COMPLETENESS IN MODAL LOGIC

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Abstract. In this paper we use canonical models to prove strong completeness for several normal modal logics. In particular, we show that a variety of normal modal logics, including **S4** and **S5**, are each strongly complete with respect to a unique class of frames. Such completeness results are possible because the axioms of these normal modal logics, $p \to \Diamond p$, $p \to \Box\Diamond p$, and $\Diamond\Diamond p \to \Diamond p$, define the classes of reflexive, symmetric, and transitive frames, respectively.

Contents

1. Introduction

Modal logic is a type of propositional logic that uses the modal operators $\Box$ and $\Diamond$ to express information about the internal features of relational structures. A certain class of modal logics called normal modal logics has been of great interest to philosophers and logicians historically. Normal modal logics are collections of well-formed modal formulas that contain the axioms $\Box(p \to q) \to (\Box p \to \Box q)$ and $\Box p \to \neg\Diamond\neg p$ and are closed under modus ponens, uniform substitution, and generalization.

One of the most important developments in modal logic concerned providing these logics with a relational semantics–mathematical structures like models and frames, as well as satisfaction and validity of modal formulas [1, p. 41]. Using these tools, which were developed around the 1960s, elegant completeness results were achieved for a variety of normal modal logics [1, p. 42][2]. Completeness, a concept perhaps best known from its usage in first-order logic, guarantees that true statements in a logic can also be deduced in that logic, and is thus an important means of characterizing the limits of a logic.

In this paper we prove a variety of these completeness results using canonical models. Canonical models are a special kind of model for which normal modal logics are always strongly complete. The lesson is not, however, that any normal modal logic has a substantive strong completeness proof. Instead, the normal modal logics that we discuss have canonical model completeness proofs because their axioms define frames with relations specifiable in first-order logic: reflexivity, symmetry, and transitivity.

This paper assumes knowledge of basic propositional and first-order logic, but no prior knowledge of modal logic. For a more detailed and comprehensive exposition of modal logic, including completeness, see [1]. For a discussion of normal modal logics and completeness only, see [3].

2. Preliminaries

In this section, we provide the primary definitions and results we will need to develop our completeness theorems. In particular, we aim to characterize the modal language, the procedure for generating modal formulas, and the relational structures that will allow us to talk about the truth and validity of modal formulas. We also introduce the concept of a modal logic, a collection of modal formulas that is in some sense self-contained. Next, we offer a more intuitive discussion of modal logics. At the end of this section, we discuss a few definitions and proofs common in propositional logic that deal with consistent sets of formulas–a key aspect of the canonical models that we will introduce later.

We begin with several definitions outlining the fundamentals of modal logic.

Definition 2.1 (Basic Modal Language)**.** The *basic modal language* contains a collection of propositional variables $(p, q, r \dots)$ as well as several symbols from propositional logic: the constant symbol $\perp$, or 'false', the negation symbol $\neg$, and the logical 'or' connective $\vee$. The distinctive addition to the modal language is the unary modal operator $\Diamond$ (called 'diamond').

Definition 2.2 (Modal Formulas)**.** A *well-formed modal formula* (or simply a modal formula) is any string of symbols in the basic modal language given by the following (recursively defined) rules:

(1) Any propositional variable p is a modal formula.
(2) The constant $\perp$ is a modal formula.
(3) If ϕ is a modal formula, then $\neg\phi$ is a modal formula.
(4) If ϕ and ψ are modal formulas, $\phi \vee \psi$ is a modal formula.
(5) If ϕ is a modal formula, $\Diamond\phi$ is a modal formula.
(6) Any finite application of the previous rules is a modal formula.

From these six requirements we can make a number of abbreviations for common formulas. Most of these carry over from propositional logic. Let ϕ and ψ be modal formulas. Then

$$
\begin{array}{rl}
(\text{'and' connective } \wedge) & \phi \wedge \psi := \neg(\neg\phi \vee \neg\psi). \\
(\text{implication } \rightarrow) & \phi \rightarrow \psi := \neg\phi \vee \psi. \\
(\text{bi-implication } \longleftrightarrow) & \phi \longleftrightarrow \psi := (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi). \\
(\text{the constant } \top) & \top := \neg\perp.
\end{array}
$$

A second modal operator can also be derived from the first.

$$(\text{'box' operator } \Box) \qquad \Box\phi := \neg\Diamond\neg\phi.$$

For the remainder of this paper we will use these substitutions.

