FLOER COHOMOLOGY AND FUKAYA CATEGORY

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Abstract. This paper aims to give an introduction to Floer Theory and relevant topics such as Arnold's conjecture and Fukaya categories. We mostly follow the survey [1]. More details on the proofs and examples are provided.

Contents

1. Introduction

Floer theory is the infinite dimensional analogue of Morse theory defined on the loop space $\mathcal{P}(L_0, L_1)$ between two Lagrangian submanifolds L_0 and L_1 which intersect transversally. The corresponding Morse function at $x \in \mathcal{P}(L_0, L_1)$ is an action functional A of a path Γ between a fixed point x_0 and x (the result is independent of the homotopy class of Γ). One can naturally see that the critical points of A are constant paths, which are intersection points of L_0 and L_1 (Non-degeneracy of critical points is guarantied by transversal intersection). Trajectories of gradient flow in this case become pseudo holomorphic strips between two intersection points. In the Morse complex, the differential is given by counting index 1 trajectories and it is the same in the Floer complex. But there are a few technical problems to be solved:

(1) For two Lagrangians which do not transversely intersect, a hamiltonian perturbation is taken. One needs to prove the Floer (co)homology is independent from the choice of perturbation.

(2) To prove $\partial^2 = 0$. This relies on some topological conditions of the moduli space of pseudo holomorphic strips.

(3) Define a $\mathbb{Z}$-grading of Floer complexes (i.e., each critical point has an index in $\mathbb{Z}$). A $\mathbb{Z}/2$-grading always exists, but $\mathbb{Z}$-grading requires two times the first Chern class of M to vanish.

In the special case of self Floer theory, an equivalence with Morse theory can be constructed. With this, Arnold's conjecture can be solved by equalizing the intersection points with critical points of the corresponding Morse function.

With Floer theory we can get (compact) Fukaya categories. Fukaya categories take compact Lagragians as their objects and Floer complexes as their morphisms. Fukaya categories have more sophisticated structure than usual dg-categories: A_∞-structure.

Fukaya categories are of great interest because of Kontsevich's homological mirror symmetry conjecture, which states a derived equivalence between the Fukaya category of a symplectic manifold and the category of coherent sheaves on its mirror.

2. Floer Theory

2.1. **Floer differential.**

Let (M,ω) be a compact symplectic manifold. Let L_0, L_1 be two compact Lagrangian submanifolds of M. For simplicity, we assume L_0 and L_1 intersect transversally for the rest of the paper (unless otherwise stated). Let

$$\chi(L_0, L_1) \coloneqq \{\text{intersection points of } L_0 \text{ and } L_1\},$$

which is a finite set. Let J be an ω-compactible almost-complex structure on M. Recall that an almost complex structure is an automorphism $J : TM \to TM$ such that $J^2 = -1$. The space of ω-compactible almost-complex structures is defined as:

$$\mathcal{J}(M,\omega) \coloneqq \left\{ J \in End(TM) \mid J^2 = -1 \text{ and } g_J = \omega(\cdot, J) \text{ is a Riemannian metric} \right\}.$$

Fix a base field $\mathbb{K}$. In this paper, unless otherwise stated, we require $\mathbb{K}$ to be of characteristic 2 (usually taken to be $\mathbb{F}_2$). The Novikov field over $\mathbb{K}$ is defined as

$$\Lambda = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{K}, (\lambda_i) \in \mathbb{R}^{\mathbb{Z}_{\geq 0}}, \lim_{i\to+\infty} \lambda_i = +\infty \right\}.$$

Definition 2.1. The Floer complex $CF(L_0, L_1)$ is the free Λ-module generated by the intersection points $\chi(L_0, L_1)$:

$$CF(L_0, L_1) \coloneqq \bigoplus_{p\in\chi(L_0,L_1)} \Lambda \cdot p.$$

Definition 2.2. Let $p, q \in \chi(L_0, L_1)$. Let $u : \mathbb{R} \times [0,1] \to M$. Then we say u is a pseudoholomorphic strip (w.r.t the almost complex structure J) with boundary points p and q if the following three conditions hold.
1) The Cauchy-Riemann equation

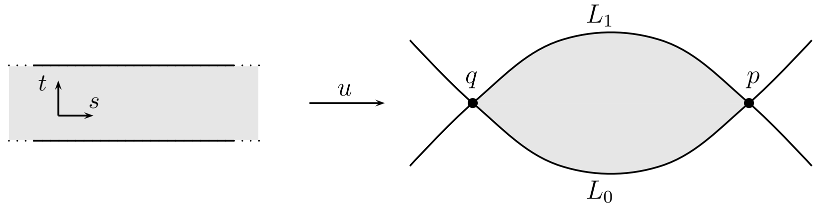
$$\frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} = 0.$$

2) The boundary conditions

$$u(s,0) \in L_0,\ u(s,1) \in L_1,\ \forall s \in \mathbb{R}.$$
$$\lim_{s\to-\infty} u(s,t) = p, \lim_{s\to+\infty} u(s,t) = q.$$

3) The finite energy condition

$$E(u) = \int u^*\omega = \int\int \left|\frac{\partial u}{\partial s}\right|^2 dsdt < \infty.$$



Given a homotopy class $[u] \in \pi_2(M, L_0 \cup L_1)$, we denote by $\widehat{\mathcal{M}}(p, q, [u], J)$ the moduli space of pseudoholomorphic strips with boundary points p and q representing the class $[u]$. Then there is an action of $\mathbb{R}$ on $\widehat{\mathcal{M}}(p, q, [u], J)$ by shifting the s parameter of u, and we denote by $\mathcal{M}(p, q, [u], J)$ the quotient of $\widehat{\mathcal{M}}(p, q, [u], J)$ by this action.

