

# PHASE TRANSITIONS IN LATTICE SYSTEMS

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ABSTRACT. We consider systems of interacting sites of a lattice, and study their behavior with respect to various parameters such as temperature, interaction strength with neighboring lattice sites, and external forces. These systems sometimes exhibit a mysteriously abrupt change, or phase transition, at some parameter values. We investigate the qualitative and quantitative behavior of the systems near the phase transitions.

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## 1. INTRODUCTION

This paper will be a survey on techniques for deducing the behavior of lattice systems near points where they exhibit abrupt changes. We will focus on the Ising model, the most commonly studied lattice system. In Section 1, we introduce some formalism and the Ising model. In Section 2, we rigorously deduce the behavior of the Ising model, to 1st order. In Section 3, we use “mean field” approximation methods to deduce the 2nd order behavior of the Ising model. In Section 4, we use a newer method to better deduce the behavior.

First we will derive the distribution of states for a system in thermal contact with its environment. Consider a system with a set of possible states  $\{a_i\}$ , each with energy  $E_i = \mathcal{H}(a_i)$ , where  $\mathcal{H}$  is called the Hamiltonian of the system. Suppose the system is in thermal contact with a large heat reservoir. We want to know the probability  $\mu(a_i)$  of being in a state  $a_i$ . We make the following assumptions:

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- (1) The total energy of the combined system (heat reservoir plus our system) is conserved.
- (2) All distinguishable microstates of the combined system are equally likely.

(1) is the 1st Law of Thermodynamics, and (2) is the Fundamental Postulate of statistical mechanics. They're empirically based.

Suppose we fix the average energy of the system  $E = \sum_i \mu(a_i) \mathcal{H}(a_i)$ . Given this constraint, if we maximize the **entropy**,  $S(\mu) \equiv -\sum_i \mu(a_i) \ln \mu(a_i)$ , we will find the most likely probability distribution given our fundamental postulate, (2). (This is explained in many statistical mechanics texts, e.g. page 20 of [2].) Maximizing using Lagrange multipliers, we find the probability distribution to be:

$$(1.1) \quad \mu(a_i) = \frac{e^{-\beta E_i}}{\sum_i e^{-\beta E_i}} \equiv \frac{e^{-\beta E_i}}{Z}.$$

This is called the **Gibbs** or **Boltzmann distribution**. The normalization constant  $Z$  is known as the partition function and is useful because derivatives of  $\ln Z$  give averages of various quantities with respect to  $\mu$ . The Lagrange multiplier  $\beta$ , because of its role in this distribution, is equivalent to our physical notion of the inverse temperature  $1/T$ .

We can now take averages of various functions  $f : \{a_i\} \rightarrow \mathbb{R}$ , called **observables**, with respect to  $\mu$ . We are interested in the dependence of these averages on the inverse temperature  $\beta$  and other parameters in the Hamiltonian. We will focus on lattice systems, primarily on the Ising model, which describes a simplified magnet on an integer lattice.

**1.1. The Ising Model.** Consider a finite  $\Lambda \subset \mathbb{Z}^d$ . At each site  $i \in \Lambda$ , suppose there is a spin  $\omega_i \in \{-1, +1\}$ , pointing either up (+1) or down (-1). Denote the configuration space  $\Omega_\Lambda = \{-1, +1\}^\Lambda$ . The energy of a configuration  $\omega = \{\omega_i \mid i \in \Lambda\}$  is

$$(1.2) \quad \mathcal{H}_{\Lambda; J, h}(\omega) = -J \sum_{\langle ij \rangle} \omega_i \omega_j - h \sum_i \omega_i$$

where  $\langle ij \rangle$  iterates over all nearest-neighbor pairs in  $\Lambda$ . This assigns an energy penalty to spins misaligned with their neighbors (interaction strength  $J$ ) and with the external magnetic field (strength  $h$ ). Since this model only actually has 2 independent parameters and we already have the variables  $\beta$ ,  $J$ , and  $h$ , we set  $J = 1$  and let  $h \mapsto h/\beta$ , for simplification purposes. So

$$(1.3) \quad -\beta \mathcal{H}_{\Lambda; \beta, h}(\omega) = \beta \sum_{\langle ij \rangle} \omega_i \omega_j + h \sum_i \omega_i$$

On a domain  $\Lambda$  and with parameters  $\beta, h$  we call the partition function

$$(1.4) \quad \sum_{\omega} e^{-\beta \mathcal{H}_{\Lambda; \beta, h}} \equiv Z_\Lambda(\beta, h).$$

Define the observable **magnetization**  $m : \Omega_\Lambda \rightarrow \mathbb{R}$  to be the average spin:

$$(1.5) \quad m(\omega) = \frac{1}{|\Lambda|} \sum_i \omega_i.$$

This has expectation value  $\langle m \rangle_{\Lambda; \beta, h} \equiv m_{\Lambda}(\beta, h)$  given by averaging with respect to the Gibbs distribution (1.1).

**1.2. Phase Transitions.** We define a  $k$ -th order phase transition as a discontinuity of the  $(k - 1)$ -th derivative of  $m_{\Lambda}(\beta, h)$  with respect to  $(\beta, h)$ . But,

$$(1.6) \quad m_{\Lambda}(\beta, h) = -\frac{1}{|\Lambda|} \frac{\partial}{\partial h} \ln Z_{\Lambda}(\beta, h)$$

and  $Z$  is a finite sum of positive analytic functions of  $(\beta, h)$ . So there can be no phase transition over finite domain  $\Lambda$ . In fact, this is true of any finite system—there is no discontinuity of any macroscopic variable with respect to any parameter, because all macroscopic variables are weighted sums of the finite terms of  $Z$ , which are analytic functions of the parameters. But we witness phase transitions in real life (e.g. discontinuity of the volume of a liquid at some boiling point  $T_c$ ).

So to mathematically describe phase transitions, we must take the limit as the system becomes infinite, i.e.  $|\Lambda| \rightarrow \infty$ , or the **thermodynamic limit**. This is justified in real life, where systems are of the order  $|\Lambda| \gtrsim 10^{23}$ . Discontinuities which appear in this limit are mimicked when  $|\Lambda|$  is large but still finite, as abrupt jumps in the derivatives of macroscopic quantities.

**1.3. Infinite-volume Gibbs States.** We now generalize our finite-domain probability measure  $\mu_{\Lambda}$  to a probability measure (or “state”)  $\mu$  over  $\mathbb{Z}^d$ . Let  $\Lambda_n$  be a sequence of increasing sets whose union is  $\mathbb{Z}^d$ , which we denote by  $\Lambda_n \uparrow \mathbb{Z}^d$ . We imposed no boundary conditions in our definition of the hamiltonian  $\mathcal{H}$ , but we could impose boundary conditions and add terms to account for interaction with the boundary, which we denote by a superscript (e.g.  $\mathcal{H}^+$  for a +1 spin boundary).

We say a function  $f : \Omega_{\mathbb{Z}^d} \rightarrow \mathbb{R}$  is local if it only depends on finitely many sites in  $\mathbb{Z}^d$ . The measures  $\mu_{\Lambda_n}^{b.c.}$  are said to converge to the **infinite-volume Gibbs measure/state**  $\mu_{(\Lambda_n)}^{b.c.}$  if, for all local functions  $f$ ,

$$(1.7) \quad \lim_{n \rightarrow \infty} \sum_{\omega \in \Omega_{\Lambda_n}} f(\omega) \mu_{\Lambda_n}^{b.c.}(\omega) = \int_{\mathbb{Z}^d} f d\mu_{(\Lambda_n)}^{b.c.} \equiv \langle f \rangle_{(\Lambda_n)}^{b.c.}$$

This coincides with the weak limit of the measures  $\mu_{\Lambda_n}^{b.c.}$  [1]. We denote the average of  $f$  with respect to an infinite-volume Gibbs measure by  $\langle f \rangle$ .

So, phase transitions occur at discontinuities in the derivatives of

$$(1.8) \quad m_{(\Lambda_n)}^{b.c.}(\beta, h) \equiv \lim_{n \rightarrow \infty} m_{\Lambda_n}^{b.c.}(\beta, h) = \lim_{n \rightarrow \infty} \sum_{\omega \in \Omega_{\Lambda_n}} m(\omega) \mu_{\Lambda_n}^{b.c.},$$

for  $\Lambda_n \uparrow \mathbb{Z}^d$ , whenever this limit exists and is independent of boundary conditions and  $(\Lambda_n)$ . Where this isn't satisfied, there is a 1st-order phase transition. We'll see that this limit exists everywhere, is independent of  $(\Lambda_n)$ , and depends on boundary conditions whenever there is a 1st-order phase transition.