It is also worth noting that the unary operators $\Box$ and $\Diamond$ have n-ary analogs. Many of the results discussed in this paper can be proved using these generalized operators, however, we do not include them since the logics we will discuss feature only the single-input operators.

We next turn to a couple of important mathematical structures.

Definition 2.3 (Frames and Models)**.** A *frame* is a pair $\mathfrak{F} = (W, R)$ where W is a nonempty set and R is a binary relation on W. We call W the *universe* and any element w in W a *world* (or a *state*). We call R the *accessibility relation.* If two worlds w and v are related by R, we write Rwv.

A *model* is a pair $\mathfrak{M} = (\mathfrak{F}, V)$ where $\mathfrak{F}$ is a frame and V is a function that assigns propositional variables to subsets of W. We regard V as a *valuation function* that determines whether a propositional variable p is 'true' at some world w in W. That is, p is true at w if $w \in V(p)$.

A collection of frames (or models) is called a *class* of frames (or models). Often we talk of the class of frames or models that share a particular relation R.

The following definition generalizes the notion of truth (or as it is usually denoted, satisfaction) at a world to arbitrary modal formulas.

Definition 2.4 (Satisfaction)**.** Let $\mathfrak{M} = (W, R, V)$ be a model and let w be a world in the universe W. If ϕ is a formula, then ϕ is *satisfied* at w (written $\mathfrak{M}, w \Vdash \phi$) in the following ways:

(1) $\mathfrak{M}, w \Vdash p$ if and only if $w \in V(p)$.
(2) $\mathfrak{M}, w \Vdash \bot$ never.
(3) $\mathfrak{M}, w \Vdash \neg\phi$ if and only if not $\mathfrak{M}, w \Vdash \phi$.
(4) $\mathfrak{M}, w \Vdash \phi \vee \psi$ if and only if $\mathfrak{M}, w \Vdash \phi$ or $\mathfrak{M}, w \Vdash \psi$.
(5) $\mathfrak{M}, w \Vdash \Diamond\phi$ if and only if for some $v \in W$ with Rwv we have $\mathfrak{M}, v \Vdash \phi$. From items (4) and (5) we may derive satisfaction conditions for $\wedge$ and $\Box$:
(6) $\mathfrak{M}, w \Vdash \phi \wedge \psi$ if and only if $\mathfrak{M}, w \Vdash \phi$ and $\mathfrak{M}, w \Vdash \psi$.
(7) $\mathfrak{M}, w \Vdash \Box\phi$ if and only if for all $v \in W$ with Rwv we have $\mathfrak{M}, v \Vdash \phi$.

If ϕ is a formula or set of formulas and $\mathfrak{M}$ is a model, we say that ϕ is *satisfiable* on $\mathfrak{M}$ (written $\mathfrak{M} \Vdash \phi$) if there exists a world w such that $\mathfrak{M}, w \Vdash \phi$.

Finally, it is worth noting that satisfaction holds for collections of formulas as well. Thus, if Γ is a collection of modal formulas, we say that $\mathfrak{M}, w \Vdash \Gamma$ if Γ is satisfied at w on $\mathfrak{M}$.

Definition 2.5 (Validity)**.** Let ϕ be a formula, $\mathfrak{F}$ a frame, and w a state in $\mathfrak{F}$. We say that ϕ is *valid at* w if ϕ is satisfied at w for all models whose frame is $\mathfrak{F}$ (written $\mathfrak{F}, w \Vdash \phi$). If F is a class of frames, then ϕ is *valid on* F if ϕ is valid at w on each frame $\mathfrak{F}$ in F (written $F, w \Vdash \phi$).

This final piece of notation will be useful for defining completeness. Let S be a class of models or a class of frames and let $\Gamma \cup \{\phi\}$ be a collection of modal formulas. We say that $\Gamma \Vdash_S \phi$ (verbally: Γ *semantically entails* ϕ on S) if for all structures $\mathfrak{S} \in S$ and all worlds w in $\mathfrak{S}$: $\mathfrak{S}, w \Vdash \Gamma$ implies that $\mathfrak{S}, w \Vdash \phi$.

The following definition allows us to talk about collections of modal formulas that have certain logically interesting features.

Definition 2.6 (Modal Logics)**.** A *modal logic* Λ is a set of modal formulas that contains all propositional tautologies and has the following closure conditions:

(modus ponens) If $\phi \in \Lambda$ and $(\phi \to \psi) \in \Lambda$, then $\psi \in \Lambda$.

