INNER PRODUCT SPACES AND FOURIER SERIES

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Abstract. The aim of this paper is to discuss the fundamentals of Fourier analysis with a focus on relevant inner product spaces. We begin with a brief introduction to inner product spaces. We then explore the space of periodic complex integrable functions and conclude with some results about the convergence of Fourier series.

Contents

Fourier analysis is the study of how general functions can be represented as weighted sums of trigonometric functions. Specifically, sine and cosine functions form a basis for the space of periodic integrable functions. Linear algebra thus plays an important role in Fourier analysis. This paper briefly introduces inner product spaces and Fourier series before culminating in some results that make clear the connection between these two areas.

1. Inner product spaces

In this section, we present the subject of inner product spaces. This will be applied to Fourier analysis in the following section.

Definition 1.1. Let V be a complex vector space. A **complex inner product** on V is a binary operation $< \cdot, \cdot >: V \times V \to V$ which satisfies the following for all $x, y, z \in V$ and $a, b \in \mathbb{C}$:

1. conjugate symmetry: $\langle x, y \rangle = \overline{\langle y, x \rangle}$
2. linearity in the first term: $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$
3. non-negativity: $\langle x, x \rangle \geq 0$
4. non-degeneracy: $\langle x, x \rangle = 0$ if and only if $x = 0$.

Inner products on real vector spaces are defined in a similar way. Going forward, "inner product" will usually mean "complex inner product."

Definition 1.2. A complex vector space with an inner product is called a (complex) **inner product space**.

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Throughout this section, V will denote an arbtirary inner product space. The following example gives valuable intuition for studying any V.

Example 1.3. If **x, y** are in the vector space $\mathbb{C}^n$, define their inner product to be $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{k=1}^{n} x_k \overline{y_k}$. This is called the (complex) standard inner product.

In fact, inner product spaces generalize the notion of the standard inner product on $\mathbb{R}^n$ or $\mathbb{C}^n$. We can therefore think of the inner product of two vectors as encoding some sense of "overlap" between the vectors.

Definition 1.4. Let $x \in V$. The **norm** of x is defined by $\|x\| = \sqrt{\langle x, x \rangle}$.

Thus, just as the norm of a vector in $\mathbb{R}^n$ is its magnitude, the norm of a vector in a general inner product space is essentially its "length". Moreover, we can think of the norm of the difference of two vectors (e.g. $\|x - y\|$) as being the distance between the vectors.

Definition 1.5. Let $x, y \in V$. If $\langle x, y \rangle = 0$, then we say that x and y are **orthogonal**.

Definition 1.6. Let $\{x_1, x_2, ..., x_n\}$ be a subset of V. If every pair of distinct x_j, x_k is orthogonal, then the set is called **orthogonal**. If, in addition, $\|x_k\| = 1$ for all k, the set is called **orthonormal**.

Example 1.7. The standard basis for the complex vector space $\mathbb{C}^n$ is

$$\{e_1 = (1, 0, ..., 0), e_2 = (0, 1, ..., 0), ..., e_n = (0, 0, ..., 1)\}.$$

This is an orthonormal set of vectors.

We will use the following familiar results in the next section.

Lemma 1.8. *Let $x, y \in V$. Then $\langle x + y, x + y \rangle = \langle x, x \rangle + \langle y, y \rangle + \langle x, y \rangle + \langle y, x \rangle$.*

Proof.
$$\begin{aligned}
\langle x + y, x + y \rangle &= \overline{\langle x + y, x + y \rangle} \\
&= \overline{\langle x, x + y \rangle + \langle y, x + y \rangle} \\
&= \overline{\langle x, x + y \rangle} + \overline{\langle y, x + y \rangle} \\
&= \langle x + y, x \rangle + \langle x + y, y \rangle \\
&= \langle x, x \rangle + \langle y, y \rangle + \langle x, y \rangle + \langle y, x \rangle.
\end{aligned}$$
□

Theorem 1.9. *(Pythagorean theorem) Suppose $x, y \in V$ are orthogonal. Then $\|x\|^2 + \|y\|^2 = \|x + y\|^2$.*

