THE BALDWIN–LACHLAN PROOF OF MORLEY'S THEOREM

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Abstract. In this paper we will provide a relatively self-contained treatment of the Baldwin–Lachlan proof of Morley's theorem. Morley's theorem is an important milestone in the history of model theory as it marks the beginnings of stability theory and more generally modern model theory. The Baldwin–Lachlan proof is distinct from the original proof, since it forgoes the topological methods of the original in favor of a more geometric and algebraic approach. All introductory model-theoretic references will be included in an appendix.

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1. Introduction and Historical Context

The idea of categoricity was introduced in 1904 by Oswald Veblen to describe a theory with exactly one model up to isomorphism [8]. This idea was not as productive as originally hoped due to work by Löwenheim and Skolem showing that any theory of first order logic with an infinite model must have infinite models of arbitrarily large infinite size [10]. Any attempt to study infinite categorical models was then restricted to studying *categoricity in power* or categoricity of all models of a given size. This is also denoted as κ*-categorical* for a given cardinal κ. Note that many theories are not categorical for any κ. Investigations into the study of κ-categorical models led to a well known conjecture of Jerzy Loś in 1954:

Conjecture 1.1 (Loś's Conjecture)**.**
There are precisely three types of κ*-categorical theories:*

1. κ*-categorical for every infinite cardinal* κ*. An example of this is a vector space over a finite field.*
2. κ*-categorical for every uncountable* κ*. An example of this is a vector space over an infinite field.*
3. κ*-categorical only for* $\kappa = \aleph_0$*. An example of this is the theory of dense linear orders without endpoints.*

This conjecture initiated a large amount of productive work in model theory, culminating in a solution given by Morley in 1965 and the proof of Morley's theorem.

Theorem 1.2 (Morley's Theorem)**.** *A complete countable theory* T *is categorical in some uncountable cardinal if and only if it is categorical for every uncountable cardinal.*

Morley's Theorem resolves Loś's conjecture by saying the three possibilities listed are in fact the only options.

2. Theorem Overview

This paper will not go into detail about the original proof of Morley's theorem and instead will cover an alternative proof given by Baldwin and Lachlan. Note that the Baldwin–Lachlan proof does not directly prove Morley's theorem but actually proves a slightly stronger result. Unless specified otherwise, assume T is countable and complete.

Theorem 2.1 (Baldwin–Lachlan)**.**
Let κ *be an uncountable cardinal. A countable* T *is* κ*-categorical if and only if* T *is* ω*-stable and has no Vaughtian pairs.*

Broadly speaking, some of the key innovations of Morley's original proof are the use of ideas from topology (Morley rank and the type space) to analyze categoricity. The Baldwin-Lachlan proof forgoes Morley rank and uses the machinery of strong minimality and a notion of geometry and dimension to perform an alternative analysis of the type space. To aid in the readability of the paper and to avoid excessive cross referencing, we will introduce machinery such as strong minimality and pregeometries as it is used. Here is an overview of the proof as it will be presented in this paper.

Let T be our theory and $\mathcal{M}, \mathcal{N}$ model T. Here are the steps we will use to prove the $\implies$ direction,

(1) T is κ-categorical $\implies$ T is ω-stable,
(2) T is κ-categorical $\implies$ T has no Vaughtian pairs,

Here are the steps we will use to prove the $\Leftarrow$ direction,

(1) An ω-stable theory T has a prime model $\mathcal{M}_0$ with a strongly minimal formula $\phi(x)$.
(2) Use the machinery of dimension for the classification of extensions of $\mathcal{M}_0$ by size.
(3) The fact that $\mathcal{M}$ is prime over $\phi(\mathcal{M})$ and the lack of Vaughtian pairs forbid any proper submodels, so any partial elementary map from $\mathcal{M}$ to $\mathcal{N}$ with $|\mathcal{M}| = |\mathcal{N}|$ can be extended to an isomorphism.

3. Types

Our primary tool in proving Morley's theorem will be analysis of the type space for a given model.

Definition 3.1 (Types)**.**
For an $\mathcal{L}$ structure $\mathcal{M}$ and a subset A of the domain M, $\mathcal{L}_A$ is the language where we add additionally constants for the elements of A. Let p be a set of $\mathcal{L}_A$-formulas in free variables $v_1, \ldots, v_n$. We call p an *n-type* (with respect to $\mathcal{M}$) if $p \cup Th_A(\mathcal{M})$ is satisfiable, aka has a model. We say that p is a complete *n-type* if $\phi \in p$ or $\neg\phi \in p$ for all $\mathcal{L}_A$-formulas ϕ with free variables from $v_1, ..., v_n$. We let $S_n^{\mathcal{M}}(A)$ be the set of all complete n-types with parameters from A.