## 2. THE 1ST-ORDER ISING MODEL PHASE DIAGRAM

We will calculate infinite-volume Gibbs measures to see under what conditions there is a continuous magnetization for the Ising model, and in which dimensions. We will show that points of discontinuity in the magnetization are equivalent to points where the expected magnetization isn't uniquely defined. First, we prove some lemmas about infinite-volume Gibbs states.

**Lemma 2.1.** *The infinite-volume Gibbs measures with  $\pm 1$  spin boundary conditions exist in all dimensions and are independent of the sequence  $\Lambda_n$ . So, we can write*

$$(2.2) \quad \mu^\pm = \lim_{n \rightarrow \infty} \mu_{\Lambda_n}^\pm$$

for all  $\Lambda_n \uparrow \mathbb{Z}^d$ . Moreover,  $\mu^\pm$  are translationally-invariant.

*Proof.* We prove this for the “+” boundary condition; it follows analogously for the “−” boundary condition. Fix  $(\beta, h)$ ; we omit these from subscripts since they are fixed. We define a partial order on  $\Omega_\Lambda$  by saying  $\eta_1 \leq \eta_2$  if  $\eta_{1,i} \leq \eta_{2,i}$  for all  $i \in \Lambda$ . We say a function is **nondecreasing** if  $\eta_1 \leq \eta_2 \implies f(\eta_1) \leq f(\eta_2)$  for all  $\eta_1, \eta_2$ .

Let  $f : \Omega_{\mathbb{Z}^d} \rightarrow \mathbb{R}$  be a local function, i.e. it only depends on finitely many sites in  $\mathbb{Z}^d$ . Applying the Jordan measure decomposition theorem to the measure defined by integrating  $f$  (which applies since  $f$  is bounded variation), we can write  $f$  as the difference of two nondecreasing functions, i.e.  $f = g - h$ .

Note that for a nondecreasing function  $g : \Omega_\Lambda \rightarrow \mathbb{R}$ , we have

$$(2.3) \quad \langle g \rangle_{\Lambda_1}^+ \geq \langle g \rangle_{\Lambda_2}^+ \quad \text{whenever } \Lambda_1 \subset \Lambda_2,$$

since  $\langle g \rangle_{\Lambda_1}^+ = \langle g \mid \omega_i = 1, \forall i \in \Lambda_2 \setminus \Lambda_1 \rangle_{\Lambda_1}^+$ , which restricts the probability to be over a strictly larger set of possibilities for  $g$ . So for a sequence  $\Lambda_n \uparrow \mathbb{Z}^d$ , we have  $\langle g \rangle_{\Lambda_n}^+$  is monotonic and thus convergent. Thus  $\langle f \rangle_{\Lambda_n}^+ = \langle g \rangle_{\Lambda_n}^+ - \langle h \rangle_{\Lambda_n}^+$  converges. So we can denote the limiting infinite-volume Gibbs state as  $\langle \cdot \rangle^+$ .

Now we show this limit is independent of the sequence  $(\Lambda_n)$  of sets. Let  $(\Lambda_n^1)$  and  $(\Lambda_n^2)$  be two sequences of sets that  $\uparrow \mathbb{Z}^d$ . We can then find a sequence of sets  $(\Lambda_n)$  such that  $\forall k \geq 1$ ,

$$(2.4) \quad \Lambda_{2k-1} \in \{\Lambda_n^1\}, \quad \Lambda_{2k} \in \{\Lambda_n^2\}, \quad \Lambda_k \subsetneq \Lambda_{k+1}.$$

Since the sequence  $\langle f \rangle_{\Lambda_n}^+$  consists of subsequences of  $\langle f \rangle_{\Lambda_n^1}^+$  and  $\langle f \rangle_{\Lambda_n^2}^+$ , it has the same limit. So  $\langle f \rangle^+$  doesn't depend on the sequence  $(\Lambda_n)$ .

This measure is translationally invariant since we could replace  $(\Lambda_n)$  with  $(i + \Lambda_n)$  for any  $i \in \mathbb{Z}^d$ , and by the above argument it would have the same limit.  $\square$

**Lemma 2.5.** *For boundary conditions  $\eta_1, \eta_2$  if  $\eta_1 \leq \eta_2$  and  $f$  is nondecreasing,*

$$(2.6) \quad \langle f \rangle_{\beta, h}^{\eta_1} \leq \langle f \rangle_{\beta, h}^{\eta_2}.$$

*Remark 2.7.* Lemma 2.5 has the consequence that, whenever the Gibbs states  $\langle \cdot \rangle_{\beta, h}^+$  and  $\langle \cdot \rangle_{\beta, h}^-$  are equal, all Gibbs states are equal.

**Lemma 2.8.** *For any sequence  $\Lambda_n \uparrow \mathbb{Z}^d$ ,*

$$(2.9) \quad m^\pm(\beta, h) \equiv \lim_{n \rightarrow \infty} m_{\Lambda_n}^\pm(\beta, h) = \langle \omega_0 \rangle_{\beta, h}^\pm.$$

We omit the proofs of Lemma 2.5, 2.8 because they follow easily by the monotonicity of  $f$  and the translational invariance of  $\mu^\pm$ , respectively. Proofs can be found in [1].

**Lemma 2.10.**

- (1)  $h \mapsto \langle \omega_0 \rangle_{\beta, h}^+$  is nondecreasing and right-continuous.  $h \mapsto \langle \omega_0 \rangle_{\beta, h}^-$  is nondecreasing and left-continuous.
- (2) If  $h \geq 0$ ,  $\beta \mapsto \langle \omega_0 \rangle_{\beta, h}^+$  is nondecreasing and right-continuous. If  $h \leq 0$ ,  $\beta \mapsto \langle \omega_0 \rangle_{\beta, h}^-$  is nonincreasing and left-continuous.

*Proof.* First we show  $h \mapsto \langle \omega_0 \rangle^+$  is nondecreasing; it will follow by symmetry that  $h \mapsto \langle \omega_0 \rangle^-$  is nonincreasing. We can calculate, for  $\Lambda \subset \mathbb{Z}^d$ ,

$$(2.11) \quad \frac{\partial}{\partial h} \langle \omega_0 \rangle_{\Lambda; \beta, h}^+ = \sum_{i \in \Lambda} \left( \langle \omega_0 \omega_i \rangle_{\Lambda; \beta, h}^+ - \langle \omega_0 \rangle_{\Lambda; \beta, h}^+ \langle \omega_i \rangle_{\Lambda; \beta, h}^+ \right).$$

This is the sum of correlations of spins in  $\Lambda$ . As expected, we can show the correlations are nonnegative since the interaction is ferromagnetic ( $J > 0$ ); see [1] for a formal proof of this. So  $\langle \omega_0 \rangle_{\Lambda; \beta, h}^+$  is nondecreasing with respect to  $h$  for any  $\Lambda \subset \mathbb{Z}^d$ ; clearly this is also true in the limit as  $\Lambda \uparrow \mathbb{Z}^d$ .

Now we show  $h \mapsto \langle \omega_0 \rangle_{\beta, h}^+$  is right-continuous; it will follow by symmetry that  $h \mapsto \langle \omega_0 \rangle^-$  is left-continuous. Let  $h_m \downarrow h$  and  $\Lambda_n \uparrow \mathbb{Z}^d$ . By (2.3) and the point above, the double sequence  $(\langle \omega_0 \rangle_{\Lambda_n; \beta, h_m}^+)$  is bounded and nonincreasing with respect to  $m$  and  $n$ ; therefore since the expectation over a finite domain is continuous in  $h$ , we can switch the limit order and obtain

$$(2.12) \quad \begin{aligned} \lim_{m \rightarrow \infty} \langle \omega_0 \rangle_{\beta, h_m}^+ &= \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \langle \omega_0 \rangle_{\Lambda_n; \beta, h_m}^+ \\ &= \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \langle \omega_0 \rangle_{\Lambda_n; \beta, h_m}^+ = \lim_{n \rightarrow \infty} \langle \omega_0 \rangle_{\Lambda_n; \beta, h}^+. \end{aligned}$$

This is  $\langle \omega_0 \rangle_{\beta, h}^+$ , so  $h \mapsto \langle \omega_0 \rangle_{\beta, h}^+$  is right-continuous.