Proof.
$$\begin{aligned}
\|x\|^2 + \|y\|^2 &= \langle x, x \rangle + \langle y, y \rangle \\
&= \langle x, x \rangle + \langle y, y \rangle + \langle x, y \rangle + \langle y, x \rangle \text{ (by orthogonality of x, y)} \\
&= \langle x + y, x + y \rangle \text{ (by Lemma 1.8)} \\
&= \|x + y\|^2.
\end{aligned}$$
□

Lemma 1.10. *(Cauchy-Schwarz) Let $x, y \in V$. Then $|\langle x, y \rangle| \leq \|x\| \|y\|$.*

Proof. If $y = 0$, we have $|\langle x, y\rangle| = |\langle x, 0\rangle| = |\overline{\langle 0, x\rangle}| = |\overline{\langle x, x\rangle + \langle -x, x\rangle} = 0$
$= \|x\|\|y\|$.

If $y \neq 0$, let $c = \frac{\langle x,y\rangle}{\langle y,y\rangle}$. Notice that

$$\begin{aligned}\langle x - cy, y\rangle &= \langle x, y\rangle - c\langle y, y\rangle \\ &= \langle x, y\rangle - \frac{\langle x, y\rangle}{\langle y, y\rangle}\langle y, y\rangle \\ &= \langle x, y\rangle - \langle x, y\rangle \\ &= 0.\end{aligned}$$

Then $x - cy$ and y are orthogonal, and we can apply Theorem 1.9 to get

$$\begin{aligned}\|x\|^2 &= \|x - cy + cy\|^2 = \|x - cy\|^2 + \|cy\|^2 \\ &\geq \|cy\|^2 = \langle cy, cy\rangle = c\langle y, cy\rangle = c\overline{\langle cy, y\rangle} = c^2\overline{\langle y, y\rangle} = |c|^2\langle y, y\rangle = |c|^2\|y\|^2 \\ &= \frac{\langle x, y\rangle^2}{\langle y, y\rangle^2}\|y\|^2.\end{aligned}$$

This implies that $\|x\|^2\langle y, y\rangle \geq \frac{\langle x,y\rangle^2}{\langle y,y\rangle}\langle y, y\rangle$. Thus, we have $\|x\|^2\|y\|^2 \geq \langle x, y\rangle^2$. Taking square roots gives the desired inequality. □

Theorem 1.11. *(Triangle inequality) Let $x, y \in V$. Then $\|x + y\| \leq \|x\| + \|y\|$.*

Proof.
$$\begin{aligned}(\|x\| + \|y\|)^2 &= \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \\ &= \langle x, x\rangle + \langle y, y\rangle + 2\|x\|\|y\| \\ &\geq \langle x, x\rangle + \langle y, y\rangle + 2|\langle x, y\rangle| \text{ (by Lemma 1.11)} \\ &= \langle x, x\rangle + \langle y, y\rangle + |\langle x, y\rangle| + |\overline{\langle y, x\rangle}| \\ &\geq \langle x, x\rangle + \langle y, y\rangle + \langle x, y\rangle + \langle y, x\rangle \\ &= \langle x + y, x + y\rangle \text{ (by Lemma 1.8)} \\ &= \|x + y\|^2.\end{aligned}$$

Taking square roots gives the desired inequality. □

We now introduce another important object.

Definition 1.12. Let W be a subspace of V. Let $x \in V$. A **projection** of x onto W is a vector $y \in W$ where $x - y$ is orthogonal to every vector in W.

If W is finite-dimensional, then a projection of a vector onto W exists and is unique. We do not give full justification here, but the proof of this fact follows from the next two propositions.

Proposition 1.13. *Let $E = \{e_1, e_2, ..., e_N\}$ be an orthonormal subset of V and let $x \in V$. Define $c_n = \langle x, e_n\rangle$ and $s = \sum_{n=1}^{N} c_n e_n$. Then s is a projection of x onto the span of E.*

Proof. Let $e_k \in E$ be arbitrary. We will start by showing that $x - s$ is orthogonal to e_k. Since $x - s = x + \sum_{n=1}^{N} -c_n e_n$, we have by linearity of the inner product that

$$\langle x - s, e_k\rangle = \langle x, e_k\rangle - c_1\langle e_1, e_k\rangle - ... - c_k\langle e_k, e_k\rangle - ... - c_N\langle e_N, e_N\rangle.$$

By the orthonormality of E, this is equal to $\langle x, e_k\rangle - c_k$. Thus, we have that $\langle x - s, e_k\rangle = c_k - c_k = 0$, so $x - s$ and e_k are orthogonal.