Let u be a pseudoholomorphic strip with some boundary points $p, q \in \chi(L_0, L_1)$. Since $\mathbb{R}\times[0,1]$ is contractible, we can fix a trivialization $\Phi : u^*TM \to \mathbb{R}\times[0,1]\times\mathbb{R}^{2n}$, where $n = \dim(M)/2$. Then $l_0 = u^*_{|\mathbb{R}\times\{0\}}TL_0$ and $l_1 = u^*_{|\mathbb{R}\times\{1\}}TL_1$ can be viewed as paths in

$$LGr(n) = \{\text{Lagrangian } n\text{-planes in symplectic vector space } \mathbb{R}^{2n}\},$$

Also both T_pL_0, T_pL_1 can be viewed as elements in $LGr(n)$. Identifying $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, there exists $A_p \in Sp(2n, \mathbb{R})$ which maps T_pL_0 to $\mathbb{R}^n \subset \mathbb{C}^n$ and T_pL_1 $(i\mathbb{R})^n \subset \mathbb{C}^n$. Define $\lambda_p(t) = A_p^{-1}((e^{i\pi t/2}\mathbb{R})^n)$, $t \in [0,1]$. Then λ_p is a path in $LGr(n)$ connecting T_pL_0 and T_pL_1. Similarly we can define a path λ_q connecting T_qL_0 and T_qL_1 in $LGr(n)$. Then we get a loop γ_u by concatenating the paths $-l_0$, λ_p, l_1, and $-\lambda_q$. Now γ_u represents an element in $\pi_1(LGr(n))$ which only depends on the homotopy class $[u]$, so we can denote it by $\gamma_{[u]} \in \pi_1(LGr(n))$. But $LGr(n) = U(n)/O(n)$, so $\pi_1(LGr(n)) \simeq \mathbb{Z}$. Now consider

$$\det{}^2 : U(n)/O(n) \to S^1$$

which induces an isomorphism between $\pi_1(LGr(n))$ and $\pi_1(S^1)$.

To prove this, consider the fibration induced by $\det^2$, with fiber $\ker \det^2_{U(n)}/O(n)$. Consider an $SU(n)$-action on it by left translation which is transitive. As the isotropy group of the element $1 \cdot O(n)$ is $SU(n) \cap O(n) = SO(n)$, we get the diffeomorphism

$$SU(n)/SO(n) \simeq \ker \det{}^2_{U(n)}/O(n).$$

Since $SU(n)$ is simply connected and $SO(n)$ is connected, using the homotopy exact sequence

$$\pi_1(SU(n)) \longrightarrow \pi_1(SU(n)/SO(n)) \longrightarrow \pi_0(SO(n))$$

we find $SU(n)/SO(n)$ is simply connected. Then using the homotopy exact sequence

$$\pi_1(SU(n)/SO(n) \longrightarrow \pi_1(U(n)/O(n)) \xrightarrow{\det^2_*} \pi_1(S^1) \longrightarrow \pi_0(SU(n)/SO(n))$$

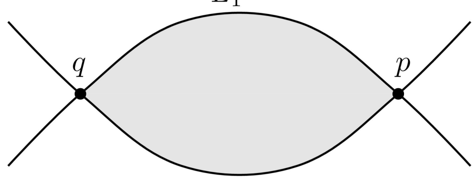
we get the isomorphism.

Define the Maslov index of $[u]$ as:

$$\operatorname{ind}([u]) \coloneqq \det{}_*^2(\gamma_{[u]}) \in \pi_1(S^1) = \mathbb{Z}.$$

Example 2.3. Let L_0 and L_1 be two Lagrangian curves in $\mathbb{C}^1$ with standard symplectic form. As in the figure, the angle at q and p are both $\pi/2$. We take the natural trivilization which maps the tangent space of each point at L_i to the corresponding tangent line at $\mathbb{C}^1$.

With these we can calculate the Maslov index of $[u]$. Let the Lagrangian corresponding to T_qL_0 be L_q and similarly define L_p. Consider the loop γ_u: L_q clockwise to L_p, clockwise to iL_p, clockwise to iL_q, counterclockwise to L_q. The loop winded πi on the $\mathbb{C}^1$ plane, which corresponds to the index 1.



We define $\bar\partial_J \coloneqq \frac{1}{2}(\frac{\partial}{\partial s} + J\frac{\partial}{\partial t})$. Then we see $\widehat{\mathcal{M}}(p,q,[u],J)$ is the zero set of $\bar\partial_J$. Consider the Fredholm index of $D_{\bar\partial_J,u}$, the linearization of $\bar\partial_J$ at u, mapping a section satisfying the boundary condition in u^*TM to another section in u^*TM. Assuming regularity of J at u, which implies the surjectivity of $D_{\bar\partial_J,u}$, we find the Fredholm index of $D_{\bar\partial_J,u}$ is the dimension of the kernel, which is also the dimension of its zero set. Then it only remains to show the Fredholm index of $D_{\bar\partial_J,u}$ is the Maslov index of u. This is due to a theorem by Floer:

Theorem 2.4. [5] *If $x, y \in L \cap L'$ are transverse intersections, then the Maslov index of u coincides with the Fredholm index of the Cauchy-Riemann operator at u.*

Then we have

Proposition 2.5.
$$\dim \widehat{\mathcal{M}}(p,q,[u],J) = \operatorname{ind}([u])$$

So if $\operatorname{ind}([u]) = 2$, then $\dim(\mathcal{M}(p,q;[u],J)) = 0$.

Definition 2.6. The Floer differential $\partial : CF(L_0,L_1) \to CF(L_0,L_0)$ is the Λ-linear map defined by

$$\partial(p) = \sum_{q\in\chi(L_0,L_1), ind([u])=1} (\#\mathcal{M}(p,q;[u],J))T^{\omega([u])}q \tag{2.7}$$

$(\#\mathcal{M}(p,q;[u],J)$ is the signed (or unsigned in $\mathbb{F}_2$-case) count of points in the moduli space of pseudo-holomorphic strips connecting p to q in the class $[u]$, and $\omega([u]) = \int u^*\omega$ is the symplectic area of those strips.

Remark 2.8. It is possible to do Floer theory over smaller coefficient fields, like the field $\mathbb{K}$, instead of Λ. We may consider exact Lagrangian submanifolds in an exact symplectic manifold. Namely, assume that $\omega = d\theta$ for some 1-form θ on M, and there exist functions $f_i \in C^\infty(L_i,\mathbb{R})$ such that $\theta|L_i = df_i$ (for $i = 0,1$). By

Stokes's theorem, any strip connecting intersection points p and q has constant area independent of the choice of strip:

$$u^*\omega = (f_1(q) - f_0(q)) - (f_1(p) - f_0(p)).$$

Thus, rescaling each generator by $p \mapsto T^{f_1(p)-f_0(p)}p$, we can eliminate the weights $T^{\omega([u])}$ from 2.7, and work directly over the coefficient field $\mathbb{K}$ instead of Λ.