Now we show  $\beta \mapsto \langle \omega_0 \rangle_{\beta, h}^+$  is nondecreasing for  $h \geq 0$ ; by symmetry it will follow that  $\beta \mapsto \langle \omega_0 \rangle_{\beta, h}^-$  is nonincreasing for  $h \leq 0$ . We can calculate, for  $\Lambda \subset \mathbb{Z}^d$ ,

$$(2.13) \quad \frac{\partial}{\partial \beta} \langle \omega_0 \rangle_{\Lambda; \beta, h}^+ = \sum_{\langle ij \rangle \subset \Lambda} \left( \langle \omega_0 \omega_i \omega_j \rangle_{\Lambda; \beta, h}^+ - \langle \omega_0 \rangle_{\Lambda; \beta, h}^+ \langle \omega_i \omega_j \rangle_{\Lambda; \beta, h}^+ \right)$$

which is nonnegative when  $h \geq 0$ , the proof of which we also defer to [1]. Hence,  $\beta \mapsto \langle \omega_0 \rangle_{\beta, h}^+$  is nondecreasing for  $h \geq 0$ .

Now show  $\beta \mapsto \langle \omega_0 \rangle_{\beta, h}^+$  is right-continuous for  $h \geq 0$ ; by symmetry left-continuity will follow for  $\beta \mapsto \langle \omega_0 \rangle_{\beta, h}^-$  and  $h \leq 0$ . Suppose  $\beta_m \downarrow \beta$  and  $\Lambda_n \uparrow \mathbb{Z}^d$ . Then by (2.3) and the above paragraphs, the double sequence  $(\langle \omega_0 \rangle_{\Lambda_n; \beta_m, h}^+)$  is bounded and

nonincreasing with respect to  $m, n$ . By the argument in (2.12),  $\beta \mapsto \langle \omega_0 \rangle_{\beta, h}^+$  is right-continuous for  $h \geq 0$ .  $\square$

**Corollary 2.14.** *There is a 1st-order phase transition at  $(\beta, h)$  if and only if  $\langle \omega_0 \rangle_{\beta, h}^+ \neq \langle \omega_0 \rangle_{\beta, h}^-$*

*Proof.* By Lemma 2.8,  $m(\beta, h) = \langle \omega_0 \rangle_{\beta, h}$ , so 1st-order phase transitions also correspond to discontinuities in  $\langle \omega_0 \rangle_{\beta, h}$  with respect to  $(\beta, h)$ , or points  $(\beta, h)$  where it isn't uniquely-defined. Suppose  $(\beta, h)$  has a 1st-order phase transition. We will find, independently of this corollary, that  $\langle \omega_0 \rangle_{\beta, h}^+ = \langle \omega_0 \rangle_{\beta, h}^-$  when  $h \neq 0$ ; therefore by Lemma 2.10,  $\langle \omega_0 \rangle_{\beta, h}^+$  and  $\langle \omega_0 \rangle_{\beta, h}^-$  are continuous and monotonic with respect to  $\beta$  at all  $h \neq 0$ ; additionally,  $\beta \mapsto \langle \omega_0 \rangle_{\beta, 0}^+$  is right-continuous and  $\beta \mapsto \langle \omega_0 \rangle_{\beta, 0}^-$  is left-continuous. We also have that  $h \mapsto \langle \omega_0 \rangle_{\beta, h}^+$  is right-continuous and  $h \mapsto \langle \omega_0 \rangle_{\beta, h}^-$  is left continuous (Lemma 2.10). Combined with the monotonicity of both magnetizations with respect to both variables (Lemma 2.5), the only possibility is  $\langle \omega_0 \rangle_{\beta, h}^+ \neq \langle \omega_0 \rangle_{\beta, h}^-$ . (We need monotonicity with respect to  $\beta, h$ ; otherwise we could have a function separately continuous in two arguments but still not continuous.)

Now if  $\langle \omega_0 \rangle_{\beta, h}^+ \neq \langle \omega_0 \rangle_{\beta, h}^-$  at  $(\beta, h)$ , by Lemma 2.8 we have  $m^+(\beta, h) \neq m^-(\beta, h)$ , therefore  $m$  is non-uniquely defined at  $(\beta, h)$ , so there is a 1st-order phase transition there.  $\square$

Now we are ready to construct the 1st-order phase diagram of the Ising model.

**Theorem 2.15.** *Consider the Ising model in dimension  $d$ .*

- (1) *In  $d = 1$ : there is no 1st-order phase transition at any  $(\beta, h)$ .*
- (2) *In  $d \geq 2$ : if  $h \neq 0$ , there is no 1st-order phase transition at any  $(\beta, h)$ .*
- (3) *In  $d \geq 2$ : there is some  $0 < \beta_c(d) < \infty$  such that there is no 1st-order phase transition at any  $(\beta < \beta_c, 0)$ , i.e. at any  $(\beta, 0)$  where  $\beta < \beta_c$ .*
- (4) *In  $d \geq 2$ , for all  $(\beta > \beta_c, 0)$ , there is a 1st-order phase transition.*

We will first discuss the transfer matrix method used to prove (1), then will develop the high-temperature expansion to prove (3), then use a Peierls argument to prove (4), and finally compare energy penalties incurred by various configurations with  $h \neq 0$  to sketch a proof of (2).

*Proof.* (1) The 1d system can be solved explicitly using a transfer matrix method. See e.g. [1] for a proof; we omit it here. The upshot is that in 1d, the magnetization is an analytic function of all parameters, independent of boundary conditions. (The lattice graph is simply too sparse for interactions to overpower thermal fluctuations.)

(3) We show there is some  $0 < \beta_c(d) < \infty$  such that there is no first-order phase transition at any  $(\beta < \beta_c, 0)$ . Let  $h = 0$  and omit this from subscripts. Fix some finite  $\Lambda \subset \mathbb{Z}^d$ . With some manipulation, we can write the difference between magnetizations as

$$(2.16) \quad \langle \omega_0 \rangle_{\Lambda; \beta}^+ - \langle \omega_0 \rangle_{\Lambda; \beta}^- = \frac{\sum_{\omega, \omega'} e^{-\beta \mathcal{H}_\Lambda^+(\omega) - \beta \mathcal{H}_\Lambda^-(\omega')} (\omega_0 - \omega'_0)}{\sum_{\omega, \omega'} e^{-\beta \mathcal{H}_\Lambda^+(\omega) - \beta \mathcal{H}_\Lambda^-(\omega')}}.$$

The high-temperature expansion constitutes rewriting this in terms of  $f_{ij}(\omega, \omega') = e^{\beta(\omega_i \omega_j + \omega'_i \omega'_j + 2)} - 1$ . We have, for any  $\omega, \omega'$ ,

$$(2.17) \quad 0 \leq f_{ij}(\omega, \omega') \leq 4\beta(1 + f_{ij}(\omega, \omega'))$$

since  $0 \leq e^z - 1 \leq ze^z$  if  $z \geq 0$ , and each spin can only be  $\pm 1$ . This is useful, since it allows us to minimize the terms in the sum explicitly by minimizing  $\beta$  (moving to high temperatures). Now if  $\bar{\mathcal{B}}_\Lambda$  is the set of nearest-neighbor pairs where at least one belongs to  $\Lambda$ , we have

$$(2.18) \quad \langle \omega_0 \rangle_{\Lambda; \beta}^+ - \langle \omega_0 \rangle_{\Lambda; \beta}^- = \frac{\sum_{B \subset \bar{\mathcal{B}}_\Lambda} \sum_{\omega, \omega'} (\omega_0 - \omega'_0) \prod_{\langle ij \rangle \in B} f_{ij}(\omega, \omega')}{\sum_{\omega, \omega'} \prod_{\langle ij \rangle \in \bar{\mathcal{B}}_\Lambda} f_{ij}(\omega, \omega')}$$

which can be deduced by switching the order of summation, looking at the exponentiated term in  $f_{ij}(\omega, \omega')$ , and noting that the exponentiated sums of proper subsets of  $\bar{\mathcal{B}}_\Lambda$  appear once negative for each time it appears positive.

