CLASSIFICATION OF BUNDLES AND ORIENTATION

GABRIELLE YANGQING LI

Abstract. In this paper, we will introduce bundle theory in a setting where fibers and maps between fibers lie in a chosen category $\mathcal{F}$. We will construct classifying spaces using the two-sided bar construction and prove a classification theorem. We will then define and classify bundles with additional structures by introducing Y-structures. We will then use orientations of bundles with respect to a cohomology theory as an example to demonstrate the Y-structure and its classification.

Contents

1. Preliminaries

Throughout this paper, we use $\mathcal{F}$ to denote a category with a faithful "underlying space" functor $\mathcal{F} \to \mathcal{U}$, where $\mathcal{U}$ denotes the category of compactly generated weak Hausdorff spaces; function spaces in $\mathcal{U}$ are given the compactly generated topology. Each object in $\mathcal{F}$ is a space, and the set of morphisms $\mathcal{F}(F, F')$ is a subspace of $\mathcal{U}(F, F')$. We can think of an object in a category $\mathcal{F}$ as a space with additional structure. For example, take $\mathcal{F}$ to be the category where objects are based spaces that are homeomorphic to the n-sphere for some fixed positive integer n and morphisms are basepoint-preserving homeomorphisms. This example will be revisited later in the paper.

Theorem 1.1 (Whitehead Theorem)**.**

(1) If $\theta : Y \to Z$ is a weak homotopy equivalence, then for a CW-complex X the induced map $\theta_* : [X, Y] \to [X, Z]$ is an isomorphism.
(2) If the spaces Y and Z above are CW-complexes, then the weak homotopy equivalence θ is a homotopy equivalence.

Proof. For (1), we first factor θ through the mapping cylinder M_θ so that we can replace Z with M_θ since the map $M_\theta \to Z$ is a homotopy equivalence. To prove both injectivity and surjectivity, we use Homotopy extension and lifting property in [3, Page 75] to induct on the skeleta of X. Substituting Y and Z for X, we can prove (2) as an immediate consequence of (1). A detailed proof can be found in [3, Page 76]. $\square$

2. $\mathcal{F}$-Spaces, $\mathcal{F}$-Maps, and $\mathcal{F}$-Homotopies

In this section, we will define some basic concepts of $\mathcal{F}$-spaces and $\mathcal{F}$-homotopies.

Definition 2.1 ($\mathcal{F}$-space)**.** An $\mathcal{F}$-space is a map $\pi : E \to B$ in $\mathcal{U}$ such that $\pi^{-1}(b) \in \mathcal{F}$ for each $b \in B$; B and E are the base space and total space respectively of π. An $\mathcal{F}$-map is a pair of maps $(g, f) : \nu \to \pi$ in $\mathcal{U}$ that makes the following diagram commute

$$\begin{array}{ccc} D & \xrightarrow{g} & E \\ {\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle \pi} \\ A & \xrightarrow[f]{} & B \end{array}$$

and is such that the restriction of $g : \nu^{-1}(a) \to \pi^{-1}(f(a))$ is in $\mathcal{F}$ for each $a \in A$.

Definition 2.2 ($\mathcal{F}$-homotopy and $\mathcal{F}$-homotopy equivalence)**.** If (g_0, f_0) and (g_1, f_1) are two $\mathcal{F}$-maps between $\mathcal{F}$-spaces $\nu : D \to A$ and $\pi : E \to B$, then an $\mathcal{F}$-homotopy between them is an $\mathcal{F}$-map (H, h) as in the following commutative diagram

$$\begin{array}{ccc} D \times I & \xrightarrow{H} & E \\ {\scriptstyle \nu\times 1}\downarrow & & \downarrow{\scriptstyle \pi} \\ A \times I & \xrightarrow[h]{} & B. \end{array}$$

If for each $s \in I$, we let $(H_s, h_s) : \nu \to \pi$ be defined by $H_s(d) = H(d, s)$ and $h_s(a) = h(a, s)$, then each (H_s, h_s) is an $\mathcal{F}$-map.

- If $A = B$ and h_s is the identity map of B for all $s \in I$, then H is said to be an $\mathcal{F}$-homotopy over B. In this case, we denote this $\mathcal{F}$-map as H instead of $(H, 1)$ since it's clear what the map on the bottom is. We then call $\nu : D \to B$ and $\pi : E \to B$ equivalent $\mathcal{F}$-spaces over B.
- An $\mathcal{F}$-map $g : D \to E$ over B is an $\mathcal{F}$-homotopy equivalence if there is an $\mathcal{F}$-map $g' : E \to D$ over B such that $g'g$ and gg' are $\mathcal{F}$-homotopic over B to identity maps.
- An $\mathcal{F}$-space $\pi : E \to B$ is said to be $\mathcal{F}$-homotopy trivial if it is $\mathcal{F}$-homotopy equivalent to the projection $\pi_1 : B \times F \to B$ for some $F \in \mathcal{F}$.

Remark 2.3. Here we can think of $\mathcal{F}$-homotopy as a continuous deformation of an $\mathcal{F}$-map through $\mathcal{F}$-maps.

