BASIC SCHUBERT CALCULUS WITHOUT COHOMOLOGY

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ABSTRACT. In this paper, we will discuss one of the founding questions in Schubert calculus and enumerate geometry: how many lines intersect four given lines in general position. Instead of the standard approach using cohomology, this investigation will use some differential topology and algebraic geometry to discuss this problem for both real and complex base fields. The main goal will be to give a lower bound for the number of intersecting lines. This method will give a sharp lower bound of 1 for the complex case, but will, unfortunately, be inconclusive in the real case and the differences are discussed. In the conclusion there is also a brief discussion of corollaries of our investigation to give the sharp upper bound of 2 in both real and complex cases.

CONTENTS

1. THE PROBLEM STATEMENT AND SOME ALGEBRAIC GEOMETRY

One of the founding questions that brought about Schubert calculus is the question of how many lines intersect four given lines in general position in three dimensions. Before we can start thinking about how to answer this question we need to be careful about the formulation. When can we even count the number of lines and how can the number change? The term "general position" is intended to make this simpler. One case, which we wish to eliminate, is when we have too many lines and can no longer count them. For example, when three lines have a common intersection point there are an uncountable number of lines that intersect all four. Another problematic case is when any three lines are parallel in different planes; no line intersects all three, regardless of the fourth. One of the first steps is to collapse

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these two cases into the same case. We do this by moving to projective 3-space instead of affine 3-space. Sets of parallel lines intersect at infinity, so when three lines are mutually parallel they have a common intersection and these two cases are identical. The work done in this paper is primarily to find a lower bound for the number of lines, as such the exact definition of "general position" is not vital to the proof, since otherwise there will be infinite intersections and not change the lower bound.

1.1. **Projective Geometry.** Given that our problem lies in projective space, we should give a brief introduction to projective geometry and some of the objects which we will be working with during our investigation. First, we should define a projective space. The most intuitive definition is that projective space is affine space, but instead, there are added points "at infinity" where any two parallel lines intersect. The set of points "at infinity" is isomorphic to projective space of one dimension lower. For our investigation, we will desire a more precise definition.

Definition 1.1 (Projective Space)**.** Given a vector space V we define $\mathbb{P}V$ to be the set of equivalence classes $V - \{0\}/\sim$ with two points $x \sim y$ if and only if there is a λ in the base field such that if $x = \lambda y$.

This can be thought of as the set of lines in V which pass through the origin. Some projective spaces are used very often and I will use some common notation where $\mathbb{P}^n$ refers to a generic n-dimensional projective space ($\mathbb{P}V \cong \mathbb{P}^n$ if and only if $\dim(V) = n+1$). When dealing with real and complex projective spaces I will use $\mathbb{RP}^n = \mathbb{P}(\mathbb{R}^{n+1})$ and similarly $\mathbb{CP}^n = \mathbb{P}(\mathbb{C}^{n+1})$. Sometimes we will look at a subspace of projective space where one coordinate is set to 1 (i.e. must be non-zero then is multiplied by an appropriate λ). This is called affine space and is denoted by $\mathbb{A}^n$. When the base field is $\mathbb{R}$ or $\mathbb{C}$ then $\mathbb{A}^n \cong \mathbb{R}^n$ or $\mathbb{A}^n \cong \mathbb{C}^n$ respectively. The point(s) where this coordinate is zero are referred to as the point(s) "at infinity" of the affine space.

One of the projective spaces we are very familiar with in daily life is $\mathbb{RP}^2$ or the real projective plane. This is because of the way our eyes work; a single eye can only see one point on each line extending from the pupil or origin. One consequence of this is the horizon effect. One good example is railroad tracks, which are parallel, yet when you look at them, they seem to converge at the horizon. This is their intersection "at infinity".

Another object in projective geometry which will be vital in our investigation is the set of lines in projective space. This space generalizes to something called the Grassmannian.

Definition 1.2 (Linear Subspace of a Projective Space)**.** An *i-dimensional linear subspace of* $\mathbb{P}V$ is set of equivalence classes, whose corresponding points in V form an $(i+1)$-dimensional linear subspace.

Definition 1.3 (Grassmannian)**.** The set of k-dimensional linear subspaces of a vector space V is called a *Grassmannian* and written as $G(k, V)$.

It should be clear that $G(1, V) = \mathbb{P}V$ and $G(k, V)$ represents the $(k-1)$-planes of $\mathbb{P}V$. There are many different notations for specific Grassmannians. For this paper $Gr(k, n) = G(k, \mathbb{R}^n)$ and $Gc(k, n) = G(k, \mathbb{C}^n)$. One particularly useful quality of these sets is that they are varieties: the common zeros of a set of polynomials. It is a fact that every Grassmannian (though we mostly care about the real and complex) can be embedded in a higher-dimensional projective space with the same base field. In this higher dimension, the Grassmannian is a projective variety. We will particularly focus on the sets of projective lines: $Gr(2,4)$ and $Gc(2,4)$. $Gr(2,4)$ can be embedded in $\mathbb{RP}^5$ as a 4-dimensional real manifold. Likewise, $Gc(2,4)$ can be embedded in $\mathbb{CP}^5$ as a 4-dimensional complex manifold. This type of embedding is called a Plücker embedding. The coordinates of the spaces we embed into are called the Plücker coordinates. To understand why this is the case see Harris [1], or Hudec[5] for more explicit computations.

Basic geometry tells us that two points define a line. Thus, the set of lines can also be thought as the set of pairs of distinct points. As such, both $Gr(2,4)$ and $Gc(2,4)$ can be represented by 2×4 matrices (with respective real and complex entries) where each row represents a distinct point in $\mathbb{RP}^3$ and $\mathbb{CP}^3$ respectively. These matrices can be embedded in $\mathbb{RP}^5$ and $\mathbb{CP}^5$.

Definition 1.4 (Plücker Coordinates)**.** The *Plücker coordinates* of a projective line are given by the minors of one of its corresponding 2×4 matrices.

Notation 1.5. Given a suitable 2×4 matrix with $0 \leq x < y \leq 3$ the coordinate function $p(xy)$ is the determinate of the 2×2 matrix which is the x and y columns in order.

