(PRO-G)-SPECTRA AND PRO-(G-SPECTRA)

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Abstract. In this paper we discuss three different proposals for a genuine model category of G-Spectra when G is compact Hausdorff (in particular, when G is profinite). We provide brief remarks on applications which are available in the literature. Particular attention is paid to a model proposed by Halvard Fausk, built upon combined categorical work with Dan Isaksen.

Contents

1. Introduction

When G is a finite group, or more generally a compact Lie group, there have long been models for a G-equivariant stable homotopy category. Due to technical problems, defining these categories for more general groups has been an uphill battle.

However when the group in question is built with groups we already understand, it is sensible that we might be able to piece together a good equivariant theory. Indeed, at least three authors—namely Halvard Fausk, Gunnar Carlsson, and Clark Barwick—have made proposals for equivariant stable homotopy categories for compact Hausdorff groups and profinite groups, which are examples of projective limits of groups we already understand: finite and compact Lie groups.

In section 2 we review the theory of compact Hausdroff groups, explicitly relating them to compact Lie groups and profinite groups. In particular, we note that every compact Hausdorff group is a projective limit of compact Lie groups.

In section 3 we review the theory of t-structures and triangulated categories before introducing the theory of pro-objects, which are certain formal limits on general categories. It will be clear that our profinite and compact Hausdorff groups are examples of this phenomenon. We will use pro-categories to develop the notion of pro-G-spectra.

Sections 4 and 5 begin the discussion of the model we are most interested, that of Halvard Fausk. We determine technical conditions required for collections $\mathcal{W}$ of subgroups of G to determine a theory of weak equivalences. After finding such conditions we create model categories for our $\mathcal{W}$-equivalences. Such a theory makes sense for G-spaces, G-spectra, and the pro-objects in G-Spectra. Often we will extend the most obvious model structures which may be missing desirable properties or not fully general. As a result we end up with several similar model structures with different strengths and capabilities.

Finally in sections 6 and 7 we briefly present alternative models and discuss some limitations and advantages.

Before beginning in earnest, we note some conventions. All topological spaces are compactly generated and weakly Hausdorff. We denote the category of based spaces by $\mathcal{T}$ and the category of based G-spaces by $\mathcal{T}_G$.

2. Groups

In this section we discuss several well known results about compact Hausdorff groups. In particular we wish to motivate the simultaneous study of profinite groups and compact Hausdorff groups. The key result is:

Theorem 2.1 ([1] A.3)**.** *Every compact Hausdorff group is a limit of compact Lie groups.*

In order to prove this theorem we will first review some more elementary facts about compact Hausdorff groups. The results in this section can be found in section 11 of [1].

Definition 2.2. A *collection* of subgroups of a given group G is a subset of the set the subgroups of G closed under conjugation.

Definition 2.3. A collection of subgroups $\mathcal{W}$ is a *family* if $K < H$ and $H \in \mathcal{W}$ implies that $K \in \mathcal{W}$. To define a *cofamily* we reverse the containment $H < K$.

Definition 2.4. The *(co)family closure* of a collection $\mathcal{W}$ is the smallest collection $\mathcal{W} \subset \mathcal{W}'$ such that $\mathcal{W}'$ is a (co)family.

Definition 2.5. For a group G, define $\mathrm{Lie}(G)$ to be the collection of normal subgroups N such that G/N is compact Lie. Let $\mathfrak{L}_G$ (omitting the group where obvious) denote the cofamily closure of $\mathrm{Lie}(G)$.

Lemma 2.6 ([1] A.1)**.** *Let $p : G \to O(V)$ be a representation of a compact Hausdorff group G. Then the induced action of G on V factors through G/N for some $N \in Lie(G)$.*

Proof. The image of p is a compact subgroup of $O(V)$ and is therefore a closed subgroup. This ensures $p(G)$ with the subspace topology is a compact Lie subgroup of $O(V)$. The conditions on G ensure that $G/\ker p \cong p(G)$ is a homeomorphism. □

Lemma 2.7. *The action of a compact Hausdorff group G on its square-integrable functions $L^2(G)$ (with respect to the Haar measure) given by:*

$$g \cdot f := f \circ g^{-1}$$

for an L^2 function f is faithful and a direct sum of finite dimensional irreducible representations.[1]

Proof. The second half of the lemma is the Peter-Weyl theorem, see e.g. [2] theorem 3.39 for a proof. To see that this action is faithful, it suffices to show that only the identity acts trivially. Let $g \neq e$ and take some neighborhood $U \ni e$ in G such that gU and U are disjoint. If f is 1 near e and supported on U then we get $g \cdot f$ is 1 near g and 0 at e. □

We are now ready to prove Theorem 2.1. This result and the proof below can be found in a more general form in [1] as proposition A.2.

Proof. The previous result implies that the intersection over all Lie Groups in $\mathrm{Lie}(G)$ is trivial as there are enough finite dimensional representations to distinguish any two elements of G. For all such $N_1 < N_2 < G$ we have projections $G \to G/N_i$ compatible with $G/N_1 \to G/N_2$, so that the universal property of the limit induces:

$$G \xrightarrow{p} \lim_{N \in \mathrm{Lie}(G)} G/N$$

Because the intersection of the N is trivial this map is injective. Fix an element x of the limit. Let x_N be its image in G/N, which corresponds to the closed coset Nx_N. For any two $L, N \in \mathrm{Lie}(G)$ we know that $Nx_N \cap Lx_L$ contains $(N \cap L)x_{N\cap L}$. Hence the total intersection $\cap_N Nx_N$ is a point whose image under p is x.

It now suffices to show p is a closed map, i.e. that $N \cdot A$ is closed for all $N \in \mathrm{Lie}(G)$ and closed $A \subset G$. N is compact by assumption, so that if A were compact, $N \cdot A$ would be compact as the image of $N \times A$ under evaluation. The topology is compactly generated, so this suffices. □

3. Categorical Preliminaries

3.1. **Triangulated Categories.** We review the theory of triangulated categories briefly and describe the notion of a t-structure on a triangulated category.