Definition 2.4 (Pullback of an $\mathcal{F}$-space)**.** Let $\pi : E \to B$ be an $\mathcal{F}$-space and let $f : A \to B$ be a map in $\mathcal{U}$. Define a space

$$f^*E = \{(a, e) \in A \times E : f(a) = \pi(e)\}$$

and a map $\tilde{f} : f^*E \to E$ by $\tilde{f}(a, e) = e$. We define the pullback $\mathcal{F}$-space of π along f to be the $\mathcal{F}$-space $f^*\pi : f^*E \to A$ where $f^*\pi(a, e) = a$. This is illustrated by the following commutative diagram

$$\begin{array}{ccc} f^*E & \xrightarrow{\tilde{f}} & E \\ {\scriptstyle f^*\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ A & \xrightarrow[f]{} & B. \end{array}$$

Definition 2.5 (Category of bundle fibers)**.** Let G be a topological group and F be a left G-space on which G acts effectively. Define a category $\mathcal{F}$ to have objects (X, x) where X is a left G-space and $x : F \to X$ is a homeomorphism of left G-spaces. Define the set of morphisms from (X, x) to (X', x') to be $\{x'gx^{-1} : g \in G\}$. Let $(F, 1)$ be the distinguished object of $\mathcal{F}$ and we call $(\mathcal{F}, (F, 1))$ a category of bundle fibers. We write $(\mathcal{F}, F)$ instead of $(\mathcal{F}, (F, 1))$ for the ease of notation when it is clear from the context.

Definition 2.6 ($\mathcal{F}$-bundle)**.** Let $(\mathcal{F}, F)$ be a category of bundle fibers. Let B be a connected space with a basepoint $b_0 \in B$ and let $p : E \to B$ be a continuous surjection, where $p^{-1}(b_0) = F \in \mathcal{F}$. We require that for each $x \in B$, there is an open neighborhood $U_x \subset B$ containing x such that there is a homeomorphism $\psi_x : p^{-1}(U_x) \to U_x \times F$, such that $p = \mathrm{proj} \circ \psi_x$, as illustrated in the following commutative diagram.

$$\begin{array}{ccc} p^{-1}(U_x) & \xrightarrow{\psi_x} & U_x \times F \\ {\scriptstyle p}\downarrow & \swarrow{\scriptstyle \mathrm{proj}} & \\ U_x & & \end{array}$$

We call $E, B,$ and F the total space, the base space, and the fiber, respectively.

Example 2.7 (Vector bundle)**.** Let n be a positive integer. Let G be the group of linear automorphisms of $\mathbb{R}^n$, denoted by $GL(n, \mathbb{R})$. Define $(\mathcal{F}, F)$ to be the category of bundle fibers where the objects are all the n-dimensional subspaces of $\mathbb{R}^\infty$ with the distinguished object $\mathbb{R}^n$. For an object X, let x be a linear isomorphism from F to X (also a homeomorphism since all the objects in $\mathcal{F}$ are given the Euclidean topology). In this example, the $\mathcal{F}$-spaces we constructed are known precisely as n-dimensional real vector bundles.

Remark 2.8. The category of fibers (instead of bundle fibers) is defined in [1, Chapter 4] for general fibrations, where category of bundle fibers is a special case that works specifically with bundles. In the category of fibers, each object $X \in \mathcal{F}$ is weakly homomopy equivalent to F instead of being homeomorphic to F. We see that it is much easier to work with category of bundle fibers because homeomophism is a much stronger condition than weak homotopy equivalence. Hence, we focus our attention to category of bundle fibers in this paper for simplicity. Note that we can generalize our main theorems 4.1 and 5.4 to fibrations and category of fibers with some more work. Details can be in [1, Chapter 9, 11].

Definition 2.9 (Associated principal category of fibers)**.** Let $(\mathcal{F}, F)$ be a category of bundle fibers. Define its associated principal category of fibers $(\mathcal{G}, G)$ by letting $\mathcal{G}$ have objects $\mathcal{F}(F, X)$ for $X \in \mathcal{F}$, with $G = \mathcal{F}(F, F)$; the product on G and the action of G on $\mathcal{F}(F, X)$ are given by composition.

Definition 2.10 (prin Functor)**.** For a category of bundle fibers $(\mathcal{F}, F)$ and its associated principal category of fibers $(\mathcal{G}, G)$, the prin functor takes an $\mathcal{F}$-space to a $\mathcal{G}$-space. For an $\mathcal{F}$-space $\pi : E \to B$, we define a space PE as a subspace of $\mathcal{U}(F, E)$ consisting of maps $\phi : F \to E$ such that $\phi(F) \subset \pi^{-1}(b)$ for some $b \in B$ and $\phi : F \to \pi^{-1}(b)$ is a map in $\mathcal{F}$. We then define a $\mathcal{G}$-space $P\pi : PE \to B$ by letting $(P\pi)(\phi) = \pi\phi(F)$.
For an $\mathcal{F}$-map $\nu : D \to A$ and $\mathcal{F}$-map $(g, f) : \nu \to \pi$, we define a $\mathcal{G}$-map $P(g, f) = (Pg, f) : P\nu \to P\pi$ where $(Pg)(\phi) = g \circ \phi$. Then we have a functor P (commonly referred as prin) from the category of $\mathcal{F}$-space to the category of $\mathcal{G}$-space.

Note that if π is an $\mathcal{F}$-bundle, then the space PE contains those maps that take F into some fiber, and that fiber is weakly equivalent to F. The bundle $P\pi$ takes a map ϕ to the point in the base of the fiber to which ϕ takes that fiber.