Now let y be an arbitrary vector in the span of E. We can represent y as a linear combination of vectors in E, $y = \sum_{n=1}^{N} b_n e_n$. Then

$$\begin{aligned}\langle x - s, y\rangle &= \overline{\langle y, x - s\rangle}\\ &= \overline{\langle b_1 e_1 + b_2 e_2 + ... + b_N e_N, x - s\rangle}\\ &= \overline{b_1\langle e_1, x - s\rangle} + \overline{b_2\langle e_2, x - s\rangle} + ... + \overline{b_N\langle e_N, x - s\rangle}\\ &= \overline{b_1}\langle x - s, e_1\rangle + \overline{b_2}\langle x - s, e_s\rangle + ... + \overline{b_N}\langle x - s, e_N\rangle\\ &= 0.\end{aligned}$$

Since y was arbitrary, we have shown that s is the projection of x onto the span of E. □

Proposition 1.14. *Let $E = \{e_1, e_2, ..., e_n\}$ be an orthonormal subset of V and let $x \in V$. Define $c_n = \langle x, e_n\rangle$ and $s = \sum_{n=1}^{N} c_n e_n$. Then for any y in the span of E, we have $\|x - s\| \leq \|x - y\|$.*

Proof. Fix an arbitrary y in the span of E. By the previous proposition, we have

$$\langle x - s, y - s\rangle = \langle x - s, y\rangle + \langle x - s, -s\rangle = 0.$$

Then we can apply Theorem 1.9 to get

$$\begin{aligned}\|x - s\|^2 + \|y - s\|^2 &= \|x - s\|^2 + \|s - y\|^2\\ &= \|x - s + s - y\|^2\\ &= \|x - y\|^2.\end{aligned}$$

This implies that $\|x - s\|^2 \leq \|x - s\|^2 + \|y - s\|^2 = \|x - y\|^2$. □

This proposition says that the projection of x onto the span of E gives a better approximation to x than any other vector in the span of E.

2. Fourier series

In this section, we will discuss the space of complex periodic integrable functions and their representation by weighted sums of trigonometric functions. We begin by describing this space before giving a definition of Fourier series. We then present some results about Fourier series in the context of the theory of inner product spaces constructed in the previous section. Bessel's inequality and Parseval's identity provide a relation between norms of functions and their Fourier coefficients. The best approximation proposition recalls a conclusion from the previous section about the distance between vectors and their projections. The most important proof we give is that of the mean square convergence theorem, which describes one sense in which the Fourier series of functions converge.

Proposition 2.1. *Let R denote the vector space of 2π-periodic complex-valued Riemann integrable functions. Then R is an inner product space with the inner product defined by*

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)}$$

and norm defined by

$$\|f\|^2 = \langle f, f \rangle = \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 dx$$

for $f, g \in R$.

Proof. Let $f, g, h \in R$, with $f(x) = a(x) + ib(x)$ and $g(x) = c(x) + id(x)$.

First, we need to check that our operation is conjugate symmetric. We have

$$\begin{aligned}
\langle f, g \rangle &= \frac{1}{2\pi} \int_0^{2\pi} f(x)\overline{g(x)} dx \\
&= \frac{1}{2\pi} \int_0^{2\pi} [(a(x) + ib(x))(c(x) - id(x))] dx \\
&= \frac{1}{2\pi} \int_0^{2\pi} [a(x)c(x) - b(x)d(x)] dx - \frac{i}{2\pi} \int_0^{2\pi} [a(x)d(x) - b(x)c(x)] dx \\
&= \frac{1}{2\pi} \int_0^{2\pi} [(c(x) + id(x)(a(x) - b(x))] dx \\
&= \frac{1}{2\pi} \int_0^{2\pi} g(x)\overline{f(x)} dx \\
&= \overline{\langle g, f \rangle}.
\end{aligned}$$