Definition 3.1. A *triangulated category* $\mathcal{C}$ is an additive category equipped with an additive translation functor $T : \mathcal{C} \xrightarrow{\simeq} \mathcal{C}$ and a set of *distinguished triangles* (denoted $\mathfrak{T}$) of the form:

$$X \to Y \to Z \to TX$$

subject to the following:

- $\mathfrak{T}$ is closed under commutative diagrams of the form:

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & TX \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & TA \end{array}$$

- $X \xrightarrow{\mathrm{id}} X \to 0 \to TX$ is in $\mathfrak{T}$.

[1]The action need not be faithful on the components of this decomposition.

- Every map $X \to Y$ can be extended to a triangle in $\mathfrak{T}$ of the form:

$$X \to Y \to Z \to TX$$

- $X \xrightarrow{f} Y \to Z \to TX$ is in $\mathfrak{T}$ if and only if $Y \to Z \to TX \xrightarrow{-T(f)} TY$ is.
- Distinguished triangles satisfy an additional 'octahedral' axiom that amounts to ensuring an analogue of the third isomorphism theorem. An account of the axioms equivalent formulations can be found in [14].

Definition 3.2. A *t-structure* on $\mathcal{C}$ is a pair of subcategories $\mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\leq 0}$ closed under isomorphism satisfying:

- $\mathcal{C}_{\geq 0}$ is closed under T and $\mathcal{C}_{\leq 0}$ is closed under T^{-1}.
- For all $X \in \mathcal{C}_{\geq 0}$ and $Y \in \mathcal{C}_{\leq 0}$ we have:

$$\mathrm{Hom}(X, T^{-1}Y) = 0$$

- Every object A in $\mathcal{C}$ is part of a triangle of the form:

$$X \to A \to T^{-1}Y \to TX$$

With $X \in \mathcal{C}_{\geq 0}$ and $Y \in \mathcal{C}_{\leq 0}$.

Define the heart of the t-structure, denoted $\mathcal{C}_{\heartsuit}$, to be the intersection of $\mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\leq 0}$. We note that the heart is always abelian.

Definition 3.3. Because $\mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\leq 0}$ are closed under T, T^{-1}, respectively, we define $\mathcal{C}_{\geq n}$ to be $T^n\mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\leq n}$ similarly with respect to T^{-n}.

If we have a t-structure on the homotopy category of some model category, we can ask about the compatibility between them. This leads us to the notion of a t-model structure. First we review some model-categorical notions.

Definition 3.4. A model category $\mathcal{C}$ is *stable* if it has a zero object, its homotopy category has a triangulated structure, and the translation lifts to and endofunctor $\mathcal{C} \to \mathcal{C}$.

Definition 3.5. A model category is *proper* if pushouts along cofibrations and pullbacks along fibrations preserve weak equivalences.

Definition 3.6. A category is *simplicial* if it is enriched in the category of simplicial sets.

Definition 3.7. An *n-equivalence* in a triangulated category with t-structure is a map $f : X \to Y$ that extends to a distinguished triangle

$$F \to X \to Y \to TF$$

for some F in $\mathcal{C}_{\geq n}$. Similarly if the maps extend as:

$$X \to Y \to C \to TX$$

for some C in $\mathcal{C}_{\leq n}$, then the map is a co-n-equivalence.

Definition 3.8. If a stable model category $\mathcal{C}$ has a t-structure on its homotopy category, we define the n-equivalences of $\mathcal{C}$ to be those morphisms sent to n-equivalences in $h\mathcal{C}$. Co-n-equivalences are defined likewise.

Definition 3.9. Let $\mathcal{C}$ be a stable simplicial model category with a t-structure on its homotopy category.

This data assembles into a *t-model structure* if the model category has a functorial factorization of maps as n-equivalences followed by co-n-equivalences for all n. That is, all maps $f : X \to Y$ extend as:

$$\begin{array}{ccccccccc} F & \longrightarrow & X & \xrightarrow{f_1} & Z & \xrightarrow{f_2} & Y & \longrightarrow & C \\ & & & & \downarrow & & & & \downarrow \\ & & & & TF & & & & TZ \end{array}$$

such that:

- $f = f_2 f_1$
- $F \to X \to Z \to TF$ is distinguished.
- $Z \to Y \to C \to TZ$ is distinguished.
- $F \in \mathcal{C}_{\geq n}$ and $C \in \mathcal{C}_{\leq n}$.
- $f \mapsto (f_1, f_2)$ is functorial in the sense of [15] definition 1.1.1.

3.2. **Pro-Categories.** Profinite groups are limits on diagrams of finite groups. This notion makes sense far more generally by the introduction of pro-objects for suitable categories.[2] Our motivation is roughly as follows. If we can constuct some theory for objects in $\mathcal{C}$, it is intuitive to extend this theory to pro-objects by taking limits over the original constructions.

Definition 3.10. A category I will be called an indexing category if it is small and cofiltered.[3]

Definition 3.11. Let $\mathcal{C}$ be any category. A *pro-object* in $\mathcal{C}$ is a functor $I \to \mathcal{C}$ from an indexing category I. The category pro-$\mathcal{C}$ has as objects pro-objects.

Morphisms between objects $F_1 : I_1 \to \mathcal{C}$ and $F_2 : I_2 \to \mathcal{C}$ are given by:

$$\mathrm{Hom}_{\text{pro-}\mathcal{C}}(F_1, F_2) := \lim \mathrm{Hom}(F_1(-), F_2(-))$$

The limit is defined over the functor $I_1^{op} \times I_2 \to \mathrm{Set}$. Alternatively one can define the hom sets by:

$$\mathrm{Hom}_{\text{pro-}\mathcal{C}} = \lim_{y \in I_2} \operatorname*{colim}_{x \in I_1} \mathrm{Hom}(F_1(x), F_2(y))$$

The composition of the limits is the limit of the compositions. Explicitly a map in the pro-category $f : X \to Y$ is given by a sequence of maps $f_{x,y} : x \to y$ indexed over I_1, I_2. To compose such maps we pick the limit of the sequence $f_{x,y} \circ g_{y,z}$, noting that the commutativity conditions in the limit definition remove dependence on the choice of y.

