'s.

Definition 3.7. Let \( \pi \) be a finite-dimensional representation of \( g \) on a vector space \( V \). We call \( \mu \in h^* \) a weight if there exists a nonzero \( v \in V \) such that
\[
\pi(H)v = \mu(H)v
\]
for all \( H \in h \). Here, \( v \) is called a weight vector for the weight \( \mu \), and the set of all vectors satisfying the equation is called the weight space with weight \( \mu \). The dimension of the weight space is called the multiplicity of the weight.

Let \( v \) be a weight vector with weight \( \mu \) and \( X_\alpha \) is an element of the root space \( g_\alpha \). Since
\[
[H,X_\alpha] = \alpha(H)X_\alpha,
\]
\[
\pi(H)\pi(X_\alpha)v = [\pi(X_\alpha)\pi(H) + \pi([H,X_\alpha])]v = (\mu + \alpha)(H)\pi(X_\alpha)v.
\]
Thus, either \( \pi(X_\alpha)v \) is zero or a weight vector with weight \( \mu + \alpha \). From this, the following theorem holds.

Proposition 3.8. Every finite-dimensional representation \( (\pi,V) \) is the direct sum of its weight spaces.

Proof. The complete reducibility of \( g \) allows \( \pi \) to decompose as a direct sum of irreducible representations, thus we can only consider the case when \( \pi \) is irreducible. Let \( U \) be a subspace of \( V \) spanned by all weight vectors. \( U \) is invariant under \( \pi(H) \) for all \( H \in h \) by the definition of weight, and we have just shown that it is also invariant under \( \pi(X_\alpha) \) for all \( X \in g_\alpha \). By the Cartan decomposition of \( g \), \( U \) is invariant under the action of \( g \). Since \( \pi \) is irreducible and \( U \neq \{0\} \), \( U = V \) holds. \( \square \)

Definition 3.9. Let \( \mu_1, \mu_2 \in h^* \). We say that \( \mu_1 \) is higher than \( \mu_2 \) if there exist non-negative real numbers \( a_1,\ldots,a_r \) such that
\[
\mu_1 - \mu_2 = \sum_i a_i \alpha_i
\]
where \( \Delta = \{\alpha_1,\ldots,\alpha_r\} \) is the set of positive simple roots. If \( \pi \) is a representation of \( g \), then a weight \( \mu_0 \) for \( \pi \) is said to be a highest weight if \( \mu_0 \) is higher than any other weights.

4. Theorem of Highest Weight

The ultimate goal of this paper is to prove the following statements for semisimple Lie algebras.

1. Every irreducible representation has a highest weight.
2. Two irreducible representations with the same weight are equivalent.
3. The highest weight of every irreducible representation is a dominant integral element.
4. Every dominant integral element occurs as the highest weight of an irreducible representation.

Definition 4.1. \((\pi,V)\) is a highest weight cyclic representation with weight \( \mu_0 \) if there exists \( v \neq 0 \) in \( V \) such that...
(1) $v$ is a weight vector with weight $\mu_0$.
(2) $\pi(X_\alpha)v = 0$ for all positive roots $\alpha$.
(3) The smallest invariant subspace of $V$ containing $v$ is $V$.

From the definition, it follows that $\pi$ has a highest weight $\mu_0$, and the weight space of $\mu_0$ is one-dimensional. To prove this, consider the subspace $U$ of $V$ spanned by the elements of the form $\pi(Y_{i\alpha_1}) \ldots \pi(Y_{i\alpha_n})v$, where $Y_{i\alpha} \in \mathfrak{g}_{-\alpha}$, and $\alpha$'s are positive roots. Using commutation relations, it is easy to prove that $U$ is invariant. Then, by definition, $V = U$. It follows that $V$ is spanned by the weight vectors of the above form, which have weights lower than $\mu_0$. $\mu_0$ is the highest weight for $V$. Also, the only weight vectors with weight $\mu_0$ are multiples of $v$ because all other weight vectors have weights lower than $\mu_0$. Thus the weight space of $\mu_0$ is one-dimensional.

**Proposition 4.2.** Every irreducible representation of $\mathfrak{g}$ is a highest weight cyclic representation, with a unique highest weight $\mu_0$.

