CHARACTERISTIC CLASSES

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Abstract. Characteristic classes are powerful tools in algebraic topology which allow one to study properties of vector bundles. They therefore have applications throughout mathematics, especially in geometry. We give an introduction to characteristic classes and some examples of their use in studying manifolds. We presume not much more than basic knowledge of the tangent bundle, and of elementary algebraic topology (namely, singular cohomology and properties of the classifying space).

Contents

1. Preliminaries: Maps and Operations of Vector Bundles

Vector bundles are fundamental objects which arise naturally in differential geometry and topology, as well as other areas of mathematics. As a reminder, a vector bundle of dimension n over a space B is a surjective map

$$\xi : E \to B$$

such that for all $b \in B$,

(i) the preimage, or "fiber" $F_b = \xi^{-1}\{b\}$ is a topological vector space of (real or complex) dimension n and

(i) There is a neighborhood U of b such that $\xi^{-1}(U)$ is homeomorphic to $U \times F_b$ and ξ restricts to the projection onto U.

We may also be interested in cases where the fiber has the structure of a module over the quaternions, although perhaps this should not really be called a "vector" bundle. We may refer to a vector bundle as the triple (ξ, E, B), or simply as ξ or E if the other data are understood.

Examples 1.1. (i) Given a vector space V, there is always the "trivial bundle" which projects $B \times V$ onto B.

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(ii) Given a smooth manifold M, we may consider the tangent bundle TM, the cotangent bundle T^*M, and the exterior powers $\Lambda^k T^*M$, among other related constructions.

(iii) Consider the real projective space $\mathbb{R}P^n$. There is canonical line bundle over $\mathbb{R}P^n$ which associates to each point the corresponding line in $\mathbb{R}^{n+1}$. We may also consider the analogous constructions for $\mathbb{C}P^n$ and $\mathbb{H}P^n$.

To study vector bundles, we need to consider maps between them. We have the following definition:

Definition 1.2. Given two vector bundles (ξ, E, B) and (η, E', B') of dimension n, a morphism or "bundle map" $(\tilde{f}, f) : \xi \to \eta$ is a pair of maps $\tilde{f} : E \to E'$ and $f : B \to B'$ such that

(i) The following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\tilde{f}} & E' \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \eta} \\ B & \xrightarrow[f]{} & B' \end{array}$$

(ii) For all points $b \in B$, $\tilde{f}$ restricts to a linear isomorphism $\xi^{-1}(b) \to \eta^{-1}(f(b))$.

Given a field (or skew-field) K we will denote the sets of isomorphism classes of n-dimensional K-vector bundles over B as $\mathrm{Vect}_n^K(B)$.

Now, given a vector bundle (ξ, E, B) and a map $f : B' \to B$ we can construct a vector bundle $(f^*\xi, f^*E, B')$, called the *pullback* of ξ by f, and a bundle map $f^*\xi \to \xi$ as follows: let $f^*E \subseteq E \times B'$ consist of those pairs (e, b) such that $e \in \xi^{-1}(f(b))$. Then, define $f^*\xi$ to be the projection onto the second coordinate, and $\tilde{f}$ to be projection onto the first, giving a bundle map $(\tilde{f}, f) : f^*\xi \to \xi$.

In order to classify vector bundles via bundle maps, we will need the following two propositions:

Proposition 1.3. *Suppose that (ξ, E, B) and (η, E', B') are vector bundles and there exists a bundle map $(\tilde{f}, f) : \xi \to \eta$. Then, ξ is isomorphic to $f^*\eta$.*

Proposition 1.4. *Suppose that (ξ, E, B) is a vector bundle and the maps $f, g : B' \to B$ are homotopic maps. Then, $f^*\xi$ is isomorphic to $g^*\xi$.*

Lastly, we may want to extend certain operations on vector spaces, such as the direct sum and the tensor product, fiberwise to vector bundles. The details of these constructions, as well as the proofs of Propositions 1.3 and 1.4 are presented, for example, in [1].

2. Grassmannians and the Universal Bundle

Here we give the definition of characteristic classes and their relationship to the classification of vector bundles.

Definition 2.1. Let R be a coefficient ring. A *characteristic class* c is choice of element $c(\xi) \in H^*(B; R)$ for each vector bundle (ξ, E, B) which obeys the following naturality property: for all bundle maps $(\tilde{f}, f) : \xi \to \eta$, we have $c(\xi) = f^*(c(\eta))$.

Now, it turns out that we can classify vector bundles as the pullbacks of a certain "universal bundle". This will allow us to realize all characteristic classes by evaluation on this bundle. To construct the universal bundle we first need some definitions.