Next, we need to show that the operation is linear. Let $\alpha, \beta \in \mathbb{C}$. Then

$$\begin{aligned}
\langle \alpha f + \beta g, h \rangle &= \frac{1}{2\pi} \int_0^{2\pi} [\alpha f(x) + \beta g(x)]\overline{h(x)} dx \\
&= \frac{1}{2\pi} \int_0^{2\pi} [\alpha f(x)\overline{h(x)} + \beta g(x)\overline{h(x)}] dx \\
&= \frac{\alpha}{2\pi} \int_0^{2\pi} f(x)\overline{h(x)} dx + \frac{\beta}{2\pi} \int_0^{2\pi} g(x)\overline{h(x)} dx \\
&= \alpha\langle f, h \rangle + \beta\langle g, h \rangle.
\end{aligned}$$

Finally, we need to show that the operation is non-negative and non-degenerate. Since $|f(x)|^2$ is always non-negative, we must also always have

$$\langle f, f \rangle = \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 dx \geq 0.$$

If $f = 0$, then

$$\langle f, f \rangle = \frac{1}{2\pi} \int_0^{2\pi} 0 dx = 0.$$

On the other hand, if

$$\langle f, f \rangle = \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 = 0,$$

the function must be 0 everywhere, since the integrand never takes on negative values. Thus, $\langle f, f \rangle = 0$ if and only if $f = 0$. □

We next introduce an orthonormal subset of R that is essential to Fourier analysis.

Proposition 2.2. *Let* $e_n(x) = e^{inx}$. *Then the set* $\{e_n\}_{n \in \mathbb{Z}}$ *is orthonormal.*

Proof. Let $j, k \in \mathbb{Z}$. Then

$$\begin{aligned}\langle e_j, e_k \rangle &= \frac{1}{2\pi}\int_0^{2\pi} e^{ijx}\overline{e^{ikx}}dx \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{ijx}e^{-ikx}dx \\ &= \frac{1}{2\pi}\int_0^{2pi} e^{(j-k)ix}dx\end{aligned}$$

If $j = k$, this expression equals $\frac{1}{2\pi}\int_0^{2\pi} e^0 dx = 1$.
If $j \neq k$, this expression equals $\frac{1}{2\pi(j-k)i}e^{(j-k)ix}\big]_0^{2\pi} = 0$.
In summary, we have

$$\langle e_j, e_k \rangle = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

and so $\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal set.

□

Definition 2.3. Let $f \in R$ and $n \in \mathbb{Z}$. The nth **Fourier coefficient** of f is

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx.$$

The **Fourier series** of f is

$$\sum_{n=-\infty}^{\infty} a_n e^{inx}.$$

The Nth **partial sum** of the Fourier series of f is

$$s_N(f, x) = \sum_{n=-N}^{N} a_n e^{inx}.$$

The central idea of Fourier series is that the set $\{e_n\}_{n\in\mathbb{Z}}$ is a basis for R; that is, every periodic complex integrable function can be uniquely represented as a linear combination of sines and cosines. The following results are a step towards demonstrating this concept.

Proposition 2.4. *Let $f \in R$ and suppose $\{o_n\}_{n\in\mathbb{Z}}$ is an orthonormal subset of R. Define $a_n = \langle f, o_n \rangle$ and $t_N(f) = \sum_{|n|\leq N} a_n o_n$. Then $t_N(f)$ is the projection of f onto the span of $\{o_n\}_{|n|\leq N}$.*

Proof. Let $o_k \in \{o_n\}_{|n|\leq N}$ be arbitrary. Then

$$\begin{aligned}\langle f - t_N(f), o_k \rangle &= \langle f - \sum_{|n|\leq N} a_n o_n, o_k \rangle \\ &= \langle f, o_k \rangle - \sum_{|n|\leq N} a_n \langle o_n, o_k \rangle \\ &= \langle f, o_k \rangle - a_k \\ &= a_k - a_k \\ &= 0.\end{aligned}$$

Now let $y = \sum_{|n|\leq N} c_n o_n$ with $c_n \in \mathbb{C}$ be some vector in the span of $\{o_n\}_{|n|\leq N}$. Then we have

$$\begin{aligned}
\langle f - t_N(f), y\rangle &= \overline{\langle \sum_{|n|\leq N} c_n o_n, f - t_N(f)\rangle} \\
&= \overline{\langle c_{-N} o_{-N}, f - t_N(f)\rangle} + \overline{\langle c_{-N+1} o_{-N+1}, f - t_N(f)\rangle} + ... + \overline{\langle c_N o_N, f - t_N(f)\rangle} \\
&= \overline{c_{-N}} \langle f - t_N(f), o_{-N}\rangle + \overline{c_{-N+1}} \langle f - t_N(f), o_{-N+1}\rangle + ... + \overline{c_N}\langle f - t_N(f), o_N\rangle \\
&= 0.
\end{aligned}$$

.