We can see how this is an embedding in $\mathbb{P}^5$, since the matrix only represents a line if the rows are linearly independent and so one minor must be non-zero. It should also be noted that the 2×4 matrix still represents the same line when transformed by any element of $GL(2)$ acting on the left. These general linear transformations represent choosing two different points on the line to define the matrix. Each of these transformations will multiply all the Plücker coordinates by the non-zero determinate of the transformation. This helps justify the Plücker coordinates embedding the Grassmannian into projective space since scaling the Plücker coordinates is the same as left multiplication of the 2×4 matrix by an appropriate element of $GL(2)$. Given that there are 6 minors of a 2×4 matrix, the Grassmannian $G(2,4)$ is embedded into $\mathbb{P}^5$. I will sometimes represent points in this space in vector notation as $\langle p(0,1), p(0,2), p(0,3), p(1,2), p(1,3), p(2,3)\rangle$. The Plücker relation, the equation whose solution set cuts out the Grassmannian in $\mathbb{P}^5$, is the following quadratic equation:

$$p(0,1)p(2,3) - p(0,2)p(1,3) + p(0,3)p(1,2) = 0.$$

1.2. **Schubert Calculus objects.** Now that some of the basic objects from projective geometry are defined, we will move on and define some of the objects which are more specialized for Schubert calculus, the most basic of which is called the Schubert Cell. These form a nice decomposition of our Grassmannian, which for those who are more topologically minded form a CW structure [3]. But, before we define them, we are going to define a flag.

Definition 1.6 (Flag)**.** A *flag* F of a projective space $\mathbb{PV}$ where $\dim(V) = n+1$ is a sequence of linear subspaces $F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n$ where each F_i is $i+1$-dimensional in V.

Any flag can be given a set of orthonormal basis vectors of V such that $\text{span}(e_1, ...e_k) = F_k$. Now, we can give a definition of a Schubert Cell.

Definition 1.7 (Schubert Cell)**.** Given a flag F of $\mathbb{P}^n$ and a tuple $\mathbf{j} = (j_0, ..., j_{k-1})$ with $0 \leq j_i < j_{i+1} \leq n$ define the *Schubert cell* $\mathcal{C}_{\mathbf{j}} := \{L \in G(k, \mathbb{P}^n) : \forall i \in [0, k-1]$ $\dim(L \cap F_{j_i}) = i$ and $\dim(L \cap F_{j_i - 1}) < i\}$.

As stated, this definition can be hard to make sense of. A Schubert cell is the set of k-dimensional linear subspaces which intersect given linear subspaces in very precise dimensions. See figure 1 for some elements of different Schubert cells with respect to a given flag.

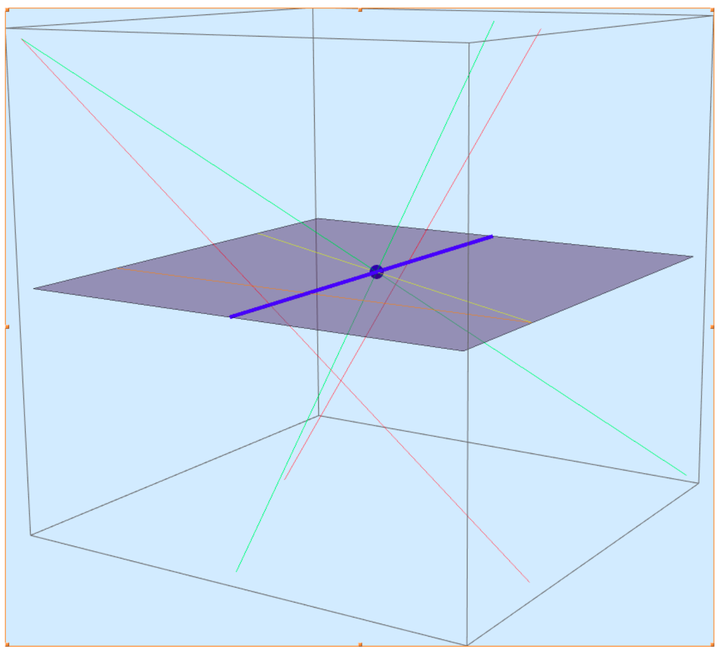

FIGURE 1. Different lines intersecting the flag (F_0, F_1, F_2 and F_3 shown with various shades of blue) in various ways. Red lines are elements of $\mathcal{C}_{1,3}$, green lines are elements of $\mathcal{C}_{0,3}$, the orange line is an element of $\mathcal{C}_{1,2}$, and the yellow line is an element of $\mathcal{C}_{0,2}$, the dark blue line is both F_1 and the sole element of $\mathcal{C}_{0,1}$

When we move to a basis defined by the flag and put the matrices of the Grassmannian in reduced row echelon form, we can get a better grasp of what these objects are. Take, for example, $\mathcal{C}_{1,3}$ in $Gr(2,4)$ which can be expressed using the matrix

$$\begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{bmatrix}.$$

The stars represent independent variables which can take any value in $\mathbb{R}$. No nontrivial combination of the top and bottom will ever give a vector of the form $(x, 0, 0, 0)$ so each line in the cell with have empty intersection with F_0 or intersection of dimension $-\infty$. The first row represents the intersection of this line with F_1 and no other nontrivial combination will have a different intersection point in $\mathbb{P}^3$ Thus each line in the cell intersects $F_{j_0=1}$ at a point, dimension 0. The intersection does not change when we move from F_1 to F_2. Finally, when we move to F_3, or the whole projective space, each line in the cell intersects F_3 in the entire line or in 1 dimension.

In general, a Schubert Cell can be expressed by a nice matrix in reduced row echelon form, with respect to the basis of the defining flag. This expression has a one in the i, j_i location, zeros on the rest of the j_i column, and zeros in the entries to the right of each 1. All the remaining entries are arbitrary, often represented with stars.