Remark 2.11*.* We claim that whenever we work with $\mathcal{F}$-bundles, it suffices to look at $\mathcal{G}$-bundles. We see that 2.10 gives us a functor from $\mathcal{F}$-bundles to $\mathcal{G}$-bundles, and we need to find some functor that acts like its inverse. Starting with a principal $\mathcal{G}$-bundle: $\pi : P \to X$, we can obtain an $\mathcal{F}$-bundle $P \times_G F :\to X$, and we claim that $- \times_G F$ is the equivalent left-adjoint functor to prin.

Let $\pi : E \to B$ be an $\mathcal{F}$-space where B is paracompact. If for every $\mathcal{F}$-space $\nu : D \to A$, $\mathcal{F}$-map $(g, f) : \nu \to \pi$, and every homotopy $h : A \times I \to B$ starting at f, there exists a homotopy $H : D \times I \to E$ starting at g such that the pair (H, h) is an $\mathcal{F}$-homotopy, then π satisfies the $\mathcal{F}$-covering homotopy property ($\mathcal{F}$-CHP). Maps that satisfy $\mathcal{F}$-CHP are called $\mathcal{F}$-fibrations, of which an $\mathcal{F}$-bundle is a special case. The object of this paper is primarily the study of bundles; generalized results for fibrations can be found in [1, Chapter 5 - 11].

Proposition 2.12. *An $\mathcal{F}$-bundle satisfies the $\mathcal{F}$-covering homotopy property.*

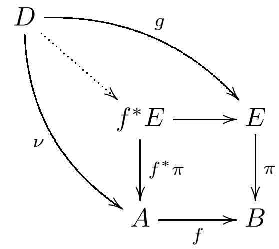
Proof. Detailed proof can be found in [5, Theorem 11.7]. □

Proposition 2.13. *Let $p \colon E \to B$ be an $\mathcal{F}$-bundle. If $f, g : A \to B$ are homotopic maps, then they induce $\mathcal{F}$-homotopy equivalent $\mathcal{F}$-bundles over A.*

Proof. We first notice that if $\nu_0 : D \to A$ and $\pi_0 : E \to B$ are trivial $\mathcal{F}$-bundles with fiber F and there is an $\mathcal{F}$-map $(g, f) : \nu_0 \to \pi_0$, then ν_0 is isomorphic to the pullback bundle $f^*\pi_0 : f^*E \to A$. Since both bundles are trivial, the bundle map (g, f) is isomorphic to the diagram

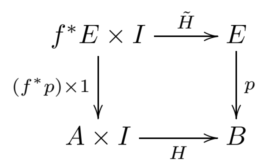
$$\begin{array}{ccc} A \times F & \longrightarrow & B \times F \\ \downarrow & & \downarrow \\ A & \xrightarrow[f]{} & B \end{array} .$$

The pullback is thus the subspace of $A \times (B \times F)$ defined by $\{(a, b, f) \in A \times (B \times F) : b = f(a)\}$, and this is homeomorphic to $A \times F$, with all the maps being the obvious ones. Then we will show that for $\mathcal{F}$-spaces ν and μ

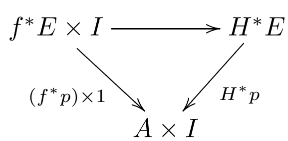


the map from ν to the pullback bundle $f^*\pi : f^*E \to A$ is an isomorphism. It is sufficient to show that there is an open cover $\{U_s\}_{s\in S}$ of A such that $\nu^{-1}U_s \to (f^*\pi)^{-1}(U_s)$ is a homeomorphism for every $s \in S$. If $\{V_t\}_{t\in T}$ is an open cover of A such that the restriction of ν to $\nu^{-1}(V_t)$ is trivial for all $t \in T$ and $\{W_p\}_{p\in P}$ is an open cover of B such that the restriction of π to $\pi^{-1}(W_p)$ is trivial for all $p \in P$, then we can let $S = T \times P$ and let $U_{(t,p)} = V_t \cap f^{-1}W_p$. By what we have shown previously, the $\mathcal{F}$-spaces ν and $f^*\pi$ are isomorphic.

To prove this proposition, let $H : A \times I \to B$ be a homotopy from f to g. Since $p : E \to B$ satisfies the $\mathcal{F}$-CHP, and so there is a map $\tilde{H} \colon f^*E \times I \to E$ that makes the diagram



commute. What we have shown in this proof implies that we have an isomorphism of bundles over A



where H^*E is the pullback of p along H and the horizontal map is the natural map to that pullback. Thus, the restriction of these bundles to $A \times \{1\}$ are isomorphic; since that restriction of $(f^*p) \times 1$ is f^*p and that restriction of H^*p is g^*p, this proves our claim. □

3. Classifying Spaces via the geometric bar construction

In this section, we introduce the geometric bar construction and classifying spaces. We also introduce some results that are useful for proving the classification theorem in Section 4.

Definition 3.1 (Geometric bar construction)**.** Let G be a topological group such that the identity e is a nondegenerate base-point. Let X and Y be left and right G-spaces respectively. The bar construction gives us a simplicial topological space $B_*(Y, G, X)$ where the space of *j-th* simplicies $Y \times G^j \times X$ with typical elements written as $y[g_1, ..., g_j]x$. Let the face and degeneracy maps be given by

$$\delta_i(y[g_1, ..., g_j]x) = \begin{cases} yg_1[g_2..., g_j]x & \text{if } i = 0 \\ y[g_1, ..., g_{i-1}, g_i g_{i+1}, g_j]x & \text{if } 1 \leq i < j \\ y[g_1, ..., g_{j-1}]g_j x & \text{if } i = j \end{cases}$$

and $s_i(y[g_1, ..., g_j]x) = y[g_1, ..., g_i, e, g_{i+1}, ..., g_j]x$.