**Proof.** Every irreducible representation is a direct sum of its weight spaces. Finite dimensional representations have only finitely many weights. Thus we can find $\mu_0$ such that no weight $\mu \neq \mu_0$ is higher than it. There exists a corresponding weight vector $v$. Then, it follows that $\pi(X_\alpha)v = 0$ for all positive roots $\alpha$. Since the representation is irreducible, the only invariant subspace containing $v$ is $V$. Thus, the representation is a highest weight cyclic representation.

Conversely, every highest weight cyclic representation of $\mathfrak{g}$ is irreducible, because the smallest invariant subspace of $V$ containing $v$ is $V$ itself.

**Proposition 4.3.** Two irreducible representations of $\mathfrak{g}$ with the same highest weight are equivalent.

**Proof.** Suppose $(\pi, V)$ and $(\sigma, W)$ are two representations of $\mathfrak{g}$ with the same highest weight $\mu_0$. Let $v \in V$ and $w \in W$ be weight vectors of $\mu_0$ in each representation. Consider the representation $V \oplus W$. It is easy to show that $\mu_0$ is also the highest weight in $V \oplus W$. Let $U$ be the smallest invariant subspace of $V \oplus W$ that contains $(v, w)$. $U$ is a highest weight cyclic representation, and therefore irreducible. Define two projection maps $P_1 : V \oplus W \to V, (v, w) \mapsto v$ and $P_2 : V \oplus W \to W, (v, w) \mapsto w$. Since $P_1|_U$ and $P_2|_U$ are intertwining maps and $U, V, W$ are irreducible, the intertwining maps are invertible by Schur’s lemma. Thus $U, V, W$ are all isomorphic to each other.

**Proposition 4.4.** If $\pi$ is an irreducible representation of $\mathfrak{g}$, then the highest weight $\mu_0$ of $\pi$ is a dominant integral element.

**Proof.** Given the highest weight $\mu_0$, its weight vector $v_0$, and a positive root $\alpha$, either $\mu_0 - \alpha$ is a root or $\pi(Y_\alpha)v_0 = 0$. Inductively, there exists an integer $m \geq 0$ such that $\mu_0 - ma$ is a root for all $n \leq m$ and $\pi(Y_\alpha)v_m = 0$ where $v_k$ is a weight vector of $\mu_0 - k\alpha$. Using commutation relations and induction, it can be proven that $\pi(X_\alpha)v_k = [k\mu(H_\alpha) - k(k - 1)]v_{k-1}$ holds for all $k > 0$. Thus, when $k = m + 1$, since $v_{m+1} = 0$, $(m + 1)(\mu_0(H_\alpha) - m) = 0$. It follows that $\mu_0(H_\alpha)$ is a nonnegative integer.

The only thing left to be shown now is that every dominant integral element is the highest weight of an irreducible representation. This is a nontrivial task. Before I finish the paper, I will briefly present Verma modules to construct such
representation. A Verma module is an infinite-dimensional representation on a vector space $V_\mu$ that has a highest weight $\mu$. It can be defined for every element $\mu$ of $\mathfrak{h}^*$, not necessarily an integral element, but it is not finite-dimensional. Thus, it doesn’t match our requirements by itself. However, if $\mu$ is a dominant integral weight, then $V_\mu$ contains a maximal invariant subspace $U_\mu$ such that $V_\mu/U_\mu$ is finite dimensional, irreducible, and has highest weight $\mu$.

We can construct $V_\mu$ by looking at its properties. Since $\mu$ is a highest weight, the representation $\pi$ of $\mathfrak{g}$ on $V_\mu$ should satisfy $\pi(X_\alpha) = 0$ for all positive roots $\alpha$. Also, since $\mu$ is a weight, $\pi(H - \mu(H)1) = 0$ for all $H \in \mathfrak{h}$. It turns out that we can always find such a representation, using the concept of universal enveloping algebra. The universal enveloping algebra of $\mathfrak{g}$ always have an ideal generated from $X_\alpha$’s and $(H - \mu(H)1$’s, so we can take the quotient of the universal enveloping algebra by the ideal, which is isomorphic to $V_\mu$.

Now we have the theorem of highest weight, and we can classify all finite-dimensional representations of $\mathfrak{g}$ according to it.

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