We will often denote a pro-object X in pro-$\mathcal{C}$ by $\{X_s\}$ where s ranges over the indexing category.

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then composition on objects extends F to a functor pro-$\mathcal{C}$ $\to$pro-$\mathcal{D}$ by sending a pro-object $A : I \to \mathcal{C}$ to a pro-object $F \circ A : I \to \mathcal{D}$, which is a pro-object of $\mathcal{D}$. In an abuse of notation we will use the same name for a functor and its extension to the pro-category.

When two pro-objects X, Y share the same indexing category I, we can represent maps in $\mathrm{Hom}(X, Y)$ via natural transformations. In this specific context, it makes

[2] In this language, compact Hausdorff groups are pro-compact-Lie-groups.

[3] I.e., every finite diagram has a cone.

sense to discuss the levelwise properties of maps. This motivates the following definition:

Definition 3.12. A map of pro-objects $X \to Y$ is said to satisfy a property *essentially levelwise* if there are isomorphisms $\hat{X} \to X$ and $Y \to \hat{Y}$ so that $\hat{X}, \hat{Y}$ have the same indexing category and the induced map $\hat{X} \to \hat{Y}$ has this property levelwise.

Pro-categories of model categories naturally inherit a model structure. This model structure will be important for discussing pro-G-spectra in section 5.

Theorem 3.13 ([4] 4.15)**.** *Given a proper model category $\mathcal{C}$, there is a strict model structure on pro-$\mathcal{C}$ such that the weak equivalences are the essentially levelwise weak equivlances with respect to $\mathcal{C}$, and likewise for cofibrations.*

4. Orthogonal G-Spectra

We give a construction of Orthogonal G-Spectra, which coincides with the one given in [1]. The reader who has never seen this construction before is encouraged to refer to [11] chapter 2 section 2 and related material. The only differences between our construction and [11] is our definition of a universe and the usage of the internal, rather than external, smash product in the creation of structure maps.

We review some (enriched) categorical notions necessary to constructing orthogonal G-spectra.

Definition 4.1. A *topological G-category* is a category enriched in $\mathcal{T}_G$. A *continuous G-functor* between topological G-categories is a functor that is a G-map on the hom spaces. A *natural G-map* of continuous G-functors $\mathcal{C} \to \mathcal{D}$ is a natural transformation whose component maps are G-maps.

Remark 4.2. We think of all objects as G-fixed. Nothing in the definition of a topological G-category implies that the morphisms are equivariant. In fact, the equivariant maps are the fixed points under the G-action on hom spaces. We will make use of both these G-categories considered with and without nonequivariant maps.

Definition 4.3. Let $\mathcal{C}$ be a topological G-category. Then the *fixed-point category*, denoted $G\mathcal{C}$, is the subcategory with all objects of $\mathcal{C}$ and for hom spaces the G-fixed points under morphism conjugation.

One can safely think of $G\mathcal{C}$ as the equivariant subcategory of $\mathcal{C}$.

Definition 4.4. Let G be a topological group. By a *G-universe* $\mathcal{U}$ we mean a countable sum $\oplus_{i=1}^{\infty}\mathcal{U}'$ of a G-inner-product space $\mathcal{U}'$

- $\mathbb{R}$ with the trivial action is a G-subspace of $\mathcal{U}'$.
- $\mathcal{U}$ is topologized as the union of its finite subspaces.

By a trivial universe we mean $\mathbb{R}^{\infty}$ with the trivial action, and by a complete universe we mean a universe containing every finite dimensional G-representation. In [1] the author requires that the action on a finite dimensional G-subspace factor through a compact Lie quotient of G, however, from here we require that G be compact Hausdorff, which ensures this. See theorem 2.6. Traditionally one asks that a Universe have countable dimension, however, the subsequent theorem fails in this case.

Lemma 4.5 ([1] Remark following 3.3)**.** *A compact Hausdorff group G has a complete universe.*

We will build a category of orthogonal G-spectra with respect to an indexing universe. Fix such a universe $\mathcal{U}$ and let $\mathcal{V}(\mathcal{U})$ be the collection of finite dimensional G-subspaces. Periodically we will add assumptions to $\mathcal{U}$ for various results, but it will be clear when we do so.

Definition 4.6. Let $\mathcal{J}$ (with respect to $G, \mathcal{U}$) be the category with objects $\mathcal{V}(\mathcal{U})$ and morphisms the (nonequivariant) linear isometric isomorphisms between them. The category is topologically enriched with $\mathrm{Hom}_{\mathcal{J}}(V, W)$ topologized as a subspace of $\mathrm{Maps}(V, W)$.

Then G acts on these hom-spaces by conjugation.

Definition 4.7. A $\mathcal{J}$-space is a continuous G-functor $\mathcal{J} \to \mathcal{T}_G$. Denote the category of $\mathcal{J}$-spaces by $\mathcal{J}\mathcal{T}$. Then for two $\mathcal{J}$-spaces X, Y we define $\mathrm{Hom}(X, Y)$ to be the (nonequivariant) natural transformations. Then

$$\mathrm{Hom}(X, Y) \subset \prod_V \mathrm{Hom}(X(V), Y(V))$$

gives the hom sets a subspace topology. G acts by conjugation index-wise. We note that the morphisms of $G\mathcal{J}\mathcal{T}$ are exactly the natural G-maps.

Definition 4.8. The category of $\mathcal{J}$-spaces has an *external smash product*:

$$\overline{\wedge} : \mathcal{J}\mathcal{T} \times \mathcal{J}\mathcal{T} \to (\mathcal{J} \times \mathcal{J})\mathcal{T}$$

where $(\mathcal{J} \times \mathcal{J})\mathcal{T}$ is the category of continuous G-functors from $\mathcal{J} \times \mathcal{J}$ to $\mathcal{T}_G$. For $\mathcal{J}$-spaces X, Y this product is defined by:

$$X \overline{\wedge} Y(V, W) := X(V) \wedge Y(W)$$

Maps are sent to their wedge product as well. This product can be internalized by a left Kan extension.

$$\begin{array}{ccc} \mathcal{J} \times \mathcal{J} & \xrightarrow{\oplus} & \mathcal{J} \\ {\scriptstyle X \overline{\wedge} Y}\downarrow & \swarrow {\scriptstyle X \wedge Y} & \\ \mathcal{T}_G & & \end{array}$$

An explicit description is given as definition 3.8 in [1]. This internalized product is called the *smash product* and is denoted $X \wedge Y$.

