VISCOSITY SOLUTIONS OF THE EIKONAL EQUATIONS

JINGHONG MIAO

Abstract. Viscosity solutions form a general theory of “weak” solutions that applies to certain nonlinear partial differential equations of first and second order. The eikonal equation is a nonlinear PDE related to wave propagation. This paper will investigate how distance functions are viscosity solutions of eikonal equations. We start by showing that the Euclidean distance is a viscosity solution of the homogeneous eikonal equation. Some properties in viscosity theory will be introduced, and a homogenization result will be studied.

Contents

1. Introduction

The eikonal equation arises from the solution for a differential equation related to wave propagation. (The word “eikonal” seems to be an interesting choice here; it derives from εἰκών, meaning image/reflection in Greek.)

Let Ω be a bounded, open set in $\mathbb{R}^n$. A general form of the eikonal equation is

$$\begin{cases} a(x)^{-1}\|Du(x)\| = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

A simple case of the eikonal equation is when $a(x)$ is a constant function. Given a bounded, open set Ω in $\mathbb{R}^n$, the distance (associated with Euclidean norm) to the boundary of Ω is not everywhere differentiable so it cannot be a solution of (1.1) in the literal sense, but it is a *viscosity solution* of the eikonal equation.

The concept of viscosity solutions was introduced by Crandall and Lions in

Date: AUGUST 29, 2020.

their 1983 paper titled "Viscosity Solutions of Hamilton-Jacobi Equations" after significant earlier contributions by Evans in 1980. In this paper, we will focus on some of the basic results regarding viscosity solutions and differential inequalities interpreted *in the viscosity sense*.

We will introduce some basic properties in the viscosity sense and will use these results in the proofs for viscosity solutions of eikonal equations. In particular, we will show that for a function $u : [a, b] \to \mathbb{R}$ that is possibly not everywhere differentiable, $u' \geq 0$ (resp. $u' \leq 0$) in the viscosity sense *if and only if* u is non-decreasing (resp. non-increasing), which indicates that in terms of monotonicity, the viscosity theory gives a parallel to the derivative of a differentiable function. Another result regards the Lipschitz constant of a continuous function $u : \mathbb{R}^d \to \mathbb{R}$. When u is once continuously differentiable and Du is bounded, the Lipschitz constant is the supremum of $\|Du\|$. For u that is not everywhere differentiable, the analog in the viscosity sense shows that for some positive real number M, $\|Du\| \leq M$ in the viscosity sense *if and only if* the Lipschitz constant of u satisfies $Lip(u) \leq M$. Another important result is that, in general, if u is a smooth function, then u is a viscosity solution of a PDE *if and only if* u is a classical solution.

Besides the homogenous eikonal equation where $a(x) = 1$, we will also study the spatially heterogeneous eikonal equation where $a(x) : \mathbb{R}^d \to (0, \infty)$ is a smooth, 1-periodic function. If we define an a-distance based on the function $a(x)$, then the a-distance to the boundary is the unique viscosity solution of (1.1).

The last part of this paper will discuss a homogenization result for an eikonal equation. We will add a small parameter ϵ to the heterogeneous eikonal equation. Let $\Omega_0 \subseteq \mathbb{R}^d$ be a bounded, open set that satisfies $\Omega = \epsilon^{-1}\Omega_0$, and we consider

$$\begin{cases} a(\epsilon^{-1}x)^{-1}\|Du^\epsilon\| = 1 & \text{in } \Omega_0 \\ u^\epsilon = 0 & \text{on } \partial\Omega_0 \end{cases} \tag{1.2}$$

The coefficient $a(\epsilon^{-1}x)$ of (1.2) oscillates with period ϵ. That means that when ϵ is very small, a oscillates rapidly. The homogenization result shows that the viscosity solutions to this equation converge to some function $\bar{u}$. In fact, there is a norm $\overline{\varphi} : \mathbb{R}^d \to [0, \infty)$ depending only on the coefficient a such that $\bar{u}$ is a viscosity solution of the anisotropic eikonal equation:

$$\begin{cases} \overline{\varphi}(D\bar{u}) = 1 & \text{in } \Omega \\ \bar{u} = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

The function $\bar{u}$ turns out to be the distance function associated with the dual norm of $\overline{\varphi}$, denoted as $\overline{\varphi}^*$.

This paper only deals with a a very small part of the theory of viscosity solutions. More advanced treatments of the subject can be found in [1] and [4].

2. Euclidean Distance and the Eikonal Equation

2.1. **Homogenous Eikonal Equation.** Let $\Omega \subseteq \mathbb{R}^d$ be a bounded, open set. Consider the following eikonal equation:

$$\begin{cases} \|Dd\| = 1 & \text{in } \Omega \\ d = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

2.2. **Viscosity Solutions.** We start by defining viscosity solutions for the eikonal equation (2.1).

Definition 2.2. A continuous function $u : \overline{\Omega} \to \mathbb{R}$ is a *viscosity sub-solution* (resp. *viscosity super-solution*) of (2.1) if

(1) $u(x) \leq 0$ (resp. $u(x) \geq 0$) for all x in $\partial\Omega$,
(2) Given $r > 0$, $x_0 \in B(x_0, r) \subseteq \Omega$, if $\varphi : B(x_0, r) \to \mathbb{R}$ is a smooth function such that $u - \varphi$ has a local maximum (resp. minimum) at x_0, then $\|D\varphi(x_0)\| \leq 1$ (resp. $\|D\varphi(x_0)\| \geq 1$).

A continuous function $u : \overline{\Omega} \to \mathbb{R}$ is a *viscosity solution* of (2.1) if it is both a viscosity sub-solution and a viscosity super-solution.

We define $d : \overline{\Omega} \to [0, \infty)$ to be the Euclidean distance to the boundary:

$$d(x) = \min\{\|x - y\| \mid y \in \partial\Omega\}$$

Remark 2.3. In general, d may not be differentiable at every point in Ω. For example, consider $B(0, 1) \subseteq \mathbb{R}^2$. The distance function d is not differentiable at the center.

We will show that the distance function $d(x)$ is a viscosity solution of (2.1).

Proposition 2.4. *The distance function d is a viscosity sub-solution of* (2.1).

Proof. Suppose $r > 0$, $x_0 \in B(x_0, r) \subseteq \Omega$, and $\varphi : B(x_0, r) \to \mathbb{R}$ is a smooth function such that $d - \varphi$ has a local maximum at x_0. We want to show that

$$\|D\varphi(x_0)\| \leq 1$$

Since $d - \varphi$ has a local maximum at x_0, for any $x \neq x_0$, we have

$$\varphi(x) - \varphi(x_0) \geq d(x) - d(x_0)$$

Since $\partial\Omega$ is compact, for some $z_x, z_y \in \overline{\Omega}$ we have

$$\begin{aligned} d(x) &= \min\{\|x - z\| \mid z \in \partial\Omega\} = \|x - z_x\| \\ d(x_0) &= \min\{\|x_0 - z\| \mid z \in \partial\Omega\} = \|x_0 - z_y\| \end{aligned}$$

By the triangle inequality,

$$\begin{aligned} |d(x) - d(x_0)| &\leq \max\{|\|x - z_x\| - \|x_0 - z_x\||, |\|x - z_y\| - \|x_0 - z_y\||\} \\ |d(x) - d(x_0)| &\leq \|x - x_0\| \\ d(x) - d(x_0) &\geq -\|x - x_0\| \end{aligned}$$