The Schubert cell $\mathcal{C}_{1,3}$ is the set of lines that intersect the given line F_1 but not the point F_0, and are not contained in plane F_2. This object is close to what we want, but we desire more than this one cell. A more general object which is simply all the lines that intersect F_1 in any way would be more suited to our problem. Furthermore, Schubert cells are not closed projective varieties, and so are not particularly nice objects to work with. Luckily, we can solve both these issues at the same time with an object called the Schubert variety.

Definition 1.8 (Schubert Variety)**.** Given a flag F of $\mathbb{P}^n$ and a tuple $\mathbf{j} = (j_0, ..., j_{k-1})$ with $0 \leq j_i < j_{i+1} \leq n$ define the *Schubert variety* $\mathcal{X}_\mathbf{j} := \{L \in G(k, \mathbb{P}^n) : \forall i \in [0, k]$ $\dim(L \cap F_{j_i}) \geq i\}$.

This definition turns out to be exactly what we want. The Schubert variety $\mathcal{X}_{1,3}$ gives the sets of lines that intersect the given line F_1 in any manner. In this paper, we will deal mostly with Schubert variteies that correspond to the tuple $\{1, 3\}$ and these will be referred to as $\{1, 3\}$ Schubert varieties. Schubert varieties also have a nice matrix representation. The change is that instead of 1 in the (i, j_i) position and zeros below it, there are stars in these places: their values are arbitrary. For example, the matrix representing $\mathcal{X}_{1,3}$ is

$$\begin{bmatrix} * & * & 0 & 0 \\ * & * & * & * \end{bmatrix}.$$

Now it should be noted that some care should be taken when writing Schubert varieties in matrix form since the first and second row must be linearly independent, otherwise the matrix doesn't represent an element of our Grassmannian. This extra care means that the number of stars does not represent the dimension of the Schubert variety as it did with Schubert cells. Another fact which can be checked is that given a Schubert cell $\mathcal{C}_\mathbf{j}$, the smallest variety which contains this Schubert cell is the Schubert variety with the same ordering: $\mathcal{X}_\mathbf{j}$.

Proposition 1.9. *Given a flag F of $\mathbb{P}^n$ and a tuple $\boldsymbol{j} = (j_0, ..., j_{k-1})$ with $0 \leq j_i < j_{i+1} \leq n$, the smallest projective variety of $\mathbb{P}^n$ containing $\mathcal{C}_\mathbf{j}$ is $\mathcal{X}_\mathbf{j}$.*

We can see that these Schubert varieties are exactly what we desire. Since $\{1, 3\}$ Schubert varieties are the set of lines which hit the one dimensional element of the respective flag, intersecting two $\{1, 3\}$ Schubert varieties with two different flags gives the set of lines which intersect both one dimensional elements of the two flags. If we are given four lines, we can then create four distinct flags such that each has one of the given lines as the one dimensional element. The intersection four of $\{1, 3\}$ Schubert varieties with respect to these four flags is precisely the set of lines that intersect all four of our given lines. Furthermore, since the intersection of varieties is a variety, we can see that our solution set is a variety which means our set of solutions will be a 'nice' algebraic object. Normally, when mathematicians do this kind of problem they use cohomology to show something about the number of intersections. In this paper, we are instead going to use some facts from intersection theory defined through differential topology.

2. Intersection numbers and differential topology

We will be using some differential topology to discuss the intersections of submanifolds of $\mathbb{P}^5$. As such, we will need to begin with a new set of definitions and assumptions before we can get the heart of our proof.

Definition 2.1 (Transverse Manifolds)**.** We say that two manifolds Y and Z in a space X are *transverse* if for every $x \in Y \cap Z$, $T_x(Y) + T_x(Z) = T_x(X)$, where $+$ indicates the span of these two vector spaces.

Put in other words, transverse manifolds have their tangent spaces "add up" to the whole space. It is important to note that any two disjoint manifolds are automatically transverse, and this fact will be important for our proof. It is also true that two transverse manifolds have an intersection which is dimension $[\dim(Y) + \dim(Z)] - \dim(X)$ when this value is non-negative and the intersection is empty when $[\dim(Y) + \dim(Z)] < \dim(X)$. One particular case we will be looking at is when $(\dim(Y) + \dim(Z)) = \dim(X)$, and the intersection is a collection of points. Sometimes it is useful to refer to maps instead of manifolds when discussing transversality, which prompts the following definition.

Definition 2.2 (Transverse Maps)**.** Let Z be a submanifold of Y. We say that a smooth map $f : X \to Y$ is *transverse* to Z if $f(X)$ is transverse to Z.

It is easy to see that this definition can be rewritten pointwise: for $y \in Z$ every $x \in f^{-1}(y)$ has the property that $\text{Image}(df_x) + T_y(Z) = T_y(Y)$. This definition can be particularly helpful to give a pre-image orientation to a manifold which we will now discuss.

Proposition 2.3. *Given the oriented manifolds X and Y with an oriented submanifold $Z \subseteq Y$ where there is a smooth map $f : X \to Y$ transverse to Z then $M := f^{-1}(Z)$ is orientable.*

Proof. First if M is the empty set, then it is automatically orientable. Thus, we can move on to the non-trivial case of the proof where $M \neq \emptyset$.

Let x be a point in M with $f(x) = y \in Z$. Since f is transverse to Z, we know that $df_x(T_x(X)) + T_y(Z) = T_y(Y)$. Furthermore, we know that $f^{-1}(Z) = M$, so the preimage under the derivative map $df^{-1}(T_y(Z)) = T_x(M)$. We then define $N_x(M, X)$ to be a complement of $T_x(M)$. Since f is transverse to Z, we know that the derivative map must send $N_x(M, X)$ to a complement of $T_y(Z)$, $N_y(Z, Y)$. Furthermore, the kernel of f is contained in M, so df_x must map $N_x(M, X)$ to $N_y(Z, Y)$ isomorphically. As such, this isomorphism induces an orientation on $N_x(M, X)$ if $N_y(Z, Y)$ has an orientation.