For additional details beyond what is below see [2],[1]. $S_n^{\mathcal{M}}$ is often referred to as either the type space or the Stone space of types. This is due to the fact that the space of types has a natural structure of a Boolean algebra, and by Stone's representation theorem for Boolean algebras we can turn the type space into a topological space.

Definition 3.2 (Stone Topology)**.**
We can define a topology on the space of complete n-types $S_n^{\mathcal{M}}(A)$ as follows: Consider an $\mathcal{L}_A$-formula ϕ with free variables $v_1, ..., v_n$. Let

$$[\phi] = \{p \in S^{\mathcal{M}}(A) : \phi \in p\}.$$

The set $[\phi]$ includes every complete type p which contains ϕ. Note that since p is complete, if $\phi \vee \psi \in p$ then either $\phi \in p$ or $\psi \in p$. Therefore as sets, $[\phi \vee \psi] = [\phi] \cup [\psi]$. Likewise, $[\phi \wedge \psi] = [\phi] \cap [\psi]$. The *Stone topology* on $S_n^{\mathcal{M}}(A)$ is the topology generated by taking $[\phi]$ as open sets.

Treating $S^{\mathcal{M}}$ as a topological space allows us to make topological arguments about types.

Definitions 3.3 (Clopen Sets and Isolated Types)**.**

(1) Sets that are both closed and open in our topology are denoted *clopen.*
(2) A type $p \in S_n^{\mathcal{M}}(A)$ is *isolated* if $\{p\}$ is an open subset of $S_n^{\mathcal{M}}(A)$.

Remark 3.4. For a complete type p, exactly one of ϕ or $\neg\phi$ is in p. Therefore, $[\phi] = S_n^{\mathcal{M}}(A) \backslash [\neg\phi]$ is closed. Furthermore any basic open set $[\phi]$ has complement $[\neg\phi]$ so is both open and closed.

Proposition 3.5.

(1) $S_n^{\mathcal{M}}(A)$ is compact.

(2) *$S_n^{\mathcal{M}}(A)$ is totally disconnected. That is, if $p, q \in S_n^{\mathcal{M}}(A)$ and $p \neq q$, then there is a clopen set X where $p \in X$ and $q \notin X$.*

Proof. (1) Follows from the compactness theorem. See [1] for details.

(2) If $p \neq q$, then there must necessarily be a formula ϕ where $\phi \in p$ and $\phi \notin q$. This is equivalent to $\neg\phi \in q$. Thus, $[\phi]$ is a clopen set generated by a single element which separates p, q.

□

One last definition needed is that of stability and κ-stability. To be complete, we will give the full definition of stability. However, in practice we will restrict ourselves to a particularly useful sub case called ω-stability.

Definition 3.6 (Stable)**.**
A theory T is *κ-stable* if (for an infinite cardinal κ) for every model $\mathcal{M}$ of T, for every set A with cardinality κ, the set of complete types over A has cardinality κ:

$$A \subseteq M,\ |M| \leq \kappa \implies |S_1(A)| < \kappa.$$

A theory is *stable* if it is κ-stable for some infinite κ. A theory is *unstable* if it is not -stable for any infinite κ. A theory is ω-stable if it is κ-stable for every infinite κ.

One may ask why the restriction to the complete 1-types, $S_1(A)$, is used in the definition. It can be shown that for the ω-stable case we obtain the additional result that $|S_n(A)| \leq \aleph_0$ for countable A and $n \in \omega$.

Example 3.7. Here are some examples of theories and their stability.

(1) The theory of $\mathbb{Q}$ considered as a dense total order is unstable. More generally any theory which is capable of encoding an infinite total order is unstable.
(2) The theory of infinite Boolean algebras is unstable.
(3) The theory of a real closed field is unstable since it can encode an infinite total order.
(4) Set theory and Peano arithmetic are unstable since they can encode within themselves other unstable theories like arithmetic on the natural numbers.

Here are some stable and ω-stable theories.

(1) The theory of the additive group of the integers is stable but not ω-stable since it has 2^ω many countable models.
(2) The theory of deferentially closed fields is ω-stable.
(3) As we shall see by the proof of Morley's theorem, any theory in a countable language which is categorical for some uncountable κ is ω-stable. This includes algebraically closed fields of a given characteristic and vector spaces.