So, the distance function is uniformly Lipschitz, and

$$\varphi(x) - \varphi(x_0) \geq -\|x - x_0\|$$

Consider $v \in B(0, 1)$. Since $\varphi(x)$ is differentiable at x_0, the directional derivative at x_0 is given by

$$D_v\varphi(x_0) = \langle D\varphi(x_0), v\rangle = \lim_{h\to 0^+} \frac{\varphi(x_0 + hv) - \varphi(x_0)}{h}$$

By the triangle inequality,

$$\langle D\varphi(x_0), v\rangle \geq \lim_{h\to 0^+} \frac{-\|x_0 + hv - x_0\|}{h}$$

Hence

$$\langle D\varphi(x_0), v\rangle \geq \lim_{h\to 0^+} \frac{-\|hv\|}{h}$$

$$\langle D\varphi(x_0), v\rangle \geq -\|v\| \tag{2.5}$$

By the Cauchy-Schwarz inequality,

$$-\|D\varphi(x_0)\|\|v\| \leq \langle D\varphi(x_0), v\rangle \leq \|D\varphi(x_0)\|\|v\|$$

Since (2.5) holds for any v,

$$-\|v\| \leq -\|D\varphi(x_0)\|\|v\|$$

and

$$\|D\varphi(x_0)\| \leq 1$$

Hence d is indeed a viscosity sub-solution. □

Proposition 2.6. *The distance function d is a viscosity super-solution of* (2.1).

Proof. Suppose $r > 0$, $x_0 \in B(x_0, r) \subseteq \Omega$, and $\varphi : B(x_0, r) \to \mathbb{R}$ is a smooth function such that $d - \varphi$ has a local minimum at x_0. We want to show that

$$\|D\varphi(x_0)\| \geq 1$$

Fix $v \in B(0, 1) \setminus \{0\}$ and consider $h \in (0, ||v||^{-1}r)$. Then, we have

$$x_0 + hv \in B(x_0, r)$$

Let $y_0 \in \partial\Omega$ be where $d(x_0)$ is attained. Since $d - \varphi$ has a local minimum at x_0, we have

$$\begin{aligned} d(x_0) - \varphi(x_0) &\leq d(x_0 + hv) - \varphi(x_0 + hv) \\ \varphi(x_0 + hv) - \varphi(x_0) &\leq d(x_0 + hv) - d(x_0) \\ \varphi(x_0 + hv) - \varphi(x_0) &\leq \|y_0 - (x_0 + hv)\| - \|y_0 - x_0\| \end{aligned}$$

Hence the directional derivative of φ at x_0 satisfies

$$\begin{aligned} D_v\varphi(x_0) = \langle D\varphi(x_0), v\rangle &= \lim_{h\to 0^+} \frac{\varphi(x_0 + hv) - \varphi(x_0)}{h} \\ &\leq \lim_{h\to 0^+} \frac{\|y_0 - (x_0 + hv)\| - \|y_0 - x_0\|}{h} \\ &= \lim_{h\to 0^+} \frac{\|x_0 - y_0 + hv\| - \|x_0 - y_0\|}{h} \end{aligned}$$

Let $f(x) = \|x - y_0\|$. A calculus exercise shows that

$$Df(x_0) = \frac{x_0 - y_0}{\|x_0 - y_0\|}$$

Therefore,

$$\langle D\varphi(x_0), v\rangle \leq \langle Df(x_0), v\rangle = \left\langle \frac{x_0 - y_0}{\|x_0 - y_0\|}, v \right\rangle$$

Note that this holds for all v in $B(0, 1) \setminus \{0\}$. We have

$$\begin{aligned} &\left\langle D\varphi(x_0) - \frac{x_0 - y_0}{\|x_0 - y_0\|}, v \right\rangle \leq 0 \\ -&\left\langle D\varphi(x_0) - \frac{x_0 - y_0}{\|x_0 - y_0\|}, v \right\rangle = \left\langle D\varphi(x_0) - \frac{x_0 - y_0}{\|x_0 - y_0\|}, -v \right\rangle \leq 0 \end{aligned}$$

Thus,

$$D\varphi(x_0) - \frac{x_0 - y_0}{\|x_0 - y_0\|} = 0$$
$$\|D\varphi(x_0)\| = 1$$

Hence d is a viscosity super solution. ☐

It follows from the two propositions that d is a viscosity solution. It is possible to show that d is the *unique* viscosity solution of (2.1), but we will not prove it here.

3. Viscosity Theory

In this section, we will introduce the properties in viscosity theory regarding monotonicity and Lipschitz continuity.

3.1. **Monotonicity.** We will show that for a continuous function $u : [a, b] \to \mathbb{R}$, the sign of u' "in the viscosity sense" is an analog of the sign of the derivative for a differentiable function. Namely, $u' \geq 0$ in the viscosity sense if and only if u is non-decreasing, and $u' \leq 0$ in the viscosity sense if and only if u is non-increasing.

Definition 3.1. A continuous function $u : [a, b] \to \mathbb{R}$ satisfies the differential inequality $u' \geq 0$ (resp. $u' \leq 0$) *in the viscosity sense* if for each $x_0 \in (x_0 - r, x_0 + r) \subseteq (a, b)$ and each smooth $\varphi : (x_0 - r, x_0 + r) \to \mathbb{R}$, if $u - \varphi$ has a local minimum (resp. maximum) at x_0, then $\varphi'(x_0) \geq 0$ (resp. $\varphi'(x_0) \leq 0$).

The following lemma will be used to prove the relationship between monotonicity and the sign of the derivative in the viscosity sense.

Lemma 3.2. *Suppose $u : [a, b] \to \mathbb{R}$ is continuous and $u' \geq 0$ in the viscosity sense in (a, b). Let $r > 0$. Assume there is an $x_0 \in (x_0 - r, x_0] \subseteq (a, b]$ and a smooth function $\varphi : (x_0 - r, x_0 + r) \to \mathbb{R}$ such that x_0 is the unique point in $(x_0 - r, x_0]$ where $u - \varphi$ attains its minimum. Then $\varphi'(x_0) \geq 0$.*

Proof. Let $\epsilon > 0$. Consider the function

$$g(x) = u(x) - \varphi(x) + \frac{\epsilon}{x_0 - x} \tag{3.3}$$

Since $g(x)$ is continuous, it attains a minimum on $(x_0 - \frac{r}{2}, x_0)$. We denote this point by x_ϵ and we want to show that x_ϵ converges to x_0.

Consider $\frac{r}{2} > \epsilon_1 > \epsilon_2 > 0$ and let $x_1, x_2 \in (x_0 - \frac{r}{2}, x_0)$ be the corresponding x_ϵ. We have

$$u(x_1) - \varphi(x_1) + \frac{\epsilon_1}{x_0 - x_1} \leq u(x_2) - \varphi(x_2) + \frac{\epsilon_1}{x_0 - x_2}$$
$$u(x_1) - \varphi(x_1) + \frac{\epsilon_2}{x_0 - x_1} \geq u(x_2) - \varphi(x_2) + \frac{\epsilon_2}{x_0 - x_2}$$

Combining,

$$\epsilon_2\Big(\frac{1}{x_0 - x_2} - \frac{1}{x_0 - x_1}\Big) \leq \epsilon_1\Big(\frac{1}{x_0 - x_2} - \frac{1}{x_0 - x_1}\Big)$$
$$\frac{1}{x_0 - x_2} - \frac{1}{x_0 - x_1} \geq 0$$

and

$$x_1 \leq x_2 \leq x_0$$

Hence, x_ϵ converges as ϵ goes to zero.