Thus, $f - t_N(f)$ is orthogonal to the span of $\{o_n\}_{|n|\leq N}$. □

Proposition 2.5. *(Bessel's inequality) Let $f \in R$ and suppose $\{o_n\}_{n\in\mathbb{Z}}$ is an orthonormal subset of R. Let $a_n = \langle f, o_n\rangle$. Then*

$$\sum_{n=-\infty}^{\infty} |a_n|^2 \leq \|f\|^2.$$

Proof. Let $t_N(f) = \sum_{|n|\leq N} a_n o_n$ as before. Since $t_N(f)$ is in the span of $\{o_n\}_{|n|\leq N}$, it is orthogonal to $f - t_N(f)$ by Proposition 2.4. Thus, we can apply Theorem 1.9 to get

$$\begin{aligned}
\|f\|^2 &= \|f - t_N(f) + t_N(f)\|^2 \\
&= \|f - t_N(f)\|^2 + \|t_N(f)\|^2.
\end{aligned}$$

Notice that

$$\begin{aligned}
\|t_N(f)\|^2 &= \| \sum_{|n|\leq N} a_n o_n\|^2 \\
&= \sum_{|n|\leq N} \langle a_n o_n, \sum_{|n|\leq N} a_n o_n\rangle \\
&= \sum_{|n|\leq N} a_n \langle o_n, \sum_{|n|\leq N} a_n o_n\rangle \\
&= \sum_{|n|\leq N} a_n \overline{\langle \sum_{|n|\leq N} a_n o_n, o_n\rangle} \\
&= \sum_{|n|\leq N} a_n \sum_{|n|\leq N} \overline{\langle a_n o_n, o_n\rangle} \\
&= \sum_{|n|\leq N} a_n \overline{a_n} \sum_{|n|\leq N} \overline{\langle o_n, o_n\rangle} \\
&= \sum_{|n|\leq N} |a_n|^2.
\end{aligned}$$

Then we have

$$\|f\|^2 = \|f - t_N(f)\|^2 + \sum_{|n|\leq N} |a_n|^2$$

.

Thus,

$$\sum_{|n|\leq N} |a_n|^2 \leq \|f - t_N(f)\|^2 + \sum_{|n|\leq N} |a_n|^2 = \|f\|^2.$$

□

Proposition 2.6. *(Best approximation) Let $f \in R$ with Fourier coefficients a_n. Then*

$$\|f - s_N(f)\| \le \|f - \sum_{|n|\le N} c_n e_n\|$$

for any $c_n \in \mathbb{C}$.

Proof. First, notice that

$$\begin{aligned}\sum_{|n|\le N} c_n e_n &= \sum_{|n|\le N} a_n e_n - \sum_{|n|\le N} (a_n e_n - c_n e_n)\\ &= s_N(f) - \sum_{|n|\le N} (a_n - c_n) e_n.\end{aligned}$$

This gives us that

$$\|f - \sum_{|n|\le N} c_n e_n\|^2 = \|f - s_N(f) + \sum_{|n|\le N} (a_n - c_n) e_n\|^2.$$

Since $f - s_N(f)$ is orthogonal to $\sum_{|n|\le N}(a_n - c_n)e_n$ by Proposition 2.4, we can use Theorem 1.9 to show that the right hand side of this equation equals

$$\|f - s_N(f)\|^2 + \|\sum_{|n|\le N} (a_n - c_n) e_n\|^2.$$

But this expression is greater than or equal to $\|f - s_N(f)\|^2$.
Thus,

$$\|f - s_N(f)\|^2 \le \|f - \sum_{|n|\le N} c_n e_n\|^2$$

and the desired inequality follows immediately. □

In words, this proposition says that the partial sum of the Fourier series of f gives a better approximation to f than any linear combination of vectors in $\{e_n\}_{n\in\mathbb{Z}}$ with non-Fourier coefficients. Notice that this is actually a special case of Proposition 1.14.