Let $B(Y, G, X)$ denote the geometric realization of $B_*(Y, G, X)$. Let $*$ denote the one-point G-space and define $BG = B(*, G, *)$ and $EG = B(*, G, G)$. These two spaces are essential to the classification of bundles, which will be introduced in Theorem 4.1.

Notation 3.2. Given a space Z, left and right G-spaces X and Y, and a map $\lambda : X \times Y \to Z$ such that $\lambda(yg, x) = \lambda(y, gx)$, we let $\epsilon(\lambda)$ denote the induced map $B(Y, G, X) \to Z$ that takes $y[g_1, g_2, \ldots, g_j]x$ to $yg_1g_2 \cdots g_j x$. In later parts of the paper, we will write ϵ or $\tilde{\epsilon}$ for $\epsilon(\lambda)$, depending on whether the map is defined on the base space or total space of a bundle, when the choice of λ is clear.

Proposition 3.3. *The map $\tilde{\epsilon} : B(Y, G, G) \to Y$ is a homotopy equivalence (here the underlying λ is the G-action $Y \times G \to Y$).*

Proof. Let Y_* denote the constant simplicial space where each level is Y and all maps are the identity. The idea of this proof is to first show that Y_* is a strong deformation retract of $B_*(Y, G, G)$ with a right inverse (analogous to [4, Proposition 9.8]. Then, using the result that homotopies between simplicial spaces are preserved by geometric realization (proved in [4, Corollary 11.1]), we can prove that $B(Y, G, G)$ is homotopy equivalent to Y. □

Proposition 3.4. *If $p : E \to B$ is a bundle with a contractible fiber F, then it is a weak homotopy equivalence.*

Proof. We know that a fiber bundle $F \to E \to B$ induces a long exact sequence $\cdots \to \pi_n F \to \pi_n E \to \pi_n B \to \pi_{n-1} F \to \cdots$. Since F is contractible, we have a sequence $\cdots \to 0 \to \pi_n E \to \pi_n B \to 0 \to \cdots$. By exactness at $\pi_n E$ and $\pi_n B$ respectively, we have that the map p_* between $\pi_n E$ and $\pi_n B$ is both injective and surjective. Therefore, we proved that p is a weak homotopy equivalence. □

Proposition 3.5. *Let G be a topological group and let $P\nu : PD \to X$ be a $(\mathcal{G}, G)$-bundle. In the following diagram, treat the map p' as a $(\mathcal{G}, G)$-bundle that takes the last coordinate to a point. Then the map ϵ is a weak homotopy equivalence (where the underlying λ is $PD \times * \to X$ that is just $P\nu$). If X is a CW-complex, then ϵ has a homotopy inverse g.*

$$\begin{array}{ccc} PD & \xleftarrow{\ \tilde{\epsilon}\ } & B(PD, G, G) \\ {\scriptstyle P\nu}\downarrow & & \downarrow{\scriptstyle p'} \\ X & \underset{\epsilon}{\overset{g}{\leftrightarrows}} & B(PD, G, *) \end{array}$$

Proof. In Proposition 3.3 we proved that $\epsilon : B(PD, G, G) \to PD$ is a homotopy equivalence, so it is a weak homotopy equivalence. We then have the following part of a long exact sequence where i denotes the identity map of G.

$$\begin{array}{ccccccccc}
\pi_n G & \longrightarrow & \pi_n B(PD,G,G) & \longrightarrow & \pi_n B(PD,G,*) & \longrightarrow & \pi_{n-1} G & \longrightarrow & \pi_{n-1} B(PD,G,G) \\
\downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle \tilde{\epsilon}^*} & & \downarrow{\scriptstyle \epsilon^*} & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle \tilde{\epsilon}^*} \\
\pi_n G & \longrightarrow & \pi_n PD & \longrightarrow & \pi_n X & \longrightarrow & \pi_{n-1} G & \longrightarrow & \pi_{n-1} PD
\end{array}$$

By the Five Lemma, we know that the middle arrow down is an isomorphism. If X is a CW-complex, then the existence of a homotopy inverse g follows from Theorem 1.1. $\square$

In fact, for any right G-space Y, we have that $p' : B(Y,G,G) \to B(Y,G,*)$ is a principal G-bundle corresponding to $q : B(Y,G,*) \to B(*,G,*)$, as illustrated in the commutative square on the right in Diagram (4.2).

4. Classification Theorem

In this section, we will introduce the classification theorem for bundles and some direct consequences.

Theorem 4.1 (Classification Theorem)**.** *Suppose that $(\mathcal{F}, F)$ is a category of bundle fibers and $(\mathcal{G}, G)$ is its associated principal category of fibers. For a CW-complex X, we have that the set $G_{\mathcal{F}}X$ of equivalence classes of $\mathcal{F}$-bundle over X is in bijection with $[X, BG]$, the set of homotopy classes of maps from X to BG.*

Proof. We begin by defining two maps $\Psi : [X, BG] \to G_{\mathcal{F}}X$ and $\Phi : G_{\mathcal{F}}X \to [X, BG]$. Define $\Psi([f]) = f^*p$, the pullback of the bundle $p : EG \to BG$. By Proposition 2.13, we know that Ψ is well-defined. Given an $\mathcal{F}$-bundle $\nu : D \to X$, we have its associated principal $\mathcal{G}$-bundle: $P\nu : PD \to X$. Define $\Phi(\nu) = [g \circ q]$, where g is the inverse of ϵ in Proposition 3.5, as shown in the commutative diagram below. Here the left square was introduced in Proposition 3.5 and the right square is a pullback diagram of the $\mathcal{G}$-bundle $p : EG \to BG$.