Definitions 2.2. Let W be a vector space with inner product (of possibly infinite dimension). Then, we have the following:

(i) The *Stiefel "manifold"* $V_n(W)$, whose points are orthonormal[1] k-frames in W.
(ii) The *Grassmannian* $Gr_n(W)$, whose points are k-dimensional subspaces of W.

There is a surjective map $p : V_n(W) \to Gr_n(W)$ given by $\gamma(x_1, x_2 \dots, x_n) = \text{span}\{x_1, x_2, \dots, x_n\}$. We topologize $V_n(W)$ as a subspace of W^n and then topologize $Gr_n(W)$ so that p is a quotient map.

Now, let $E_n(\mathbb{R})$ be the subspace of $Gr_n(\mathbb{R}^\infty) \times \mathbb{R}^\infty$ consisting of those pairs (X, x) such that $x \in X$. We have a vector bundle

$$\begin{aligned} \gamma^n_{\mathbb{R}} : E_n(\mathbb{R}) &\to Gr_n(\mathbb{R}^\infty) \\ (X, x) &\mapsto X. \end{aligned}$$

Then, consider the following:

Theorem 2.3. *Let (ξ, E, B) be any real n-dimensional vector bundle, where B is paracompact. Then, there exists a bundle map $\xi \to \gamma^n_{\mathbb{R}}$.*

The proof of Theorem 2.3 relies on a lemma, proved in [1]:

Lemma 2.4. *Let B be paracompact and let (ξ, E, B) be a fiber bundle. Then, there exists a countable open cover $U_1, U_2, \dots$ of B such that ξ is trivial on each U_i.*

Now we can prove Theorem 2.3.

Proof of Theorem 2.3. Let $U_1, U_2, \dots$ be a countable open cover of B such that ξ is trivial on each U_i. Since B is paracompact, B is normal, so there exist open covers $V_1, V_2, \dots$ and $W_1, W_2, \dots$ of B such that

$$\overline{W_i} \subseteq V_i \subseteq \overline{V_i} \subseteq U_i,$$

and there exist continuous functions $\phi_i : B \to [0, 1]$ such that $\phi_i(\overline{W_i}) = \{1\}$ and $\phi_i(B \setminus U_i) = \{0\}$ for all i. Since ξ is trivial over each U_i, there are projection maps $pi_i : \xi^{-1}(U_i) \to \mathbb{R}^n$. Using the fact that ϕ_i is zero outside U_i, we may define

$$h_i(e) = \begin{cases} \phi_i(\xi(e))\pi_i(e), & e \in \xi^{-1}(U_i) \\ 0, & \text{else.} \end{cases}$$

Then, since at any point, h_i is zero for all but finitely many i, we have a map

$$\begin{aligned} h : E &\to (\mathbb{R}^n)^\infty \approx \mathbb{R}^\infty \\ e &\mapsto (h_1(e), h_2(e), \dots) \end{aligned}$$

[1]in lieu of an inner product, we may replace "orthonormal" with "linearly independent" and the space will be the same up to homotopy equivalence.

Then, the bundle map $(\tilde{f}, f)$ is given by

$$\tilde{f}(e) = (h(e), h(\xi^{-1}(\xi(e)))$$
$$f(e) = \gamma^n_{\mathbb{R}}(\tilde{f}(e)).$$

□

In the case of the universal bundle, the homotopy-invariance of the pullback is strengthened to the following theorem from [1]:

Theorem 2.5. *A real n-dimensional vector bundle (ξ, E, B) determines a unique homotopy class of maps $B \to Gr_n(\mathbb{R}^\infty)$. Two such bundles are isomorphic if and only if they determine the same class of maps.*

Remark 2.6. Theorem 2.5 can be restated as follows: there exists a natural isomorphism between the set-valued functors $\mathrm{Vect}^{\mathbb{R}}_n(-)$ and $[-, Gr_n(\mathbb{R}^\infty)]$. We may therefore realize Theorem 2.5 for a more fundamental reason: $(p, V_n(\mathbb{R}^\infty), Gr_n(\mathbb{R}^\infty))$ is a principal $O(n)$-bundle, and $V_n(\mathbb{R}^\infty)$ is contractible. Therefore, $Gr_n(\mathbb{R}^\infty)$ is the classifying space $BO(n)$. This perspective has the advantage that we can utilize the functorality of B. Likewise, we can realize $BU(n)$ as $Gr_n(\mathbb{C}^\infty)$ and $BSp(n)$ as $Gr_n(\mathbb{H}^\infty)$.