The following theorem gives the condition needed for the well-definedness of the Floer differential over $\mathbb{K}$ (of character 2).

Theorem 2.9. [1] *Assume that* $[\omega] \cdot \pi_2(M, L_0) = 0$ *and* $[\omega] \cdot \pi_2(M, L_1) = 0$. *If* $\mathbb{K} = \mathbb{F}_2$, *then the Floer differential* ∂ *is well-defined and satisfies* $\partial^2 = 0$.

Proof. We assume that L_0 and L_1 intersect transversally and that the $\bar{\partial}_J$-operator is surjective at u. This assumption can be satified by a generic choice of Hamitonian pertubation H and time-dependent almost complex strucure $J(t)$. We later prove Floer cohomology is independent of the choice of these data.

Consider the compactification (by taking limits in Banach space of all maps from strip to M) of the one dimensional moduli space $\mathcal{M}(p, q; [u], J)$, for $\mathrm{ind}([u]) = 2$. We show that the result of ∂^2 can be calculated through (signed) counting of boundary points, which goes to zero in $\mathbb{K}$.

Using Gromov's compactness theorem, we can depict the boundary points of such moduli spaces.

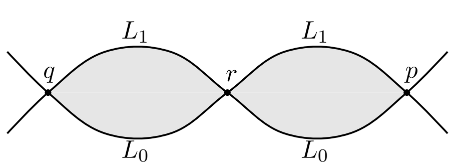
Lemma 2.10. *(Gromov's compactness theorem) A sequence of pseudo holomorphic curves in an almost complex manifoldwith a uniform energy bound must have a subsequence which limits to a pseudoholomorphic curve which may have nodes or (a finite tree of) 'bubbles'. A bubble is a holomorphic sphere which has a transverse intersection with the rest of the curve.*

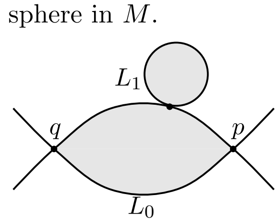
To be concrete, the boundary points are of the following three kinds:

(1) strip breaking: energy concentrates at either end $s \to \pm\infty$, i.e. there is a sequence $a_n \to \pm\infty$ such that the translated strips $u_n(s - a_n, t)$ converge to a 'broken strip' (left).

(2) disc bubbling: energy concentrates at a point on the boundary of the strip ($t \in \{0, 1\}$), where suitable rescalings of u_n converge to a J-holomorphic disc in M with boundary entirely contained in either L_0 or L_1 (right).

(3) sphere bubbling: energy concentrates at an interior point of the strip, where suitable rescalings of u_n converge to a J-holomorphic sphere in M.





By observation, coefficients of ∂^2 can be expressed in terms of case (1) part. So all we need is to exclude case (2) and (3).

In our case, the absence of disc and sphere bubbles is ensured by the assumption that $[\omega] \cdot \pi_2(M, L_i) = 0$. This is because in the bubbling cases, the bubbles of sphere or disks with whole boundary on L_i takes values in $\pi_2(M, L_i) = 0$. The assumption requires that the volume of such disks/spheres goes to zero. All pseudo holomorphic strips with volume zero are constant strips, thus reducing it to the trivial case with no bubble.

Given two generators p, q of the Floer complex, and a homotopy class $[u]$ with $\operatorname{ind}([u]) = 2$, the moduli space $\mathcal{M}(p, q; [u], J)$ is a 1-dimensional manifold. Since our assumptions exclude the possibilities of disc or sphere bubbling, Gromov compactness implies that this moduli space can be compactified to a space $\bar{\mathcal{M}}(p, q; [u], J)$ whose boundary elements are broken strips connecting p to q and representing the total class $[u]$.

Now, consider the boundary of compactified moduli space $\bar{\mathcal{M}}(p, q; [u], J)$. Two-component broken strips of case (1) correspond to $\mathcal{M}(p, r; [u'], J) \times \mathcal{M}(r, q; [u''], J)$, where r is any generator of the Floer complex and $[u'] + [u''] = [u]$. Observe that the index is additive under such decompositions; moreover, the dimension property implies that the index of any non-constant strip must be at least 1. Thus, the only possibility is $\operatorname{ind}([u']) = \operatorname{ind}([u'']) = 1$, and broken configurations with more than two components cannot occur.

Conversely, section 4 of [6] shows that for any broken strip with two parts $\mathcal{M}(p, r; [u'], J)$ and $\mathcal{M}(r, q; [u''], J)$, there exists a class of gluing maps:

$$\# : \mathbb{R}_{\geq 0} \times \mathcal{M}(p, r; [u'], J) \times \mathcal{M}(r, q; [u''], J) \to \mathcal{M}(p, q; [u], J)$$

which approximates $\partial \bar{\mathcal{M}}(p, q; [u], J)$ at infinity. From this we conclude the whole boundary is made up of broken strips:

$$\partial \bar{\mathcal{M}}(p, q; [u], J) = \coprod_{\substack{r \in \chi(L_0, L_1) \\ [u'] + [u''] = [u] \\ \operatorname{ind}([u']) = \operatorname{ind}([u'']) = 1}} \mathcal{M}(p, r; [u'], J) \times \mathcal{M}(r, q; [u''], J). \tag{2.11}$$

Since the total number of boundary points of a compact 1-manifold with boundary is always zero (in $\mathbb{K}$), we conclude that, for each $[u]$

$$\sum_{\substack{r \in \chi(L_0, L_1) \\ [u'] + [u''] = [u] \\ \operatorname{ind}([u']) = \operatorname{ind}([u'']) = 1}} (\#\mathcal{M}(p, r; [u'], J))(\#\mathcal{M}(r, q; [u''], J)) = 0 \tag{2.12}$$

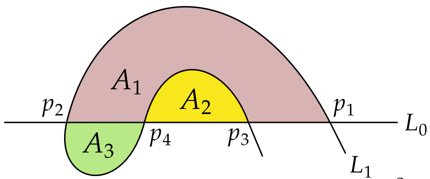
This shows that, for the result of ∂^2, the $\mathbb{K}$-coefficient for each $T^{\omega([u])}$ goes to zero. $\square$

Remark 2.13. The theorem also holds for general base field $\mathbb{K}$ with characteristic that is not 2, provided there is a choice of orientation and spin structure for each Lagrangian.