Now, note that for all nonzero terms in the numerator,  $B$  has a path of nearest neighbor pairs connecting 0 to  $\Lambda^c$ , since each  $f_{ij}(\omega, \omega')$  is symmetric under interchange of  $\omega, \omega'$  while  $(\omega_0 - \omega'_0)$  is antisymmetric. For each such  $B$  connected to both 0 and  $\Lambda^c$  by a path  $P$ , we can write

$$(2.19) \quad \begin{aligned} |(\omega_0 - \omega'_0) \prod_{\langle ij \rangle \in B} f_{ij}(\omega, \omega')| &\leq 2 \prod_{\langle ij \rangle \in P} f_{ij}(\omega, \omega') \prod_{\langle kl \rangle \in B \setminus P} f_{kl}(\omega, \omega') \\ &\leq 2(4\beta)^{|P|} \prod_{\langle ij \rangle \in P} (1 + f_{ij}(\omega, \omega')) \prod_{\langle kl \rangle \in B \setminus P} (1 + f_{kl}(\omega, \omega')). \end{aligned}$$

Assuming  $4\beta < 1$ , we can then bound the absolute value of the numerator of (2.18), (\*), by

$$(2.20) \quad \begin{aligned} (*) &\leq 2 \sum_P (4\beta)^{|P|} \sum_{\omega, \omega'} \prod_{\langle ij \rangle \in P} (1 + f_{ij}(\omega, \omega')) \prod_{\langle kl \rangle \in \bar{\mathcal{B}}_\Lambda \setminus P} (1 + f_{kl}(\omega, \omega')) \\ &= 2 \sum_P (4\beta)^{|P|} \sum_{\omega, \omega'} \prod_{\langle ij \rangle \in \bar{\mathcal{B}}_\Lambda} (1 + f_{ij}(\omega, \omega')), \end{aligned}$$

where  $P$  iterates over all paths connecting 0 to  $\Lambda^c$ . Since the sum over  $\omega, \omega'$  cancels with the denominator, we have

$$(2.21) \quad \begin{aligned} |\langle \omega_0 \rangle_{\Lambda; \beta}^+ - \langle \omega_0 \rangle_{\Lambda; \beta}^-| &\leq 2 \sum_P (4\beta)^{|P|} = 2 \sum_{k=d(0, \Lambda^c)}^{\infty} \#\{P : |P| = k\} (4\beta)^k \\ &\leq 2 \sum_{k=d(0, \Lambda^c)}^{\infty} 2d \cdot (2d-1)^{k-1} (4\beta)^k = \frac{4d}{2d-1} \cdot \frac{[4\beta(2d-1)]^{d(0, \Lambda^c)}}{1 - 4\beta(2d-1)}, \end{aligned}$$

if  $4\beta(2d-1) < 1$ . The 3rd inequality comes from estimating the number of paths by allowing the first step to go in  $2d$  directions, then the rest of the steps to go in  $2d-1$  directions (just preventing it from backtracking). Requiring  $4\beta(2d-1) < 1$ , this approaches zero as  $|\Lambda| \rightarrow \infty$ . So the Gibbs state for  $\beta < 1/4(2d-1)$  and  $h = 0$  is unique and there is no 1st-order phase transition here.

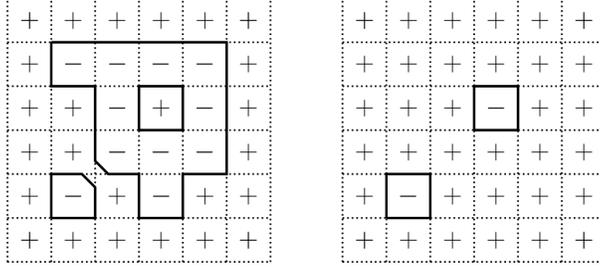


FIGURE 1. Constructing the “droplet representation” of a spin configuration. This procedure and argument is known as the Peierls argument. Figure from [4], where the Peierls argument is described in greater detail.

(4) We will show there is some  $0 < \beta_c(d) < \infty$  such that there is a first-order phase transition for all  $(\beta > \beta_c, 0)$ . (It will follow that this is the same  $\beta_c(d)$  as in part (3)). We first show this in 2d; then we will generalize to  $d \geq 2$ . If we show that there is some  $\delta(\beta) < 1/2$  such that  $\mu_{\Lambda; \beta, 0}^+(\omega_0 = -1) \leq \delta(\beta)$  for all  $\Lambda$ , it will follow that

$$(2.22) \quad \langle \omega_0 \rangle_{\beta, 0}^+ = \left[ 1 - \mu_{\beta, 0}^+(\omega_0 = -1) \right] - \mu_{\beta, 0}^+(\omega_0 = -1) \geq 1 - 2\delta(\beta),$$

and by symmetry  $\langle \omega_0 \rangle_{\Lambda; \beta, 0}^+ \leq -(1 - 2\delta(\beta))$ , so they will be unequal.

Working in the “+” boundary condition, we use the following algorithm to define a contour representation  $\Gamma(\omega)$  for each configuration  $\omega$ , where each contour  $\gamma \in \Gamma(\omega)$  surrounds a “droplet” of negative spins. For a configuration  $\omega$ , draw a boundary between different spins. Turn the set of contours into closed contours using a deformation rule: when there is a crossroads, split along the  $y = -x$  diagonal. This is illustrated in Fig. 1. We now use the sets  $\Lambda_n = B(n) = [-n, n]^2$  to take the thermodynamic limit. Note that, for any contour  $\gamma_*$ ,

$$(2.23) \quad \begin{aligned} \mu_{\Lambda_n; \beta, 0}^+(\Gamma \ni \gamma_*) &= \sum_{\omega: \Gamma(\omega) \ni \gamma_*} \mu_{\Lambda_n; \beta, 0}^+(\omega) \\ &= e^{-2\beta|\gamma_*|} \frac{\sum_{\omega: \Gamma(\omega) \ni \gamma_*} \prod_{\gamma \in \Gamma(\omega) \setminus \{\gamma_*\}} e^{-2\beta|\gamma|}}{\sum_{\omega} \prod_{\gamma \in \Gamma(\omega)} e^{-2\beta|\gamma|}} \leq e^{-2\beta|\gamma_*|}. \end{aligned}$$

The ratio is bounded above by 1 since the above sum can be expressed as the sum of  $\prod_{\gamma \in \Gamma(\omega)} e^{-2\beta|\gamma|}$  over configurations  $\omega$  that don’t contain  $\gamma_*$  in the contour representation but would if we flipped all the signs in  $\text{Int}(\gamma_*)$ . So, we have

$$\begin{aligned}
(2.24) \quad \mu_{\Lambda_n, \beta, 0}^+(\omega_0 = -1) &\leq \sum_{\gamma_*: \text{Int}(\gamma_*) \ni 0} e^{-2\beta|\gamma_*|} \\
&= \sum_{k \geq 4} \sum_{\substack{\gamma_*: \text{Int}(\gamma_*) \ni 0 \\ |\gamma_*| = k}} e^{-2\beta|\gamma_*|} \\
&= \sum_{k \geq 4} e^{-2\beta k} \#\{\gamma_* \mid \text{Int}(\gamma_*) \ni 0, |\gamma_*| = k\} \\
&\leq \sum_{k \geq 4} e^{-2\beta k} \frac{k}{2} \cdot 3^{k-1}.
\end{aligned}$$

The last inequality comes from placing a generous lower bound on the number of contours of length  $k$  surrounding 0: the contour must go through  $(0, l)$  where  $0 < l \leq k/2$ , can go in 4 directions in the first step, and in 3 directions in the next  $k - 1$  steps. This converges to some  $\delta(\beta) > 0$  when  $3e^{-2\beta} < 1$ , where  $\delta(\beta) \downarrow 0$  as  $\beta \rightarrow \infty$  and  $\delta(\beta)$  doesn't depend on  $\Lambda_n$ . So the infinite-volume Gibbs state is non-unique, and there is a 1st-order phase transition for these parameters.

(2) Suppose  $d \geq 2$  and  $h > 0$ . We want to show  $\langle \omega_0 \rangle_{\beta, h}^+ = \langle \omega_0 \rangle_{\beta, h}^-$ ; by symmetry the same will hold with  $h < 0$ . We provide a rough sketch of why this is true; a full proof can be found in [6]. Consider the “-” boundary condition state with  $h > 0$ , over domain  $\Lambda$ . The energy penalty for having all “-” spins is  $h|\Lambda|$ , and the energy penalty for having all “+” spins is  $\beta|\partial\Lambda| - h|\Lambda|$ . So if we make  $\Lambda$  large enough so that

$$(2.25) \quad \frac{|\partial\Lambda|}{|\Lambda|} < \frac{2h}{\beta},$$

the all “+” spin configuration becomes more likely than the all “-” configuration, and the effects of the magnetic field dominate the effects of the boundary condition. Thus we can expect that, in the thermodynamic limit, the boundary conditions will become irrelevant compared to the presence of the magnetic field, and  $\langle \omega_0 \rangle_{\beta, h}^+$  will equal  $\langle \omega_0 \rangle_{\beta, h}^-$ .  $\square$

So, we know when there are first-order phase transitions for the Ising model of all dimensions. But that doesn't give any information about any higher-order phase transitions, e.g. it doesn't make any quantitative predictions about the scaling of the magnetization near the critical points. Currently, we need to use less precise methods to get these quantitative results.