The punchline to all of this is that by naturality, every characteristic class of real, complex, or quaternionic vector bundles is uniquely determined by its evaluation on the universal bundle, hence there is a bijection between the set of all characteristic classes and the cohomologies of a Grassmannian. Therefore, with the goal of describing characteristic classes, we can make significant progress by calculating the cohomology of Grassmannians.

3. The Chern and Stiefel-Whitney Classes.

We state here the cohomolgies of $BU(n)$ and $BO(n)$:

Theorem 3.1 (Universal Chern Classes)**.** *$H^*(BU(n); \mathbb{Z}) = \mathbb{Z}[c_0, c_1, \ldots, c_n]$ where the classes $c_i = (p^*)^{-1}(\sigma_i)$ lie in degree $2i$. They satisfy and are uniquely characterized by the following axioms:*

(i) $c_0 = 1$ *and* $c_i = 0$ *for* $i > n$
(ii) c_1 *is the canonical generator for* $H^2(\mathbb{C}P^\infty; \mathbb{Z})$
(iii) $i^*_n(c_i) = c_i$
(iv) $p^*_{m,n}(c_k) = \sum_{i+j=k} c_i \otimes c_j$.

The analogous classes for real vector bundles are the *Stiefel-Whitney classes*, which lie in the mod 2 cohomology, and their existence and uniqueness is proved similarly. We have:

Theorem 3.2 (Universal Stiefel-Whitney Classes)**.** *$H^*(BO(n); \mathbb{Z}/2) = \mathbb{Z}/2[w_0, w_1, \ldots, w_n]$ where the classes w_i lie in degree i. They satisfy and are uniquely characterized by the following axioms:*

(i) $w_0 = 1$ *and* $w_i = 0$ *if* $i > n$
(ii) w_1 *is the unique nonzero element of* $H^2(\mathbb{R}P^\infty; \mathbb{Z}/2)$
(iii) $i^*_n(w_i) = w_i$
(iv) $p^*_{m,n}(w_k) = \sum_{i+j=k} w_i \otimes w_j$.

The same method of proof verifies both cases. The proof can take on many "flavors," although they all boil down to the same essential ingredients. An approach using only elementary methods (i.e. only basic cohomology) can be found in [2]. However, we can amp this up to a more modern and general approach. The main point of this approach is the same as that of Hatcher's proof: we can obtain an injective homomorphism whose domain is $H^*(BU(n);\mathbb{Z})$ and whose image is the ring generated by the elementary symmetric polynomials in even degrees, but the methods to obtain this map and to prove its injectivity and image are different. We'll outline the idea of this approach here.

First, one can use the fact that $H^*(U(n),\mathbb{Z})$ is an exterior algebra on n generators in degree 2 to obtain the rank of $H^*(BU(n);\mathbb{Z})$. The details of this are handled in [3]. Thus, we only need to obtain our desired map and show its injectivity. In fact, this map arises naturally in the following way: consider the inclusion of a maximal torus $T^n = U(1)^n \hookrightarrow U(n)$. We obtain a classifying map $BU(1)^n \to BU(n)$. The *generalized splitting principle* (see [4]) then tells us that this map induces an injective map i^* on cohomology.

Now, using the fact that $H^*(BU(1);\mathbb{Z})$ is a polynomial ring with one generator in degree 2, the Kunneth theorem tells us that $H^*(BU(1)^n;\mathbb{Z})$ is a polynomial ring with n generators in degree 2. We have an action of the symmetric group Σ_n on $U(n)$ by permutation of rows. This action fixes $U(1)$, so in fact our map i^* lands in those elements of $H^*(BU(1)^n;\mathbb{Z})$ which are invariant under the action. That is, the symmetric polynomials. Since we know i^* is injective and we know the rank of $H^*(BU(n);\mathbb{Z})$, the map must be an isomorphism.

Thus we have two important examples of characteristic classes. Given a complex vector bundle ξ, we will denote by $c_i(\xi)$ the pullback of c_i by the classifying map corresponding to ξ. Likewise, if ξ is a real vector bundle, then we may consider $w_i(\xi)$. We can also define the *total* Chern and Stiefel-Whitney classes as

$$c(\xi) = \sum_{i=0}^{n} c_i(\xi)$$
$$w(\xi) = \sum_{i=0}^{n} w_i(\xi).$$

In the following sections, we will see some examples which demonstrate the use of characteristic classes.

4. The Euler Class

The Euler class is a particular example of a characteristic class which encodes information about the existence (or non-existence) of non-zero sections of an oriented vector bundle. To define the Euler class, we need to get some preliminaries out of the way. Throughout this section, we will consider only real vector bundles.