We now need to find a way to give a tangent space an orientation given an orientation of a complement and an orientation of the ambient space. This is not too difficult to do. Given a positively oriented basis β_1 of $N_y(Z, Y)$ and any basis β_2 of $T_y(Z)$ consider the combined basis $\beta = (\beta_1, \beta_2)$ of Y. If β is positively oriented then β_2 is positively oriented, β_2 is negatively oriented otherwise. As such, the following relation holds: $\text{sign}(\beta_1, \beta_2) = \text{sign}(\beta_1)\text{sign}(\beta_2)$. It is worth noting that the order here matters and should be kept consistent when an orientation is defined. Fixing $\text{sign}(\beta_2, \beta_1) = \text{sign}(\beta_2)\text{sign}(\beta_1)$ to be true also orients Z.

Given the orientations on Z and Y, we can orient $N_y(Z, Y)$. Via our bijection we can orient $N_x(M, X)$. Now that we have an orientation of $N_x(M, X)$ and X we can apply the same trick to orient $T_x(M)$. □

This orientation is what we call a pre-image orientation. With this new machinery, we are ready to discuss oriented intersection numbers. To discuss intersection numbers we need to set the scene somewhat with the following notation.

Notation 2.4. We say that $f : X \to Y$ *and* Z *are appropriate for intersection theory* if X and Y are oriented manifolds without boundary with $Z \subseteq Y$ a closed submanifold of Y without boundary. We require X to be compact and equipped with a smooth map $f : X \to Y$ which is transverse to Z, and $\dim(X) + \dim(Z) = \dim(Y)$.

When $f : X \to Y$ and Z are appropriate for intersection theory, define $M := f^{-1}(Z)$. Observe that M is a finite set of points. Furthermore, the requirements for proposition 2.3 have been met, so we can give M a pre-image orientation. Since $\dim(M) = 0$ the orientation is a ± 1 at each point.

Definition 2.5 (Orientation Number)**.** When $f : X \to Y$ and Z are appropriate for intersection theory, each point $x \in f^{-1}(Z)$ has an *orientation number*, either $+1$ or -1, which is its orientation from a pre-image orientation.

Each orientation number on its own is arbitrary since the reverse orientation of any manifold is perfectly valid. What is more important is the relative value of these orientation numbers over the whole manifold M. We encapsulate this meaning with the idea of an intersection number.

Definition 2.6 (Intersection Number)**.** When $f : X \to Y$ and Z are appropriate for intersection theory, define the *intersection number* over the points in $f^{-1}(Z)$, $I(f, Z)$, as the sum of the orientation numbers.

Intersection numbers deal with some of the arbitrary decisions of orientation between different points, but it should be noted that the positive and negative values of intersection numbers are of no consequence. It is the absolute value of the intersection number which is particularly important. Intersection numbers turn out to be exactly what we want to discuss the number of lines that intersect our given four lines embedded in $\mathbb{P}^5$. We will now state a few facts about orientation numbers which will be helpful in our proof.

Proposition 2.7. *Suppose f and g are homotopic. If $f, g : X \to Y$ and Z are both appropriate for intersection theory, then $I(f, Z) = I(g, Z)$.*

Proof sketch

This proposition comes from a combination of factors, most importantly that if X is the boundary of a compact manifold, W, and f extends smoothly to W, it can be shown that $I(f, Z) = 0$. This comes largely from the idea that compact one-dimensional manifolds with boundary have an even number of boundary points and each boundary point is connected to another boundary point with the opposite orientation. Thus, the total intersection number of the boundary of these manifolds is zero. Consider a homotopy, $F : [0, 1] \times X \to Y$, between the two maps f and g with $[0, 1] \times X$ compact. By the first fact we stated letting $[0, 1] \times X = W$ then ∂F extends smoothly to all of W, so $0 = I(\partial F, Z)$. Thus, there must be an even number of boundary points which we can connect with one-dimensional submanifolds of W and see that

$$I(\partial F, Z) = I(f, Z) - I(g, Z) = 0.$$

Those interested in a more complete explanation of this proposition should look in Guillemin and Pollack [4].

We can then improve our definition of intersection number to non-transversal smooth maps by defining their intersection number in relation to a transversal map they are homotopic to. This transversal map always exists due to the following consequence of Sard's Theorem.

Theorem 2.8 (Transversality Homotopy Theorem)**.** *Given two manifolds without boundary X and Y with any smooth map $f : X \to Y$, and Z a closed submanifold without boundary of Y, then there exists a map $g : X \to Y$ which homotopic to f and traversal to Z.*

Definition 2.9 (Intersection Number)**.** Given oriented manifolds without boundary X and Y where X is compact. Let Z be a closed submanifold of Y such that $\dim(X) + \dim(Z) = \dim(Y)$. Let $f : X \to Y$ be a smooth map. We know that there is another smooth map $g : X \to Y$ homotopic to f such that $g : X \to Y$ and Z are appropriate for intersection theory. We define the *intersection number of f and Z*, $I(f, Z) := I(g, Z)$.

It should be reasonably clear that this definition is well defined and agrees with the earlier definition of intersection number. This definition is convenient since it enables us to loosen some of the restrictions on our maps and make them more general theorems. In addition, we can extend proposition 2.7 to the following lemma.

Lemma 2.10. *Given oriented manifolds without boundary X and Y, with $Z \subseteq Y$ a closed submanifold of Y without boundary, suppose further that X is compact and $\dim(X) + \dim(Z) = \dim(Y)$, then any two homotopic smooth maps $f : X \to Y$ and $g : X \to Y$ have the same intersection number: $I(f, Z) = I(g, Z)$.*

This lemma is quite useful because it enables us to find the intersection number for an entire family of maps or manifolds by finding a single intersection number which is easy to compute. This concept is exactly the heart of our proof.

Another important observation that we will use is the intersection number of non-intersecting manifolds. If $f : X \to Y$ and Z are appropriate for intersection theory, then we can allow $f(X) \cap Z = \emptyset$ since f is still transverse to Z. The sum of the orientation numbers, in this case, is zero. Thus, if we have a map f whose intersection number is non-zero, then it must intersect Z. The entire family of homotopic functions which have non-zero intersection numbers must intersect Z. Note that an intersection number greater than 1 does not change the minimum number of intersections of homotopic maps. There could be a homotopic map which is not transversal map intersecting Z only once. This is a potential drawback of this strategy since the lower bound may not be sharp. Yet, this possible single intersection can be considered to have multiplicity since there is a small perturbation that will make the map transverse and witness at least as many intersections as the intersection number.