Now, we want to show that it converges uniquely to x_0. Suppose for contradiction that there is some $\bar{x} \neq x_0$ such that

$$\lim_{\epsilon \to 0^+} x_\epsilon = \bar{x}$$

So, for any $x \in (x_0 - r, x_0), x \neq \bar{x}$,

$$\lim_{\epsilon \to 0} g(\bar{x}) \leq \lim_{\epsilon \to 0} g(x)$$

$$\lim_{\epsilon \to 0} [u(\bar{x}) - \varphi(\bar{x}) + \frac{\epsilon}{x_0 - \bar{x}}] \leq \lim_{\epsilon \to 0} [u(x) - \varphi(x) + \frac{\epsilon}{x_0 - x}]$$

$$u(\bar{x}) - \varphi(\bar{x}) \leq u(x) - \varphi(x)$$

Hence, $u - \varphi$ attains a minimum at $\bar{x}$. Since x_0 is the unique point in $(x_0 - r, x_0]$ where the minimum of $u - \varphi$ is attained,

$$x_0 = \bar{x} = \lim_{\epsilon \to 0^+} x_\epsilon$$

Fix $\epsilon_0 > 0$ and let $\epsilon \in (0, \epsilon_0)$. Consider

$$\varphi_\epsilon(x) = \varphi(x) - \frac{\epsilon}{x_0 - x} \tag{3.4}$$

and note that by (3.3),

$$g(x) = u(x) - \varphi_\epsilon(x)$$

Since $u' \geq 0$ in the viscosity sense, $\varphi_\epsilon(x)$ is continuous and $g(x)$ attains a minimum at x_ϵ, we have

$$\varphi_\epsilon'(x_\epsilon) \geq 0$$

We take the derivative of (3.4) and get

$$\varphi'(x) = \varphi_\epsilon'(x) + \left(\frac{\epsilon}{x_0 - x}\right)'$$

$$\varphi'(x) = \varphi_\epsilon'(x) + \frac{\epsilon}{(x_0 - x)^2}$$

Therefore,

$$\varphi'(x_0) \geq \lim_{\epsilon \to 0^+} \varphi_\epsilon'(x_\epsilon) \geq 0$$

□

Now, we present the main results regarding monotonicity.

Proposition 3.5. *$u' \geq 0$ in the viscosity sense if and only if u is non-decreasing.*

Proof. First, suppose u is non-decreasing. Let $r > 0$, $x_0 \in (a, b)$. Consider the interval $(x_0 - r, x_0 + r) \subseteq (a, b)$ and a smooth function $\varphi : (x_0 - r, x_0 + r) \to \mathbb{R}$. Suppose $u - \varphi$ has a local minimum at x_0. We want to show that $\varphi'(x_0) \geq 0$.

Suppose for contradiction that the derivative satisfies

$$\varphi'(x_0) = \lim_{h \to 0^-} \frac{\varphi(x_0 + h) - \varphi(x_0)}{h} < 0$$

Since $u(x)$ is non-decreasing, for $h < 0$ we have

$$u(x_0) \geq u(x_0 + h)$$

Therefore, if h is sufficiently small,

$$\begin{aligned} \varphi(x_0 + h) - \varphi(x_0) > 0 \\ u(x_0) - \varphi(x_0) > u(x_0 + h) - \varphi(x_0 + h) \end{aligned}$$

This contradicts the fact that $u - \varphi$ attains a local minimum at x_0. Thus,

$$\varphi'(x_0) \geq 0$$

Next, suppose $u' \geq 0$ in the viscosity sense. To show that u is non-decreasing, we argue by contradiction. Assume that there are $x_1, x_2 \in (a, b)$ with $x_1 < x_2$ and $u(x_1) > u(x_2)$. Fix $\epsilon > 0$ and let $u_\epsilon(x) = u(x) + \epsilon x$. Assume that ϵ is sufficiently small such that $u_\epsilon(x_1) > u_\epsilon(x_2)$. Let

$$x_* = \inf \left\{ x \in (x_1, x_2) \mid u_\epsilon(x) \leq \frac{u_\epsilon(x_1) + u_\epsilon(x_2)}{2} \right\}$$

Since u_ϵ is continuous, by the intermediate value theorem

$$u_\epsilon(x_*) = \frac{u_\epsilon(x_1) + u_\epsilon(x_2)}{2}$$

Let $r = \frac{x_* - x_1}{2}$. Note that x_* is the infimum, so $u_\epsilon(x)$ attains its minimum in $(x_* - r, x_*]$ uniquely at x_*.

Since $u' \geq 0$ in the viscosity sense, and $u_\epsilon(x) = u(x) - (-\epsilon x)$ attains a unique minimum at x_* on $(x_* - r, x_*]$, by Lemma 3.2, we have

$$-\epsilon = -(\epsilon x)' \geq 0$$

This is a contradiction. There is no $x_1, x_2 \in (a, b)$ with $x_1 < x_2$ and $u(x_1) > u(x_2)$. Therefore, u is non-decreasing. □

Proposition 3.6. *$u' \leq 0$ in the viscosity sense if and only if u is non-increasing.*

Proof. First, suppose $u' \leq 0$ in the viscosity sense in (a, b). Let

$$v = -u$$

We want to show that $v' \geq 0$ in the viscosity sense. Let φ_v be a smooth function such that $v - \varphi_v$ attains local minimum at $x_0 \in (x_0 - r, x_0 + r) \subseteq (a, b)$. Let

$$\varphi = -\varphi_v$$

Note that

$$v - \varphi_v = -(u - \varphi)$$

Hence, $u - \varphi$ attains a local maximum at x_0. Since $u' \leq 0$ in the viscosity sense,

$$\varphi_v'(x_0) = -\varphi'(x_0) \geq 0$$

Therefore, $v' \geq 0$ in the viscosity sense. By Proposition 3.5, v is non-decreasing, so $u = -v$ is non-increasing. The proof of the other implication is similar. Thus, $u' \leq 0$ if and only if u is non-increasing. □

3.2. **Lipschitz Continuity.** Let M be a positive constant. We will show that for a continuous function $u : \mathbb{R}^d \to \mathbb{R}$, $\|Du\| \leq M$ in the viscosity sense if and only if the Lipschitz constant of u satisfies $Lip(u) \leq M$.

First, we recall the definition of Lipschitz continuity and Lipschitz constant.

Definition 3.7. A function $u : \mathbb{R}^d \to \mathbb{R}$ is called *uniformly Lipschitz continuous* if

$$\sup \left\{ \frac{|u(x) - u(y)|}{\|x - y\|} \mid x, y \in \mathbb{R}^d \right\} < \infty$$

Given a uniformly Lipschitz continuous function, we define its Lipschitz constant $Lip(u)$ by

$$Lip(u) = \sup \left\{ \frac{|u(x) - u(y)|}{||x - y||} \mid x, y \in \mathbb{R}^d \right\}$$

For u to satisfy a differential inequality in the viscosity sense, u does not need to be everywhere differentiable. However, if u is once continuously differentiable and it gradient is bounded, then $\|Du\|$ is well-defined, and the following lemma shows that $\|Du\| \leq M$ if and only if $Lip(u) \leq M$. The proof is a calculus exercise.

Lemma 3.8. *If $u : \mathbb{R}^d \to \mathbb{R}$ is once continuously differentiable and Du is bounded, then*

$$Lip(u) = \sup\{\|Du(x)\| \mid x \in \mathbb{R}^d\}$$

We now define the differential inequality $\|Du\| \leq M$ in the viscosity sense.