Definition 4.1. An orientation on a vector space is an equivalence class of ordered bases related by positive-determinant coordinate transformations. An orientation on a vector bundle (ξ, E, B) is a choice of orientation for each fiber such that every point of the base space has a neighborhood U on which ξ is trivial and where the associated homeomorphism $U \times \mathbb{R}^n \to \xi^{-1}(U)$ preserves the orientation of the fibers.

The classical example is, of course, an oriented n-manifold M. In this case, an orientation on M is equivalent to an orientation on TM, which is in turn equivalent to an orientation on $\Lambda^n T^* M$.

Proposition 4.2. *A real vector bundle (ξ, E, B) is orientable if and only if the first Stiefel-Whitney class $w_1(\xi)$ is zero.*

Proof. Oriented real vector bundles are classified by $BSO(n)$. Consider the exact sequence of groups

$$SO(n) \xrightarrow{i} O(n) \xrightarrow{\det} O(1).$$

We have an induced fibration

$$BSO(n) \xrightarrow{Bi} BO(n) \xrightarrow{B\det} BO(1)$$

and hence an exact sequence

$$[B, BSO(n)] \xrightarrow{Bi_*} [B, BO(n)] \xrightarrow{B\det_*} [B, BO(1)].$$

Now, ξ is classified by a map $f_\xi \in [B, BO(n)]$, and is orientable if and only if f_ξ comes from $[B, BSO(n)]$, which is to say that $B\det_*(f_\xi) = 0$. Now, since $[-,-]$ is contravariant in the first variable and covariant in the second, we have a commutative diagram

$$\begin{array}{ccc} [BO(n), BO(n)] & \xrightarrow{B\det^*} & [BO(n), BO(1)] \approx H^1(BO(n); \mathbb{Z}/2) \\ \downarrow {f_\xi}_* & & \downarrow {f_\xi}^* \\ [B, BO(n)] & \xrightarrow{B\det_*} & [B, BO(1)] \approx H^1(B; \mathbb{Z}/2) \end{array} .$$

The isomorphisms on the right come from the general fact that $H^n(X; A)$ is naturally isomorphic to $[X, K(A, n)]$ for abelian groups A, a special case of Brown's representability theorem. Consider the element $\mathrm{id}_{BO(n)} \in [BO(n), BO(n)]$. Its image $B\det_*(\mathrm{id}_{BO(n)}) \in H^1(BO(n); \mathbb{Z}/2)$ restricts under inclusion to a nonzero element of $H^1(BO(1); \mathbb{Z}/2)$ and is therefore equal to $w_1(\gamma^n_{\mathbb{R}})$. Thus, we have

$${f_\xi}_*(B\det_*(\mathrm{id}_{BO(n)})) = w_1(\xi).$$

On the other hand, going the other way around the diagram gives $B\det_*(f_\xi)$, which is necessarily zero. Therefore, ξ is orientable if and only if $w_1(\xi) = 0$. □

There is an alternative, algebraic formulation of an orientation on a vector bundle which we will use to define the Euler class. We first note that an orientation on a vector space W is equivalent to a choice of generator for $H^n(W, W \setminus \{0\}; \mathbb{Z}) \approx \mathbb{Z}$, since an orientation-preserving isomorphism of W induces the identity on said cohomology group. In general, an orientation on W gives rise to a choice of generator of $H^n(W, W \setminus \{0\}; R) \approx R$ for an arbitrary ring R, which in turn gives rise to a preferred generator for the cohomology $H^n(Sph(W); R)$, where $Sph(W)$ is the sphere obtained from W by applying one-point compactification.

In order to extend our new definition of orientation to vector bundles, we can define the *Thom Space* of a vector bundle. Given a vector bundle (ξ, E, B) we can apply one-point compactification to each fiber in E to obtain the associated bundle $Sph(E)$ whose fibers are n-spheres, and whose local trivializations are inherited from those of E. Then, we obtain the Thom space $T(\xi)$ as the quotient of $Sph(E)$ by the compactification points. As a geometric example, if E is the trivial bundle on S^1, (i.e. an infinite cylinder), then $Sph(E)$ is a torus and $T(\xi)$ is the quotient of the torus by its equatorial circle.

Now, we have the following definition:

Definition 4.3. A *Thom class* of an R-oriented vector bundle (ξ, E, B) is an element $\mu \in \tilde{H}^n(T(\xi); R)$ which pulls back under the composition

$$Sph(F_b) \xrightarrow{\iota} Sph(E) \xrightarrow{q} T(\xi)$$

to the preferred generator for the cohomology of each fiber of $Sph(E)$.