Definition 4.9. Let $\mathcal{S}$ be the $\mathcal{J}$-space given by $\mathcal{S}(V) := S^V$, the one point compactification of V, with $\mathcal{S}(V \to W)$ given by sending points at infinity to eachother in $S^V \to S^W$.

With respect to this product the $\mathcal{J}$-space $\mathcal{S}$ is a commutative monoid (where $\mathcal{S} = \mathcal{S}$ it understood to be the unit). This allows us to discuss $\mathcal{S}$-modules, by which we mean $\mathcal{S}$-objects in the category of $\mathcal{J}$-spaces.

Definition 4.10. An *orthogonal G-spectrum* is a right $G\mathcal{J}\mathcal{T}$-module over $\mathcal{S}$. The category of orthogonal G-Spectra is the subcategory of $G\mathcal{J}\mathcal{T}$ spanned by these objects and morphisms of $\mathcal{S}$-modules, i.e. maps that commute with the $\mathcal{S}$ action.

We note that the natural transformation in the definition of a left Kan extension, combined with an $\mathcal{S}$-module structure, grants us structure maps:

$$X(W) \wedge S^V \to X(W \oplus V)$$

with adjoints:

$$X(W) \to \Omega^V X(W \oplus V)$$

Definition 4.11. Let X be any G-space. Then we can associate a *suspension spectrum* to X as follows:

$$\Sigma^\infty X(W) := X \wedge S^W$$

Further if V is an indexing representation we can define the Vth suspension spectrum as:

$$\Sigma_V^\infty X(W) := \begin{cases} X \wedge S^W & V \subset W \\ * & \text{Otherwise} \end{cases}$$

4.1. **Model Structures.** The details of this model structure are proven in [1] sections 3, 4, and 5. We will state the results as needed for the remainder of the paper, but will not prove any model category axioms. We do not review the basic theory of model categories, the unfamiliar reader is encouraged to see [15].

4.1.1. *Unstable Model Structure.* Our model structures on the category of G-Spectra will be induced by a model structure on G-Spaces. We quickly review the latter.

In a natural generalization from the non-equivariant settings, If X is a G-Space, we can define:

$$\pi_n^H(X) := \pi_n(X^H)$$

Where the right hand side is the standard homotopy group of the H-fixed space.

Definition 4.12. Let $\mathcal{W}$ be a collection of G subgroups. A $\mathcal{W}$*-equivalence* of G-spaces $X \to Y$ is a G-Map whose underlying maps on fixed-point spaces $X^H \to Y^H$ induce isomorphisms $\pi_n^H(X) \cong \pi_n^H(Y)$ for all $H \in \mathcal{W}$ and $n \geq 0$.

The following classes of maps, with respect to some collection $\mathcal{W}$ of subgroups, will be our generating cofibrations:

$$\mathcal{W}\mathcal{I} := \{(G/H \times S^n)_+ \to (G/H \times D^n)_+ \mid H \in \mathcal{W}\}$$

Where the actions on S^n, D^n, and $[0,1]$ are all trivial. These maps are given by $\mathrm{id} \times i$ and $\mathrm{id} \times i_0$ respectively, where i is the boundary inclusion and i_0 is inclusion at the base.

Definition 4.13. Given a class $\mathcal{I}$ of maps, a *relative $\mathcal{I}$-cell complex* is a composition of pushouts of maps in $\mathcal{I}$. An $\mathcal{I}$-complex is the codomain of a relative $\mathcal{I}$-cell complex with domain the initial object.

Theorem 4.14 ([1] 2.11)**.** *There is a proper model structure on $G\mathcal{T}$ such that:*

- *The weak equivalences are $\mathcal{W}$-equivalences*
- *$X \to Y$ is a fibration if $X^H \to Y^H$ is a Serre fibration for all $H \in \mathcal{W}$.*
- *The cofibrations are retracts of relative $\mathcal{W}\mathcal{I}$-complexes.*

4.1.2. *Strict Stable Model Structure.* The unstable model structure can be used to construct a model structure on the stable category. First we define the stable equivariant homotopy groups for orthogonal spectra. For an orthogonal G-Spectrum X they are:

$$\pi_n^H(X) := \mathrm{colim}_V\, \pi_n^H(\Omega^V X(V))$$

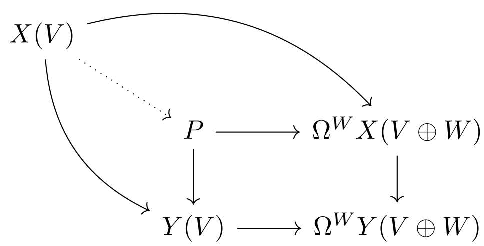
$$\pi_{-n}^H(X) := \mathrm{colim}_{\mathbb{R}^n \subset V}\, \pi_0(\Omega^{V-\mathbb{R}^n} X(V))$$

For $n \geq 0$. If a map of G-Spectra induces isomorphisms on all stable homotopy groups with respect to a collection $\mathcal{W}$, we say that it is a stable $\mathcal{W}$-equivalence.

We want to develop a model structure for general collections of subgroups $\mathcal{W}$ on the stable model category. We give the following technical definition which will allow us to develop such a theory.

Definition 4.15. Given two collections of subgroups $\mathcal{W}$ and $\mathcal{E}$, we say that $\mathcal{E}$ is $\mathcal{W}$-*Illman* if $(G/H \times G/U)_+$ is a $\mathcal{W}\mathcal{I}$-complex for all $H \in \mathcal{W}$ and $U \in \mathcal{E}$. If $\mathcal{E} = \mathcal{W}$ and the condition holds we say that $\mathcal{W}$ is *Illman*.

Before giving a characterization of the model structure, we note that a map of G-Spectra $X \to Y$ induces:



Where P is the pullback and $X(V) \to P$ is induced by its universal property.