$$\begin{array}{ccccc}
PD & \xleftarrow{\ \tilde{\epsilon}\ } & B(PD,G,G) & \xrightarrow{\ \tilde{q}\ } & B(*,G,G) = EG \\
\downarrow{\scriptstyle P\nu} & & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\
X & \underset{\epsilon}{\overset{g}{\rightleftarrows}} & B(PD,G,*) & \xrightarrow[q]{} & B(*,G,*) = BG
\end{array} \tag{4.2}$$

We first show that the composition $\Psi\Phi$ is the identity. Note that we can view $p' : B(PD,G,G) \to B(PD,G,*)$ as a $\mathcal{G}$-bundle where the last coordinate gets mapped to a single point of G, and we get a pullback principal $\mathcal{G}$-bundle $g^*p' : g^*B(PD,G,G) \to X$ and a lift $\tilde{g} : g^*B(PD,G,G) \to B(PD,G,G)$ as in the following diagram.

$$\begin{array}{ccccc}
g^*B(PD,G,G) & \xrightarrow{\ \tilde{g}\ } & B(PD,G,G) & \xrightarrow{\ \epsilon\ } & PD \\
\downarrow{\scriptstyle g^*p'} & & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle P\nu} \\
X & \xrightarrow[g]{} & B(PD,G,*) & \xrightarrow[\epsilon]{} & X
\end{array}$$

Since $g \circ \epsilon$ is homotopic to 1_X, by lifting the homotopy as in Proposition 2.12 we know that the bundle map from g^*p' to $P\nu$ is homotopic to the identity bundle map on g^*p'. Therefore, we have equivalent bundles $P\nu$ and $g^*B(P,G,G)$ over X, and so $\Psi\Phi(\nu) = [(q \circ g)^*p] = [(g^*(q^*p))] = [(g^*p')] = [\nu]$, as desired.

To prove that $\Phi\Psi$ is the identity, consider the following diagram where f^*EG is the pullback of $p : EG \to BG$ along $f : X \to BG$ and

$$\begin{array}{ccccc} X & \overset{g}{\underset{\epsilon}{\rightleftarrows}} & B(f^*EG, G, *) & \xrightarrow{q} & B(*, G, *) = BG \\ \Big\downarrow f & & \Big\downarrow B\tilde{f} & \nearrow q' & \\ BG & \xleftarrow{\epsilon'} & B(EG, G, *) & & \end{array}$$

The bottom left map ϵ' is a weak homotopy equivalence as $B(EG, G, G) \to EG$ is a homotopy equivalence and there is G-bundle $EG \to BG$ and $B(EG, G, G) \to B(EG, G, *)$ (similar to proof of Proposition 3.5). The map q' can be understood as a bundle with fiber EG. It is also a weak homotopy equivalence by Proposition 3.4. Given two weak homotopy equivalences, by Theorem 1.1, we have bijections $[X, BG] \cong [X, B(f^*EG, G, *)] \cong [X, BG]$. Then $\Phi\Psi([f]) = \Phi([f^*p]) = [q \circ g]$ is an automorphism, and so Ψ is an injection. We know that $\Psi\Phi\Psi([f]) = \Psi([f])$ since $\Psi\Phi$ is the identity. By associativity of composition of maps, we know that $\Psi\Phi\Psi([f]) = \Psi([q \circ g])$. By injectivity of Ψ, we have $[f] = [q \circ g]$, so $\Phi\Psi$ is actually an isomorphism, as desired. Therefore, we have proved that the set of homotopy classes of maps $[X, BG]$ is in one-to-one correspondence to the equivalence classes of $\mathcal{G}$-principal bundles over X. □

Remark 4.3. Though we started with $\mathcal{F}$-bundles, in much of the proof we worked with their associated principal $\mathcal{G}$-bundles. In Section 2, we showed that there is a pair of equivalent adjoint functors between $(\mathcal{F}, F)$-bundles and their associated principal $(\mathcal{G}, G)$-bundles over X. Therefore, it suffices to classify the equivalence classes of principal $\mathcal{G}$-bundle over X.

5. Y-structure: definition, example, and classification

We will introduce how additional structures on bundles are defined and provide two motivating examples. We will also classify bundles with additional structure, using an approach similar to the one in Theorem 4.1.

Definition 5.1 (Y-structure)**.** Let $(\mathcal{F}, F)$ be a category of bundle fibers, let Z be an auxiliary space, and suppose we have an inclusion of Y into the function space $\mathcal{U}(F, Z)$ such that the right action of $G = \mathcal{U}(F, F)$ on Y is the action G on $\mathcal{U}(F, Z)$ by pre-composition. Define a Y-structure θ on an $\mathcal{F}$-space $\nu : D \to A$ to be a map $\theta : D \to Z$ such that, for an inclusion of fiber $\psi : F \to D$ (like the map θ in Definition 2.10), the composition $\theta \cdot \psi : F \to Z$ is in Y. Similarly, define an $\mathcal{F}$-map $(\nu, \theta) \to (\nu', \theta')$ between two $\mathcal{F}$-map ν, ν' with Y-structure θ, θ' to be an $\mathcal{F}$-map $(g, f) : \nu \to \nu'$ such that $\theta \circ g$ is homotopic to θ' via homotopy $H : D \times I \to E$. For all $t \in I$, we require that $H_t\psi : F \to Z$ is in Y for all $\psi \in PD$.