(uniform substitution) If $\phi \in \Lambda$, then any complete substitution of propositional variables of ϕ is also a formula in Λ.

If $\phi \in \Lambda$ we may say ϕ is a *theorem* of Λ or, equivalently, $\vdash_\Lambda \phi$. Otherwise we have that $\nvdash_\Lambda \phi$.

Now that we have a basic mathematical perspective on modal logics, it may be helpful to understand their role more intuitively. Part of the motivation to find useful modal logics arises from issues in ordinary language. Consider, for instance, a statement like "it could have rained on Tuesday" (suppose that it did not, in fact, rain on Tuesday). Depending on the weather where you live at this time of year, this statement may seem plausible. But what does it *mean* to say that it *could* have rained on Tuesday? Or, a related question: what could the *truth conditions* of this statement possibly be? One answer that has been proposed in philosophy is that a statement of this kind refers to a nearby "possible world," a hypothetical universe similar to ours.[1] In other words, to say that "it could have rained on Tuesday" is simply to suggest that there is a close possible world in which it did, in fact, rain on Tuesday.

Thus, we can think of one application of modal logic as an attempt to formalize this kind of analysis of our ordinary language. In this case, $\Diamond p$ means "it is possible that p," or "p is true at some accessible, possible world." Similarly, $\Box p$ means "it is necessary that p," or "p is true at all accessible, possible worlds." The possibility and neccessity interpretation of modal logic has also been deployed to shed light on a variety of philosophical issues, including essential (as opposed to accidental) properties [6], the character of the laws of nature [7], and proper names and identity [8]. Other interpretations of modal logic also arise in philosophy and related fields. For example, modal logics can model states of belief and knowledge, past and future events, and obligatory and permissible actions [1].

Returning to the topic at hand, as mentioned in Definition 2.6, modal logics are particular collections of formulas with helpful deductive relationships. Eventually we would like to know whether, given a plausible set of axioms in a modal logic, every satisfiable formula also ends up being deducible in the logic. This would be a proof of completeness, and it would allow us to use the modal logic (and know it is well-behaved) in applications, for example, to clarify particular philosophical questions. The aim of this paper, as discussed previously, will be to show that a certain class of modal logics are in fact complete.

Armed with this philosophical perspective on modal logic, we can proceed once more with the mathematics.

Definition 2.7 (Deducibility)**.** Let $\Gamma \cup \{\phi\}$ be a set of modal formulas in some modal logic Λ. Then ϕ is Λ-deducible from Γ (written $\Gamma \vdash_\Lambda \phi$) if $\vdash_\Lambda \phi$ or there exist formulas $\psi_1, \dots \psi_n \in \Gamma$ where $\vdash_\Lambda (\psi_1 \wedge \cdots \wedge \psi_n) \to \phi$.

Definition 2.8 (Consistency)**.** Let Γ be a set of modal formulas in some modal logic Λ. Then Γ is Λ-*consistent* if $\Gamma \nvdash_\Lambda \bot$. We say that Γ is Λ-*inconsistent* otherwise. It can be shown that Γ is Λ-inconsistent if and only if any formula of Λ is Λ-deducible from Γ. Likewise, Γ is Λ-inconsistent if and only if there exists a formula ϕ such that $\Gamma \vdash_\Lambda \phi$ and $\Gamma \vdash_\Lambda \neg\phi$.

[1]Some philosophers hold that possible worlds are not hypothetical, but just as real as our world [5]. This view, known as "modal realism," is of course highly controversial.

Given that proofs in modal logic are finite, it also follows by a straightforward argument, familiar from propositional logic, that Γ is Λ-consistent if and only if every finite subset of Γ is Λ-consistent.

A special kind of consistency will be useful for our purposes.

Definition 2.9 (Maximal Consistency)**.** Let Γ be a set of modal formulas in some modal logic Λ. We say that Γ is *maximal* Λ*-consistent* if Γ is Λ-consistent and any proper superset of Γ is Λ-inconsistent.

We will also make use of several key features of maximal consistent sets. As the next two proofs are familiar from propositional logic and lie outside the scope of this paper, we do not provide the complete arguments.