By chapter 11 of [9], the choice of orientations and spin structures on L_0 and L_1 equips all these moduli spaces with natural orientations, and (2.11) is compatible with these orientations (up to an overall sign). Then the counting in (2.12) becomes the signed counting of the boundary points, which also goes to zero.

Example 2.14. Calculate ∂ in this case:

$$\begin{aligned}
\partial p_1 &= T^{A_1 + A_2} p_2 + T^{A_1 + A_3} p_3 \\
\partial p_2 &= T^{A_3} p_4 \\
\partial p_3 &= T^{A_2} p_4 \\
\partial p_4 &= 0.
\end{aligned}$$



The only non-trivial part of ∂^2 is p_1 part, which is cancelled by two $T^{A_1+A_2+A_3}p_4$.

2.2. **$\mathbb{Z}$-grading of Floer complexes.** Let u be a pseudoholomorphic strip with some boundary points $p, q \in \chi(L_0, L_1)$. There arises a natural question: is it possible to give each intersection point an index (independent of u), so that

$$\mathrm{ind}([u]) = \mathrm{ind}\ q - \mathrm{ind}\ p$$

is always satisfied for all such pseudoholomorphic strips u?

The answer is positive, provided some conditions on M and L_0, L_1 are given so that the ind($[u]$) is independent of $[u]$. In this section, we will investigate what these conditions are.

Let J be a ω-compatible almost complex structure on M , and g the corresponding Riemannian Metric. Let $LGr(TM)$ be the Grassmanian of Lagrangian planes (fiberwise) in TM, which is an $LGr(n)$-bundle over M.

Proposition 2.15. *Suppose the first Chern Class of M is 2-torsion, i.e. $2c_1(TM) = 0$. Then we can construct a fiberwise universal cover $\widetilde{LGr}(TM)$ of $LGr(TM)$.*

Proof. Let $\Delta := \Lambda^n_{\mathbb{C}}(TM, J)^{\otimes 2}$, then it is a complex line bundle on M with first Chern class $c_1(\Delta) = 2c_1(TM) = 0$. Therefore Δ is trival, so is its dual line bundle $\Delta^* = \Lambda^n_{\mathbb{C}}(T^*M, J)^{\otimes 2}$. Fix a unit length global section Θ of Δ^*. Let $S\Delta$ be the circle bundle of unit circles (with respect to the metric induced by g) in each fiber of Δ. Now we can define

$$\det{}^2_\Theta : LGr(TM) \to S^1$$

fiberwise as follows: for each $x \in M$, and $\Lambda \in LGr(T_xM)$, we define

$$\det{}^2(\Lambda) = (e_1 \wedge \cdots \wedge e_n)^{\otimes 2} \in (S\Delta)_x = S^1$$

where (e_i) is any orthonormal basis of $(\Lambda, g|\Lambda)$, and $(S\Delta)_x$ is identified with S^1 via Θ. Then we define

$$\widetilde{LGr}(TM) := \left\{(\Lambda, t) \in LGr(TM) \times \mathbb{R} \,|\, \det{}^2_\Theta(\Lambda) = e^{2\pi i t}\right\},$$

which is the pullback of the universal covering $\mathbb{R} \to S^1$. □

The map $\det^2_\Theta$ constructed in the proof will also help us define the so called Maslov class. Let $L \subset M$ be a Lagrangian submanifold. Then there is a natural section $s_L : L \to LGr(TM)|_L, s_L(x) = TL_x \in LGr(T_xM, \omega_x)$. Now we get a map

$$\varphi_L := \det{}^2_\Theta \circ s_L : L \to S^1.$$

Definition 2.16. The Maslov class $\mu_L \in \mathrm{Hom}(\pi_1(L), \mathbb{Z}) = H^1(L, \mathbb{Z})$ is defined to be the homotopy class $[\varphi_L] \in [L, S^1] = H^1(L, \mathbb{Z})$.

Then the Maslov class is the obstruction of lifting the section s_L in $LGr(TM)$ to a section of the universal cover $\widetilde{LGr}(TM)$. If $\mu_L = 0$, then we can get a lift of the map φ_L to some $\widetilde{\varphi_L} : L \to \mathbb{R}$. Then we have a graded lift $\widetilde{L}$ which corresponds to a map $\widetilde{s}_L : L \to \widetilde{LGr}(TM)$ lifting s_L.

Definition 2.17. Suppose $2c_1(TM) = 0$. Let L_0, L_1 be two Lagrangian submanifolds of M with vanishing Maslov classes. Fix graded lifts $\widetilde{L}_0, \widetilde{L}_1$. Then for all $p \in \chi(L_0, L_1)$, consider a loop γ_p of the following form in $LGr(n)$: First take any path from $\widetilde{L}_0(p)$ to $\widetilde{L}_1(p)$, then project it to $LGr(n)$, and concatenate with the canonical short path $-\lambda_p$. Then the the index of p is defined by

$$\text{ind } p \coloneqq \det{}^2_*[\gamma_p] \in \pi_1(S^1) = \mathbb{Z}.$$

One can then check that if the conditions in the definiton hold, the starting point and ending point of the lift of the loop do not depend on $[u]$. Then

$$\text{ind}([u]) = \text{ind } q - \text{ind } p$$

and $\text{ind}([u])$ is independent of $[u]$.

3. Perturbation, self Floer complex and Arnold's conjecture

When we drop the transversality assumption and the regularity of $\bar{\partial}_J$, we should make some 'correction' to the Cauchy-Riemann equation to do Floer theory.