Definition 3.9. Let $M > 0$. A continuous function $u : \mathbb{R}^d \to \mathbb{R}$ satisfies $\|Du\| \leq M$ in the viscosity sense if for each $x_0 \in B(x_0, r) \subseteq \mathbb{R}^d$ and each smooth function $\varphi : B(x_0, r) \to \mathbb{R}$, if $u - \varphi$ has a local maximum at x_0, then

$$\|D\varphi(x_0)\| \leq M$$

The main result regarding the Lipschitz constant is the following proposition.

Proposition 3.10. *Let $M > 0$. Suppose $u : \mathbb{R}^d \to \mathbb{R}$ is continuous. The Lipschitz constant $Lip(u) \leq M$ if and only if $\|Du\| \leq M$ in the viscosity sense.*

Proof. First, suppose $Lip(u) \leq M$. Consider $v \in B(0, 1) \subseteq \mathbb{R}^d$, $h \in \mathbb{R}$. Suppose φ is a smooth function such that $u - \varphi$ has a local maximum at x_0. We have

$$u(x_0 + hv) - u(x_0) \leq \varphi(x_0 + hv) - \varphi(x_0)$$

By the definition of Lipschitz continuity,

$$|u(x_0 + hv) - u(x_0)| \leq M||hv||$$

Combining, we have

$$-M||hv|| \leq \varphi(x_0 + hv) - \varphi(x_0)$$

Consider the directional derivative

$$\begin{aligned} D_v\varphi(x_0) = \langle D\varphi(x_0), v\rangle &= \lim_{h \to 0^+} \frac{\varphi(x_0 + hv) - \varphi(x_0)}{h} \\ &\geq \lim_{h \to 0^+} \frac{-M||hv||}{h} \\ &= -M||v|| \end{aligned}$$

By the Cauchy-Shwarz inequality,

$$-\|D\varphi(x_0)\|\|v\| \leq \langle D\varphi(x_0), v\rangle$$

Thus, we have

$$\begin{aligned} -M\|v\| &\leq -\|D\varphi(x_0)\|\|v\| \\ \|D\varphi(x_0)\| &\leq M \end{aligned}$$

Let us suppose $\|Du\| \leq M$ in the viscosity sense. Assume that $u : \mathbb{R}^d \to \mathbb{R}$ is bounded. Let

$$K = \sup\left\{|u(x)| \mid x \in \mathbb{R}^d\right\}$$

Fix $x_0 \in \mathbb{R}^d$, $\epsilon > 0$, and let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be defined as

$$\varphi(x) = u(x_0) + (M + \epsilon)\|x - x_0\|$$

Then

$$u - \varphi = u(x) - u(x_0) - (M + \epsilon)\|x - x_0\|$$

and

$$u(x_0) - \varphi(x_0) = 0$$

Consider $B(x_0, \frac{2K}{M})$. The maximum of $u - \varphi$ is not attained on the boundary, since if $x \in \partial B(x_0, \frac{2K}{M})$,

$$u(x) - \varphi(x) \leq 2K - (M + \epsilon)\frac{2K}{M} < 0$$

Since $\|Du\| \leq M$ in the viscosity sense, if $u - \varphi$ attains a local maximum at some $x^* \in B(x_0, \frac{2K}{M})$, $x^* \neq x_0$,

$$\|D\varphi(x^*)\| \leq M$$

However, it contradicts the fact that

$$\|D\varphi(x^*)\| = M + \epsilon > M$$

Thus, $u - \varphi$ attains its local maximum in $B(x_0, R)$ at x_0, and $u - \varphi \leq 0$ with the equality attained at x_0 only. Taking $\epsilon \to 0^+$, we have

$$\begin{aligned} u(x) - u(x_0) - (M + \epsilon)\|x - x_0\| &\leq 0 \\ u(x) - u(x_0) &\leq M\|x - x_0\| \end{aligned}$$

Since we can define a φ for any $x_0 \in \mathbb{R}^d$, we have

$$|u(x) - u(y)| \leq M\|x - y\|$$

for $x, y \in \mathbb{R}^d$. So $Lip(u) \leq M$.

A technical argument that we omit can be used to extend from the bounded case to arbitrary continuous functions u satisfying $\|Du\| \leq M$. □

4. Homogenization of the Spatially Heterogeneous Eikonal Equation

Let Ω be a bounded, open subset of $\mathbb{R}^d$ and let $a : \mathbb{R} \to (0, \infty)$ be a smooth, 1-periodic function. Assume without loss of generality that there are constants λ, $\Lambda > 0$ such that $\lambda \leq a(x) \leq \Lambda$ for all $x \in \mathbb{R}^d$. Consider the following spatially heterogeneous eikonal equation

$$\begin{cases} a(x)^{-1}\|Du\| = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

We define the viscosity solution for (4.1) as follows:

Definition 4.2. A continuous function $u : \overline{\Omega} \to \mathbb{R}$ is a *viscosity sub-solution* (resp. *viscosity super-solution*) of (4.1) if $u \leq 0$ (resp. $u \geq 0$) for all $x \in \partial\Omega$, and for each $x_0 \in B(x_0, r) \subseteq \Omega$ and each smooth function $\varphi : B(x_0, r) \to \mathbb{R}$, if $u - \varphi$ has a local maximum (resp. minimum) at x_0, then

$$a(x_0)^{-1}\|D\varphi(x_0)\| \leq 1$$

(resp.

$$a(x_0)^{-1}\|D\varphi(x_0)\| \geq 1)$$

We have shown that the Euclidean distance solves the homogenous eikonal equation when $a(x) = 1$. Similarly, we can define a distance function based on a more general $a(x)$. It turns out that the new distance function solves (4.1). We define a length and a corresponding distance function as follows:

Definition 4.3. Given a path $\gamma \in C^1([0,\infty), \mathbb{R}^d)$ and $T > 0$, the a-length $L_a(\gamma, T)$ in $[0, T]$ is defined by

$$L_a(\gamma, T) = \int_0^T a(\gamma(t))\|\dot{\gamma}(t)\|dt$$

Definition 4.4. We define the distance function $d_a : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ as

$$d_a(x, y) = \inf\{L_a(\gamma, T) \mid T > 0, \gamma \in C^1([0,\infty), \mathbb{R}^d), \gamma(0) = x, \gamma(T) = y\}$$

Proposition 4.5. *The function $u : \overline{\Omega} \to \mathbb{R}$ defined by*

$$u(x) = \min\{d_a(x, y) \mid y \in \partial\Omega\}$$

is a viscosity solution of (4.1).

Remark 4.6. Let the function $d_\Omega : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ denote the Euclidean distance. It is easy to show that d_Ω is a special case of a distance function based on a-length. That is, when $a(x) = 1$, we have

$$L(\gamma, T) = \int_0^T \|\dot{\gamma}(t)\|dt$$

$$d_\Omega(x, y) = \inf\{L(\gamma, T) \mid T > 0, \gamma \in C^1([0,\infty), \mathbb{R}^d), \gamma(0) = x, \gamma(T) = y\}$$

4.1. **Microscopic view.** Let $\epsilon > 0$. Let $\Omega_0 \subseteq \mathbb{R}^d$ be a bounded, open set such that $\Omega = \epsilon^{-1}\Omega_0$. Consider the following eikonal equation

$$\begin{cases} a(\epsilon^{-1}x)^{-1}\|Du^\epsilon\| = 1 & \text{in } \Omega_0 \\ u^\epsilon = 0 & \text{on } \partial\Omega_0 \end{cases} \tag{4.7}$$

Comparing (4.1) and (4.7), we notice that when $\epsilon \to 0^+$, the set Ω gets large and larger, and the coefficient $a(x)$ remains 1-periodic. For (4.7), it can be showed that $a(\epsilon^{-1}x)$ oscillates with period ϵ, which gets finer and finer as $\epsilon \to 0^+$. The relationship between the viscosity solutions for (4.1) and (4.7) is presented in the next proposition.