### 3. MEAN FIELD THEORY

Instead of treating a spin configuration as a lattice of discrete values  $\omega$ , we can treat it as a continuous field  $m(\vec{x})$ , allowing us to use calculus of variations and other techniques to estimate the most probable configurations and other macroscopic quantities. It is important to keep in mind that the following calculations, because of their very general assumptions, provide rough descriptions of many different lattice systems (not just the Ising model), and apply to all dimensions. However, we will show in section 3.3 that mean field theory tends to fail in low dimensions.

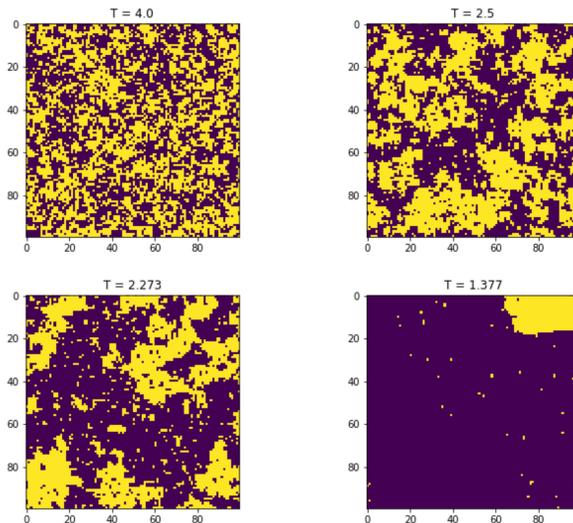


FIGURE 2. Equilibrium configurations for the 2d Ising model with  $h = 0$ , at various temperatures ( $T = 1/\beta$ ), from a freezing simulation. This simulation was performed using the numerical methods outlined in [9]. The bottom left is close to the critical temperature, and the symmetry breaks around here, “freezing” the configuration into mostly-down spins.

We change variables from  $\omega$  to  $m(\vec{x})$  using a coarse-graining approach: for each  $\vec{x} \in \mathbb{R}^d$ , let  $m(\vec{x})$  be the average of  $\omega$  over all lattice sites within a distance  $b$  of  $\vec{x}$ , where  $b \gg a$  but is small relative to the size of the system. Then in the partition function when we change variables from  $\omega$  to  $m(\vec{x})$ , we obtain

$$(3.1) \quad Z = \sum_{\omega \in \Omega} e^{-\beta \mathcal{H}(\omega)} = \sum_{m(\vec{x})} \sum_{\omega|_{m(\vec{x})}} e^{-\beta \mathcal{H}(\omega)} \equiv \sum_{m(\vec{x})} e^{-\beta F[m(\vec{x})]}$$

where  $F[m(\vec{x})]$  is known as the **free energy** of the system and is like the energy  $\mathcal{H}$  but accounts for excess degrees of freedom accumulated in coarse graining. (In statistical physics language, it is like the Helmholtz free energy:  $F[m(\vec{x})] = E[m(\vec{x})] - TS[m(\vec{x})]$ .) On a finite domain,  $m(\vec{x})$  can take on only finite values, but in the limit as  $\Lambda \uparrow \mathbb{Z}^d$ , we can write this as the functional integral

$$(3.2) \quad Z = \int \mathcal{D}m(\vec{x}) e^{-\beta F[m(\vec{x})]}$$

which is convergent because variation in  $m(\vec{x})$  is restricted due to the finite spacing between lattice sites. The terms in this sum with maximum probability are those where  $F[m(\vec{x})]$  is minimized.

We can write a very general form of  $\beta F[m(\vec{x})]$  based on a few general assumptions about the systems:

- (1) **Locality**: since interactions are nearest-neighbor, we can write  $F$  as the integral over  $\mathbb{Z}^d$  of some local function  $f(\vec{x}, m, \nabla m)$ .

- (2)  **$\mathbb{Z}_2$  symmetry**:  $f$  must be symmetric with respect to  $\{m \mapsto -m, h \mapsto -h\}$ .
- (3) **Translational/rotational symmetry**:  $\mathcal{H}$  has enough translational and rotational symmetries that it is reasonable to assume  $f$  have no  $\vec{x}$ -dependence.
- (4) **Analyticity/polynomial**: since microscopic non-analyticities are washed out in coarse-graining procedure, we assume  $f$  is an analytic function of  $(m, \nabla m)$  and can be polynomial expanded in small powers of these variables.
- (5) **Stability**: the leading coefficient with respect to  $m$  must be positive since  $F$  is minimized in the maximum-probability  $m(\vec{x})$  (otherwise the optimal value of  $m$  will diverge); also, the leading coefficient of  $\nabla m$  must be positive since interactions are ferromagnetic (variation is penalized).

Based on these assumptions, we can write

$$\begin{aligned}
 \beta F[m(\vec{x})] &= \beta \int d^d \vec{x} f(m, \nabla m) \\
 (3.3) \quad &= \int d^d \vec{x} \left[ \frac{t}{2} m^2(\vec{x}) + um^4(\vec{x}) + \frac{K}{2} (\nabla m(\vec{x}))^2 + \dots - hm(\vec{x}) \right].
 \end{aligned}$$

This general form is known as the **Landau-Ginzberg Free Energy**.

**3.1. Saddle Point Approximation.** As mentioned above, the maximum-probability configurations will have minimal  $F[m(\vec{x})]$ . The saddle point method (or method of steepest descent) assumes, when calculating the functional integral for  $Z$  in (3.2), that the dominant contribution is from this maximum-probability term, and ignores the rest. We will work with this assumption in this subsection.

To find which configuration  $m$  minimizes  $F[m(\vec{x})]$ , calculate the functional derivative and set it to zero:

$$\begin{aligned}
 \beta \delta F &= \int d^d \vec{x} [tm + 4um^3 - K\nabla^2 m - h] \delta m \\
 (3.4) \quad &\implies \beta \frac{\delta F}{\delta m(\vec{x})} = tm(\vec{x}) + 4um^3(\vec{x}) - K\nabla^2 m(\vec{x}) - h = 0.
 \end{aligned}$$

The simplest solutions to this are when  $m(\vec{x}) = m_0$  everywhere, satisfying

$$(3.5) \quad tm_0 + 4um_0^3 - h = 0.$$

We make some inferences about the behavior of the coefficients with respect to  $\beta, h$ . Note that we often speak in terms of the temperature,  $T = 1/\beta$ . We have  $u, K > 0$  by stability. From our knowledge about the qualitative behavior of  $m$  with respect to  $(\beta, h)$  (Section 2), and stability, we can conclude that  $t > 0$  when  $T > T_c$  and  $t < 0$  when  $T < T_c$ . In fact, the dependence on  $T$  is so weak that we can assume  $t = a(T - T_c)$  for a constant  $a$  near  $T_c$ . We could also deduce this fact using the technique in [8], by expanding explicitly.

From Section 2, we know the qualitative behavior of  $m_0$  for different parameters  $(\beta, h)$ . If  $h = 0$ , then  $m_0 = 0$  when  $T > T_c$ ; and  $m_0$  is nonzero and multi-valued when  $T < T_c$ . If  $h \neq 0$ , then for all values of  $T$ , the magnetization is monotonic with  $h$ , positive when  $h > 0$  and negative when  $h < 0$ . We solve (3.5) in these different regimes:

- If  $h = 0$  and  $T > T_c$ ,  $m_0 = 0$ ; this is also a solution of (3.5).

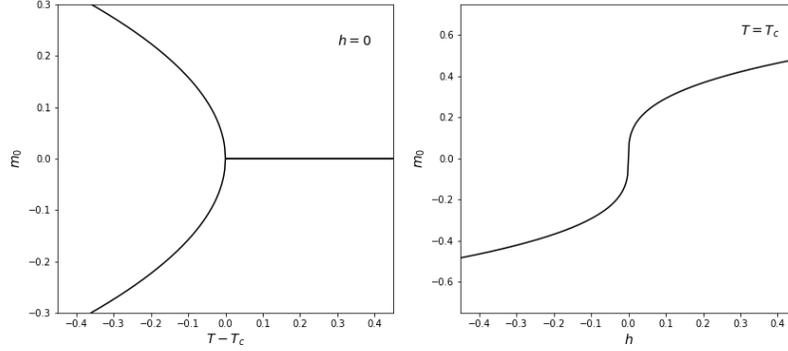


FIGURE 3. The magnetization given by the saddle point approximation from mean field theory, found by solving (3.5).