Every oriented vector bundle has a Thom class for arbitrary coefficients. See [1] for the proof. We can state the Thom isomorphism theorem:

Theorem 4.4 (Thom Isomorphism)**.** *Let (ξ, E, B) be an R-oriented n-plane bundle and let $\mu \in \tilde{H}^n(T(\xi); R)$ be its Thom class. Then the map*

$$H^i(B; r) \to \tilde{H}^{i+n}(T(\xi); R)$$
$$x \mapsto x \smile \mu$$

is an isomorphism.

Of course, we have to be precise about what the cup product here (really, a map $H^i(B; R) \otimes \tilde{H}^j(T(\xi); R) \to \tilde{H}^{i+j}(T(\xi); R)$ means. The details, as well as the proof of 4.4 can be found in [5]. Now, we can define the Euler class:

Definition 4.5. Let (ξ, E, B) be an oriented vector bundle with Thom class $\mu \in \tilde{H}^n(T(\xi); \mathbb{Z})$. Then, the *Euler Class* $e(\xi)$ is the inverse image of $\mu \smile \mu$ under the Thom isomorphism.

In fact, we have already seen a particular case of the Euler class: when ξ has the structure of a complex vector bundle, and therefore an oriented real $2n$-plane vector bundle, $e(\xi)$ is just the Chern class $c_n(\xi)$. In fact, one can construct the Chern classes using the Euler class. See [1] for this approach.

We list some important properties of the Euler class:

Theorem 4.6. *The Euler class e satisfies the following:*

(i) *Naturality, i.e. $f^*e(\xi) = e(f^*(\xi)$. That is, e is a characteristic class.*
(ii) *Let ξ' be the same bundle as ξ but with opposite orientation on the total space. Then, $e(\xi') = -e(\xi)$. As a consequence, if the dimension of ξ is odd, the $e(\xi) + e(\xi) = 0$.*
(iii) *Reduction to mod 2 coefficients takes $e(\xi)$ to $w_n(\xi)$.*
(iv) *For bundles (ξ, E, B) and (η, E', B) we have*

$$e(\xi \oplus \eta) = e(\xi) \smile e(\eta).$$

Likewise, for cartesian products

$$e(\xi \times \eta) = e(\xi) \times e(\eta).$$

The proof of 4.6 can be found in [1]. The primary significance of the Euler class lies in the following theorem:

Theorem 4.7. *Let (ξ, E, B) be an oriented vector bundle. If ξ admits a non-vanishing section, then $e(\xi) = 0$.*

Proof. For the proof of this theorem, we will need to make use of the isomorphism $\tilde{H}^*(T(\xi); R) \approx H^*(E, E_0; R)$, where E_0 is the collection of nonzero vectors in E (see [5]). We can obtain this isomorphism as a sequence of isomorphisms

$$\tilde{H}^*(T\xi) \approx H^*(Sph(E), B) \approx H^*(Sph(E), Sph(E)_0) \approx H^*(E, E_0).$$

For the first isomorphism we make use of the fact that for pairs (X, A) where A is a neighborhood deformation retract (i.e. the inclusion is a cofibration), $\tilde{H}^n(X/A) \approx H^(X, A)$, where in this case $X = Sph(E)$ and A is the zero section, which is the homeomorphic image of B. For the second isomorphism, we can make use of an alternate embedding of B in $Sph(E)$ as the "points at infinity". Then, we can consider the triple $(Sph(E), Sph(E)_0, B)$. We can deformation retract $Sph(E)_0$ onto B, so the long exact sequence of this triple gives the desired isomorphism. The last isomorphism is obtained by excising a neighborhood of the section at infinity.

Then, the Thom class can be thought of as an element $\mu \in H^n(E, E_0; \mathbb{Z})$, and the Thom isomorphism is the map $\Phi : x \mapsto (\xi_r^* x) \smile \mu$.

Consider a section σ of ξ, i.e. a right inverse. If σ is nonzero, then the following composition gives the identity:

$$B \xrightarrow{\sigma} E_0 \xrightarrow{i} E \xrightarrow{\xi} B,$$

where i is the inclusion. Let $j : (E, \emptyset) \to (E, E_0)$ be the inclusion of pairs. Now, $j^*(\mu) \smile \mu = \mu^2$, so by definition we have $\xi^* e(\xi) = j^*(\mu)$. But the composition $i^* j^*$ is zero, so we have $e(\xi) = \sigma^* i^* \xi^* e(\xi) = 0$. □

As a corollary, we have the classical "hairy ball theorem":

Theorem 4.8. *Let n be even. Then, TS^n has no non-trivial sub-bundles. In particular, there is no non-vanishing vector field on S^n.*