Proposition 4.8. *Let u^ϵ be defined as*

$$u^\epsilon(x) = \epsilon u(\epsilon^{-1}x)$$

The function u^ϵ is a viscosity solution of (4.7) *if and only if u is a viscosity solution of* (4.1).

4.2. **Homogenization: Main Results.** In this section, we will introduce a theorem about the convergence of the viscosity solutions u^ϵ to the eikonal equation (4.7) and its limit point $\bar{u}$. We will prove the following theorem in the next subsection.

Theorem 4.9. *Let Ω_0 be a bounded, open subset in $\mathbb{R}^d$, and let (u^ϵ) be the viscosity solutions of* (4.7)

$$\begin{cases} a(\epsilon^{-1}x)^{-1}\|Du^\epsilon\| = 1 & \text{in } \Omega_0 \\ u^\epsilon = 0 & \text{on } \partial\Omega_0 \end{cases}$$

There is a function $\bar{u} : \overline{\Omega}_0 \to [0, \infty)$ such that u^ϵ converges to $\bar{u}$ uniformly in $\overline{\Omega}_0$ as $\epsilon \to 0^+$. There is a norm $\overline{\varphi} : \mathbb{R}^d \to [0, \infty)$ depending only on the coefficient a such that $\bar{u}$ is the unique viscosity solution of the anisotropic eikonal equation

$$\begin{cases} \overline{\varphi}(D\bar{u}) = 1 & \text{in } \overline{\Omega}_0 \\ \bar{u} = 0 & \text{on } \partial\Omega_0 \end{cases} \tag{4.10}$$

To prove Theorem 4.9, we consider the following "cell problem." Let $p \in \mathbb{R}^d$. There is a 1-periodic continuous function $\chi_p : \mathbb{R}^d \to \mathbb{R}$, called a "corrector," and a constant $\overline{\varphi}(p)$ such that χ_p is a viscosity solution of

$$a(y)^{-1}\|p + D\chi_p\| = \overline{\varphi}(p) \tag{4.11}$$

Remark 4.12. We define the viscosity solutions for (4.11) in a similar way as the previous ones. A continuous function $\chi : \mathbb{R}^d \to \mathbb{R}$ is a viscosity sub-solution (resp. super-solution) of (4.11) if for each $y_0 \in B(y_0, r) \subseteq \mathbb{R}^d$ and each smooth function $\varphi : B(y_0, r) \to \mathbb{R}$, if $\chi - \varphi$ has a local maximum (resp. minimum) at y_0, then

$$a(y_0)^{-1}\|p + D\varphi(y_0)\| \leq \overline{\varphi}(p)$$

(resp.

$$a(y_0)^{-1}\|p + D\varphi(y_0)\| \geq \overline{\varphi}(p)\ )$$

The existence of correctors is a nontrivial result that is beyond the scope of this paper, so we will state the following theorem without proof.

Theorem 4.13. *Given $p \in \mathbb{R}^d$, there is a $\overline{\varphi}(p) \in \mathbb{R}$ such that* (4.11) *has a 1-periodic continuous solution $\chi_p : \mathbb{R}^d \to \mathbb{R}$.*

The following proposition uses a typical kind of proof in viscosity theory. This is an important result that will be used to prove the properties of $\overline{\varphi}$.

Proposition 4.14. *(Comparison Principle) Fix $p \in \mathbb{R}^d$. If $\chi, \tilde{\chi}$ are 1-periodic continuous functions such that*

$$a(y)^{-1}\|p + D\chi\| \leq A$$

in the viscosity sense and

$$a(y)^{-1}\|p + D\tilde{\chi}\| \geq B$$

in the viscosity sense, then $B \leq A$.

Using the comparison principle, a short proof shows that $\overline{\varphi}$ is a well-defined function. We will show that $\overline{\varphi}$ is a norm.

Proposition 4.15. *$\overline{\varphi}$ is well defined. That is, given $p \in \mathbb{R}^d$, if there are two 1-periodic, continuous functions $\chi_p, \tilde{\chi}_p : \mathbb{R}^d \to \mathbb{R}$, and two constants $\overline{\varphi}(p), \tilde{\varphi}(p) \in \mathbb{R}$ such that $\chi_p, \tilde{\chi}_p$ are viscosity solutions of*

$$a(y)^{-1}\|p + D\chi_p\| = \overline{\varphi}(p)$$

and

$$a(y)^{-1}\|p + D\tilde{\chi}_p\| = \tilde{\varphi}(p)$$

then $\overline{\varphi}(p) = \tilde{\varphi}(p)$.

Definition 4.16. A function $\varphi : \mathbb{R}^d \to \mathbb{R}$ is a norm if

(1) $\varphi(v) \geq 0$ for all $v \in \mathbb{R}^d$
(2) $\varphi(v) = 0$ if and only if $v = 0$
(3) $\varphi(\alpha v) = |\alpha|\varphi(v)$ if $v \in \mathbb{R}^d$ and $\alpha \in \mathbb{R}$
(4) $\varphi(v + w) \leq \varphi(v) + \varphi(w)$

Proposition 4.17. *$\overline{\varphi}$ is a norm.*

Proof. Fix $p \in \mathbb{R}^d$. Let $\tilde{\chi}$ be a continuous, 1-periodic function.

(1) Let $\tilde{\chi}$ be a viscosity solution of (4.11). Suppose φ is a smooth function such that $\tilde{\chi} - \varphi$ attains a local minimum at y_0. Since $0 < \lambda \leq a(y) \leq \Lambda$,

$$a(y_0)^{-1}\|p + D\varphi(y_0)\| \geq \Lambda^{-1}\|p\|$$

Hence,

$$a(y)^{-1}\|p + D\tilde{\chi}\| \geq \Lambda^{-1}\|p\|$$

in the viscosity sense. Now, consider a different 1-periodic, continuous function χ_p such that χ_p is a viscosity solution of (4.11). In particular,

$$a(y)^{-1}\|p + D\chi_p\| \leq \overline{\varphi}(p)$$

By Proposition 4.14,

$$\overline{\varphi}(p) \geq \Lambda^{-1}\|p\| \geq 0$$

(2) Since $\overline{\varphi}(p) \geq \Lambda^{-1}\|p\|$, we have showed that $\overline{\varphi}(p) = 0$ only when $p = 0$. Let $\tilde{\chi} = 0$, $p = 0$. Since $\tilde{\chi}$ is a viscosity solution of

$$a(y)^{-1}\|p + D\tilde{\chi}\| = \overline{\varphi}(p)$$

we have $\overline{\varphi}(p) = 0$ if $p = 0$, and hence $\overline{\varphi}(p) = 0$ if and only if $p = 0$.