3. The Proof

3.1. **The proof outline.** The main goal of this proof is to find a minimum number of lines which intersect four projective lines in general position. Call these lines L_1, L_2, L_3, and L_4. Recall (from section 1.2) that the set of lines which intersect a given line is a $\{1, 3\}$ Schubert variety. Label these Schubert varieties S_1, S_2, S_3, and S_4. If the four Schubert varieties have a non-empty intersection then there is a line which intersects L_1, L_2, L_3, and L_4. Thus our goal is to show that $S_1 \cap S_2 \cap S_3 \cap S_4 \neq \emptyset$.

As a disclaimer, for the proof we use a few lemmas which I state and prove in other sections. Those who are less familiar with some of the topics may find it useful to jump down and read the lemma and its proof before finishing this outline. Those already familiar with some of these concepts may wish to skip some or all the lemmas entirely, which is why they are set to the side.

The proof will rely mostly on a computation of an intersection number which we will first justify. By Lemma 3.2 we know that characteristic $\{1, 3\}$ Schubert varieties can be written as the intersection of hyperplanes in Plücker coordinates with the Grassmannian, label these hyperplanes H_i accordingly. Written in symbols we can say $H_i \cap G(2, 4) = S_i$. As such we can modify our goal to show that $H_1 \cap H_2 \cap H_3 \cap H_4 \cap G(2, 4) \neq \emptyset$. When the original four lines represent linearly independent hyperplanes, the intersection will be a projective line: $H_1 \cap H_2 \cap H_3 \cap H_4 = H$. Notice H is compact (by Lemma 3.3) and is one-dimensional meaning we can apply intersection theory for $H \cap G(2, 4)$.

Definition 3.1 (General Position)**.** We say that four lines L_1, L_2, L_3, and L_4 are in *general position* if H_1, H_2, H_3, and H_4 are linearly independent.

This definition of general position is a touch abstract and partly why it was left until here to define. Note that for finding a lower bound this definition isn't terribly important since lines that are not in general position will simply define a higher dimensional subspace of $\mathbb{P}^5$ and contain uncountably many candidates for H, thus uncountably many times more intersections than the lower bound.

A projective line is exactly the type of object we want to perform intersection theory with the Grassmannian. Recall $G(2, 4)$ is a closed manifold of dimension 4 and H is a compact manifold of dimension 1. If we define a function f_H to be an isomorphism between the projective line to this common intersection, H. Thus $f_H : \mathbb{P}^1 \to \mathbb{P}^5$ is transverse to $G(2, 4)$ if and only if H is transverse to $G(2, 4)$. When f_H is transverse to $G(2, 4)$ then $f_H : \mathbb{P}^1 \to \mathbb{P}^5$ and $G(2, 4)$ are appropriate for intersection theory. The elements of $H \cap G(2, 4)$ are precisely the lines which intersect L_1, L_2, L_3, and L_4. Our goal is to show that this intersection is non-empty; this is implied when the intersection number is non-zero. Lemma 3.4 tells us that f_H is homotopic to f_L for every other projective line $L \in \mathbb{P}^5$. Thus, by Lemma 2.10 the intersection number $I(f_L, G(2, 4))$ is also non-zero for every L. As such, if we find that one intersection number is non-zero, then every set of four lines in $\mathbb{P}^3$ has at least one line of common intersection. Thus, if we compute an intersection number for a particular set of four lines to be non-zero we are done.

This lower bound is sharp. Examples can be produced fairly easily by choosing three skew of lines. Draw the two planes defined by a point p and each of the other lines. Choose the fourth line to be the intersection of these planes. See figure 2 for an example. Note, since small perturbations can make intersections transverse, there is always a small perturbation that will realize at least as many intersections as the intersection number.

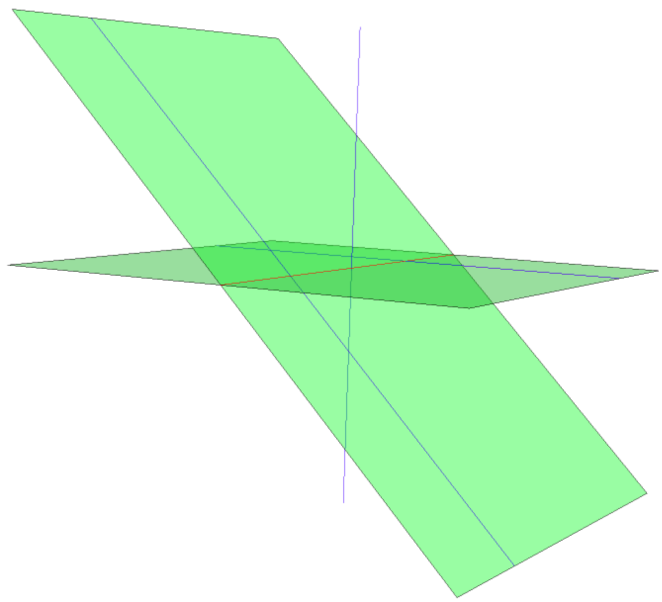

FIGURE 2. The red line is the only line which intersects the three blue lines and red line.

3.2. **Lemmas.**

Lemma 3.2 (Schubert varieties extend to hyperplanes)**.** *Given a flag and the corresponding Schubert variety $\mathcal{X}_{1,3}$, there is a hyperplane in $\mathbb{P}^5$, H, such that $G(2,4) \cap H = \mathcal{X}_{1,3}$.*

Proof. Recall that, with the basis of the flag, the elements of the Grassmannian are in the Schubert variety $\mathcal{X}_{1,3}$ if and only if can be represented by the matrix:

$$\begin{bmatrix} \mathrm{a} & \mathrm{b} & 0 & 0 \\ \mathrm{c} & \mathrm{d} & \mathrm{e} & \mathrm{f} \end{bmatrix}.$$

Where the two rows are linearly independent.