Proof. Let all cohomology here have coefficients in $\mathbb{Z}$. First, we can describe the geometric tangent bundle of S^n. For each $p \in S^n$, we can naturally identify $T_p S^n$ with the affine space $p + \operatorname{span}\{p\}^\perp \subseteq \mathbb{R}^{n+1}$. Then, we can consider TS^n as

$$TS^n = \{(p, v) \in S^n \times \mathbb{R}^{n+1} : v \in T_p S^n\}.$$

Let $A = \{(p, -p) : p \in S^n\}$. With our description of TS^n in mind, we can stereographically project $S^n \setminus \{-p\}$ onto $T_p S^n$ for each p, and this extends to a homeomorphism $S^n \times S^n \setminus A \to TS^n$. Notice that under this homeomorphism, the image of the zero section is taken to the diagonal $\Delta \subseteq S^n \times S^n$. Thus, we have an isomorphism $H^*(TS^n, TS^n_0) \approx H^*(S^n \times S^n \setminus A, S^n \times S^n \setminus (A \cup \Delta))$.

By excising A from the pair $(S^n, S^n \setminus \Delta)$, we get the isomorphism

$$H^*(S^n \times S^n \setminus A, S^n \times S^n \setminus (A \cup \Delta)) \approx H^*(S^n \times S^n, S^n \times S^n \setminus \Delta).$$

Now, for any $p \in S^n$, the identity $\mathrm{id}_{S^n \setminus \{p\}}$ is homotopic to the constant map at $-p$. To see this, let $\pi : S^n \setminus \{p\} \to T_{-p} S^n$ be stereographic projection, and let h be the straight-line homotopy $T_{-p} S^n \simeq c_{-p}$. The homotopy we are interested in is

$$\begin{aligned} H : S^n \setminus \{p\} \times I &\to S^n \setminus \{p\} \\ (x, t) &\mapsto \pi^{-1}(h(\pi(x), t)). \end{aligned}$$

Then, H extends to a homotopy $\mathrm{id}_{S^n \times S^n \setminus \Delta} \simeq r$ where r is the retraction onto A. Lastly, the map

$$\begin{aligned} (S^n \times S^n, A) &\to (S^n \times S^n, \Delta) \\ ((x, y), (p - p)) &\mapsto ((x, -y), (p, p)) \end{aligned}$$

Is a homeomorphism of pairs. Thus, we have a chain of isomorphisms giving

$$H^*(TS^n, TS^n_0) \approx H^*(S^n \times S^n, \Delta). \tag{4.9}$$

Let $\mu \in H^n(TS^n, TS^n_0)$ be the Thom class (using the identification of $H^*(T(\xi))$ and $H^*(E, E_0)$ discussed in the previous proof). We can show that μ^2 is twice a generator of $H^{2n}(TS^n, TS^n_0)$ and hence that $e(TS^n)$ is twice a generator of $H^n(S^n)$. The long exact sequence of the pair $(S^n \times S^n, \Delta)$ gives a short exact sequence

$$0 \to H^n(S^n \times S^n, \Delta) \to H^n(S^n \times S^n) \to H^n(\Delta) \to 0.$$

The Kunneth theorem tells us that $H^n(S^n \times S^n)$ is generated by a pair of elements α and β, the pullbacks of the generator of $H^n(S^n)$ under each projection. The images of α and β in $H^n(\Delta)$ are the same, and this element is a generator, so $\alpha - \beta$ generates $H^n(S^n \times S^n, \Delta)$. Therefore, $\alpha - \beta$ corresponds (up to sign) to the Thom class under the isomorphism (4.9).

Since n is even, the cup product is commutative in degree n and hence $(\alpha-\beta)^2 = -2\alpha\beta$ is twice a generator of $H^{2n}(S^n \times S^n, \Delta)$. Thus, the Euler class is twice a generator of $H^n(S^n)$.

Now, suppose we can decompose TS^n as the direct sum $\xi \oplus \eta$. By naturality and Proposition 4.2, we find that ξ and η are both orientable. Thus, we have $e(TS^n) = e(\xi) \smile e(\eta)$ by Theorem 4.6 (iv). But $H^i(S^n) = 0$ for $0 < i < n$, so $e(TS^n) = 0$, and we have a contradiction. Therefore, no such decomposition exists. Therefore, no non-trivial sub-bundles exist, since S^n is a Riemannian manifold and hence we can always find the orthogonal complement to a sub-bundle of the tangent bundle. Moreover, since two trivial bundles sum to a trivial bundle, there are no non-zero sub-bundles of TS^n. Hence, there is no non-vanishing vector field on S^n since such a section would span a line bundle. □

In fact, the hairy ball theorem is a corollary to another classical theorem, the Poincare-Hopf theorem, a statement about the Euler characteristic of a space. We should expect the Euler class to arise in these sorts of situations. Given a compact oriented manifold M, there exists a fundamental class (preferred generator) $[M] \in H_n(M, M \setminus \{x\}; \mathbb{Z})$, and it turns out that $\langle e(TM), [M]\rangle = \xi(M)$. See, for example, Theorem 3.3 in [6]. Notice that the proof employs intersection theory. In particular, given a vector field $X \in \Gamma(TM)$ whose image I in TM intersects the zero section Z transversally, the euler class $e(TM)$ is Poincare-dual to the fundamental class $[I \cap Z]$.