4. Ehrenfeucht-Mostowski Models

Before proving the first part of the Baldwin–Lachlan proof, we must introduce a bit of technical machinery. The strategy of the proof is to suppose to the contrary and then exhibit two separate models which contradict categoricity. The existence of the first of such models depends upon the fundamental model-theoretic tools of indiscernibles and Ehrenfeucht–Mostowski models. Here in proving Morley's theorem we will only use the existence of a given model with minimal type space and thus will simply state the needed theorem without proof and refer the reader to [1] for details.

Theorem 4.1 (Consequence of Ehrenfeucht–Mostowski Models)**.**
Let $\mathcal{L}$ be countable and T be an $\mathcal{L}$-theory with infinite models. For all $\kappa \geq \aleph_0$, there is $\mathcal{M} \vDash T$ with $|\mathcal{M}| = \kappa$ such that if $A \subseteq M$, then $\mathcal{M}$ realizes at most $|A| + \aleph_0$ types over A.

Now we can move on to proving our desired theorem.

4.1. κ-categorical implies ω-stable.

Theorem 4.2 (κ-categorical $\implies$ ω-stable)**.**
Let T be a countable theory with infinite models and $\kappa \geq \aleph_1$. If T is κ-categorical then it is ω-stable.

Proof. Assume T is not ω-stable. Then there must exist $\mathcal{M} \vDash T$ and $A \subseteq M$ countable with $|S_1(A)| > \aleph_0$. By a compactness argument we can find $\mathcal{N}_0$ such that $\mathcal{M} \preceq \mathcal{N}_0$ and $\mathcal{N}_0$ realizes uncountably many types from $S_1(A)$. Now, by Theorem 4.1, we can find a similar model $\mathcal{N}_1$ which realizes at most $\aleph_0$ types from $S_1(B)$ for all $B \subseteq N_1$. Then $\mathcal{N}_0 \not\cong \mathcal{N}_1$, contradicting κ-categoricity. □

Remark 4.3. For a complete countable theory T, ω-stable implies κ stable for all infinite κ.

5. Vaughtian Pairs and (κ, λ)-Models

Prior to proving the fact that κ-categoricity implies no Vaughtian pairs we will first define Vaughtian pairs.

Definition 5.1 (Vaughtian Pair)**.** A theory T has a *Vaughtian pair* if there are $\mathcal{M} \preceq \mathcal{N} \vDash T$ and $\phi(x) \in \mathcal{L}(M)$ such that

(1) $M \neq N$;
(2) $\phi(M)$ is infinite;
(3) $\phi(M) = \phi(N)$.

One of the hidden purposes of this definition is that it provides a very easy test for another important model theoretic property, that of the (κ, λ)-model.

Definition 5.2 ((κ, λ)-model)**.** Let $\kappa > \lambda \geq \aleph_0$ be cardinals. Then $\mathcal{M} \vDash T$ is a (κ, λ) model if $|M| = \kappa$ and for some $\phi(x) \in \mathcal{L}$, $|\phi(M)| = \lambda$.

The connection between the existence (or lack thereof) of a Vaughtian pair and that of (κ, λ)-models is quite strong. We will now provide a couple of lemmas and theorems from which the desired proof will follow. The details for the proof of Vaught's Theorem are beyond the scope of this paper and the proof can be found in [1],[2].

Theorem 5.3 (Vaught's Two Cardinal Theorem)**.** *If T has a (κ, λ)-model with $\kappa > \lambda \geq \aleph_0$, then T has an $(\aleph_1, \aleph_0)$-model.*

Lemma 5.4. *If T has a Vaughtian pair, then it has an $(\aleph_1, \aleph_0)$-model.*

Lemma 5.5. *Suppose T is ω-stable and T has an $(\aleph_1, \aleph_0)$-model. If $\kappa > \aleph_1$, then T has a $(\kappa, \aleph_1)$-model.*

5.1. **κ-categorical implies no Vaughtian Pairs.**
Given the previously mentioned lemmas, we can conclude our desired proof quite quickly.

Theorem 5.6 (κ-categorical $\implies$ no Vaughtian pairs)**.**
Let T be a complete theory in a countable language with infinite models. If $\kappa \geq \aleph_1$ and T is κ-categorical, then T has no Vaughtian pairs and hence no (κ, λ)-models for $\kappa > \lambda \geq \aleph_0$.