Example 5.2 (Reduction of structure group)**.** Suppose that H is a subgroup of G with an inclusion $i : H \to G$. Given a G-bundle $\nu : D \to A$ with fiber F, we let $Z = B(*, H, F)$ and $Y = B(*, H, G)$. By Remark 2.11, we know that Y is homeomorphic to $PB(*, H, F)$. Recalling Definition 2.10, we have that Y lies in $\mathcal{U}(F, Z)$. Take $\theta : D \to B(*, H, F)$ as a G/H-structure. Let $E \to A$ be the principal H-bundle induced from the universal bundle $EH \to BH$ by $A \to BH$.

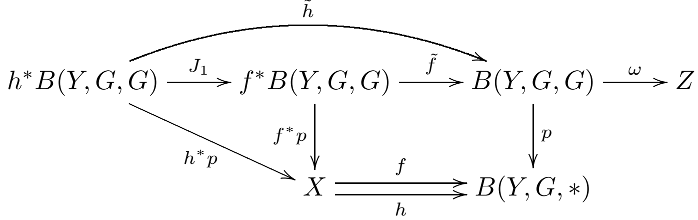
Then θ determines an equivalence of G-bundles from D to the bundle $E \times_H F$ (recall Remark 2.11). This Y-structure is characterized by the reduction from a G-bundle to a H-bundle.

Example 5.3 (Trivialization of bundles)**.** Let $(\mathcal{F}, F)$ and $(\mathcal{F}', F)$ be two categories of fibers together with their associated principal categories $(\mathcal{G}, G)$ and $(\mathcal{G}', G')$ respectively. Note that here we are changing the category $\mathcal{F}$ to $\mathcal{F}'$ but the distinguished object F stays the same. Assume that $(\mathcal{F}, F)$ and $(\mathcal{F}', F)$ also have the structure of bundle fibers. We require that $G \subset G'$, which means $\mathcal{F}(F, F) \subset \mathcal{F}'(F, F)$, i.e., the category $\mathcal{F}$ allows more homeomorphisms from F to itself. Given a G-bundle $\nu : D \to X$, we take $Y = G'$ and $Z = F$. Then we can view the G'-structure $\theta : D \to F$ as the second coordinate of an $\mathcal{F}'$-map $D \to X \times F$ over X. In other words, the bundle ν (as a G'-bundle) is equivalent to the trivial bundle. This Y-structure is characterized by the trivialization of a G-bundle as a G'-bundle.

This example will be revisited shortly with an alternative explanation using Theorem 5.4.

Theorem 5.4 (Classification for bundles with Y-structure)**.** *Suppose that $(\mathcal{F}, F)$ is a category of bundle fibers and $(\mathcal{G}, G)$ is its associated principal category of fibers. Assume that $(\mathcal{F}, F)$ also has the structure of a category of bundle fibers. For CW-complex X, we have that the set $G_{\mathcal{F}}(X, Y)$ of equivalence classes of $\mathcal{F}$-bundles with Y-structure over X is in bijection to $[X, B(Y, G, *)]$ the homotopy classes of maps from X to $B(Y, G, *)$.*

Proof. The structure of this proof is similar to that of Theorem 4.1, despite the fact that now we have to define where Φ and Ψ takes the Y-structure ω and check if the maps are well-defined with respect to the Y-structure. As in the proof of Theorem 4.1, we define maps $\Psi : [X, B(Y, G, *)] \to G_{\mathcal{F}}(X, Y)$ and $\Phi : G_{\mathcal{F}}(X, Y) \to [X, B(Y, G, *)]$. Suppose we have $p : B(Y, G, G) \to B(Y, G, *)$ with Y-structure $\omega : B(Y, G, G) \to Z$. We define $\Psi([f]) = \{f^* p, \omega \tilde{f}\}$ where $\tilde{f} : f^* B(Y, G, G) \to B(Y, G, G)$. To prove that this is well-defined, suppose we have two homotopic maps $f, h : X \to B(Y, G, *)$.



By Proposition 2.13, we know that there is a bundle-covering homotopy $J : h^*(Y, G, G) \times I \to f^*(Y, G, G)$ that starts at the identity on $h^* p$. The composition

$$h^* B(Y, G, G) \xrightarrow{J_1} f^* B(Y, G, G) \xrightarrow{\tilde{f}} B(Y, G, G) \xrightarrow{\omega} Z$$

is homotopic to

$$h^* B(Y, G, G) \xrightarrow{\tilde{h}} B(Y, G, G) \xrightarrow{\omega} Z,$$

which shows that Ψ is well-defined with respect to ω.