- If  $h = 0$  and  $T < T_c$ , we find  $m_0 = \pm(-t/4u)^{1/2} = \pm(a(T_c - T)/4u)^{1/2}$ .
- If  $h \neq 0$  and  $T = T_c$ , we find  $m_0 = (h/4u)^{1/3}$ .

We plot this behavior in Fig. 3. We have found, using this somewhat crude approximation, the presence of a 2nd-order phase transition with respect to  $\beta$  at  $(\beta_c, 0)$ . The scaling behavior near the phase transitions is typically phrased in the language of several **critical exponents**. We can immediately infer  $\alpha, \beta, \gamma, \delta$ . We include all their definitions and saddle point estimates in the following table:

	describes	regime	formula	estimate
$\alpha$	specific heat	$(\sim T_c, 0)$	$C \propto \frac{\partial^2(\beta f_{thermo})}{\partial T^2}$ <sup>1</sup>	0
$\beta$	magnetization	$(\lesssim T_c, 0)$	$m_0 \propto (T_c - T)^\beta$	1/2
$\gamma$	magnetic susceptibility	$(\sim T_c, 0)$	$\chi \propto \frac{\partial m_0}{\partial h} _{h=0} \propto  T - T_c ^{-\gamma}$	1
$\delta$	magnetization	$(T_c, \sim 0)$	$m_0 \propto  h ^{1/\delta}$	3
$\nu$	correlation length	$(\sim T_c, 0)$	$\xi \propto  T - T_c ^{-\nu}$	1/2
$\eta$	correlation function	$(T_c, 0)$	$G(r) \propto 1/r^{d-2+\eta}$	0

For  $\nu$ , the correlation length  $\xi$  is defined so that the spin correlation function  $G(r)$  behaves as  $G(r) \propto e^{-r/\xi}/r^{(d-1)/2}$  whenever  $r \gg \xi$ , which we will soon justify. To show that  $\nu = 1/2$  and  $\eta = 0$ , we need to allow for the presence of fluctuations about the average magnetization  $m_0$ . We make these calculations below.

**3.2. Fluctuations about the Saddle Point.** We replace the  $m(\vec{x})$  in our Landau-Ginzberg free energy (3.3) with  $m_0 + \phi(\vec{x})$ , where  $m_0$  is the same average magnetization and  $\phi(\vec{x})$  is a small fluctuation about  $m_0$ . Let  $h = 0$ . This gives us

$$(3.6) \quad \beta F[m(\vec{x})] = V \left( \frac{t}{2} m_0^2 + u m_0^4 \right) + \int d^d \vec{x} \frac{K}{2} [(\nabla \phi)^2 + \xi^{-2} \phi^2],$$

where  $t + 12u m_0^2 \equiv \frac{K}{\xi^2}$ .

<sup>1</sup>The thermodynamic free energy  $f_{thermo}$  is defined as  $\beta f_{thermo} \equiv (\ln Z)/V$ , where  $V$  is the volume of the system.

2 The quantity in brackets contains an operator, applied to  $\phi$ , which is diagonal in the Fourier basis over  $\phi$ , where the change of variables is accomplished by  $\phi(\vec{x}) = \frac{1}{\sqrt{V}} \sum_{\vec{q}} \phi_{\vec{q}} e^{i\vec{q}\cdot\vec{x}}$ , where  $V$  is the Fourier space volume being integrated over. We place a cap on modes, setting a minimal wavelength  $2a$ , where  $a$  is the spacing between lattice sites. Substituting this into (3.6), we find the probability of a mode  $\phi_{\vec{q}}$  by noting that this is proportional to the mode's exponentiated contribution to  $-\beta F[m(\vec{x})]$ , i.e.

$$(3.7) \quad P[\phi_{\vec{q}}] \propto \exp \left[ -\frac{K}{2} (q^2 + \xi^{-2}) |\phi_{\vec{q}}|^2 \right].$$

So the fluctuation mode amplitudes have a Gaussian distribution about zero, with

$$(3.8) \quad \langle \phi_{\vec{q}} \phi_{\vec{q}'} \rangle = \frac{\delta_{\vec{q}, -\vec{q}'}}{K(q^2 + \xi^{-2})}$$

and thus

$$(3.9) \quad G(\vec{x}, \vec{x}') = \langle \phi(\vec{x}) \phi(\vec{x}') \rangle = \frac{1}{V} \sum_{\vec{q}} e^{i\vec{q}\cdot(\vec{x}-\vec{x}')} \frac{1}{K(q^2 + \xi^{-2})} \sim \begin{cases} r^{2-d}, & r \ll \xi \\ \frac{e^{-r/\xi}}{r^{(d-1)/2}}, & r \gg \xi \end{cases}$$

where  $r = |\vec{x} - \vec{x}'|$ . We call this  $G(r)$ . Now, we know that  $\xi$  is the same correlation length given in the definition of our critical exponents, and thus by our definition of  $\xi$  (3.6) and our calculations for  $m_0$ , we have

$$(3.10) \quad \xi = \sqrt{\frac{K}{t + 12um_0^2}} = \begin{cases} |-2a(T - T_c)|^{-1/2}, & T < T_c \\ |a(T - T_c)|^{-1/2}, & T > T_c, \end{cases}$$

which confirms  $\nu = 1/2$  and  $\eta = 0$ .

Note also that  $\xi$  approaches  $\infty$  near  $T = T_c$ ; therefore  $r \ll \xi$  for all distances  $r$  at the critical temperature, so  $G(r)$  is always a power law at the critical temperature. Call this function  $G_c(r)$ , and note then that

$$(3.11) \quad G_c(\lambda r) = \lambda^{2-d} G_c(r).$$

We can see that the correlation between spins displays a statistically self-similar behavior at the critical temperature, and fractal patterns appear in the spin configuration. We demonstrate this in Figure 4, which shows a spin configuration at different scales. This is an important thing to remember when we look at the renormalization group later, a technique for solving these systems that takes advantage of the critical system's statistical self-similarity.

**3.3. Ginzburg Criterion.** We have obtained estimates for the critical exponents from the saddle point approximation, but it is unclear in which regime of the parameters  $(\beta, h)$  these estimates are accurate. Consider the Landau-Ginzberg free energy, expressed in terms of fluctuation corrections to the saddle point (3.6). In the saddle point approximation, we have assumed the 1st term of (3.6) makes the dominant contribution to  $Z$ , i.e.

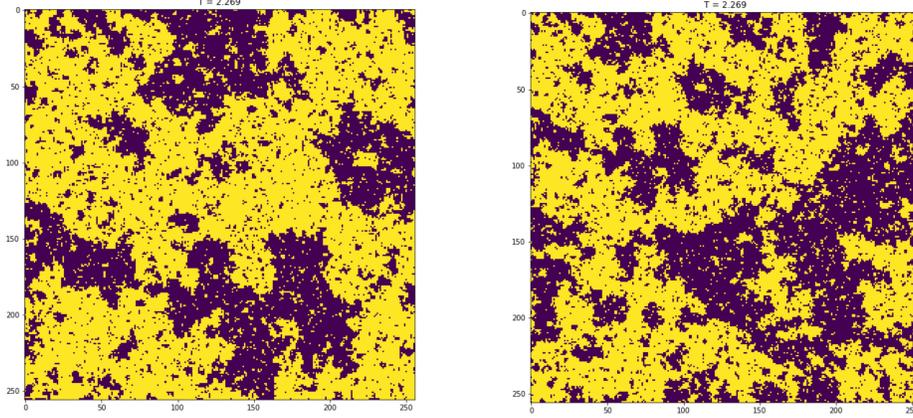


FIGURE 4. An equilibrium spin configuration at  $T = T_c$  (analytically computed by Onsager), and the same thing zoomed out by a factor of 3 and coarse-grained with a box width of 3 (i.e. we have applied the renormalization group action  $\mathcal{R}_3$ ). The different scales display qualitatively similar behavior.

$$\begin{aligned}
 (3.12) \quad \exp(-\beta F_{MFT}[m_0]) &\equiv \exp\left[-V\left(\frac{t}{2}m_0^2 + um_0^4\right)\right] \\
 &\gg \int \mathcal{D}\phi \exp\left[-\int d^d\vec{x} \frac{K}{2} [(\nabla\phi)^2 + \xi^{-2}\phi^2]\right] \equiv \int \mathcal{D}\phi \exp(-\beta F_{fluct}[\phi]).
 \end{aligned}$$