Proof. Since T is κ-categorical then it must be ω-stable. Suppose for the sake of contradiction there is a Vaughtian Pair. Then by Lemma 5.4 there must be a $(\aleph_1, \aleph_0)$-model. By Lemma 5.5 there is a $(\kappa, \aleph_0)$-model. However we can find a model of cardinality κ where every definable set has cardinality κ. $\square$

6. Existence of a Prime model

We have finished the $\implies$ direction of the proof of Theorem 2.1 through Theorems 5.6 and 4.2. Now we shall introduce the required machinery to prove the converse. The first part of the converse proof is to show the existence of a prime model of a κ-categorical theory with a strongly minimal formula. We will proceed in two parts, first to show the existence of the prime model then to introduce the language of minimality and show that our prime model possesses a strongly minimal formula.

6.1. **Remark on isolated types.** Prior to our discussion of prime models, recall the following definition:

Definition 6.1 (Isolated Types)**.** $p \in S_n^{\mathcal{M}}(A)$ is *isolated* if $\{p\}$ is an open subset of $S_n^{\mathcal{M}}(A)$.

Our prime model will be constructed by realizing all appropriate isolated types over A. This can be seen from the connection types carry between the global behavior of a model (an element as represented by types) and local behavior (individual formulas satisfied by a single element). Finite subsets of types are always realized by compactness, so the finitary behavior of a type being isolated ensures that it is always realized in every elementary substructure or extension. This means that if we want to find the *smallest* substructure of a given theory which is contained in every model (loosely speaking, a prime model), then it must realize every isolated type.

It is important to note that prime models do not always exist, but, given certain nice conditions on theories, they do. In our case, we will be considering ω-stable theories. In particular the restriction of the size of the type space given by ω-stability ensures that isolated types are in fact dense by a proof by contradiction and binary tree construction. In this situation by dense we mean that for any open set in the stone space, it must contain an isolated type.

6.2. **Prime model existence.**

Theorem 6.2 (Prime Model Existence)**.** *Let T be a countable ω-stable theory, $\mathcal{M} \vDash T$ and $A \subseteq M$. Then there exists $\mathcal{M}_0 \preceq \mathcal{M}$ which is a prime model over A.*

Proof. First we find an ordinal δ and subsets $(A_\alpha : \alpha :\leq \delta)$ of M such that the following properties hold:

(1) $A_0 = A$.
(2) If α is a limit cardinal, then $A_\alpha = \cup_{\beta<\alpha} A_\beta$.
(3) If no element of $M \backslash A_\alpha$ realizes an isolated type over A_α then let $\alpha = \delta$;
(4) otherwise, pick a_α with $tp(a_\alpha \backslash A_\alpha)$ isolated and let $A_{\alpha+1} = A_\alpha \cup \{a_\alpha\}$.

Let $\mathcal{M}_0$ be the substructure of $\mathcal{M}$ with $M_0 = A_\delta$. Now we can prove the two claims of our theorem:

(1) $\mathcal{M}_0 \preceq \mathcal{M}$;
(2) $\mathcal{M}_0$ is a prime model extension of A.

Claim 1, $\mathcal{M}_0 \preceq \mathcal{M}$: By the Tarski–Vaught test, suppose $\mathcal{M} \models \phi(x)$ with $\phi(x) \in \mathcal{L}(A_\delta)$. By the density of isolated types mentioned earlier, there must exist $b \in \mathcal{M}$ such that $tp(b \backslash A_\delta)$ is isolated and $\mathcal{M} \models \phi(b)$.
Claim 2, $\mathcal{M}_0$ is a prime extension of A: Since we have restricted ourselves to isolated types and built A_δ, $\mathcal{M}_0$ by induction, then we can follow a similar induction to extend any partial elementary map $f : A \to \mathcal{N}$ to $f_\delta : \mathcal{M}_0 \to \mathcal{N}$, where f_δ is elementary. □

7. Minimality

Before showing that the prime model described in Theorem 6.2 has a strongly minimal formula, we define minimality and strong minimality.

Take $\mathcal{M}$ as an $\mathcal{L}$-structure and $\phi(\bar{v})$ as an $\mathcal{L}_M$-formula. Let $\phi(\mathcal{M})$ be the set of elements of M that satisfy ϕ. Formally we define it as

$$\phi(\mathcal{M}) := \{a \in M : x \models \phi(a)\}.$$

Definitions 7.1 (Minimality and Strong Minimality)**.**
In an $\mathcal{L}$-structure $\mathcal{M}$, consider an infinite definable set $D \subseteq M^n$. We say that D is *minimal* in $\mathcal{M}$ if for any definable $Y \subseteq D$, Y is either finite or cofinite (in D). If $\phi(\bar{v}, \bar{a})$ is the formula that defines D, then we say that $\phi(\bar{v}, \bar{a})$ is *minimal*.