Using the fact that the Euler class coincides with the top Chern class for complex vector bundles, we have another simple application, the degree-genus formula from algebraic geometry:

Theorem 4.10. *Let $X \subseteq \mathbb{C}P^2$ be a smooth algebraic curve of degree d. Then, the topological genus g of X is given by*

$$g = \frac{1}{2}(d-1)(d-2).$$

To prove Theorem 4.10 we first need the following theorem from [1]:

Theorem 4.11. *There exists a generator $a \in H^2(\mathbb{C}P^n; \mathbb{Z})$ so that $c(T\mathbb{C}P^n) = (1+a)^{n+1}$.*

In particular, $c_1(T\mathbb{C}P^n; \mathbb{Z}) = (n+1)a$. In fact, a is Poincaré dual to the fundamental class of $\mathbb{C}P^n$.

Proof of the Degree-Genus Formula. Throughout this proof denote $\mathbb{C}P^2$ by P^2. Consider the three bundles TX, NX, and $TP^2|_X$ over X. We have $TP^2|_X =$

$TX \oplus NX$, so

$$c(TP^2|_X) = c(TX) \smile c(NX).$$

In particular, since TX and NX have complex dimension one, we have $c_1(TP^2|_X) = e(TX) + e(NX)$. We know $\langle e(TX), [X] \rangle = \chi(X) = 2 - 2g$. By naturality, $c_1(TP^2|_X) = i^* c_1(TP^2)$, where i is the inclusion of X in P^2, and from the previous theorem we find that $\langle c_1(TP^2|_X), [X] \rangle = 3d$. Lastly, we have $e(NX) = PD[NX]^2$, where PD is Poincaré duality, so we have $\langle e(NX), [X] \rangle = d^2$ □

5. Embeddings of Manifolds in $\mathbb{R}^n$

Throughout this section, any "manifold" mentioned is assumed to be a real smooth manifold. We have so far seen some basic applications of characteristic classes, mostly expressing in some form or another the relationship between the Euler class and Euler characteristic. However, there is a broader scope of information which characteristic classes can contain.

For example, what sort of information do the Stiefel-Whitney classes encode? We saw in the previous section that the first Stiefel-Whitney class is the primary obstruction to a real vector bundle being orientable. In particular, consider the application of w_1 to the normal bundle NM of a manifold M. The tubular neighborhood theorem tells us that for a closed embedded submanifold $M^n \subseteq \mathbb{R}^{n+1}$, NM is orientable, and so w_1 is zero. In fact, this generalizes as:

Proposition 5.1. *Let $M = M^n \subseteq \mathbb{R}^{n+k}$ be a closed embedded submanifold. Then, $w_k(NM) = 0$.*

Proposition 5.1 follows as an immediate corollary to a more technical theorem in [1] (namely, Theorem 11.3). We can see that the Stiefel-Whitney classes contain information about the embeddability of a manifold in Euclidean space. We would like to be able to say something about when, *a priori*, M^n can be embedded as a closed submanifold of $\mathbb{R}^{n+k}$. Unfortunately, Proposition 5.1 only tells us information about the normal bundle, which only exists after an embedding has already been given. To make any real use of Proposition 5.1, we first need to convert the statement to one which is intrinsic to the manifold M, i.e. a statement in terms of the tangent bundle. Theorem 3.2 (iv) tells us generally that $w(\xi)w(\eta) = w(\xi \oplus \eta)$, and in particular $w(TM)w(NM) = 1$. So, in order to calculate $w(NM)$ using information about TM, we just need a way to calculate the multiplicative inverse of $w(TM)$. In fact, the group of units in mod 2 cohomology is a subgroup of a nice group, and we can do this fairly easily.