Proposition 2.10 (Properties of maximal consistent sets)**.** *Let Λ be a modal logic and let Γ be a maximal Λ-consistent set of modal formulas. Then*

(1) Γ is closed under modus ponens: if $\phi \in \Gamma$ and $\phi \to \psi \in \Gamma$, then $\psi \in \Gamma$.
(2) $\Lambda \subseteq \Gamma$.
(3) for all formulas ϕ, either $\phi \in \Gamma$ or $\neg\phi \in \Gamma$.
(4) if $\phi, \psi \in \Gamma$, then $\phi \wedge \psi \in \Gamma$.

Proof. These properties are straightforward consequences of maximal consistency. □

The following construction is a useful way to extend a consistent set of formulas into a maximal consistent set.

Lemma 2.11 (Lindenbaum's Lemma)**.** *Let Λ be a logic and let Σ be a Λ-consistent set of formulas. Then there exists a maximal Λ-consistent set Σ' such that $\Sigma \subseteq \Sigma'$.*

Proof. We first enumerate the formulas of our modal language such that each formula has a natural number index like so: $\phi_1, \phi_2, \ldots$. Then we denote

$$\Sigma_0 = \Sigma,$$

$$\Sigma_{n+1} = \begin{cases} \Sigma_n \cup \{\phi_n\} & \text{if this set is } \Lambda\text{-consistent} \\ \Sigma_n \cup \{\neg\phi_n\} & \text{if } \Sigma_n \cup \{\phi_n\} \text{ is not } \Lambda\text{-consistent,} \end{cases}$$

and

$$\Sigma' = \bigcup_{n \geq 0} \Sigma_n.$$

It can be shown that $\Sigma \subseteq \Sigma'$ and Σ' is maximal Λ-consistent. □

3. The Road to Completeness

In this section, we prove the main results we will need to establish the completeness of our modal logics of interest. The key concepts are completeness, normal modal logics, and the canonical model. Along the way, we also prove a number of useful lemmas that will allow us to argue for a strong form of completeness using canonical models.

We first define soundness and completeness for modal logics.

Definition 3.1 (Soundness)**.** Let S be a class of frames or a class of models. Let Λ be a modal logic. Then Λ is *sound* with respect to S if for every formula ϕ and structure $\mathfrak{S} \in S$, if $\vdash_\Lambda \phi$ then there is a world w with $\mathfrak{S}, w \Vdash \phi$.

We do not include soundness proofs in our exposition, though it is worth noting that they can be proved straightfowardly.

Definition 3.2 (Completeness)**.** Let S be a class of frames or a class of models. Let Λ be a modal logic. Then Λ is *strongly complete* with respect to S if for any set of formulas $\Gamma \cup \{\phi\}$, if $\Gamma \Vdash_S \phi$ then $\Gamma \vdash_\Lambda \phi$.

We may contrast strong completeness with weak completeness: if Λ is a modal logic, then Λ is *weakly complete* on S if $S, w \Vdash \phi$ implies $\vdash_\Lambda \phi$ for any formula ϕ.

Notice that weak completeness is an exact converse of our soundness definition, while strong completeness applies to whole sets of formulas at once. As it happens, all of the normal modal logics investigated in this paper have strong completeness proofs. Moreover, strong completeness always implies weak completeness, but the converse does not hold. At the end of this paper, we will briefly discuss an example of a normal modal logic that is weakly complete, but not strongly complete.

This next proposition aligns consistency with completeness, a useful connection when one is working with logics. We will use this result directly in our completeness proofs.

Proposition 3.3. *Let S be a class of frames or a class of models. Let Λ be a modal logic. Then Λ is strongly complete with respect to S if and only if for every Λ-consistent set of formulas Γ there is a structure $\mathfrak{S} \in S$ and a world w such that $\mathfrak{S}, w \Vdash \Gamma$.*

Proof. For the right to left direction, we prove the contrapositive. Thus, we assume that Λ is not strongly complete with respect to S. Then there exists a set of formulas $\Gamma \cup \{\phi\}$ where $\Gamma \Vdash_S \phi$ but $\Gamma \nvdash_\Lambda \phi$. Since $\Gamma \nvdash_\Lambda \phi$, it follows immediately that Γ is Λ-consistent (else every formula would be deducible from Γ). By extension, $\Gamma \cup \{\neg\phi\}$ must also be Λ-consistent, since $\Gamma \cup \{\neg\phi\} \vdash_\Lambda \neg\phi$ but $\Gamma \cup \{\neg\phi\} \nvdash_\Lambda \phi$. However, $\Gamma \cup \{\neg\phi\}$ cannot be satisfiable on any $\mathfrak{S} \in S$. For suppose otherwise. Then there exists a structure $\mathfrak{S}$ and a world w with $\mathfrak{S}, w \Vdash \Gamma \cup \{\neg\phi\}$, and it follows that $\mathfrak{S}, w \Vdash \phi$ and $\mathfrak{S}, w \Vdash \neg\phi$, which is impossible. Therefore, $\Gamma \cup \{\neg\phi\}$ is a Λ-consistent set that is not satisfiable on any structure $\mathfrak{S} \in S$, and so we have shown the contrapositive.