Define the hyperplane of $\mathbb{P}^5$ where the coordinate representing the $\{2,3\}$ minor is zero; $H = \{x \in \mathbb{P}^5 : p_{(2,3)}(x) = 0\}$. Fix A, an element of the intersection of H and the Grassmannian. As an element of the Grassmannian, A can be represented by a 2×4 matrix. As an element of H, every matrix representing A has $\{2,3\}$ minor which is zero. Given that the $\{2,3\}$ minor is zero, there is a general linear transformation, $G \in GL(2)$, such that the $G \cdot A$ is of the form

$$\begin{bmatrix} \mathrm{a} & \mathrm{b} & 0 & 0 \\ \mathrm{c} & \mathrm{d} & \mathrm{e} & \mathrm{f} \end{bmatrix}.$$

We have just shown that $H \cap G(2,4) \subseteq \mathcal{X}_{1,3}$. The opposite set inclusion is immediate since the Schubert variety lies in the Grassmannian and in the hyperplane, H.

□

Lemma 3.3. *Projective lines are compact manifolds.*

Proof. Given a projective space $\mathbb{P}^n$ with a base field $\mathbb{F}$ which is either $\mathbb{R}$ or $\mathbb{C}$. Consider any projective line $L \subseteq \mathbb{P}^n$. Note that there is a surjective continuous map from the n-sphere (in $\mathbb{F}^{n+1}$) to $\mathbb{P}^n$ sending points to the corresponding points in $\mathbb{P}^n$. The preimage of L under this map is a 1-sphere which is compact. We know that the image of a compact set under a surjective continuous map is compact so L is compact.

□

Lemma 3.4 (f_H and f_L are homotopic)**.** *For every two projective lines in $\mathbb{P}^5$ H and L, f_H and f_L are homotopic.*

Proof. Note that both f_L and f_H are linear embeddings. Define the function $G : \mathbb{P}^1 \times [0,1] \to \mathbb{P}^5$ with $G(x,t) = (1-t)f_H(x) + t f_L(x)$. This is a linear combination of linear maps, therefore it is linear. We know that linear maps are continuous, so G is a homotopy of f_L and f_H.

□

3.3. **The Computation for the Real Case.** Given the four lines

$$L_1 : y = 1,\ x = -z \qquad L_2 : z = 0,\ y = 1-x \qquad L_3 : x = y = 1 \qquad L_4 : y = z,\ x = 1,$$

there are exactly two lines which intersect them:

$$R : x = z,\ y = 1 \qquad B : y = -z,\ 1 = x.$$

This can be seen in figure 3, where R is colored red and B blue.

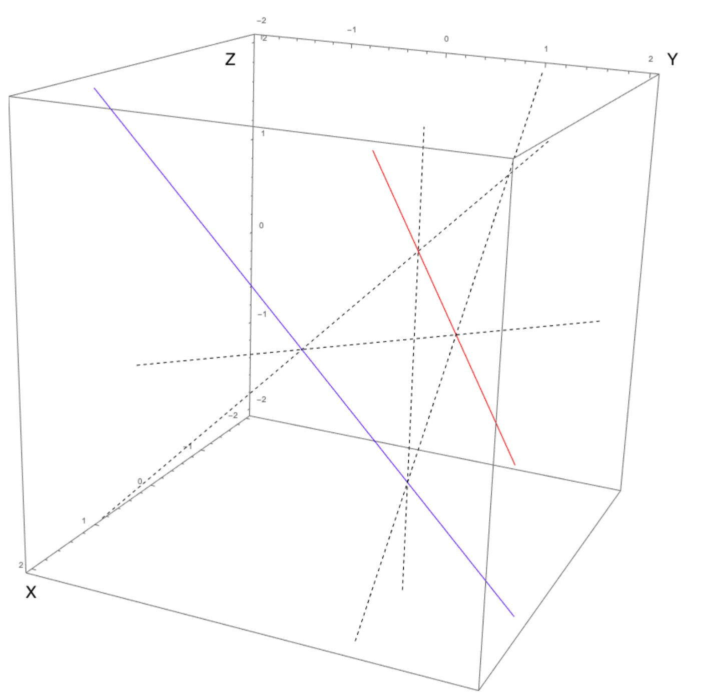

FIGURE 3. The lines R and B intersect dotted lines L_1, L_2, L_3, and L_4

Define $H_i \subset \mathbb{RP}^5$ to be the corresponding hyperplane of $\mathbb{RP}^5$ associated with the set of lines that intersect L_i for $i = \{1, 2, 3, 4\}$. We know that any k-dimensional linear subspace intersected by an independent hyperplane in projective space has a common intersection of a $k-1$-dimensional linear subspace of the projective space. In other words, $H = H_1 \cap H_2 \cap H_3 \cap H_4$ is a line in $\mathbb{RP}^5$. As such $\dim(H) + \dim(Gr(2,4)) = \dim(\mathbb{RP}^5)$ and so H and $Gr(2,4)$ can be given an intersection number (note this uses Lemma 3.3).

Proposition 3.5. *The intersection number,* $I(H, Gr(2,4)) = 0$.

Proof. To see that R and B are the only two lines of that intersect L_1, L_2, L_3, and L_4, first note that L_1 and L_3 are in the plane $y = 1$ and have a single intersection point at $(1, 1, -1)$. Likewise note that L_2 and L_4 are in the plane $z = x + y - 1$ and have a single intersection point at $(1, 0, 0)$. Any line which is to intersect L_1 and L_3 must either go through $(1, 1, -1)$ or lie in the plane $y = 1$ and any line to intersect L_2 and L_4 must lie in the plane $z = x + y - 1$ or go through the point $(1, 0, 0)$. Since $(1, 0, 0)$ does not lie in the plane $y = 1$ and $(1, 1, -1)$ does not lie in the plane $z = x + y - 1$, the only lines which intersect L_1, L_2, L_3, and L_4 must go either through both intersection points, which defines B, or be in the intersection of the planes $z = x + y - 1$ and $y = 1$, which defines the line R.