Let $H \in C^\infty(M \times [0,1], \mathbb{R})$, which we call a time-dependent Hamiltonian. Let X_H be the (time-dependent) Hamiltonian vector field on M associated to H. For all $t \in [0,1]$, let $\phi_H^t \in \text{Ham}(M, \omega)$ be the flow of X_H over the interval $[0,t]$. Let $J(t)$ be a generic family of ω-compatible almost complex structures on M, so that the regularity condition holds (the linearalized $\bar{\partial}_{J,u}$ operator is surjective at any solution u). Let L_0, L_1 be two Lagrangian submanifolds of M, which do not intersect transversally. Take a generic time-dependent Hamiltonian H. Then L_0 intersect $(\phi_H^1)^{-1}(L_1)$ transversally. Then we will replace the original CR-equation by

$$\frac{\partial \tilde{u}}{\partial s} + \tilde{J}(t, \tilde{u}) \frac{\partial \tilde{u}}{\partial t} = 0,$$

where $\tilde{J}(t) = (\phi_H^t)_*^{-1} J(t)$, and $\tilde{u}(s,t) = (\phi_H^t)^{-1}(u(s,t))$.

Definition 3.1. The Floer complex of (L_0, L_1) with perturbation $(H, J(t))$ is defined to be the usual Floer complex of two transverse Lagrangians:

$$CF(L_0, L_1, J(t), H(t)) \coloneqq CF(L_0, (\phi_H^t)^{-1} L_1, \tilde{J}(t)). \tag{3.2}$$

Example 3.3. One important example of a 'perturbed Floer complex' is defining the Floer complex of L with itself, which shows the necessity of applying 'perturbation'.

Consider T^*S^1 and a cotangent fiber $L = \{0\} \times \mathbb{R}$ as a Lagrangian. Take $H = \frac{1}{2} r^2$ as the perturbation datum. Then we will see

$$\phi_H^1(L) = \{(\theta, r) \,|\, \theta = r, \text{up to } 2\pi\}.$$

Then L and $\phi_H^1(L)$ intersect transversally.

All the intersection points are of degree zero, which means the differential is trivial.

The following theorem shows that the Floer cohomology is invariant under the choice of 'perturbation'.

Theorem 3.4. *The Floer cohomology $HF(L_0, L_1) = H^*(CF(L_0, L_1), \partial)$ is, up to isomorphism, independent of the chosen almost-complex structure J and invariant under Hamiltonian isotopies of L_0 or L_1.*

Proof. For any two Hamiltonian perturbations H_0 and H_1, we can take $H_t = (1-t)H_0 + tH_1$ as a class of Hamiltonian perturbation. The space of compatible almost complex structures is also contractible. With these we conclude that the space of such choices are contractible. Thus, given two choices (H, J) and (H', J') (for which we assume transversality holds), let $(H(\tau), J(\tau))$, $\tau \in [0,1]$ be a homotopy from (H, J) and (H', J'). One can then construct a continuation map $F : CF(L_0, L_1; H, J) \to CF(L_0, L_1; H', J')$ by counting solutions to the equation

$$\frac{\partial u}{\partial s} + J(\tau(s), t, u)\left(\frac{\partial u}{\partial t} - X_{H(\tau)}\right) = 0 \tag{3.5}$$

where $\tau(s)$ is a smooth function of s which equals 1 for $s \gg 0$ and 0 for $s \ll 0$. Given generators $p \in \chi(L_0, L_1; H)$ and $p' \in \chi(L_0, L_1; H')$ of the respective Floer complexes, the coefficient of p' in $F(p)$ is defined as a sum of $T^{\omega([u_i])}$. Each u_i is an index 0 solution to 3.5 which converges to p as $s \to -\infty$ and to p' as $s \to +\infty$.

We now prove that F is a homotopy of chain complexes. We construct a reverse homotopy F' by considering equation 3.5 by letting $\tau(s)$ equal 0 for $s \gg 0$ and 1 for $s \ll 0$.

We first prove F is a chain map. We study spaces of index 1 solutions to (3.5), going from p to p', with class $[u]$. These spaces are 1-dimensional manifolds, whose end points correspond to broken trajectories where the main component is an index 0 solution to (3.5), either preceded by a J'-holomorphic strip of index 1 with perturbation data H (if energy concentrates at $s \to +\infty$), or followed by a J-holomorphic strip of index 1 with perturbation data H' (if energy concentrates at $s \to -\infty$). The $T^{\omega[u]}p'$ coefficient of composition $F \circ \partial$ counts the first type of limit configuration, while that of $\partial \circ F$ counts the second type of limit configuration, and the equality between these two maps follows again from the statement that the total (signed) number of end points of a compact 1-manifold with boundary is zero. The choice of p' and $[u]$ is arbitrary, thus concluding $\partial \circ F = F \circ \partial$.

To show $F' \circ F$(and is the same for $F \circ F'$) is homotopic to the identity, we can construct an explicit homotopy $K(K')$ by counting index (-1) solutions for 3.5 where $\tau(s)$ is 0 (and 1 for K') near $\pm\infty$ and is nonzero (non-one for K') over an interval of values of s of varying width.

To prove $\partial \circ K - K\partial = \mathrm{id} - F \circ F'$, consider the boundary of the following moduli space $\mathcal{M}'(p, q; [u], J(\tau'))$, which counts the index 0 solutions of:

$$\frac{\partial u}{\partial s} + J(\tau'(as), t, u)\left(\frac{\partial u}{\partial t} - X_{H(\tau)}\right) = 0 \tag{3.6}$$

where $\tau'(s)$ is 0 for $s \ll 0$ and $s \gg 0$, and is nonzero over an interval of values of s of varying width, and u goes to p at $-\infty$ and q at $+\infty$, with a varying from $(0, +\infty)$ as a scaling of width. When $p \neq q$, the boundary of $\mathcal{M}'(p, q; [u], J(\tau'))$ is made up three parts:

(1) an index (-1) strip solving (3.6) from p to r and an index 1 strip solving the perturbed CR-equation with respect to (H, J) from r to q

(2) an index 1 strip solving the perturbed CR-equation with respect to (H, J) from p to r and an index (-1) strip solving (3.6) from r to q

(3) an index 0 strip solving the reversed (3.5) from p to r and an index 0 strip solving (3.5) from r to q.

The first part is the $T^{\omega([u])}q$ coefficient of $\partial \circ K$, and the second is that of $K \circ \partial$, and the third is that of $F' \circ F$. For the $p = q$ case, the extra fourth part of the boundary is the constant strip at p, which counts the p coefficient of id.