For the left to right direction, we let $\Gamma \cup \{\phi\}$ be a Λ-consistent set of formulas and assume that there is no structure $\mathfrak{S} \in S$ and world w with $\mathfrak{S}, w \Vdash \Gamma \cup \{\phi\}$. It follows that Λ is not strongly complete with respect to S. For suppose otherwise. Since $\Gamma \cup \{\phi\}$ is Λ-consistent, $\Gamma \nvdash_\Lambda \neg\phi$. By the strong completeness of Λ, $\Gamma \nVdash_S \neg\phi$, and so there exists a structure $\mathfrak{S}$ and world w where $\mathfrak{S}, w \Vdash \Gamma$ but not $\mathfrak{S}, w \Vdash \neg\phi$. Then $\mathfrak{S}, w \Vdash \phi$, and so $\mathfrak{S}, w \Vdash \Gamma \cup \{\phi\}$. But this contradicts the fact that $\Gamma \cup \{\phi\}$ is not satisfiable on any structure $\mathfrak{S} \in S$. Therefore, Λ is not strongly complete on S, and this again proves the contrapositive. $\square$

We now turn to a particular kind of modal logic.

Definition 3.4 (Normal Modal Logics)**.** A *normal modal logic* Λ is a modal logic that contains

(K) $\quad \Box(p \to q) \to (\Box p \to \Box q)$

as a formula and the closure condition

(Generalization) $\quad$ If $\vdash_\Lambda \phi$ then $\vdash_\Lambda \Box\phi$.

Normal modal logics have a number of nice features which enable them to contain plausible modal axioms and have simple completeness proofs. We demonstrate one such feature that will be used later on.

Lemma 3.5. *Let Λ be a normal modal logic and let $\psi_1, \ldots, \psi_n, \phi$ be a collection of modal formulas in Λ where $\vdash_\Lambda (\psi_1 \wedge \cdots \wedge \psi_n) \to \phi$. Then $\vdash_\Lambda (\Box\psi_1 \wedge \cdots \wedge \Box\psi_n) \to \Box\phi$.*

Proof. We proceed by induction. For the base case, we assume $\vdash_\Lambda \psi_1 \to \phi$. By generalization, $\vdash_\Lambda \Box(\psi_1 \to \phi)$, and by the K axiom, $\vdash_\Lambda \Box(\psi_1 \to \phi) \to (\Box\psi_1 \to \Box\phi)$. By modus ponens, $\vdash_\Lambda \Box\psi_1 \to \Box\phi$, and this completes the base case.

Next, fix $n \in \mathbb{N}$ and let $A_1, \ldots, A_n, B \in \Lambda$ be modal formulas. For the inductive hypothesis, assume that

$$\text{if } \vdash_\Lambda (A_1 \wedge \cdots \wedge A_n) \to B, \text{ then } \vdash_\Lambda (\Box A_1 \wedge \cdots \wedge \Box A_n) \to \Box B.$$

Now, let $\psi_1, \ldots, \psi_n, \psi_{n+1}, \phi$ be any collection of modal formulas in Λ with

$$\vdash_\Lambda (\psi_1 \wedge \cdots \wedge \psi_n \wedge \psi_{n+1}) \to \phi.$$

By uniform substitution into the propositional tautology

$$(p \wedge q \to r) \to (p \to (q \to r)),$$

we have

$$\vdash_\Lambda (\psi_1 \wedge \cdots \wedge \psi_n \wedge \psi_{n+1} \to \phi) \to (\psi_1 \wedge \cdots \wedge \psi_n) \to (\psi_{n+1} \to \phi).$$

By modus ponens,

$$\vdash_\Lambda (\psi_1 \wedge \cdots \wedge \psi_n) \to (\psi_{n+1} \to \phi).$$