We say that D and ϕ are *strongly minimal* if ϕ is minimal in any elementary extension $\mathcal{N}$ of $\mathcal{M}$. Furthermore, a theory T is *strongly minimal* if the formula $v = v$ is strongly minimal.

To understand the difference between a minimal and strongly minimal theory we consider the following example:

Example 7.2.
Let $\mathcal{L} = \{E\}$ and let $\mathcal{M}$ be an $\mathcal{L}$-structure in which E is an equivalence relation with one class of size n for each finite n and no infinite classes. Then $v = v$ is a minimal formula by definition. However, using the compactness theorem, there is an extension $\mathcal{N}$ of $\mathcal{M}$ where there exists an element $a \in N$ such that the equivalence class of a is infinite. The formula vEa defines an infinite coinfinite subset of our model, and the formula $x = x$ is not strongly minimal.

Lemma 7.3 (There exists a minimal formula $\phi(x)$)**.** *For an ω-stable theory T, if $\mathcal{M} \models T$, then there exists a minimal formula in $\mathcal{M}$.*

The proof for this is similar to the proof of the density of isolated types. For the sake of contradiction, if there were not a minimal formula, then we could build a binary tree starting with the formula $v = v$. This tree would have size $2^{\aleph_0}$ in contradiction with ω-stability.

Lemma 7.4 ($\exists$ strongly minimal formula)**.** *If T has no Vaughtian pairs then any minimal formula is strongly minimal.*

The proof of this stronger lemma results from the following theorem:

Theorem 7.5 (Elimination of $\exists^\infty$)**.** *If T has no Vaughtian pairs, then T eliminates $\exists^\infty$. That is, for for each $\phi(\bar{v}, \bar{w}) \in \mathcal{L}$, $\exists n_\phi \in \omega$ such that if $\mathcal{M} \models T, \bar{a} \in M$, and $|\phi(M, \bar{a})| > n$, then $\phi(M, \bar{a})$ is infinite.*

$\exists^\infty$ stands for "exists infinitely many". Intuitively, this can be thought of as saying that the model has the expressive capability to say that if a formula has a definable set greater than a given size, then it is infinite. Now, the fact that any minimal formula is strongly minimal follows from the fact that, given the elimination of $\exists^\infty$, we can express minimality as an elementary property, so it must be true in every extension. Specifically, here is the expression for a minimal formula $\phi(x, \bar{a})$ and any formula $\psi(x, \bar{z})$:

$$\neg\left(\exists^\infty x(\phi(x, \bar{a}) \wedge \psi(x, \bar{z})) \wedge \exists^\infty x(\phi(x, \bar{a}) \wedge \neg\psi(x, \bar{z}))\right)$$

8. Pregeometries and Algebraic Closure

Now we will use the geometry inherent to strongly minimal sets and algebraic formulas to develop a notion of dimension for our formulas. This sense of dimension as mentioned in the overview of Theorem 2.1 will allow for a characterization of extensions of our prime model $\mathcal{M}_0$ by size.

Definitions 8.1 (Algebraic closure)**.** Let $\mathcal{M}$ be a structure and A a subset of M.

1. A formula $\phi(x) \in \mathcal{L}_A$ is *algebraic* if $\phi(M)$ is finite.
2. An element $a \in M$ is *algebraic over A* if it satisfies an algebraic $\mathcal{L}(A)$ formula. We say that a is algebraic when a is algebraic over the empty set.
3. The *algebraic closure* of A is the set
$$acl(A) = \{a \in M : a \text{ is algebraic over } A\}.$$
4. A is *algebraically closed* if $acl(A) = A$.

Definition 8.2 (Closure Operator)**.**

Take $\mathcal{M}$ to be an $\mathcal{L}$-structure and $D(x) \in \mathcal{L}(M)$ to be a strongly minimal formula. Now consider an abstract closure operator $cl : 2^{D(M)} \to 2^{D(M)}$ defined by

$$cl(A) := acl(A) \cap D(M).$$

Proposition 8.3.