In conclusion, the $T^{\omega([u])}q$ coefficient of $\partial \circ K - K \circ \partial - \mathrm{id} + F \circ F'$ is always zero in $\mathbb{F}_2$, for any choice of q and $[u]$. Therefore we have proved K is a homotopy connecting Id and $F' \circ F$. The similar property of K' can be proved in the same way. □

An appplication of Floer cohomology is solving the Arnold's conjecture:

Theorem 3.7. *(Arnold's conjecture) Let L be a compact Lagrangian submanifold of M. Assume that the symplectic area of any topological disc in M with boundary in L vanishes. Let ψ be the Hamiltonian diffeomorphism generated by H. Assume moreover that $\psi(L)$ and L intersect transversely. Then the number of intersection points of L and $\psi(L)$ satisfies the lower bound $|\psi(L) \cap L| \geq \sum_i \dim H^i(L, \mathbb{Z}/2\mathbb{Z})$.*

Proof. First consider T^*L, with its standard exact symplectic form. Let L be the zero section, and suppose given a Morse function $f : L \to \mathbb{R}$ and a small $\varepsilon > 0$. Define $H = \varepsilon \circ f \circ \pi$. A lemma shows the correspondence between self Floer theory and Morse theory in this case:

Lemma 3.8. *(Floer) If f is C^2-small (which means its first and second derivative is bounded by a small constant), there is an isomorphism of cochain complexes:*

$$CF(L, L, H) \simeq C^{Morse}_{n-*}(L, f)$$

Outline of proof: The proof is done by showing the bijection of the following pairs:

(1) Nondegenerate critical points and transverse intersection points.

(2) Index k intersection points of L and $\psi(L)$ and index $n - k$ critical points of f.

(3) Pseudo-holomorphic strips and gradient flows with respect to $g = \omega(\cdot, J\cdot)$.

With these bijections, we identify the following two complexes.

For the general case of $L \subset M$, note that for a small perturbation, the tubular neighboorhood of L can be identified with T^*L. Since the Floer cohomology is independent of the choice of perturbation, we can take H to be sufficiently small. Therefore a more general conclusion can be achieved:

Theorem 3.9. [1] *If $[\omega] \cdot \pi_2(M, L) = 0$, then $HF^*(L, L) \simeq H^*(L, \Lambda)$.*

This directly gets Arnold's conjecture, since both the left hand side and the right hand side is isomorphic to Morse cohomology. □

We can calculate the case of $S^1 \times \{0\} \subset T^*S^1$ as an example:

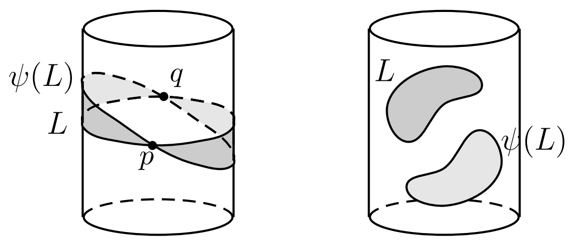
Example 3.10. Consider the cylinder $M = T^*S^1$, with the standard area form, and let $L = S^1 \times \{0\}$ be its zero section. Take the Hamiltonian $H = \sin\theta$ and Morse function f as its restriction on S^1. Then $\psi(L)$ is a closed curve around the cylinder satisfying:

$$\psi(L) = \{(\theta, r) \,|\, r = \cos\theta\}.$$

The two intersection points are of index 1 and zero respectively (left). The Floer differential is trivial in $\mathbb{K}$ (of character 2), since the volumes of the two strips are equal. Therefore, $HF^*(L, L) = H^*(L)$, which are both $\mathbb{F}_2$ at degree 0 and 1, and vanish in other degrees.

The zero volume condition and Hamiltonian diffeomorphism (not only symplectic) are both essential. Continuing the study of T^*S^1 we will find more examples and counter-examples.

Let L be a simple closed curve going around the cylinder once. Then $\psi(L)$ is also a simple closed curve going around the cylinder once. The assumption that $\psi \in Ham(M)$ means that the total signed area of the 2-chain bounded by L and $\psi(L)$ is zero. Then we get $|\psi(L) \cap L| \geq 2$, which satisfies Arnold's conjecture (left). On the other hand, the result becomes false if we only assume that ψ is a symplectomorphism (consider a vertical translation taking L away from itself); or if we take L to be a homotopically trivial simple closed curve, which bounds a disc of non-zero area (right).



4. Fukaya categories

4.1. **Product Operations.** Let (M,ω) be a symplectic manifold. Let L_0, L_1, L_2 be three Lagrangian submanifolds of M. For simplicity we assume they intersect each other transversely and for each i,

$$[\omega] \cdot \pi_2(M, L_i) = 0.$$

In this section, we will define a product operation

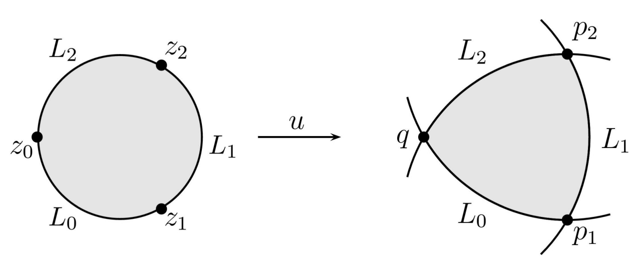
$$CF(L_1, L_2) \otimes CF(L_0, L_1) \to CF(L_0, L_2),$$

and check it satisfies the Leibniz rule. Let D be the closed unit disk $\{z \in \mathbb{C} \mid |z| \leq 1\}$.

Definition 4.1. Given an almost complex structure J on M, and a homotopy class $[u] \in \pi_2(M, L_0 \cup L_1 \cup L_2)$, we denote by

$$\mathcal{M}(p_1, p_2, q; [u], J)$$

the space of J-holomorphic maps $u : D \setminus \{z_0, z_1, z_2\} \to M$ with finite energy, which extend continuously to the closed disk. Here z_0, z_1, z_2 are any three points on the boundary ∂D. The extension $D \to M$ maps z_0, z_1, z_2 to q, p_1, p_2 respectively, and each of three arcs in ∂D (seperated by z_0, z_1, z_2) are mapped to L_0, L_1, L_2 respectively (as the figure shows).