(3) Let χ_p be a 1-periodic, continuous viscosity solution of

$$a(y)^{-1}\|p + D\chi_p\| = \overline{\varphi}(p)$$

Let $\alpha \in \mathbb{R}$. Consider the 1-periodic, continuous function $\alpha\chi_p$. We want to show $\alpha\chi_p$ is a viscosity solution of

$$a(y)^{-1}\|\alpha p + D(\alpha\chi_p)\| = |\alpha|\overline{\varphi}(p)$$

First, we show that $\alpha\chi_p$ is a viscosity super-solution. Suppose ψ_p is a smooth function such that $\alpha\chi_p - \psi_p$ attains a local minimum at y_0, and suppose without loss of generality that $\alpha\chi_p(y_0) - \psi_p(y_0) = 0$. We want to show

$$a(y)^{-1}\|\alpha p + D\psi_p\| \geq |\alpha|\overline{\varphi}(p)$$

Let $\psi = \psi_p/\alpha$. Then, $\chi_p - \psi$ also attains a local minimum at y_0. Since χ_p is a viscosity solution of (4.11), we have

$$a(y)^{-1}\|p + D\psi\| \geq \overline{\varphi}(p)$$
$$a(y)^{-1}\|\alpha p + D\psi_p\| \geq |\alpha|\overline{\varphi}(p)$$

The case for viscosity sub-solution is similar. Notice that $\alpha p \in \mathbb{R}^d$ satisfies

$$a(y)^{-1}\|\alpha p + D\chi_{\alpha p}\| = \overline{\varphi}(\alpha p)$$

Therefore, by Proposition 4.15,

$$\overline{\varphi}(\alpha p) = |\alpha|\overline{\varphi}(p)$$

(4) Fix $p, q \in \mathbb{R}^d$. Let χ_p, χ_q be 1-periodic, continuous solutions of

$$a(y)^{-1}\|p + D\chi_p\| = \overline{\varphi}(p)$$
$$a(y)^{-1}\|q + D\chi_q\| = \overline{\varphi}(q)$$

Let

$$\chi = \chi_p + \chi_q$$

A technical argument that we omit shows that

$$a(y)^{-1}\|p + q + D\chi\| \leq \overline{\varphi}(p) + \overline{\varphi}(q)$$

in the viscosity sense. Note that χ_{p+q} is a viscosity solution of

$$a(y)^{-1}\|p + q + D\chi\| = \overline{\varphi}(p + q)$$

By Proposition 4.15, we have

$$\overline{\varphi}(p + q) \leq \overline{\varphi}(p) + \overline{\varphi}(q)$$

□

4.3. **Proof of Homogenization.** The main goal of this section is to prove Theorem 4.9. We will show that (u^ϵ), the viscosity solutions of (4.7), is pre-compact and converges to some $\bar{u}$. Moreover, $\bar{u}$ solves the anisotropic eikonal equation (4.10).

In the proof of pre-compactness, we will invoke the following theorem.

Theorem 4.18. *(Arzelà-Ascoli) If $\mathcal{F} \subseteq C(\overline{\Omega})$, then $\mathcal{F}$ is pre-compact if and only if it satisfies*

(1) (uniform boundedness) There is an $M > 0$ such that

$$\sup\{\|u\|_\infty \mid u \in \mathcal{F}\} \leq M$$

(2) (equi-continuity) For each $\zeta > 0$, there is a $\delta > 0$ such that if $x, y \in \overline{\Omega}$ satisfy $\|x - y\| < \delta$, then

$$\sup\{|u(x) - u(y)| \mid u \in \mathcal{F}\} < \zeta$$

Proposition 4.19. *(Pre-Compactness) Let Ω_0 be an open, bounded subset of $\mathbb{R}^d$ and let $(u^\epsilon)_{\epsilon>0}$ denote the solutions of (4.7). The set $\{u^\epsilon \mid \epsilon > 0\}$ is pre-compact in $C(\overline{\Omega}_0)$.*

Proof. We define the Euclidean distance to the boundary as

$$d_{\Omega_0} = \inf\{\|x-y\| \mid y \in \partial\Omega_0\}$$

Since $0 < \lambda \leq a(x) \leq \Lambda$, an application of Proposition 4.5 shows that

$$0 \leq u^\epsilon(x) \leq \Lambda d_{\Omega_0}$$

and

$$\|u^\epsilon\|_\infty \leq \Lambda d_{\Omega_0} \leq \Lambda \|d_{\Omega_0}\|_\infty$$
$$\sup\{\|u^\epsilon\|_\infty \mid \epsilon > 0\} \leq \Lambda \|d_{\Omega_0}\|_\infty$$

Hence, the set $\{u^\epsilon \mid \epsilon > 0\}$ satisfies uniform boundedness.

Next, we want to show equi-continuity. Fix $\zeta > 0$. If $x \in \overline{\Omega}_0 \setminus \Omega_0((2\Lambda)^{-1}\zeta)$, then

$$0 \leq u^\epsilon(x) \leq \Lambda d_{\Omega_0}(x) \leq \frac{\zeta}{2}$$

Let $\delta > 0$ be given by

$$\delta = \min\left\{\left(\max\left\{\Lambda, \frac{4M\Lambda}{\zeta}\right\}\right)^{-1}, \frac{1}{4\Lambda}\right\} \cdot \zeta$$

We want to show that if $x, y \in \overline{\Omega}_0$ and $\|x-y\| < \delta$, then

$$\sup\left\{|u^\epsilon(x) - u^\epsilon(y)| \mid \epsilon > 0\right\} \leq \zeta$$

Suppose $x, y \in \overline{\Omega}_0$, $\|x-y\| < \delta$, and $\epsilon > 0$. If $x \in \overline{\Omega}_0 \setminus \Omega_0((2\Lambda)^{-1}\zeta)$, then

$$d_{\Omega_0}(y) \leq d_{\Omega_0}(x) + \|y-x\| < \frac{\zeta}{2\Lambda} + \delta \leq \frac{\zeta}{\Lambda}$$

Note that

$$|u^\epsilon(x) - u^\epsilon(y)| \leq \max\left\{u^\epsilon(x), u^\epsilon(y)\right\} \leq \Lambda \max\left\{d_{\Omega_0}(x), d_{\Omega_0}(y)\right\} < \zeta$$

If $x \in \Omega_0((2\Lambda)^{-1}\zeta)$, then

$$d_{\Omega_0}(y) \geq d_{\Omega_0}(x) - \|y-x\| > \frac{\zeta}{2\Lambda} - \delta > \frac{\zeta}{4\Lambda}$$

Let $d > 0$ and let

$$\Omega_0(d) = \left\{x \in \mathbb{R}^d \mid d_{\Omega_0}(x) > d\right\}$$

and

$$M = \max\left\{|u^\epsilon(x)| \mid x \in \overline{\Omega}\right\}$$

Then, the proof of Proposition 3.10 shows that

$$\sup\left\{\frac{|u^\epsilon(x) - u^\epsilon(y)|}{\|x-y\|} \mid x, y \in \overline{\Omega_0(d)},\ \epsilon > 0\right\} \leq \max\left\{\Lambda, \frac{M}{d}\right\}$$

Hence if $x, y \in \overline{\Omega_0((4\Lambda)^{-1}\zeta)}$, we have

$$\frac{|u^\epsilon(x) - u^\epsilon(y)|}{\|x-y\|} \leq \max\left\{\Lambda, \frac{4M\Lambda}{\zeta}\right\}$$
$$|u^\epsilon(x) - u^\epsilon(y)| \leq \max\left\{\Lambda, \frac{4M\Lambda}{\zeta}\right\}\|x-y\| \leq \max\left\{\Lambda, \frac{4M\Lambda}{\zeta}\right\}\delta < \zeta$$

Thus,

$$\sup\left\{|u^\epsilon(x) - u^\epsilon(y)| \mid \epsilon > 0\right\} < \zeta$$

Therefore, by Theorem 4.18, the set $\{u^\epsilon \mid \epsilon > 0\}$ is pre-compact in $C(\overline{\Omega}_0)$. □

We now introduce a property of viscosity solutions that will be used in the next proof. So far, we have been working with the fact that the viscosity solution does not need to be everywhere differentiable. In the next proposition, we will show that if the viscosity solution of the cell problem (4.11) is smooth, then it is also a classical solution.