Denote by $st(\mathcal{U})$ the collection of stabilizers of elements of a given G-universe $\mathcal{U}$.

Theorem 4.16 ([1] 4.4)**.** *Let $\mathcal{W}$ be a collection such that $st(\mathcal{U})$ is $\mathcal{W}$-Illman. Then there exists a compactly generated proper model structure on the orthogonal G-spectra such that:*

- *The weak equivalences are the stable $\mathcal{W}$-equivalences.*
- *The fibrations are those level-wise fibrations of G-spaces whose induced maps $X(V) \to P$ are weak equivalences.*
- *The cofibrations are retracts of $\Sigma^\infty \mathcal{W}\mathcal{I}$-complexes.*

We will refer to this model structure as the $\mathcal{W}$-model structure.

For utility we now review a few cases in which the hypotheses for theorem 4.16 are satisfied. They can be found as exposition following definition 4.1 in [1].

Theorem 4.17. *If $\mathcal{U}$ is a trivial universe then $st(\mathcal{U})$ is always $\mathcal{W}$-Illman for any collection of closed subgroups.*

Theorem 4.18. *If $\mathcal{U}$ is a complete universe then for any family $\mathcal{W}$ of subgroups in $\mathfrak{L}_G$, we have that $st(\mathcal{U})$ is $\mathcal{W}$-Illman.*

Theorem 4.19. *If G is in $\mathcal{W}$ then $st(\mathcal{U})$ is $\mathcal{W}$-Illman if and only if $st(\mathcal{U}) \subset \mathcal{W}$.*

Going forward we will fix a universe $\mathcal{U}$ and let $\mathcal{W}$ be a collection of subgroups with $st(\mathcal{U})$ being $\mathcal{W}$-Illman. We now introduce a more general model structure in which one collection $\mathcal{W}$ is used to detect weak equivalences, and another, $\mathcal{C}$ generates the cofibrations.

Definition 4.20. Suppose that K is a closed subgroup of G such that the closure $\overline{HK}$ is a subgroup in $\mathcal{C}$ whenever H is. We define the *H-homotopy groups* by:

$$\Pi_*^K X := \operatorname{colim}_{H \in \mathcal{C}} \pi_*^{\overline{HK}} X$$

Where the morphisms in the colimit are inclusions. The colimit is directed so long as $\mathcal{W}$ is Illman. For $K \in \mathcal{C}$ there is a canonical isomorphism $\Pi_*^K \cong \pi_*^K$.

We denote by $\mathcal{CW}$ the collection of $\overline{HK}$ for $H \in \mathcal{C}, K \in \mathcal{W}$. If $\mathcal{CW} \subset \mathcal{C}$, which always holds if $\mathcal{C}$ is a cofamily, we call a map of G-Spectra a $\mathcal{W}$-equivalence if it induces isomorphisms on all Π_*^K with $K \in \mathcal{W}$.

Theorem 4.21 ([1] 5.4)**.** *Let $\mathcal{C}$ be an Illman collection containing $st(\mathcal{U})$. Let $\mathcal{W}$ be such that $\mathcal{CW} \subset \mathcal{C}$. Then there is a cofibrantly generated proper simplicial model structure on G-Spectra such that:*

- *The weak equivalences are $\mathcal{W}$-equivalences.*
- *The cofibrations are retracts of $\mathcal{C}$-complexes.*

4.1.3. *Postnikov Stable Model Structure.* We wish to refine the $\mathcal{W} - \mathcal{C}$ model structure so that it satisfies the axioms of a t-model category. We will denote the category of orthogonal G-Spectra with $\mathcal{SP}_G$ and its homotopy category $h\mathcal{SP}_G$. We must first define the triangulation $h\mathcal{SP}_G$ and its t-structure. The details of this construction can be found in section 8.2 of [1].

Construction 4.22. Let $h\mathcal{D}$ be the homotopy category of some proper simplicial stable model category. Let $D_{\geq 0}$ be a strictly full subcategory closed under suspension. Put $D_{\geq n} := \Sigma^n D_{\geq 0}$. Let W_n be the n-equivalences, lifted to $\mathcal{D}$. Let cof be the cofibrations in $\mathcal{D}$ and put $\mathrm{cof}_n := \mathrm{cof} \cap W_n$. Let F_n be those maps with right lifting against cof_n. Denote by $D_{\leq n-1}$ the full subcategory spanned by the homotopy fibers of the maps in F_n.

Lemma 4.23. *Under these assumptions on $\mathcal{D}$ and $h\mathcal{D}$, the construction above gives a t-model structure on $\mathcal{D}$.*

Theorem 4.24. *$\mathcal{SP}_G$ has a t-model structure.*

Proof. Apply construction 4.22 to the spectra with trivial negative stable homotopy groups, i.e., the connective spectra. □

From here we will denote by $\mathcal{SP}_G$ the $\mathcal{W} - \mathcal{C}$ model structure on orthogonal G spectra equipped with this factorization. This factorization induces a functorial Postnikov system on $\mathcal{SP}_G$.

4.2. **Coefficient Systems and EM Spectra.** We describe coefficient systems for the $\mathcal{W} - \mathcal{C}$ model structure and their equivalence to Eilenberg Mac Lane Spectra. Denote, abusively, by G/H_+ the spectrum $\Sigma^\infty G/H_+$.

Definition 4.25. A G-spectrum X is an *Eilenberg Mac Lane spectrum* if $\Pi_n^H X$ is trivial for all $H \in \mathcal{W}$ and $n \neq 0$.

Lemma 4.26 ([1] 8.22)**.** *The heart of the t-structure on $h\mathcal{SP}_G$ is a subcategory of the full homotopy subcategory of Eilenberg Mac Lane Spectra. If $\mathcal{W} \subset \mathcal{C}$ then we get all Eilenberg Mac Lane spectra.*

We wish to have some equivalence between Eilenberg Mac Lane spectra and some suitably defined categories of coefficient systems.

Definition 4.27. Denote by $\mathcal{O}$ the full subcategory of $\mathcal{SP}_G$ spanned by the objects G/H_+ for $H \in \mathcal{W}$.