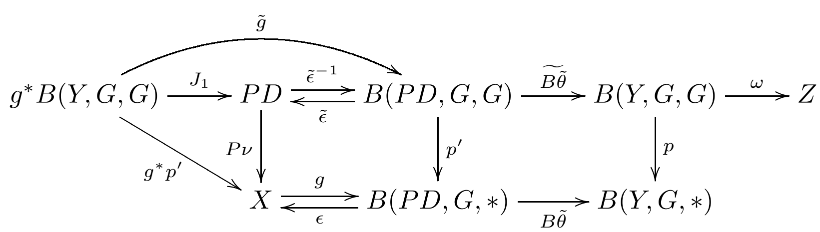
Given an $\mathcal{F}$-bundle $\nu : D \to X$ with Y-structure $\theta : D \to Z$, we have its associated principal G-bundle: $P\nu : PD \to X$ with Y-structure $\tilde{\theta} : PD \to Z$ where

$\tilde{\theta} : PD \to Y$ is defined as $\tilde{\phi}(\theta) = \theta \circ \phi$. Consider the composition

$$A \xrightarrow{g} B(PD, G, *) \xrightarrow{B\tilde{\theta}} B(Y, G, *)$$

where g is an inverse of $\epsilon : B(PD, G, *) \to A$ as in Theorem 4.1, and define $\Phi(\nu) = [B\tilde{\theta} \circ g]$. To prove well-definedness of Φ, for two equivalent bundles with equivalent Y-structure, we want to prove that their images under Φ are homotopic. Given another bundle (ν', θ'), an $\mathcal{F}$-map $k : D \to D'$ over X, and a homotopy $h : D \times I \to Z$ through Y-structure from θ to $\theta' k$, we define $Ph : PD \times I \to Y$ by $(Ph)_t(\psi) = h_t \circ \psi$. For $\phi \in G$, we have $(Ph)_t(\psi \circ \phi) = (Ph)_t(\psi) \circ \phi$; hence $Ph : PD \times I \to PZ$ is an induced G-equivariant homotopy from $\tilde{\theta}$ to $\tilde{\theta} \circ Pk$. It therefore induces a homotopy from $B\tilde{\theta}$ to $B\tilde{\theta}' \circ BPk$ after the bar construction. After composing each one with g, the homotopy is preserved. Thus, we proved that Φ is well-defined, as desired.

Using the argument in Theorem 4.1, we are able to show that $\Psi\Phi$ is the identity with respect to bundle. Let's now prove that $\Psi\Phi$ is also the identity with respect to the Y-structure. In the following diagram, we have proved this part of in Theorem 4.1 that $g^* p'$ and $P\nu$ are equivalent bundles over X.



Using exactly the same argument as the well-definedness of Ψ, we see that the composition

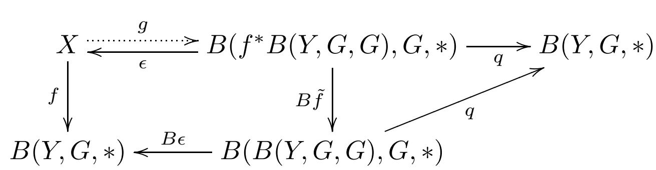
$$g^*B(Y, G, G) \xrightarrow{\tilde{g}} PD \xrightarrow{\tilde{\epsilon}^{-1}} B(PD, G, G) \xrightarrow{\widetilde{B\tilde{\theta}}} h \xrightarrow{\omega} Z$$

is homotopic to

$$g^*B(Y, G, G) \xrightarrow{\tilde{g}} B(PD, G, G) \xrightarrow{\widetilde{B\tilde{\theta}}} h \xrightarrow{\omega} Z.$$

Therefore, the Y-structures $PD \to Z$ and $g^*B(Y, Y, G) \to Z$ are equivalent, and so $\Psi\Phi$ is the identity on Y-structures as well.

To verify that $\Phi\Psi$ is an automorphism and therefore the identity, we can construct a diagram similar to the one in the proof of Theorem 4.1.



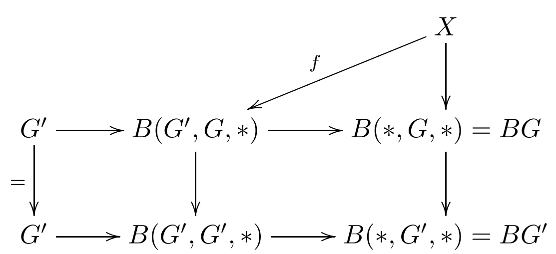
Here the map q is a homotopy equivalence since it is the map induced by $B(Y, G, G) \to Y$. The map $B\epsilon$ is a weak homotopy equivalence using the proof of Proposition 3.5. Therefore, similar to Theorem 4.1, we can show that $\Phi\Psi$ is also the identity. $\square$

Here are some immediate consequences of the proofs of Theorem 5.4 and Theorem 4.1.

Corollary 5.5.

1. *The map $q_* : [X, B(Y, G, *)] \to [X, BG]$ represents the forgetful transformation from $G_{\mathcal{F}}(X, Y) \to G_{\mathcal{F}}(X)$ taking (ν, θ) to ν.*
2. *$[X, Y]$ is isomorphic to the set of equivalence classes of Y-structures on the trivial $\mathcal{F}$-bundle $\epsilon : X \times F \to X$ as follows: for $f : X \to Y$, the corresponding Y-structure is given by its adjoint $X \times F \to Z$.*
3. *$[X, G]$ is naturally isomorphic to the set of $\mathcal{F}$-homotopy classes of $\mathcal{F}$-maps over A from the trivial bundle ϵ to itself. Given $f : X \to G$, its adjoint $X \times F \to F$ is the second coordinate of the corresponding $\mathcal{F}$-map over X.*
4. *Let ι be a map from G to Y; then $\iota_* : [X, G] \to [X, Y]$ represents the transformation that sends an $\mathcal{F}$-map $g : X \times F \to X \times F$ to the Y-structure $\theta_0 \circ g$, where $\theta_0 : X \times F \to Z$ is the Y-structure on ϵ with adjoint being the constant map $X \to \iota(e)$.*

Remark 5.6. We now have another explanation of Example 5.3 using the proof of Theorem 5.4. We know that the map f below $X \to B(G', G, *)$ corresponds to an equivalence class of G-bundle with G'-structure over X.