The operator cl satisfies the following properties for all $A \subseteq D(M)$ and $a, b \in D(M)$:

(1) Reflexivity: $A \subset cl(A)$;
(2) Finite Character: $cl(A) = \bigcup\{cl(A') : A' \subset A \ finite\}$;
(3) Transitivity: $cl(cl(A)) = cl(A)$;
(4) Exchange: $a \in cl(A \cup \{b\}) \backslash cl(A) \implies b \in cl(A \cup \{a\})$.

Note that $A \cup \{a\}$ is often written Aa for convenience.

A *pregeometry* is a set X along with a closure operator which fulfills the criteria in Proposition 8.3.

Proof. See [12, 4] for reference, pages 87,75 respectively. $\square$

Upon reading the definition of a pregeometry, the connection to any intuitive notion of geometry appears obscure. The connection becomes more clear when we see that one can develop notions of independence, basis, and dimension which follow the intuition developed in linear algebra and the theory of algebraically closed fields.

Definitions 8.4 (Independence, Basis, Generating Set)**.**
Let (D, cl) be a pregeometry. A subset $A \subseteq D$ is called

(1) *independent over* C if $a \notin cl(C \cup (A \backslash \{a\})$ for all $a \in A$;
(2) a *generating set* if $D = cl(A)$.;
(3) a *basis* for $Y \subseteq D$ if $A \subseteq Y$ is an independent generating set.

Note that any maximally independent subset of Y is a basis.

Lemma 8.5 (Replacement lemma)**.**
Let $A, B \subset D$ be independent from A, with $A \subseteq cl(B)$.

(1) Suppose $A_0 \subseteq A$, $B_0 \subseteq B$, and $A_0 \cup B_0$ is a basis for $acl(B)$ and $a \in A \backslash A_0$. Then $\exists b \in B_0$ such that $A_0 \cup \{a\} \cup (B_0 \backslash \{b\})$ is a basis for $acl(B)$.
(2) $|A| \leq |B|$.
(3) If A, B are bases for $Y \subseteq D$, then $|A| = |B|$.

Proof. See [12] page 88. □

Replacement allows us to know that the cardinality of a basis is well-defined so that we have an intuitive notion of dimension.

Definition 8.6 (Dimension)**.**
If $Y \subseteq D$, then $\dim(Y)$, the *dimension* of Y, is the cardinality of a basis for Y.

9. Application of Pregeometries

We now begin the description of our model in terms of independence and geometry. This will give us two propositions useful in the final proof of Theorem 2.1.

Proposition 9.1. *Suppose $\mathcal{M}, \mathcal{N} \models T$, $\phi(x) \in \mathcal{L}(A)$ is a strongly minimal formula and either $A = \emptyset$ or $A \subseteq M_0$, where $M_0 \preceq \mathcal{M}, \mathcal{N}$.*

If $a_0, ..., a_n \in \phi(M)$ are independent over A and $b_0, ..., b_n \in \phi(N)$ are independent over A, then $tp^{\mathcal{M}}(\bar{a} \backslash A) = tp^{\mathcal{N}}(\bar{b} \backslash A)$.

Intuitively, this says that given $\bar{a}, \bar{b}$ which are independent over the subset in different extensions of $\mathcal{M}_0$, then the type of those elements over A is the same.

Proof. Begin with the assumption that $A \subset M_0$. The case where $A = \emptyset$ will follow. Now we will induct on the value of n in $a_1, ..., a_n$. For the case where $n = 1$, we assume $a \in \phi(M) \backslash cl(A)$ and $b \in \phi(N) \backslash cl(B)$. Now let $\psi(x) \in \mathcal{L}(A)$ and suppose $M \models \psi(a)$. Since $a \notin cl(A)$ by independence, then $\phi(M) \cap \psi(M)$ is cofinite in $\phi(M)$. So, there exists an $n \in \mathbb{N}$ such that

$$\mathcal{M} \models |\{x : \phi(x) \wedge \neg\psi(x)\}| = n.$$

Because we have $\mathcal{M}_0 \preceq \mathcal{M}, \mathcal{N}$ and $b \notin cl(A)$, then $\mathcal{N} \models \psi(b)$. Since our original $\psi \in \mathcal{L}(A)$ was arbitrary, then $tp^{\mathcal{M}}(a \backslash A) = tp^{\mathcal{N}}(b \backslash A)$. Now, since we have proved our claim for $n = 1$, assume that it is true for n and $a_1, ..., a_{n+1} \in \phi(M)$, $b_1, ..., b_{n+1} \in \phi(N)$ are independent sequences over A. Let $\bar{a} = (a_1, ..., a_n)$, $\bar{b} = (b_1, ..., b_n)$. By induction, we know $tp^{\mathcal{M}}(\bar{a} \backslash A) = tp^{\mathcal{N}}(\bar{b} \backslash B)$. Now we can repeat a variant of our argument for the base case.