Thus, $H \cap Gr(2,4)$ consists of two points corresponding to the lines R and B. The orientation number at R is 1 if the orientation of $T_R(H) \bigoplus T_R(Gr(2,4))$ has the same orientation as $T_R(\mathbb{RP}^5)$ and -1 if not. The orientation number at B equivalently defined. Note this requires that $\mathbb{RP}^5$ be orientable which is true for real projective space of odd dimension. This also requires an orientation on both H and $G(2,4)$. The fact that the line H can be oriented is fairly simple because it is topologically a circle which is orientable.

Recall, the quadratic relation for the Plücker embedding for the real Grassmannian $Gr(2,4)$, $F : \mathbb{P}^5 \to \mathbb{R}$,

$$F = p(0,1)p(2,3) - p(0,2)p(1,3) + p(0,3)p(1,2)$$

where $p(xy)$ is the coordinate function for the xy minor of the 2×4 matrix representing a projective line. We know that $F^{-1}(0) = Gr(2,4)$. The image of F is all of $\mathbb{R}$ which is transverse to 0. Therefore, $F^{-1}(0) = Gr(2,4)$ can be given a preimage orientation. Let N_x be a complement of $T_x(Gr(2,4))$ in $\mathbb{P}^5$ at x. Note that the derivative of F at X maps N_x to $\mathbb{R}$ linearly, and so N_x inherits an orientation from $\mathbb{R}$ that is just the gradient of F. We then can give $Gr(2,4)$ an orientation that makes the equation $T_x(N_x) \bigoplus T_x(Gr(2,4)) = T_x(\mathbb{P}^5)$. With this orientation we can see that the orientation number at an intersection point R or B is $+1$ if $T_x(N_x)$ and $T_x(H)$ have orientations when their positive basis vectors "point in the same direction" i.e. have a positive dot product (the dot product is zero when they are not transverse).

Now what is left is to find these basis vectors. First, we will work with H. To find H we must find the linear relation of $\mathbb{P}^5$ that defines lines which intersect our given lines. To do this, we will write these sets of lines in matrix form. Note that we need to use the same coordinates for each relation, so from now on we will use the standard basis. To understand how these matrices are obtained, the first row represents the set of possible intersection points with L_1. When $b \neq 0$, divide through by b to see the coordinates in affine space. When $b = 0$ the intersection is the 'infinity' point of the line. The set of lines which hit $L_1 : y = 1\ x = -z$ can be represented by the variable matrix:

$$\begin{bmatrix} \text{a} & \text{b} & \text{-a} & \text{b} \\ \text{c} & \text{d} & \text{e} & \text{f} \end{bmatrix}.$$

The minor relation which this matrix defines is

$$p(0,1) - p(0,3) - p(1,2) - p(2,3) = 0.$$

Lines which intersect $L_2 : z = 0\ y = 1 - x$ are of the form:

$$\begin{bmatrix} \text{b-a} & \text{a} & 0 & \text{b} \\ \text{c} & \text{d} & \text{e} & \text{f} \end{bmatrix}.$$

Whose corresponding relation is

$$p(0,2) + p(1,2) + p(2,3) = 0.$$

For lines intersecting $L_3 : x = y = 1$ the matrix takes the form:

$$\begin{bmatrix} \text{b} & \text{b} & \text{a} & \text{b} \\ \text{c} & \text{d} & \text{e} & \text{f} \end{bmatrix}.$$

The minor relation which is then defined is

$$p(0,1) - p(0,3) + p(1,3) = 0.$$

Finally, for lines intersecting $L_4 : y = z\ x = 1$ the matrix is

$$\begin{bmatrix} \text{b} & \text{a} & \text{a} & \text{b} \\ \text{c} & \text{d} & \text{e} & \text{f} \end{bmatrix}$$

and the corresponding relation is

$$p(0,1) - p(0,2) + p(1,3) - p(2,3) = 0.$$

Solving these four equations gives us solutions of the form: $p(0,1) = p(2,3) = X$, $p(0,3) = -p(1,2) = Y$, and $p(0,2) = p(1,3) = Y - X$. Do not be alarmed that our line is defined with two unknowns. This is projective space, so X and Y are not simultaneously zero. When $X \neq 0$, we can divide each term by X and get the line in affine space, $\mathbb{A}^5$. In the case where $X \neq 0$, we can orient this line by calling the basis vector $\langle 0, 1, 1, -1, 1\rangle$ the positive basis.

Now we can move on to the orientations of the normal vectors of the Grassmannian, at R and B. First, we are going to write out R and B in matrix form so we can easily find their minors. R in its reduced row echelon form is

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

B in its reduced row echelon form is

$$\begin{bmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

The Plücker coordinates of R in $\mathbb{P}^5$ are $\langle 1, 0, 1, -1, 0, 1\rangle$ and the coordinates of B are $\langle 1, -1, 0, 0, -1, 1\rangle$. The direction of the gradient of F at R is $\langle 1, 0, -1, 1, 0, 1\rangle$ and the direction of the gradient of F at B is $\langle 1, 1, 0, 0, 1, 1\rangle$. Now, in order to have a well defined dot-product, we should move to affine space and each of these vectors has already a 1 in the final coordinate, so if we set $\mathbb{A}^5$ to be the affine space where the last coordinate is 1 means that all that is left is to compute some dot products. The gradient of F at R in affine space is $\langle 1, 0, 1, -1, 0\rangle$ and this dotted with the direction of H, $\langle 0, 1, 1, -1, 1\rangle$, is 2. The gradient of F at B in affine space is $\langle 1, -1, 0, 0, -1\rangle$ and this dotted with the direction of H, $\langle 0, 1, 1, -1, 1\rangle$, is -2. As such, the two points of intersection have opposite orientation numbers, so the intersection number is zero. □

When dealing with a real base feild our proof method is inconclusive since the intersection number is zero. With some thought, we can actually show that this must be the case since there is a line in $\mathbb{RP}^5$ which is disjoint from $Gr(2, 4)$ which is the line $\langle a, 0, b, b, 0, a\rangle$. Plugging points of this line into the quadratic relations, we get that this line intersects our Grassmannian when $a^2 + b^2 = 0$ and we know that a and b cannot be simultaneously zero. This leaves then the question: can four hyperplanes defined from $\{1, 3\}$ Schubert varieties can define this line and so have no common line of intersection. Or if not, why? Regardless of the result of this question, we can use this line, $\langle a, 0, b, b, 0, a\rangle$, as motivation to move to complex numbers where it does intersect the Grassmannian.