Definition 4.28. A coefficient system is a contravariant additive functor of the form $\mathcal{O} \to \text{Ab}$. Denote this functor category as $\mathcal{G}$. An object Y in $h\mathcal{SP}_G$ naturally gives a coefficient system via:

$$G/H_+ \mapsto [G/H_+, Y]$$

In particular, if X is spectrum, let $\pi_n^{\mathcal{W}} X$ be the functor represented by $X \wedge \Sigma^\infty_{\mathbb{R}^n} S^0$.

Theorem 4.29 ([1] 8.27)**.** *The functor which takes a spectrum X to $\pi_0^{\mathcal{W}} X$ induces an equivalence from the Eilenberg Mac Lane spectra to $\mathcal{G}$.*

Before introducing pro-G-spectra, we wish to give a brief overview some applications of Fausk's model structures on G-spectra when G is a pro-group.

4.3. **Example: Rational $\mathbb{Z}_p$-Spectra.** In [10] David Barnes uses this theory of profinite equivariant spectra to create a algebraic model for the category of rationalized $\mathbb{Z}_p$-spectra. We give their basic setup and note some key results obtained in this paper.

Theorem 4.30. *There exists a complete $\mathbb{Z}_p$-universe $\mathcal{U}$. Let $\mathcal{O}$ be the collection of open $\mathbb{Z}_p$ subgroups. Then by theorem 4.14 we have a model structure on orthogonal $\mathbb{Z}_p$-spaces with respect to $\mathcal{O}$. Further, this extends to an $\mathcal{O}$-model structure on $\mathbb{Z}_p$-spectra via theorem 4.16.*

Proof. The universe $\mathcal{U}$ can be constructed as a direct sum of irreducible $\mathbb{Z}_p$ representations. The second statement is immediate from the cited theorem. The final statement follows from theorem 4.18 by noting that an open subgroup of $\mathbb{Z}_p$ has finite index and therefore $\mathcal{O}$ is a family in $\text{Lie}(\mathbb{Z}_p)$. □

We denote by $\mathcal{SP}_{\mathbb{Z}_p}$ the model category described above. There is a localization corresponding to the inclusion of $H\mathbb{Q}$-module spectra that gives a rationalization functor and allows us to restrict attention to the full subcatgory $\mathbb{Q}\mathcal{SP}_{\mathbb{Z}_p}$ of rational $\mathbb{Z}_p$-spectra.

Definition 4.31. The algebraic category $\mathcal{A}(\mathbb{Z}_p)$ is defined as follows. An object of $\mathcal{A}(\mathbb{Z}_p)$ is for all k a $\mathbb{Q}[\mathbb{Z}_p/p^k\mathbb{Z}_p]$-module M_k and a $\mathbb{Q}[\mathbb{Z}_p]$-module M_∞ equipped with structure maps $\sigma : M_\infty \to \text{colim}_n \prod_{k \geq n} M_k$. Morphisms are defined levelwise and must commute with the structure maps.

We can then take the category $dg\mathcal{A}(\mathbb{Z}_p)$ of differential graded objects in $\mathcal{A}(\mathbb{Z}_p)$. Barnes gives this category a model structure and proves that it is Quillen equivalent to $\mathbb{Q}\mathcal{SP}_{\mathbb{Z}_p}$.

Theorem 4.32 ([10] 4.7 and 7.1)**.** *There is a proper model structure $dg\mathcal{A}(\mathbb{Z}_p)$ whose weak equivalences are the homology isomorphisms and whose fibrations are those maps that are levelwise surjections. Moreover, there is a zig-zag of Quillen equivalences from $dg\mathcal{A}(\mathbb{Z}_p)$ to $\mathbb{Q}\mathcal{SP}_{\mathbb{Z}_p}$.*

The Quillen equivalence is proven via a zig-zag of equivalences using results developed in [10] and [12]. Because $\mathbb{Q}\mathcal{SP}_{\mathbb{Z}_p}$ does not have a monoidal fibrant replacement functor, the authors do not prove a monoidal Quillen equivalence, however, they conjecture that it holds. To see some of the added computational power of this algebraic model, we define an invariant on $h\mathbb{Q}\mathcal{SP}_{\mathbb{Z}_p}$ which is a graded object in $\mathcal{A}(\mathbb{Z}_p)$ as follows:

Definition 4.33. Let X be an object in $\mathcal{SP}_{\mathbb{Z}_p}$. Let $\pi_*^i X$ denote the rationalization of the homotopy groups $\pi_*^{p^i\mathbb{Z}_p} X$. Then we define $(\varpi_* X)_n := e_n \pi_*^n X$ and $(\varpi_* X)_\infty := \operatorname{colim}_n e_n \pi_*^n X$. The structure map is given by:

$$\operatorname{colim}_k \left(\pi_*^k X \to \prod_{m \geq k} e_m \pi_*^m X \right)$$

Details of the maps above can be found as definitions 3.2, 6.1, and related discussion in [10].[4]

Further, this algebraic model category give us the following calculational tool:

Theorem 4.34 ([10] 6.5)**.** *Let X, Y be objects in $\mathbb{Q}\mathcal{SP}_{\mathbb{Z}_p}$. Then there is a short exact sequence:*

$$0 \to \operatorname{Ext}_*(\varpi_* \Sigma X, \varpi_* Y)) \to [X, Y] \to \operatorname{Hom}(\varpi_* X, \varpi_* Y) \to 0$$

Where Ext and Hom are taken in $\mathcal{A}(\mathbb{Z}_p)$ and $[X, Y]$ is taken in $h\mathbb{Q}\mathcal{SP}_{\mathbb{Z}_p}$.

This computation tool is essential in proving theorem 4.32; however, the intermediate categories must be carefully defined so that giving the details would be a lengthy digression for our purposes. These details and other interesting results about this algebraic model can be found in [10].

5. Pro-G-Spectra

The category of pro-G-Spectra is, as the name suggests, the pro-category on orthogonal G-Spectra in the sense of §3.2. Our particular interest in this setting is the study of homotopy fixed points, for which we provide several examples from the literature. There is an obvious candidate for a model structure on this category, achieved by applying theorem 3.13 to the $\mathcal{W} - \mathcal{C}$ model structure introduced by theorem 4.21.