Assume $\mathcal{M} \models \psi(\bar{a}, a_{n+1})$ for some $\psi(\bar{w}, v) \in \mathcal{L}(A)$. By the strong minimality of ψ and the fact that $a_{n+1} \notin cl(A\bar{a})$ by independence, we get that $\phi(M) \setminus \psi(\bar{a}, M)$ is finite. Therefore, there must exist an $n \in \mathbb{N}$ such that

$$\mathcal{M} \models |\{v : \phi(v) \wedge \neg\psi(\bar{a}, v)\}| = n.$$

Because we have $\mathcal{M}_0 \preceq \mathcal{M}, \mathcal{N}$ and $tp^{\mathcal{M}}(\bar{a} \setminus A) = tp^{\mathcal{N}}(\bar{b} \setminus A)$,

$$\mathcal{N} \models |\{v : \phi(v) \wedge \neg\psi(\bar{b}, v)\}| = n.$$

Finally, since $b_{n+1} \notin cl(A\bar{b})$, $\mathcal{N} \models \psi(\bar{b}, b_{n+1})$ as desired. □

Proposition 9.2 (Dimension as relates to isomorphism of models)**.**

If $\mathcal{M}$ and $\mathcal{N}$ are as above and $\dim(\phi(M)) = \dim(\phi(N)$*, then there is a bijective partial elementary map $f : \phi(M) \to \phi(N)$.*

In particular we have if T is a strongly minimal theory and $\mathcal{M}, \mathcal{N} \models T$ then

$$\mathcal{M} \simeq \mathcal{N} \iff \dim(M) = dim(N).$$

This says that strongly minimal theories are characterized by their dimensions.

Proof. Let B be a basis for $\phi(M)$ and C a basis for $\phi(N)$. Since $|B| = |C|$ by assumption, let $f : B \to C$ be a bijection. By Proposition 9.1, this map is elementary. Let I be the set

$$I = \{g : B' \to C' : B \subseteq B' \subseteq \phi(M), C \subseteq C' \subseteq \phi(N), f \subseteq g \textit{ is partial elementary.}\}$$

Using Zorn's Lemma, we can find a maximal such $g : B' \to C'$ in I. Suppose for the sake of contradiction that there is a $b \in \phi(M) \setminus B'$. As $b \in cl(B')$, there is a formula $\psi(v, \bar{d})$ which isolates $tp^{\mathcal{M}}(b \setminus B')$. Since our map is elementary, we can find a $c \in \phi(N)$ such that $\mathcal{N} \models \psi(c, g(\bar{d}))$. Then we have $tp^{\mathcal{M}}(b \setminus B') = tp^{\mathcal{N}}(c \setminus C')$, and we can extend g by sending b to c. This contradicts the maximality of g ensured by Zorn's Lemma. Therefore, $\phi(M) = B'$ as desired. A similar argument shows that $C' = \phi(N)$. □

Using the above results allows us to get the following extra result with almost no extra effort.

Theorem 9.3 (Strongly minimal implies uncountably categorical)**.**

If T is a countable strongly minimal theory, then T is κ-categorical for all uncountable κ. Furthermore, T has at most $\aleph_0$ many models of cardinality $\aleph_0$.

Proof. If T is countable, then $|acl(A)| \leq |A| + \aleph_0$, so any basis of $\mathcal{M}$ with $|M| = \kappa > \aleph_0$ has cardinality κ. Therefore, if $|M| \leq \aleph_0$, then $\dim(M) \leq \aleph_0$. □

10. Baldwin–Lachlan Proof

For clarity, we will recap the proof of the converse direction of Theorem 2.1 so far and then finish the argument using propositions from Section 9.

Proof. Let T be our ω-stable theory with no Vaughtian pairs. From the ω-stability of T we know it has a prime model $\mathcal{M}_0$ and there must exist a strongly minimal formula $\phi(x) \in \mathcal{L}(M_0)$. Let $\kappa \geq \aleph_1$ denote the cardinality of $\mathcal{M}, \mathcal{N} \models T$. Taking $\mathcal{M}_0 \preceq \mathcal{M}, \mathcal{N}$, we know $|\phi(M)| = |\phi(N)| = \kappa$, since T has no (κ, λ)-models. This implies that $\dim(\phi(M)) = \dim(\phi(N)) = \kappa$. By Proposition 9.2, there must exist $f : \phi(M) \to \phi(N)$ a partial elementary map.