Calculate the “fluctuation” term to determine when this is true. It is useful again to consider the Fourier basis of the space of fluctuations, and again calculate that

$$(3.13) \quad \beta F_{fluct}[\phi] = \int d^d\vec{x} \frac{K}{2} \left[ (\nabla\phi)^2 + \frac{\phi^2}{\xi^2} \right] = \int d^d\vec{x} \int \frac{d^d q}{(2\pi)^d} \frac{K}{2} (q^2 + \xi^{-2}) |\phi_{\vec{q}}|^2$$

thanks to the diagonalization of the bracketed operator with respect to the Fourier basis. We now also replace the sum  $\int \mathcal{D}\phi$  with an explicit sum with respect to the Fourier basis coefficients, times some normalization, noting that  $F$  ignores complex conjugation, i.e.  $F[\phi_{\vec{q}}] = F[\phi_{\vec{q}}^*]$ :

$$(3.14) \quad \int \mathcal{D}\phi \exp[-\beta F_{fluct}[\phi(\vec{x})]] = \prod_{\vec{q}} \mathcal{N} \int d\phi_{\vec{q}} d\phi_{\vec{q}}^* \exp[-\beta F_{fluct}[\phi_{\vec{q}}]],$$

where the integration in the exponential gets absorbed into the overall product. It now becomes convenient to work with  $\ln Z$ , and change the product over  $\vec{q}$  into an integral:

$$\begin{aligned}
\ln Z_{fluct} &= \int d^d \vec{q} \left[ \ln \mathcal{N} + \ln \left( \int d\phi_{\vec{q}} d\phi_{\vec{q}}^* \exp \left[ -\frac{1}{V} \frac{K}{2} (q^2 + \xi^{-2}) |\phi_{\vec{q}}|^2 \right] \right) \right] \\
(3.15) \quad &= \int d^d \vec{q} \left[ \ln \mathcal{N} + \ln \sqrt{\frac{2\pi V}{q^2 + \xi^{-2}}} \right].
\end{aligned}$$

where in the 2nd equality we have computed a Gaussian integral. Since we have picked up an unknown constant  $\mathcal{N}$  in our calculations, we need to compare quantities which this constant doesn't affect. One example is the heat capacity,  $C \propto \frac{\partial^2 (\beta f_{thermo})}{\partial t^2}$  where again  $\beta f_{thermo} = -(\ln Z)/V$ . We have

$$\begin{aligned}
\beta f_{thermo}^{MFT} &= -(\ln Z_{MFT})/V = \frac{t}{2} m_0^2 + u m_0^4 \\
(3.16) \quad \implies C_{MFT} &= \begin{cases} 0, & T > T_c \\ -\frac{1}{8u}, & T < T_c. \end{cases}
\end{aligned}$$

(confirming our previous calculation of  $\alpha = 0$ ), and for the fluctuation component

$$\begin{aligned}
\beta f_{thermo}^{fluct} &= -(\ln Z_{fluct})/V \\
(3.17) \quad \implies C_{fluct} &= \frac{1}{VK^2} \int d^d \vec{q} \frac{1}{(q^2 + \xi^{-2})^2} = \frac{1}{VK^2} \int d^d \vec{q} \frac{1}{(q^2 + t(t)/K)^2},
\end{aligned}$$

where  $b(t)$  is 1 if  $T > T_c$  and  $-2$  if  $T < T_c$ . We see that if  $d > 4$ , the integral doesn't diverge with respect to  $t = a(T - T_c)$  at any value of  $t$  (since the overall dimension of  $q$  is  $\geq 1$ ). However, if  $d \leq 4$ , the integral diverges with respect to  $t$  at  $t = 0$  after integration (since the overall dimension of  $q$  is  $\leq 0$ , and  $t$  is in the denominator). Therefore in  $d \leq 4$ , the heat capacity is wrong by arbitrarily large amounts near the critical temperature in the saddle point approximation. Importantly, the dimensions we care about are all  $< 4$ , so our mean field theory estimates are all wrong near the critical point (for example, our estimated  $\alpha = 0$  is no longer correct, as  $C$  is no longer constant with respect to  $\xi$  near  $t = 0$ ).

We numerically solve this integral and plot heat capacity for  $d = 3$  and  $d = 5$  to show the qualitative differences (Fig. 5).

#### 4. THE RENORMALIZATION GROUP

We deduced in Section 3.2 that there is statistical self-similarity in the correlation function  $G(r)$  at the critical temperature. Kadanoff had the idea to disregard statistically self-similar scales ( $r \ll \xi$ ) with correlated degrees of freedom by zooming out and looking at the system on larger and larger scales, until you reach a scale where degrees of freedom are simpler/uncorrelated (i.e. the  $r \gg \xi$  regime, which is reached since  $\xi$  decreases when you zoom out). For example, if  $T > T_c$ , zooming out will reveal smaller and smaller structures until they are on the pixel scale (entirely random and uncorrelated) [7].

We work with the old discrete  $\omega$  configuration over  $\mathbb{Z}^d$ , and define a procedure to zoom out by a factor of  $b > 1$ ,  $b \in \mathbb{Z}$ :

- (1) Divide  $\mathbb{Z}^d$  into  $b \times b \times \dots \times b$  boxes. Each box will be a lattice site in the new configuration  $\omega'$ . In each new box centered on index  $i$ , define a new spin

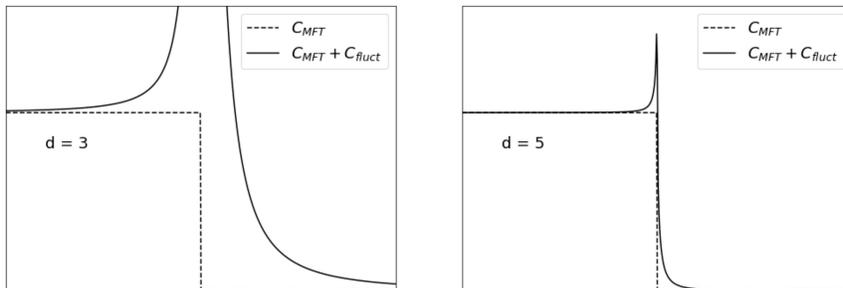


FIGURE 5. The heat capacity in the saddle point approximation (dotted line) and saddle point plus fluctuation approximation (solid line), in  $d = 3$  and  $d = 5$ . The behavior is qualitatively similar for all  $d \leq 4$ , and for all  $d > 4$ .

$\omega'_i$  by some consensus operation on the spins  $\omega_j$  with  $j$  in the box around  $i$ . For example, we could take the average.

- (2) Rescale the new domain by a factor of  $b$ : let  $i' = i/b$ , so the effective lattice spacing remains the same. The new configuration is  $\omega' = \{\omega'_{i'} \mid i' \in \mathbb{Z}^d\}$
- (3) Renormalize  $\omega'$  to maintain the same level of contrast as before coarse-graining. For example, if we took an average over the spins in the box, we could set positive  $\omega'_{i'}$  to be 1, and non-positive  $\omega'_{i'}$  to be -1.
- (4) Adjust all parameters (i.e.  $\beta, h$ ) so the new system obeys the physics of a  $d$ -dimensional Ising model with configuration  $\omega'$ . Formally, if the vector of parameters is  $\vec{K}$ , send them to a new vector  $\vec{K}'$  so that

$$(4.1) \quad e^{-\beta \mathcal{H}_{\vec{K}'}(s')} = \sum_{s: s \rightarrow s'} e^{-\beta \mathcal{H}_{\vec{K}}(s)}.$$

(This is similar to changing  $\mathcal{H}(\omega)$  to the free energy  $F[m(\vec{x})]$  when coarse-graining in mean field theory.) If this is exactly possible, we say the system is **renormalizable**; typically though, this is inexact.

We are interested in the map  $\vec{K} \mapsto \vec{K}'$  on the space of parameters. Denote this by  $\mathcal{R}_b$ , so  $\mathcal{R}_b(\vec{K}) = \vec{K}'$ . We will see that parameters which flow away from fixed points under  $\mathcal{R}_b$  are the physically relevant ones, since they exhibit changes at macroscopic scales. Note that this is almost a group action: we have  $\mathcal{R}_{b_1} \mathcal{R}_{b_2} = \mathcal{R}_{b_1 b_2}$ ; likewise associativity holds. However, the coarse-graining process is typically non-invertible, so  $\mathcal{R}_b$  has no inverse and it is actually a semi-group.