Definition 5.1. We will refer to the model structure on pro-G-spectra obtained via theorem 3.13 from the $\mathcal{W} - \mathcal{C}$ model structure of theorem 4.21 as the *strict $\mathcal{W} - \mathcal{C}$ model structure*, omitting $\mathcal{W}, \mathcal{C}$ where there is no risk of confusion.

We will introduce a refinement of this model structure.

5.1. **Postnikov Model Structure.**

The Postnikov system on $\mathcal{SP}_G$ extends as functors P_n to the pro-category. However, if we use the strict model structure, the map $X \to \operatorname{holim}_n P_n X$ is not in general a weak equivalence. To solve this problem, we add more weak equivalences. To be exact, we have the following structure:

Theorem 5.2 ([1] 9.4)**.** *Let $\mathcal{C}$ be $\mathcal{U}$-Illman and $\mathcal{CW} \subset \mathcal{C}$. There exists a proper simplicial stable model structure on pro-$\mathcal{SP}_G$ with the following:*

[4] We make changes to the notation of [10] here in order to avoid any confusion. Our ϖ_* corresponds to their $\underline{\pi}_*^{\mathcal{A}}$.

- *The cofibrations are retracts of $\mathcal{C}$-cell complexes.*
- *The weak equivalences are those maps that are n-equivalences for all n.*[5]

Let X, Y be pro-G-spectra indexed on categories S, T. The category pro-$\mathcal{SP}_G$ has a smash product, however, it is not as well behaved as the product in $\mathcal{SP}_G$. It is given by:

$$X \wedge Y := \{X_s \wedge Y_t\}_{s \times t \in S \times T}$$

In particular, pro-$\mathcal{SP}_G$ is not closed monoidal in general (see the discussion following definition 9.12 in [1]). In fact, the product does not commute with direct sums in general. A counterexample in a more abstract setting can be found as example 11.2 of [5].

Similarly one can try to extend the internal hom functor F on $\mathcal{SP}_G$ to pro-$\mathcal{SP}_G$ by putting:

$$F(X, Y) := \{\mathrm{colim}_{s \in S} F(X_s, Y_t)\}_{t \in T}$$

But this functor does not satisfy the requirements of being an internal hom functor for pro-$\mathcal{SP}_G$. With additional assumptions on X, Y, it can, however, act like an internal hom functor up to homotopy. See [1] proposition 9.15.

5.2. **Homotopy Fixed Points.** We wish to develop a theory of homotopy fixed points for $\mathcal{SP}_G$. From here we specialize $\mathcal{U}$ to be a trivial universe, $\mathcal{W}$ to be the trivial group, and $\mathcal{C}$ to be $\mathfrak{L}$. We will specialize to pro-$\mathcal{SP}_G$ with the Postnikov model structure with respect to this choice of $\mathcal{C}$ and $\mathcal{W}$.[6]

Definition 5.3. For a pro-G-spectrum X define the *homotopy fixed points* X^{hG} to be the levelwise G fixed points of a fibrant replacement of X. A choice of functorial fibrant replacement yields a homotopy fixed point functor.

In analogy with the standard homotopy fixed points for a G-Space it follows that

$$X^{hG} \simeq \{\mathrm{holim}_N F(\Sigma^\infty EG/N_+, X_s)\}$$

where the N range over Lie(G). We wish also to define the homotopy fixed points for a closed subgroup of G.

Definition 5.4. Let X be a pro-G-spectrum with a fibrant replacement X_f. Then the H *homotopy fixed points pro-G-spectrum* $X^{h_G H}$ for a closed subgroup $H < G$ is defined to be:

$$X^{h_G H} := \mathrm{holim}_N \{X_f^{HN}\}$$

Where the colimit is over $N \in \mathrm{Lie}(G)$. Note that in the case $H = G$ we get the standard definition.

Then $X^{h_G H}$ has an action of the Weyl group of H. If H itself is in $\mathfrak{L}$ then $X^{h_G H} \to X_f^H$ is weak equivalence in the strict model structure for pro-Weyl(H)-spectra.

Theorem 5.5 ([1] 9.23)**.** *For a pro-G-spectrum X there is a spectral sequence with:*

$$E_2^{p,q} = H^p(G, \Pi^1_{-q}(X))$$

Converging conditionally to $\pi_{-p-q}(X^{hG})$. This spectral sequence is multiplicative for homotopy monoids in the strict pro-model-category.

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[5]This property is detected by $\mathcal{W}$ due to construction 4.22 and theorem 4.24.

[6]This particular setup is described by Fausk as the $\mathcal{C}$-cofree model structure.

5.3. **Example: Profinite Galois Extensions.** Independently of Fausk, Mark Behrens and Daniel Davis develop in [13] a theory of pro-G-Spectra when for a slightly more restricted class of groups. As Fausk points out in [1] §11.3, their theory of homotopy fixed points for pro-G-spectra corresponds to ours up to strict-pro-$\mathcal{SP}_G$ equivalence. Moreover, their pro-G-spectra roughly coincide with the cofibrant objects of the $\mathfrak{L}$-Postnikov model structure on pro-$\mathcal{SP}_G$. Their alternate construction is based on Bousfield-Friedlander Spectra.

The authors define k-local profinite G-Galois extensions of an E_∞-ring spectrum produce their version of pro-G-spectra. This machinery is then used to prove that if E is such a spectrum associated G-Galois extension of A and N is a closed normal subgroup, then the associated fixed-point spectra are G/N-Galois extensions of A. This is result 7.2.1 in [13].

6. Carlsson's Model

In [3] Gunnar Carlsson puts forth an alternative notion of G-spectrum for profinite G. In fact, his construction generalizes to the compact Hausdorff case as well with a small modification, given below.

Definition 6.1. Let G be compact Hausdorff represented by the pro-object constructed in the proof of theorem 2.1. Then for the subgroups N indexing the limit G/N is compact Lie, for which we already have a good category of G/N-Spectra.