We will use this result to prove the following theorem.

Theorem 2.7. *(Mean square convergence) Suppose $f \in R$. Then*

$$\lim_{N\to\infty} \|f - s_N(f)\| = 0.$$

Proof. Let $f \in R$. If f is continuous on $[0, 2\pi]$, then we can use Lemma 3.1. Fix $\epsilon > 0$. There exists a function $P(x) = \sum_{-M}^{M} c_n e^{inx}$ such that $|f(x) - P(x)| < \epsilon$ for all x. Then we have

$$\begin{gathered}|f(x) - P(x)|^2 < \epsilon^2\\ \frac{1}{2\pi}\int_0^{2\pi} |f(x) - P(x)|^2 dx < \frac{1}{2\pi}\int_0^{2\pi} \epsilon^2 dx\\ \|f - P\| < \epsilon.\end{gathered}$$

The previous proposition tells us that for all $N \ge M$, we have

$$\|f - s_N(f)\| \le \|f - P\| < \epsilon$$

so the limit is 0 when f is continuous.

If f is not continuous, then we need to use Lemma 3.2. Let B denote the least upper bound of $|f|$. Then there exists a continuous function g such that

$$\sup_{x\in[0,2\pi]} |g(x)| \leq B$$

and

$$\lim_{k\to\infty} \int_0^{2\pi} |f(x) - g(x)|dx < \epsilon^2.$$

This gives us that

$$\begin{aligned} \|f - g\|^2 &= \frac{1}{2\pi} \int_0^{2\pi} |f(x) - g(x)|^2 dx \\ &\leq \frac{2B}{2\pi} \int_0^{2\pi} |f(x) - g(x)|dx \\ &\leq C\epsilon^2, \end{aligned}$$

where C is a constant. So we have

$$\|f - g\| \leq \sqrt{C}\epsilon.$$

Using Lemma 3.1 again, there exists a function $Q(x) = \sum_{-M}^{M} d_n e^{inx}$ with $d_n \in \mathbb{C}$ such that $|g(x) - Q(x)| < \epsilon$. Following the same logic as before, this implies that

$$\|g - Q\| < \epsilon.$$

Applying the triangle inequality, we have

$$\|f - Q\| \leq \|f - g\| + \|g - Q\| < \sqrt{C}\epsilon + \epsilon = \epsilon(\sqrt{C} + 1).$$

Finally, the previous proposition gives us that

$$\|f - s_N(f)\| \leq \|f - Q\| < \epsilon(\sqrt{C} + 1).$$

□

This theorem says that the partial sum of the Fourier series of f approximates f with increasing accuracy as the degree of the sum approaches infinity. This is one sense in which the Fourier series of a function converges to the function. It implies a special case of Bessel's inequality that highlights a connection between the respective norms in $\mathbb{C}^n$ and R.

Corollary 2.8. *(Parseval's identity) If $f \in R$, then*

$$\sum_{n=-\infty}^{\infty} |a_n|^2 = \|f\|^2.$$

Proof. From the proof of Proposition 2.4, we have the equality

$$\|f - s_N(f)\|^2 + \sum_{|n|\leq N} |a_n|^2 = \|f\|^2.$$

The identity follows from taking the limit of both sides as N approaches infinity and applying the previous theorem. □

Appendix

The proofs of the following lemmas can be found in [1] or [3].

Lemma 2.9. *Suppose f is a continuous 2π-periodic function. Then for all $\epsilon > 0$, there exists a function of the form $P(x) = \sum_{-M}^{M} c_n e^{inx}$ with $c_n \in \mathbb{C}$ such that $|f(x) - P(x)| < \epsilon$ for all x.*

Lemma 2.10. *Let $f \in R$. Suppose f is bounded by B. Then there exists a sequence $\{f_k\}_{k=1}^{\infty}$ of continuous 2π-periodic functions such that*

$$\sup_{x\in[0,2\pi]} |f_k(x)| \leq B$$

for all positive $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} \int_0^{2\pi} |f(x) - f_k(x)| dx = 0.$$

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