Proposition 4.20. *Suppose a smooth function $\chi : \mathbb{R}^d \to \mathbb{R}$ is a viscosity solution of the cell problem,*

$$a(y)^{-1}\|p + D\chi\| = \overline{\varphi}(p)$$

Then, χ is a classical solution of the differential equation.

Proof. Let $r > 0, y_0 \in \mathbb{R}^d$ and let $\chi : B(y_0, r) \to \mathbb{R}$ be a smooth function that solves the equation in the viscosity sense. Then, $\chi - \chi$ attains a local maximum at y_0. We have

$$a(y_0)^{-1}\|p + D\chi(y_0)\| \leq \overline{\varphi}(p)$$

Similarly, $\chi - \chi$ also attains a local minimum at y_0. Hence

$$a(y_0)^{-1}\|p + D\chi'(y_0)\| \geq \overline{\varphi}(p)$$

Thus, for all $y \in \mathbb{R}^d$, χ satisfies

$$a(y)^{-1}\|p + D\chi(y)\| = \overline{\varphi}(p)$$

□

In general, a smooth function is a viscosity solution of the given PDE *if and only if* it is a classical solution. We will not include the proof, which follows a similar idea as the proof for Proposition 4.20.

We now assume the convergence of a sequence of solutions for (4.7), (u^{ϵ_n}), and show that the limit point $\tilde{u}$ solves the anisotropic eikonal equation (4.10) in the viscosity sense.

Proposition 4.21. *Suppose that $(\epsilon_n)_{n\in\mathbb{N}} \subseteq (0, \infty)$ converges to zero and $(u^{\epsilon_n})_{n\in\mathbb{N}}$ converges in $C(\overline{\Omega}_0)$ to some function $\tilde{u}$. Then, $\tilde{u}$ is a viscosity solution of* (4.10).

Proof. To avoid technicalities, we assume that χ_p is a smooth function. For the boundary condition, notice that (u^{ϵ_n}) satisfies

$$u^{\epsilon_n} = 0 \text{ on } \partial\Omega$$

Since (u^{ϵ_n}) converges uniformly to $\tilde{u}$,

$$\tilde{u} = 0 \text{ on } \partial\Omega$$

We will show that $\tilde{u}$ is a viscosity sub-solution of (4.10). Let $r > 0$, $x_0 \in B(x_0, r) \subseteq \Omega_0$, and $\psi : B(x_0, r) \to \mathbb{R}$ be a smooth function. Suppose that $\tilde{u} - \psi$ has a local maximum at x_0. We want to show

$$\overline{\varphi}(D\psi(x_0)) \leq 1$$

Fix $r_1 \in (0, r)$ such that $\tilde{u} - \psi$ attains its maximum in $\overline{B(x_0, r_1)}$ at x_0. Let $\delta > 0$ and define

$$\psi_\delta(x) = \psi(x) + \frac{\delta\|x - x_0\|^2}{2}$$

Notice that

$$\psi_\delta(x) \geq \psi(x)$$

with the equality attained at $x = x_0$ only. Hence, $\tilde{u} - \psi_\delta$ attains its maximum in $\overline{B(x_0, r_1)}$ uniquely at x_0.

Let $p = D\psi_\delta(x_0)$ and let $\chi_p : \mathbb{R}^d \to \mathbb{R}$ be a 1-periodic, continuous viscosity solution of

$$a(y)^{-1}\|p + D\chi_p\| = \overline{\varphi}(p)$$

By Proposition 4.20, χ_p is also a solution of the equation. We define $\psi_\delta^\epsilon(x) : \overline{B(x_0, r_1)} \to \mathbb{R}$ by

$$\psi_\delta^\epsilon(x) = \psi_\delta + \epsilon\chi_p(\epsilon^{-1}x)$$

For each $\epsilon > 0$, let x_{ϵ_n} be a point where $u^{\epsilon_n} - \psi_\delta^{\epsilon_n}$ attains a maximum in $\overline{B(x_0, r_1)}$. We have

$$u^{\epsilon_n} - \psi_\delta^{\epsilon_n} = u^\epsilon - \psi_\delta - \epsilon\chi_p(\epsilon^{-1}x)$$

Therefore, $u^{\epsilon_n} - \psi_\delta^{\epsilon_n}$ converges to $u^\epsilon - \psi_\delta$ as ϵ converges to zero. Since $u^\epsilon - \psi_\delta$ attains a maximum at x_0,

$$x_0 = \lim_{n\to\infty} x_{\epsilon_n}$$

Let $\epsilon > 0$ be small enough, and $u^{\epsilon_n} - \psi_\delta^{\epsilon_n}$ attains its maximum at x_ϵ. Since u^ϵ is the viscosity solution of (4.7), in particular, u^ϵ is a viscosity sub-solution. Hence,

$$\begin{aligned}
a(\epsilon^{-1}x_\epsilon)^{-1}\|D\psi_\delta^\epsilon(x_\epsilon)\| &\leq 1 \\
a(\epsilon^{-1}x_\epsilon)^{-1}\|D\psi_\delta(x_\epsilon) + \epsilon\epsilon^{-1}D\chi_p(\epsilon^{-1}x_\epsilon)\| &\leq 1 \\
a(\epsilon^{-1}x_\epsilon)^{-1}\|D\psi_\delta(x_\epsilon) + D\chi_p(\epsilon^{-1}x_\epsilon)\| &\leq 1
\end{aligned}$$

Let $p_\epsilon = D\psi_\delta(x_\epsilon)$. Note that χ_p is a solution of

$$\begin{aligned}
a(y)^{-1}\|p_\epsilon + D\chi_p(y)\| &= \overline{\varphi}(p_\epsilon) \\
a(y)^{-1}\|D\psi_\delta(x_\epsilon) + D\chi_p(y)\| &= \overline{\varphi}(D\psi_\delta(x_\epsilon))
\end{aligned}$$

Let $y = \epsilon^{-1}x_\epsilon$, then

$$\overline{\varphi}(D\psi_\delta(x_\epsilon)) = a(\epsilon^{-1}x_\epsilon)^{-1}\|D\psi_\delta(x_\epsilon) + D\chi_p(\epsilon^{-1}x_\epsilon)\| \leq 1$$

Since x_ϵ converges to x_0 as ϵ_n converges to zero, we have

$$\overline{\varphi}(D\psi(x_0)) = \overline{\varphi}(D\psi_\delta(x_0)) \leq 1$$

The proof for viscosity super-solution is similar and is thus omitted here. □

So far, we have showed that (u^ϵ), the sequence of viscosity solutions for (4.7), is pre-compact. We have the following proposition about convergence of (u^ϵ). The proof is an elementary analysis exercise.

Proposition 4.22. *Let X be a metric space with metric d and let $\{x\}$, $(x_\epsilon)_{\epsilon>0} \subseteq X$. Assume that $(x_\epsilon)_{\epsilon>0}$ is pre-compact. Then, $x = \lim_{\epsilon\to 0^+} x_\epsilon$ if and only if for each $(\epsilon_n)_{n\in\mathbb{N}} \subseteq (0, \infty)$ such that $\lim_{n\to\infty} \epsilon_n = 0$ and $(x_{\epsilon_n})_{n\in\mathbb{N}}$ converges to some $\tilde{x} \in X$, we have $\tilde{x} = x$.*

The following uniqueness result regarding the anisotropic eikonal equation is nontrivial, and the proof of it is beyond the scope of this paper. We will state the proposition without proof.