Similar to the case of strips, we can define the index $\mathrm{ind}[u]$. And given the regularity condition on J, we have that $\mathcal{M}(p_1, p_2, q; [u], J)$ is a smooth manifold with dimension $\mathrm{ind}[u]$. We define:

Definition 4.2. The Floer product is the Λ-linear map

$$CF(L_1, L_2) \otimes CF(L_0, L_1) \to CF(L_0, L_2)$$

defined by

$$p_2 \cdot p_1 = \sum_{\substack{q \in \chi(L_0, L_2) \\ \mathrm{ind}([u])=0}} (\#\mathcal{M}(p_1, p_2, q; [u], J)) T^{\omega([u])} q.$$

Then we can check

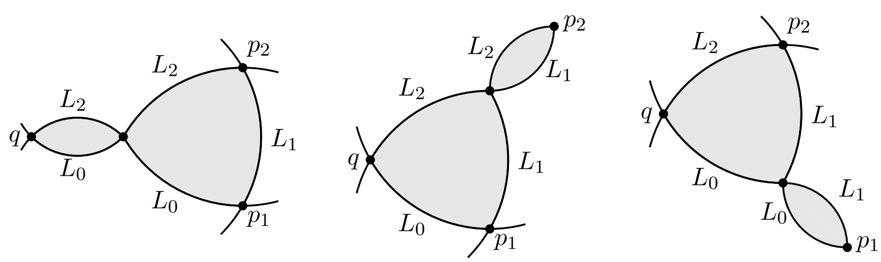
Theorem 4.3. *The Floer product satisfies the Leibniz rule (with suitable signs) with respect to the Floer differential (the sign problem can be omitted if we are in characteristic 2 case):*

$$\partial(p_2 \cdot p_1) = (-1)^{\deg p_1} (\partial p_2) \cdot p_1 + p_2 \cdot (\partial p_1).$$

We give a proof in character 2 case:

Proof. We solve this problem by considering the boundary of a 1-dimensional moduli space $\mathcal{M}(p_1, p_2, q; [u], J)$. By the same reasoning as in the proof of Theorem 2.9, $[\omega] \cdot \pi_2(M, L_i) = 0$ insures that the boundary is only made up of broken strips with no boundaries. We enumerate all the possible cases of broken strips, like it was done in 2.9. They are the following three cases:

(1) one part in $\mathcal{M}(p_1, p_2, r; [u], J)$ (index 0) and the other in $\mathcal{M}(r, q; [u], J)$ (index 1).
(2) one part in $\mathcal{M}(p_1, r; [u], J)$ (index 1) and the other in $\mathcal{M}(r, p_2, q; [u], J)$ (index 0).
(3) one part in $\mathcal{M}(p_2, r; [u], J)$ (index 1) and the other in $\mathcal{M}(p_1, r, q; [u], J)$ (index 0).



As we know, the $\mathbb{F}_2$ counting of boundary points sums up to zero. The counting of the first part corresponds to the $T^{\omega([u])}q$ coefficient of $\partial(p_2 \cdot p_1)$, and the second corresponds to the $T^{\omega([u])}q$ coefficient of $p_2 \cdot (\partial p_1)$, and the third corresponds to the $T^{\omega([u])}q$ coefficient of $(\partial p_2) \cdot p_1$. By summing them up together, we get that the $T^{\omega([u])}q$ coefficient of all terms of 4.3 sum up to zero. Since the choice of $[u]$ and q is arbitrary, we get the Lebniz rule. □

4.2. **Higher Product.** We have already defined the differential structure and product structure of $CF(L_0, L_1)$. To get Fukaya category, we need higher structure to make it an A_∞-category consists of the following data.

Definition 4.4. An A_∞-category $\mathcal{A}$ consists of a set of objects $Ob\ \mathcal{A}$, a graded vector space $\hom_{\mathcal{A}}(X_0, X_1)$ for any pair of objects, and composition maps of every order $k \geq 1$,

$$\begin{aligned}\mu^k_{\mathcal{A}} &: \hom_{\mathcal{A}}(X_{k-1}, X_k) \otimes \hom_{\mathcal{A}}(X_{k-2}, X_{k-1}) \otimes \dots \hom_{\mathcal{A}}(X_0, X_1) \\ &\to \hom_{\mathcal{A}}(X_0, X_k)[2-k].\end{aligned} \tag{4.5}$$

Here, $[k]$ means shifting the grading of a vector space down by $k \in \mathbb{Z}$. The maps (4.5) must satisfy the (quadratic) A_∞-associativity equations

$$\sum_{l=1}^{k} \sum_{j=0}^{k-l} (-1)^* \mu^{k+1-l}(p_k, \dots, p_{j+l+1}, \mu^l(p_{j+l}, \dots, p_{j+1}), p_j, \dots, p_1) = 0 \tag{4.6}$$

where $* = j + deg(p_1) + \cdots + deg(p_j)$, $p_i \in \hom_{\mathcal{A}}(X_{i-1}, X_i)$.

Given an almost-complex structure J on M and a homotopy class $[u]$, similarly to Definition 4.1, we denote by $\mathcal{M}(p_1, \dots, p_k, q; [u], J)$ the space of J-holomorphic finite energy maps $u : D \setminus \{z_0, \dots, z_k\} \to M$. Hhere the positions of $z_0, \dots, z_k$ are not fixed a priori. It extends continuously to the closed disc, mapping the boundary arcs from z_i to z_{i+1} (or z_0 for $i = k$) to L_i, and the boundary punctures $z_1, \dots, z_k, z_0$ to $p_1, \dots, p_k, q$ respectively, in the given homotopy class $[u]$, up to the action of $\mathrm{Aut}(D^2)$ by reparametrization.