Finishing the proof: We claim that $\mathcal{M}$ is prime over $\phi(M)$. This can be seen since no proper $\mathcal{K} \prec \mathcal{M}$ contains $\phi(M)$, as otherwise $\phi(\kappa) = \phi(M)$ and $\phi(M)$ is infinite, so $(\mathcal{M}, \mathcal{K})$ would make a Vaughtian pair. We still do know that there must exist a $\mathcal{K}$ such that $\mathcal{K} \preceq \mathcal{M}$ is a prime model over $\phi(M)$. We conclude that $\mathcal{K} = \mathcal{M}$.

Now, by the definition of primality, f extends to an elementary $f' : M \to N$. However, as claimed above, $\mathcal{N}$ has no proper elementary submodels containing $\phi(M)$. Therefore, f' is a surjection and hence an isomorphism. □

11. Further Reading

11.1. **Classification Theory.** The original proof of Morley's theorem set off an explosion of productivity in model theory. The ideas inherent in the Baldwin-Lachlan proof show the power of analyzing a "core structure" of a theory to look for combinatoric and geometric invariants which allow for robust classification. Consider the following quote by Saharon Shelah upon receiving the Steele prize in 2013:

"I am grateful for this great honour. While it is great to find full understanding of that for which we have considerable knowledge, I have been attracted to trying to find some order in the darkness, more specifically, finding meaningful dividing lines among general families of structures. This means that there are meaningful things to be said on both sides of the divide: characteristically, understanding the tame ones and giving evidence of being complicated for the chaotic ones. It is expected that this will eventually help in understanding even specific classes and even specific structures. Some others see this as the aim of model theory, not so for me. Still I expect and welcome such applications and interactions. It is a happy day for me that this line of thought has received such honourable recognition. Thank you."

This philosophy helped bring a sense of unity to applied versus pure model theory. Models could be studied for their own inherent structural properties, which in turn allowed for great applied results. This particular branch of model theory is known as classification theory and is extensively covered in [5].

11.2. **Geometric Stability Theory.** After it was shown that unaccountably categorical structures have a rich sense of structure, model theorists (notably Boris Zilber and Greg Cherlin) began applying techniques from the study of algebraic groups and algebraic geometry within model theory. One notable example is the theorem that no totally categorical structure can be finitely axiomatized. Additionaly there is the well known (false but important) Zilber Trichotomy conjecture,

Conjecture 11.1 (Zilber Trichotomy)**.** *If X is a strongly minimal set then exactly one of the following is true:*

(1) X has a trivial geometry; or
(2) X is essentially a vector space (aka its geometry forms a modular lattice); or
(3) X is essentially an algebraically closed field (aka its geometry forms a non-modular lattice).

Furthermore the methods that arose in the study of geometric model theory went on to be used to prove non-logical statements such as Hrushovski's and others' work on the Mordell-Lang conjecture [9]). For a foundational treatment of model theory and the significance of geometry in model theory geometry, see [11].

Appendix: Basic Model Theory

Here are some basic theorems and definitions used repeatedly in the text:

Definition 11.2 (Prime Model)**.** For a complete theory T, a model P is *prime* if it admits an elementary embedding into any $\mathcal{M}$ to which it is elementary equivalent.

Theorem 11.3 (Compactness)**.** *A set of sentences Σ has a model if and only if every finite subset of it has a model.*

Theorem 11.4 (Löwenheim–Skolem)**.** *If a countable theory has an infinite model model then it has a countable model. Furthermore, it has a model of every size κ for $\kappa \geq \aleph_0$*

The Tarski-Vaught test is a well-known necessary and sufficient test for elementary substructures. Its statement goes as follows:

Theorem 11.5 (Tarski-Vaught Test)**.** *$\mathcal{M}$ is an elementary substructure of $\mathcal{N}$ if and only if for every $\mathcal{L}$-formula $\phi(x, \bar{v})$ and for every $\bar{a}$ in $\mathcal{M}$: if there exists an n in $\mathcal{N}$ such that $\mathcal{N} \vDash \phi(n, \bar{a})$, then there exists an m in $\mathcal{M}$ such that $\mathcal{N} \vDash \phi(m, \bar{a})$.*

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