The first row represents a principal G-bundle with a G'-structure. The second row represents the universal G'-bundle. By commutativity of this diagram, the map $f : X \to BG'$ factors through the contractable space $B(G', G', *)$. It is thus null-homotopic, and the $\mathcal{F}'$ bundle associated with it is trivial. In other words, a G'-structure for an G-bundle is characterized by its triviality as a G' bundle.

6. Application: orientations of bundles

In this section, we introduce orientations of fiber bundles with respect to a cohomology theory E. We show why an orientation is an example of a Y-structure and revisit the classification theorem of Y-structures in the context of orientations.

We first introduce some preliminary concepts that are essential for understanding orientations and classification.

Definition 6.1 (Thom space)**.** For a vector bundle $\xi : P \to A$, we construct the Thom space $T\xi$ by applying one point compactification to each fiber V (so at each point of X we have a spherical fiber S^V) and identifying the infinity points of the spherical fibers we just obtained.

Definition 6.2 (E-orientation)**.** Given a cohomology theory E with cup products, a G-bundle ξ with fiber V is E-orientable if there exists a class $\mu \in E^n(T\xi)$ (where n = dimV) such that μ restricts to a generator of $E^n(S^v) \cong E^0 S^0$ for each fiber V along $E^n(T\xi) \to E^n(S^v)$. We call μ the orientation class.

We denote a bundle ξ with an orientation μ as (ξ, μ), and we say that (ξ, μ) is E-oriented.

Proposition 6.3 (Properties of E-orientations)**.**

(1) The trivial spherical bundle $\epsilon : X \times S^V \to X$ has an E-orientation of ϵ that is the suspension of $1 \in E^0X$, called the canonical orientation μ_0.
(2) Preservation by pullback: for an E-oriented G-bundle (ψ, ν) over Y and $f : X \to Y$, the pullback bundle $(f^*\psi, (Tf)^*(\nu))$ is an E-oriented G-bundle over X where $Tf : Tf^*\psi \to T\psi$ is the induced map on the Thom spaces.
(3) Preservation by product: for E-oriented G-bundles (ψ, ν) over X and (ξ, μ) over Y, the product of the bundles is an E-orientable G-bundle over $X \times Y$.

Further properties of orientations can be found in [2, III, Remark 1.5].

Here, instead of using the common notion of spectra where each level E_i is indexed by a natural number i, we use coordinate-free spectra where each level E_V is indexed by some finite-dimensional subspace V of $\mathbb{R}^\infty$. Analogously, for an inclusion $V \subset W$, there is a homeomorphism $\sigma_{V,W} : E_V \to \Omega^{W-V}E_W$ where $W - V$ denotes the orthogonal complement of V in W.

Let E_0 denote the 0-*th* level of the spectrum that is associated with the cohomology theory E. Let $GL(1, E)$ denote the union of components of E_0 that contain units in $\pi_0 E_0$.

We claim that given a cohomology theory E, an E-orientation of a bundle is actually an example of a Y-structure if we take the Y in Definition 5.1 to be $GL(1, E)$. To see this, we describe an E-orientation in the language of Y-structures. First, for an E-oriented G-bundle: $(\xi, \mu) : D \to X$ with fiber S^V, take $E_V = Z$. From the definition of spectrum we have a map $\tilde{\delta} : E_0 \to F(S^V, E_V)$, so we can identify $GL(1, E)$ with a subspace of $F(S^V, E_V)$, which satisfies how Y is defined in Definition 5.1 as it lies in $\mathcal{U}(S^V, Z)$. We think of the orientation class μ as a homotopy class of maps $D \to E_V$ such that for any $\psi : S^V \to D$ that is a based homotopy equivalence into a fiber, the composition $\mu\psi : S^V \to E_V$ is in $GL(1, E)$. We know that μ factor through $T\xi = D/X$ because for $\chi : \xi^{-1} \to D$, the composition $\mu\chi$ takes each fiber to a single unit of E_V. This ensures that the restriction of μ from $E^nT\xi \to T^nT\chi$ is a generator. More details can be found [2, Page 56]

The proof of Theorem 6.4 comes directly from Theorem 5.4.

Theorem 6.4 (Classification of E-oriented bundles)**.** *For a CW-complex X, we have that the set of equivalence classes of E-oriented G-bundles over X with fiber S^V (under the equivalence relation of orientation preserving G-bundle equivalence) is in bijection with* $[X, B(GL(1, E), G, *)]$.

Acknowledgments

I would like to thank my mentor Hana Jia Kong for her help throughout the REU, without which I would not have made as much out of this experience as I did. I would also like to thank Professor May, first and foremost, for running this remote REU during this difficult time, suggesting this interesting topic, and answering all of my questions.

References

[1] J Peter May. *Classifying spaces and fibrations*, volume 155. American Mathematical Soc., 1975.
[2] J Peter May. *E_∞ Ring Spaces and E_∞ Ring Spectra*. Springer, 1977.
[3] J Peter May. *A concise course in algebraic topology*. University of Chicago press, 1999.
[4] J Peter May. *The geometry of iterated loop spaces*, volume 271. Springer, 2006.
[5] Norman Steenrod. *The topology of fibre bundles*. Princeton university press, 1999.