Assuming transversality, for a fixed marked disk with $k + 1$ marked points $z_1, \dots, z_k, z_0$, the dimension of the space of solutions is $\mathrm{ind}([u])$, and the total dimension of the moduli space with non-fixed marked points is:

$$\dim(\mathcal{M}(p_1, \cdots, p_k, q; [u], J)) = k - 2 + \mathrm{ind}([u]). \tag{4.7}$$

Definition 4.8. Define

$$\mu^k : CF(L_{k-1}, L_k) \otimes CF(L_{k-2}, L_{k-1}) \cdots CF(L_0, L_1) \to CF(L_0, L_k)$$

to be the Λ-linear map defined by

$$\mu^k(p_k, p_{k-1}, \cdots, p_1) = \sum_{\substack{q \in \chi(L_0, L_k) \\ \mathrm{ind}([u]) = 2-k}} (\#\mathcal{M}(p_1, p_2, \cdots, p_k, q; [u], J)) T^{\omega([u])} q. \tag{4.9}$$

Remark 4.10. For low dimensional cases, μ^1 is ∂ and μ^2 is product. The case $k = 1$ of this proposition is the identity $\partial^2 = 1$, while $k = 2$ corresponds to the Leibniz rule

$$\partial(p_1 \cdot p_2) = (-1)^{\deg p_1} (\partial p_1) \cdot p_2 + p_1 \cdot (\partial p_2). \tag{4.11}$$

For k=3, the proposition gives an associativity up to a homotopy given by μ^3:
(4.12)

$$\begin{aligned}&(-1)^{1+\deg(p_1)}(p_3 \cdot p_2) \cdot p_1 + p_3 \cdot (p_2 \cdot p_1) = \partial \mu^3(p_3, p_2, p_1) + \\ &(-1)^{(\deg(p_2)+\deg(p_1))} \mu^3(\partial p_3, p_2, p_1) + (-1)^{(1+\deg(p_1))} \mu^3(p_3, \partial p_2, p_1) + \mu^3(p_3, p_2, \partial p_1).\end{aligned}$$

On the level of cohomology, this shows the associativity law of products of Floer cohomology up to sign. For the special case of $\mu^k = 0$ for $k \geq 3$, it is the strict associativity law on the chain level.

We can prove that μ^k satisfies the A_∞-relations.

Theorem 4.13. [1] *If $[\omega] \cdot \pi_2(M, L_i) = 0$ for all i, then the operations μ^k satisfy the A_∞-relations.*

Proof. The sketch of the proof is similar to that of Theorem 2.9. We only consider $\mathbb{K} = \mathbb{F}_2$ case, in which signs and orientations do not matter.

Study the boundary of the 1 dimensional moduli space $\mathcal{M}(p_k, p_{k-1}, \ldots, p_1, q; [u], J)$, with the index of $[u]$ being $3-k$. For the same reason as that of Theorem 2.9, the boundary points are all 'broken strips' of the following two parts:

one part is:

$$u_1 \in \mathcal{M}(p_{j+l}, p_{j+l-1}, \ldots, p_{j+1}, r; [u], J)$$

with the index of $[u_1]$ being $2-l$,

and the other part is

$$u_2 \in \mathcal{M}(p_k, p_{k-1}, \ldots, p_{j+l+1}, r, p_j, \ldots, p_1, q; [u], J)$$

with the index of $[u_2]$ being $1-k+l$. The counting of points of this kinds corresponds to

$$\mu^{k+1-l}(p_k, \ldots, p_{j+l+1}, \mu^l(p_{j+l}, \ldots, p_{j+1}), p_j, \ldots, p_1).$$

By summing them up, we get the A_∞-relation. □

With these set-ups we can give the definition of (compact) Fukaya categories (only over $\mathbb{F}_2$) as A_∞-categories.

Definition 4.14. Let (M, ω) be a symplectic manifold with $2c_1(TM) = 0$. The objects of the (compact) Fukaya category $\mathcal{F}(M, \omega)$ are compact closed Lagrangian submanifolds $L \subset M$ such that $[\omega] \cdot \pi_2(M, L_0) = 0$ and with vanishing Maslov class $\mu_L = 0 \in H_1(L, Z)$, together with a graded lift of L. (We will usually omit it from the notation and simply denote the object by L.)

For every pair of objects (L, L') (not necessarily distinct), we choose a time dependent Hamiltonian $H_{L,L'} \in C^\infty([0,1] \times M, R)$ and one-parameter family of complex structures $J_{L,L'} \in C^\infty([0,1], \mathcal{J}(M, \omega))$ (we call these Floer data); and for all tuples of objects $(L_0, \ldots, L_k)$ and all moduli spaces of discs, we choose consistent perturbation data (H, J) compatible with the choices made for the pairs of objects (L_i, L_j), so as to achieve transversality for all moduli spaces of perturbed J-holomorphic discs.

Given this, the morphism space of the pair is:

$$\hom(L, L') \coloneqq CF(L, L'; J_{L,L'}, H_{L,L'}).$$

The A_∞ structure μ^k is given by the counting of moduli space in Definition 4.8, and they satisfies the A_∞ relation by (4.13).

Remark 4.15. When we want to construct Fukaya categories over a general base field $\mathbb{K}$ with characteristic not equalling 2, we also need to contain spin structure and orientation for each L to guarantee the coherent orientation of the moduli space.

Remark 4.16. The A_∞-relation between non-transversal Lagrangians is more complicated because of the choice of 'perturbation data', which means the composition of the fixed choices of perturbation in a Fukaya category. This problem also involves the well-definedness of the Fukaya category. The answer to this question is: for various choices of perturbation and their composition, a Fukaya category is well-defined up to A_∞ quasi-equivalence.

5. Closing remarks—on other versions of Fukaya categories

When we generalize our discussion to non-compact cases, we will get other versions of Fukaya categories, namely wrapped Fukaya categories and partially wrapped Fukaya categories. Wrapped Fukaya categories includes exact non-compact Lagrangians in non-compact Liouville manifolds which are conical at infinity. By taking a proper(fixed) perturbation, they have a 'wrapping' behaviour at infinity. To define an A_∞-structure for wrapped Fukaya categories, a 'recaling trick' of perturbation is developed by M. Abouzaid in [2]. Cotangent bundles are objects of interest to study, since M. Abouzaid proved the A_∞ quasi-equivalence between the wrapped Floer complex of a cotangent fiber and the chain complex of a based loop space. He also proved that a cotangent fiber generates the wrapped Fukaya category, thus giving an explicit depiction of wrapped Fukaya categories in cotangent bundles.

As for partially wrapped Fukaya categories, they are defined in Liouville (or Weinstein) sectors, which are manifolds with contact boundaries. The contact boundary 'stops' the wrapping of a conical Lagrangian. Weinstein manifolds (sectors) are a class of symplectic manifolds with a Morse function. By studying Morse theory on it one can find some interesting geometric properties: the maximal index of critical points is n, and the unstable manifolds of those points are Lagrangian submanifolds, which are called cocores. A result in [8] claims the cocores generates partially wrapped Fukaya categories.

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