3.4. **Moving to the Complex Case.** Now, since every real line can be extended to a complex line, we can pick the same four lines which will have the same intersection in $\mathbb{CP}^3$. So, what is left to figure out is what part of the computation is different. First, we need to notice that complex manifolds are effectively twice the dimension of their real counterparts. There is no longer a single basis vector of a complex line, but two: v and iv. Determining if two complex lines have the same orientation is rather simple because the complex operator i provides an orientation where (v, iv) is positively oriented and (iv, v) is negatively oriented. Since all complex manifolds are of the form (v, iv), they all have the same orientation. This leads to the following proposition.

Proposition 3.6. *Any two complex manifolds which meet the requirements for intersection theory have only positive orientation numbers.*

Consider the four complex lines:

$$L_1' : y = 1,\ x = -z \qquad L_2' : z = 0,\ y = 1-x \qquad L_3' : x = y = 1 \qquad L_4' : y = z,\ x = 1.$$

There are exactly two lines which intersect them: $R' : x = z\ y = 1$, $B' : y = -z\ 1 = x$. Note that x, y, z are complex coordinates. Thus, these equalities actually represent two equations: one for the real components the other for the complex components.

Define $H_i' \subset \mathbb{CP}^5$ to be the corresponding hyperplane of $\mathbb{CP}^5$ associated with the set of lines that intersect L_i' for $i = \{1, 2, 3, 4\}$. We know that any k-dimensional linear subspace and an independent hyperplane in projective space have a common intersection of a $(k-1)$-dimensional linear subspace of the projective space. In other words, $H' := H_1' \cap H_2' \cap H_3' \cap H_4'$ is a line in $\mathbb{CP}^5$. As such $\dim(H') + \dim(Gc(2, 4)) = \dim(\mathbb{CP}^5)$ and so H' and $Gc(2, 4)$ can be given an intersection number (note this uses Lemma 3.3).

Proposition 3.7. *The intersection number of H' and $Gc(2, 4)$ is 2.*

Proof. First, note that R' and B' are the only lines that intersect L_1', L_2', L_3', and L_4' by the same logic as in the proof in the real case. These lines have the same intersection points and the same equations of their planes which yield only two solutions R' and B'. Note that since H was transverse to $Gr(2, 4)$, we also know that H' is transverse to $Gc(2, 4)$. Thus, we can apply our new proposition.

By proposition 2.7, both of these intersections have a positive orientation number, and so the intersection number is 2. □

Since the complex version of our two solution lines still intersects the Grassmannian twice, we can say that the computation gives a value of 2 and not zero. As such, our proof technique will give a lower bound of 1 in the complex case.

4. Conclusion

Now that we have completed this computation and our lemmas, we are prepared to draw some conclusions. First of all we will write a formal proof giving the lower bound for number of lines intersecting four given lines in complex projective 3-space.

Theorem 4.1. *Given any four lines in* $\mathbb{CP}^3$*, there is a line in* $\mathbb{CP}^3$ *that intersects all four.*

Proof. The set of lines intersecting each of the four given lines is the intersection of a hyperplane in $\mathbb{CP}^5$ with $Gc(2,4)$ by Lemma 3.2.

If the four lines are in general position, then these four hyperplanes have an intersection A, a line in $\mathbb{CP}^5$. If the four lines are not in general position, then this intersection of hyperplanes is larger than a line, thus choose any line in the intersection to be A.

By Lemma 3.4 f_A and $f_{H'}$ are homotopic. Thus, they have the same intersection number with $Gc(2,4)$ by Lemma 2.10. By proposition 3.7 this intersection number is 2, which is not zero, so $A \cap Gc(2,4) \neq \emptyset$. Any element of this intersection is a line that intersects all given four lines. □

This is actually not all that we can conclude. Since orientation numbers are always positive with complex manifolds an intersection number of 2 implies there can be at most 2 points of intersection. Thus, we can get the following corollary.

Corollary 4.2. *Four lines in general position within* $\mathbb{CP}^3$ *have at most* 2 *complex lines which intersect all four.*

Since real lines extend to complex lines this upper-bound also holds in real space. Thus, the following corollary can also be shown.

Corollary 4.3. *Four lines in general position within* $\mathbb{RP}^3$ *have at most* 2 *lines which intersect all four.*

Despite approaching the question in an unconventional manner without the powerful machinery of cohomology, our investigation has found some good results. Over the complex numbers, we have defined the condition for when the set of intersecting lines is countable: general position. When we can count these lines we have given sharp lower and upper bounds of 1 and 2.

The real case was less successful, which is largely due to the fact that the real numbers are not algebraically closed. Thankfully, some of our work in complex numbers extends back to the real numbers, mainly the upper-bound for the number of lines which intersect four lines in general position, 2, which we also know is sharp.

There is also a question of where do these lines go. Every real line can be extended to a complex line, but the converse is not true. As such, four real lines can be extended to complex lines which can gave two distinct complex lines intersecting all four, but these two complex lines might not be capable of being turned into real lines.

We can see why this method is not the best for more general problems in enumerative geometry. First of all, it did not give a conclusive result for the real case. Furthermore, the orientability of a Grassmannian or of a projective space is not something which is always true with a real base field, but was true in our problem. The nice extension of $\{1, 3\}$ Schubert varieties to hyperplanes may not have a clear analog in other problems. In addition, this technique can only give a non-trivial lower bound of 1, which may not be helpful when attempting to count objects which have more lines intersecting them. Finally, our technique hinted at a type of multiplicity of solutions. An ideal method would have a more robust way of counting multiplicity of solutions.

This method has its drawbacks, but it is nonetheless interesting since we were able to discuss a problem with less machinery than is typically done. Although, it did require some computation the proof sketch was fairly quick and simple. The results were good since the bounds we gave were sharp and non-trivial.

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