Note that the correlation length scales like  $\xi' = \xi/b$ . Since  $\xi$  is a function of our  $\vec{K}$ , fixed points occur only when the critical length is zero or infinity. We deduced, from mean field theory, that the correlation length should grow infinite near the critical point of the Ising model. In fact, we expect the critical point  $(T_c, 0)$  to be a fixed point of  $\mathcal{R}_b$ . We showed, in Figure 4, the effect of  $\mathcal{R}_3$  on the 2d Ising model at  $(T_c, 0)$  ( $T_c$  is exactly known). The correlation length is indeed very large in these spin configurations, and they exhibit similar behavior at both scales, as is required at a fixed point.

Consider the behavior of  $\mathcal{R}_b$  for the Ising model in the vicinity of a fixed point. Our parameters are  $(\beta, h)$ , but we will change  $\beta$  to  $t \equiv (T - T_c)/T_c$ . So our parameters are  $(t, h)$  and the critical point occurs at  $(0, 0)$ . From the transformation in (4.1), we expect  $\mathcal{R}_b$  to have first-order partial derivatives in the vicinity of  $(0, 0)$ , so we expand about  $(0, 0)$ :

$$(4.2) \quad \mathcal{R}_b(t, h) = (A(b)t + B(b)h, C(b)t + D(b)h).$$

However, to obey the symmetry  $\{m \mapsto -m, h \mapsto -h, t \mapsto t\}$ , we must have  $B(b) = C(b) = 0$ . Thus by the semigroup structure of this map,

$$(4.3) \quad A(b_1 b_2)t = \mathcal{R}_{b_1 b_2}(t) = \mathcal{R}_{b_1} \mathcal{R}_{b_2}(t) = A(b_1)A(b_2)t, \quad A(1) = 1$$

so we can infer that  $A(b) = b^{y_t}$  for some **anomalous dimension**  $y_t$ . Similarly,  $D(b) = b^{y_h}$ . By explicitly computing the transformation  $\mathcal{R}_b$ 's action on the parameters from part (4) of the renormalization group procedure, we can explicitly compute these anomalous dimensions.

**4.1. Renormalization for Critical Exponents.** We can also work in mean field theory, applying the procedure to  $m(\vec{x})$  by applying the same steps (1)-(4) but to the field. The renormalization group in the vicinity of the critical point can help us infer the critical exponents. Consider the local Landau-Ginzberg free energy  $f$  (which is integrated to get  $F$ ). Since we require  $Z = Z'$ , and since  $f(t, h) \sim (1/V) \ln Z(t, h)$ , we have  $V f(t, h) = V' f(t', h')$ . Since  $V$  scales like  $b^d$ , we have

$$(4.4) \quad f(t, h) = b^{-d} f(b^{y_t} t, b^{y_h} h).$$

But then we can choose  $t = b^{1/y_t}$  so that  $b^{y_t} t = 1$ , i.e. so that

$$(4.5) \quad f(t, h) = t^{d/y_t} f(1, h/t^{y_h/y_t}) \equiv t^{d/y_t} g(h/t^{y_h/y_t}),$$

giving us a ‘‘homogenous form’’ for the free energy. Note that since  $C \propto \frac{\partial^2 f}{\partial t^2} |_{h=0} \propto |t|^{-\alpha}$ , we immediately have

$$(4.6) \quad \alpha = 2 - d/y_t.$$

Starting from this homogenous form, one can derive all the critical exponents in terms of the anomalous dimensions  $y_t, y_h$ . This is done in some texts, e.g. [7, 10]. If  $y_t, y_h$  are explicitly determined in step (4) of the renormalization group procedure, this gives us all critical exponents. We can also eliminate  $y_t, y_h$  to give relations between the exponents (‘‘scaling relations’’). These are consistent with all experimental observations. Except for the one involving the dimension  $d$ , they are also consistent with mean field theory.

Exponents in terms of $y_t, y_h$	Scaling relations
$\alpha = 2 - d/y_t$	$\alpha + 2\beta + \gamma = 2$
$\beta = \frac{d-y_h}{y_t}$	$\delta - 1 = \gamma/\beta$
$\gamma = \frac{2y_h-d}{y_t}$	$2 - \alpha = \nu d$
$\delta = \frac{y_h}{d-y_h}$	$\gamma = \nu(2 - \eta)$
$\nu = 1/y_t$	
$\eta = d - 2(y_h + 1)$	

**4.2. Renormalization Group, Formal.** We conclude with a formal discussion of the renormalization group's action on a general lattice system, and a discussion of the phenomenon of universality. Consider a system with an arbitrary vector of parameters  $\vec{K}$ , that transforms like  $\vec{K}' = \mathcal{R}_b \vec{K}$ . Suppose this vector has a fixed point  $\vec{K}^*$ . Near the fixed point,  $\mathcal{R}_b$  has behavior

$$(4.7) \quad K'_\alpha + \delta K'_\alpha = K_\alpha + (\mathcal{R}_b)_{\alpha,\beta} \delta K_\beta + \dots$$

where we sum over repeated indices, and where

$$(4.8) \quad (\mathcal{R}_b)_{\alpha,\beta} = \frac{\partial K'_\alpha}{\partial K_\beta} = \frac{\partial \mathcal{R}_b(K)_\alpha}{\partial K_\beta}.$$

We can diagonalize the matrix  $(\mathcal{R}_b)_{\alpha,\beta}$  to get eigenvalues  $\lambda_i(b)$  and eigenvectors  $\mathcal{O}_i$ . Note that for different scaling factors  $b_1, b_2$ ; we have by the semigroup property  $\mathcal{R}_{b_1} \mathcal{R}_{b_2} = \mathcal{R}_{b_1 b_2} = \mathcal{R}_{b_2} \mathcal{R}_{b_1}$  so that the  $b_1$  and  $b_2$  Jacobian matrices commute, and thus can be simultaneously diagonalized; therefore the eigenvectors  $\mathcal{O}_i$  are independent of the scaling factor  $b$ . Also because of the semigroup property, we have (as in the 2-parameter case above),

$$(4.9) \quad \lambda_i(b_1) \lambda_i(b_2) \mathcal{O}_i = \mathcal{R}_{b_1} \mathcal{R}_{b_2} \mathcal{O}_i = \mathcal{R}_{b_1 b_2} \mathcal{O}_i = \lambda_i(b_1 b_2) \mathcal{O}_i.$$

and  $\lambda_i(1) = 1$ ; therefore we have  $\lambda_i(b) = b^{y_i}$  for some anomalous dimension  $y_i$ . We classify the eigendirections of  $\mathcal{R}_b$  in the vicinity of a critical point as follows:

- (1) If  $y_i > 0$ ,  $\mathcal{O}_i$  is a **relevant** operator and the parameters are stretched in the  $\mathcal{O}_i$  direction under scaling.
- (2) If  $y_i < 0$ ,  $\mathcal{O}_i$  is an **irrelevant** operator and the parameters are compressed in the  $\mathcal{O}_i$  direction under scaling.
- (3) If  $y_i = 0$ ,  $\mathcal{O}_i$  is a **marginal** operator and the parameters are, to first order, unchanged in the  $\mathcal{O}_i$  direction under scaling.

Relevant operators are, in fact, the physically relevant operators, because they are the ones that exhibit changes at large, macroscopic scales. They are often thought of as “knobs” an experimentalist is able to physically turn to change the system from their macroscopic-scale vantage point. In most physical systems, there are very few relevant operators. This is key to understanding the phenomenon of **universality**, where systems described by different microscopic details have the same macroscopic properties, e.g. phase diagrams that are qualitatively the same. This is demonstrated in Fig. 6 for the Ising model and a gas.

Looking at  $\mathcal{R}_b$  as a dynamical system, consider the basin of attraction of  $\vec{K}^*$ . This is the space spanned by irrelevant operators. Call the codimension of this basin of attraction the “codimension of  $\vec{K}^*$ .” It is the number of relevant operators.



FIGURE 6. The phase diagrams for the  $d \geq 2$  Ising model, and for a liquid/gas. These systems belong to the same universality class, which can be deduced by studying their renormalization group flows. Figure from [3].

For each possible codimension, there is a limited number of possible types of fixed points; these are tabulated in [10] (e.g. if the codimension is 2, the fixed point can either be a triple point or a critical fixed point). Therefore, if the critical points of two systems have the same codimension (same number of relevant operators), it's likely that the phase behavior near this point is qualitatively the same, and that they even share the same critical exponents. Thus, the renormalization group ends up being a very useful technique for identifying different systems that share similar macroscopic behavior.

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