Given two subgroups $N_1 < N_2 < G$, denote by f the projection $G/N_1 \to G/N_2$ and f^* the pullback functor. We can then define a *Carlsson G-Spectrum* X to be a choice of G/N-spectrum X_N for each N indexing the limit, equipped with isomorphisms $X_{N_1} \to f^* X_{N_2}$ whenever $N_1 < N_2$.[7]

The notion of morphism is the immediate one.

7. Barwick's Model

A third model is described by Clark Barwick in [6]. It is an ∞-categorification of a 2-categorical theory developed by Guillou and May in [9]. We will give the construction of this model. Unlike our previous constructions, however, this model will involve higher category theory. We refrain from providing an introduction to the subject and instead suggest the unfamiliar reader refer to [16] which approaches the subject through the same lens as is used in the following.

By ∞-category we mean a weak $(\infty, 1)$-category. For this model the basic input is a disjunctive triple of ∞-categories. The motivation to keep in mind with respect to earlier work is that we want our morphisms $A \to C$ to eventually be spans $A \leftarrow B \to C$, but we want to be careful about which side an arrow is allowed to be on. This is the data provided by the following definition.

Definition 7.1. A *disjunctive triple* is an ∞-category $\mathcal{C}$ with finite coproducts, an *ingressive* subcategory $C_\dagger$ and an *egressive* subcategory $C^\dagger$. Additionally the in/egressive categories must be stable under pullbacks.

Definition 7.2. A category is *disjunctive* if $\mathcal{C}, \mathcal{C}, \mathcal{C}$ forms a disjunctive triple.

Taking inspiration from one of many standard constructions of Mackey Functors, we define an (effective) Burnside category for $\mathcal{C}$.

[7] The only modification we make to the original definition in [3] is that we do not require that G/N be finite, but instead compact Lie

Definition 7.3. The *effective Burnside category* of $\mathcal{C}, \mathcal{C}^\dagger, \mathcal{C}_\dagger$ has the same objects as $\mathcal{C}$. A morphism $X \to Y$ is an object Z with maps:

$$X \leftarrow Z \to Y$$

Where the first arrow is egressive and the second is ingressive. Composition is given by pullbacks and is only defined up to contractible choice.

The term effective Burnside category is used because in general our category is only enriched in monoids and the traditional Burnside category is constructed by taking a local group completion. Our functors are valued in additive categories, however, so this is not a problem . In constructing Mackey functors, we will need to use the fact that this category has a direct sum coming from the coproduct in C.

We now describe sufficient technical conditions on topological groups G required to use disjunctive triples for a stable equivariant homotopy theory.

Definition 7.4. A topological group G is *coherent* if, for a fixed open subgroup $H$of G, there are only finitely many distinct double cosets HgH.

The original paper notes that this condition holds for profinite groups, but we further observe that it applies to compact Hausdorff groups as well: $\{HgH\}_{g\in G}$ is an open cover and the double cosets are disjoint or the same.

Definition 7.5. For a topological group G define B_G to be the full subcategory of G-spaces with discrete topologies. Let B_G^c be the subcategory generated by coherent objects.[8]

The category B_G is a 1-topos. If G is coherent B_G is the nerve of a coherent topos. We refrain from providing an introduction to (higher) topos theory as it would be a lengthy digression. The interested reader should see a reference such as [17], where specific discussion of the topos B_G can be found in section III.9. The important takeaway is that for a coherent topological group (such as a compact Hausdorff group), we can talk about Mackey functors associated to B_G^c

Theorem 7.6 ([6] B.2)**.** *If G is coherent then B_G^c has a natural disjunctive structure.*

Let $A(G)$ be the effective Burnside category for B_G^c. We can now define spectral Mackey functors for G.[9]

Definition 7.7. A *spectral G-Mackey functor* is a coproduct preserving additive ∞-functor:

$$A(G) \to \mathcal{SP}^\infty$$

where the target is the stable infinity category of (nonequivariant) Spectra.

The category of such functors is the model of stable equivariant G-Spectra. We will denote it with $\mathcal{SP}_G^\infty$. It is somewhat technical to construct, but in [7] a symmetric monoidal product is constructed for $\mathcal{SP}_G^\infty$. In the spirit of [9], the

[8] Barwick uses the notation B_G^{fin} because the coherent objects for profinite G are the G-sets with finitely many orbits.

[9] When G is finite $A(G)$ is the 2-category described in [9] and its homotopy category is the genuine effective Burnside category. For more comparisons to the classical theory, see [6] section B.6.

authors prove that when G is finite there is an equivalence between $\mathcal{SP}_G^\infty$ and an appropriate higher analogue of orthogonal G spectra, although in this paper the equivalence is not monoidal.

References

[1] Fausk, Halvard. *Equivariant Homotopy Theory for Pro–spectra.*
[2] Adams, J.F. *Lectures on Lie Groups.*
[3] Carlsson, Gunnar. *Derived Representation theory and the algebraic K-theory of fields.*
[4] Isaksen, D.C. *Strict model Structures for Pro-categories.*
[5] Fausk, Halvard and Isaksen, D.C. *t-Model Categories.*
[6] Barwick, Clark. *Spectral Mackey functors and equivariant algebraic K-theory (I).*
[7] —. *Spectral Mackey functors and equivariant algebraic K-theory (II).*
[8] —. *Parametrized Higher Category Theory and Higher Algebra.*
[9] Guillou, Bert and May, Peter. *Models of G-Spectra as Presheaves of Spectra.*
[10] Barnes, David. *Rational* $\mathbb{Z}_p$*-Equivariant Spectra.*
[11] Mandel, Michael and May, Peter. *Equivariant Orthogonal Spectra and S-Modules.*
[12] Schwede, Stefan and Shipley, Brooke. *Stable Model Categories are Categories of Modules.*
[13] Behrens, Mark and Davis, Daniel. *Homotopy Fixed Point Spectra of Profinite Galois Extensions.*
[14] Hubrey, *Notes on the Octahedral Axiom.*
[15] Hovey, *Model Categories.*
[16] Lurie, Jacob. *Higher Topos Theory.*
[17] Mac Lane, Sanders. *Sheaves in Geometry and Logic.*