Let $H^{\Pi}(M; \mathbb{Z}/2)$ denote the collection of formal infinite series $a = \sum_{i=0}^{\infty} a_i$ where $a_i \in H^i(M; \mathbb{Z}/2)$. Then, $H^{\Pi}(M; \mathbb{Z}/2)$ is endowed with the structure of a ring by taking addition to be formal addition and distributing the cup product. Consider the following theorem:

Proposition 5.2. *Let $H_1^{\Pi}(M; \mathbb{Z}/2) \subseteq H^{\Pi}(M; \mathbb{Z}/2)$ consist of those series a for which $a_0 = 1$. Then, $H_1^{\Pi}(M; \mathbb{Z}/2)$ forms a group under the given multiplication.*

Proof. That $H_1^{\Pi}(M; \mathbb{Z}/2)$ is closed under multiplication is immediate from the definition, so we just have to check the existence of inverses. Let $a_1^{\Pi}(M; \mathbb{Z}/2)$ and let $\overline{a}_0 = 1$. Inductively, define $\overline{a}_{n+1} = \sum_{i=1}^{n-1} a_{n-i}\overline{a}_i + a_{n+1}$. Then, $a\overline{a} = 1$. □

Now, we define the normal Stiefel-Whitney classes of a manifold to be the classes $\overline{w}_i(TM)$. Now that we have a way of actually calculating the multiplicative inverses in $H^{\Pi}(M;\mathbb{Z}/2\mathbb{Z})$, we can determine the embeddability of M in $\mathbb{R}^{n+k}$ by knowing the Stiefel-Whitney classes of TM. As an example, we can give some partial results for when $\mathbb{R}P^n$ can be embedded in $\mathbb{R}^{n+k}$. Similar to the total Chern class for complex projective space, there exists a generator $a \in H^1(\mathbb{R}P^n;\mathbb{Z}/2)$ so that $w(T\mathbb{R}P^n) = (1+a)^{n+1}$. In particular, when $n = 2^r$, we have $w(T\mathbb{R}P^n) = 1 + a + a^n$, and this gives

$$\overline{w}(T\mathbb{R}P^n) = 1 + a + a^2 + \ldots + a^{n-1}.$$

Therefore, $\mathbb{R}P^{2^r}$ cannot be embedded in $\mathbb{R}^{2^r-1}$. In this case, the Whitney embedding theorem therefore gives the best possible lower bound for dimensions in which $\mathbb{R}P^{2^r}$ embeds.

In fact, we can say more. For example, consider the following theorem, originally due to Thomas:

Theorem 5.3. *Let $n = 8s + t$ where s is a positive integer, not a power of 2, and $0 \leq t \leq 7$. Let $\alpha(n)$ be the number of ones in the binary expansion of n, and assume that either t is neither 1 nor 2, or that $\alpha(n) \geq 4$. Then, $\mathbb{R}P^n$ embeds in $\mathbb{R}^{2n-6}$.*

See [7]. The proof still ultimately comes down to determining normal Stiefel-Whitney classes, but is much more computationally involved than in the proof for the 2^r case. In general, it is still an open question as to what the best lower bound is for dimensions which admit embeddings of $\mathbb{R}P^n$.

We can use the Stiefel-Whitney classes to study not only embeddings, but immersions of manifolds. Suppose that $M = M^n$ embeds in $\mathbb{R}^{n+k}$. Then, M has a well-defined, but not necessarily trivial, normal bundle NM, with fiber dimension k, and the Whitney duality theorem still holds. Thus, $\overline{w_i}(TM) = 0$ for $i > k$. This tells us, for example, that $\mathbb{R}P^{2^r}$ cannot be immersed in $\mathbb{R}^N$ for $N < 2^r - 1$. In fact, any n-manifold can be immersed in $\mathbb{R}^{2n-1}$, so this is the best possible estimate.

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References

[1] Milnor, J. W., Stasheff, J. D. (2009). Characteristic classes. Beijing: World Publishing Corporation.

[2] Hatcher, A. (2002). Algebraic topology. New York: Cambridge University Press.

[3] Bruner, R. R., Catanzaro, M. J., May, J. P. Characteristic Classes. http://math.uchicago.edu/ may/REU2020/charclasses.pdf

[4] May, J. (2005). A note on the splitting principle. Topology and its Applications, 153(4), 605–609. doi:10.1016/j.topol.2005.02.007

[5] May, J. P. A Concise Course in Algebraic Topology. https://www.math.uchicago.edu/ may/CONCISE/ConciseRevised.pdf

[6] Görg, M. The Euler Class and Intersection Theory. https://www.mathematik.hu-berlin.de/ wendl/Sommer2018/Topologie3/20180621_Goerg.pdf

[7] Thomas, E. Embedding manifolds in Euclidean space. Osaka Journal of Mathematics. Osaka University and Osaka City University, Departments of Mathematics. https://projecteuclid.org/euclid.ojm/1200769312. Accessed 15 August 2020