Proposition 4.23. *There is a unique viscosity solution* $\overline{u} : \overline{\Omega}_0 \to [0, \infty)$ *of* (4.10).

Now, we will use the previous propositions to prove Theorem 4.9.

Proof. We want to show that $(u^\epsilon)_{\epsilon>0}$ converges to $\bar{u}$, where $\bar{u}$ is the unique viscosity solution of (4.10). Suppose that (u^ϵ) converges to some function $\tilde{u}$ in $C(\overline{\Omega}_0)$. Then, by Proposition 4.21, $\tilde{u}$ is a viscosity solution of (4.9). By Proposition (4.23), $\bar{u}$ is the unique viscosity solution, so $\tilde{u} = \bar{u}$. By Proposition (4.19), (u^ϵ) is pre-compact in $C(\overline{\Omega}_0)$, and by Proposition 4.22, $\lim_{n\to\infty} u^\epsilon = \bar{u}$ in $C(\overline{\Omega}_0)$. □

Recall that in Section 2, we showed that the Euclidean distance to the boundary solves the homogenous eikonal equation. In the beginning of Section 4, we remarked that a-distance to the boundary solves the heterogeneous eikonal equation. Now, we will give a more general result concerning the anisotropic eikonal equation, which uses a different norm $\overline{\varphi}$. We will show that if we define a distance function d^* based on $\overline{\varphi}^*$, the dual norm of $\overline{\varphi}$, then the d^*-distance to the boundary solves the anisotropic eikonal equation.

Definition 4.24. The dual norm $\overline{\varphi}^*$ associated with $\overline{\varphi}$ is

$$\overline{\varphi}^*(q) = \sup\left\{ \frac{\langle p, q\rangle}{\overline{\varphi}(p)} \mid p \in \mathbb{R}^d \setminus \{0\} \right\}$$

Proposition 4.25. *The dual norm* $\overline{\varphi}^*$ *is a norm.*

Proof. (1) Note that

$$\overline{\varphi}^*(q) \geq \frac{\langle q, q\rangle}{\overline{\varphi}(q)} \geq 0$$

(2) Suppose that $q = 0$. Then, for all $p \in \mathbb{R}^d \setminus \{0\}$,

$$\langle p, q\rangle = 0$$

Thus, $\overline{\varphi}^*(q) = 0$.
Now, suppose that $\overline{\varphi}^*(q) = 0$. Since there exists $p_1, p_2 \in \mathbb{R}^d$ such that

$$\frac{\langle p_1, q\rangle}{\overline{\varphi}(p_1)} = -\frac{\langle p_2, q\rangle}{\overline{\varphi}(p_2)}$$

If $q \neq 0$, we have

$$\overline{\varphi}^*(q) = \sup\left\{ \frac{\langle p, q\rangle}{\overline{\varphi}(p)} \mid p \in \mathbb{R}^d \setminus \{0\} \right\} > 0$$

Hence $q = 0$ if $\overline{\varphi}^*(q) = 0$.
Combining, we have $\overline{\varphi}^*(q) = 0$ if and only if $q = 0$.

(3) Consider $\overline{\varphi}^*(\alpha q)$.

$$\begin{aligned}\overline{\varphi}^*(\alpha q) &= \sup\left\{ \frac{\langle p, \alpha q\rangle}{\overline{\varphi}(p)} \mid p \in \mathbb{R}^d \setminus \{0\} \right\} \\ &= |\alpha| \sup\left\{ \frac{\langle p, q\rangle}{\overline{\varphi}(p)} \mid p \in \mathbb{R}^d \setminus \{0\} \right\} \\ &= |\alpha|\overline{\varphi}^*(q)\end{aligned}$$

(4) Consider $\overline{\varphi}^*(q_1+q_2)$. Notice that

$$\frac{\langle p, q_1+q_2\rangle}{\overline{\varphi}(p)} = \frac{\langle p, q_1\rangle}{\overline{\varphi}(p)} + \frac{\langle p, q_2\rangle}{\overline{\varphi}(p)}$$

$$\sup\left\{\frac{\langle p, q_1+q_2\rangle}{\overline{\varphi}(p)}\right\} \le \sup\left\{\frac{\langle p, q_1\rangle}{\overline{\varphi}(p)}\right\} + \sup\left\{\frac{\langle p, q_2\rangle}{\overline{\varphi}(p)}\right\}$$

$$\overline{\varphi}^*(q_1+q_2) \le \overline{\varphi}^*(q_1) + \overline{\varphi}^*(q_2)$$

□

Finally, we define a distance function d^* associated with the norm $\overline{\varphi}^*$, and we have that the d^*-distance to the boundary solves the anisotropic eikonal equation.

Definition 4.26. We define the distance function $d^* : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ as

$$d^*(x, y) = \overline{\varphi}^*(x - y)$$

Proposition 4.27. *The viscosity solution $\bar{u}$ of* (4.10)

$$\begin{cases} \overline{\varphi}(D\bar{u}) = 1 & \text{in } \overline{\Omega}_0 \\ \bar{u} = 0 & \text{on } \partial\Omega_0 \end{cases}$$

equals the d^*-distance to $\partial\Omega_0$ in Ω_0. That is,

$$\bar{u}(x) = \min\{d^*(x, y) \mid y \in \partial\Omega_0\} = \min\{\overline{\varphi}^*(x-y) \mid y \in \partial\Omega_0\}$$

Recall that the Euclidean distance to the boundary solves (2.1), which can be seen as a special case of (4.10). The following proposition shows that this result is consistent with Proposition 4.27.

Proposition 4.28. *The Euclidean norm is the dual norm of itself.*

Proof. By Cauchy-Schwarz inequality,

$$\frac{|\langle x, y\rangle|}{\|y\|} \le \|x\|$$

The equality is attained when $y = x$. Therefore, the dual norm $\|\cdot\|^* : \mathbb{R}^d \to \mathbb{R}$ of the Euclidean norm satisfies

$$\|x\|^* = \sup\left\{\frac{\langle x, y\rangle}{\|y\|} \mid y \in \mathbb{R}^d \setminus \{0\}\right\} = \|x\|$$

□

Acknowledgments

I would like to thank my mentor, Peter Morfe, for his guidance, advice and support throughout the program. I would also like to thank Prof. Peter May for organizing the UChicago REU program during this challenging time.

References

[1] Guy Barles. An Introduction to the Theory of Viscosity Solutions for First-Order Hamilton-Jacobi Equations and Applications. Springer-Verlag Berlin Heidelberg. 2013.

[2] Lawrence C. Evans. On Solving Certain Nonlinear Partial Differential Equations By Accretive Operator Methods. Israel Journal of Mathematics, Vol. 36, Nos. 3-4. 1980.

[3] Lawrence C. Evans. Periodic Homogenisation of certain fully nonlinear partial differential equations. Proceedings of the Royal Society of Edinburgh, 120A, 245-265. 1992.

[4] Michael G. Crandall, Hitoshi Ishii, and Pierre-Louis Lions. User's Guide to Viscosity Solutions of Second Order Partial Differential Equations. Bulletin of the American Mathematical Society, Vol. 27, No. 1. 1992

[5] Vasudevan Lakshminarayanan, Ajoy K. Ghatak, and K. Thyagarajan. Lagrangian Optics. Kluwer Academic Publisher. 2002.