Applying the inductive hypothesis, we have

$$\vdash_\Lambda (\Box\psi_1 \wedge \cdots \wedge \Box\psi_n) \to \Box(\psi_{n+1} \to \phi).$$

Using the K axiom,

$$\vdash_\Lambda \Box(\psi_{n+1} \to \phi) \to (\Box\psi_{n+1} \to \Box\phi).$$

By propositional logic,

$$\vdash_\Lambda (\Box\psi_1 \wedge \cdots \wedge \Box\psi_n) \to (\Box\psi_{n+1} \to \Box\phi).$$

Finally, by uniform substitution into the propositional tautology

$$(p \to (q \to r)) \to (p \wedge q \to r)$$

and modus ponens,

$$\vdash_\Lambda (\Box\psi_1 \wedge \cdots \wedge \Box\psi_n \wedge \Box\psi_{n+1}) \to \Box\phi,$$

and this completes the induction. $\Box$

We next introduce the canonical model, a type of model that will be useful for several completeness arguments.

Definition 3.6 (Canonical Models)**.** We associate a *canonical model*

$$\mathfrak{M}^\Lambda = (W^\Lambda, R^\Lambda, V^\Lambda)$$

with a normal modal logic Λ. In particular, we define

(1) W^Λ as the set of all maximal Λ-consistent sets,
(2) R^Λ as the relation $R^\Lambda wv$ if for all formulas ψ, if $\psi \in v$ then $\Diamond\psi \in w$, and
(3) V^Λ as the valuation function $V^\Lambda(p) = \{w \in W^\Lambda \mid p \in w\}$.

The frame on which the canonical model is based, $\mathfrak{F}^\Lambda = (W^\Lambda, R^\Lambda)$, is called the *canonical frame.*

The canonical model is the primary tool that will enable us to generate completeness proofs for a variety of normal modal logics. However, we first need to show that any consistent set of formulas can be satisfied in some world in the canonical model. This requires a number of important lemmas. The next lemma allows us to use the $\Box$ operator in the context of the relation R^Λ. The two lemmas after together show that a maximal consistent world will always exist that can satisfy a consistent set of formulas.

Lemma 3.7. *Let Λ be a normal modal logic. Then $R^\Lambda wv$ if and only if for all formulas ψ, if $\Box\psi \in w$, then $\psi \in v$.*

Proof. For the forwards direction, suppose that $R^\Lambda wv$ and $\psi \notin v$. Since v is maximal Λ-consistent, by Proposition 2.10 $\neg\psi \in v$. Since $R^\Lambda wv$, $\Diamond\neg\psi \in w$. By the Λ-consistency of w, $\neg\Diamond\neg\psi \notin w$. By substitution, $\Box\psi \notin w$, and this proves the contrapositive.

For the backwards direction, suppose that for all formulas ψ, if $\Box\psi \in w$, then $\psi \in v$. Suppose as well that $\Diamond\psi \notin w$. Since w is maximal Λ-consistent, by Proposition 2.10 $\neg\Diamond\psi \in w$. By substitution, $\neg\neg\Box\neg\psi \in w$, and so $\Box\neg\psi \in w$. Hence by our initial assumption $\neg\psi \in v$. Since v is Λ-consistent, $\psi \notin v$, and this proves the contrapositive once more. $\square$

Lemma 3.8 (Existence Lemma)**.** *Let Λ be a normal modal logic, let w be a world in W^Λ, and let ϕ be an arbitrary modal formula. If $\Diamond\phi \in w$, then there exists a world v in W^Λ such that $R^\Lambda wv$ and $\phi \in v$.*

Proof. Say $\Diamond\phi \in w$. Let v^- be a set of modal formulas with

$$v^- = \{\phi\} \cup \{\psi \mid \Box\psi \in w\}.$$

We first show that v^- is Λ-consistent. Suppose, for the sake of contradiction, that v^- is Λ-inconsistent. Then $v^- \vdash_\Lambda \neg\phi$, and so by Definition 2.7 there exist $\psi_1, \dots, \psi_n$ such that

$$\vdash_\Lambda (\psi_1 \wedge \cdots \wedge \psi_n) \to \neg\phi.$$

By generalization,

$$\vdash_\Lambda \Box((\psi_1 \wedge \cdots \wedge \psi_n) \to \neg\phi).$$

Applying the K axiom,

$$\vdash_\Lambda \Box((\psi_1 \wedge \cdots \wedge \psi_n) \to \neg\phi) \to (\Box(\psi_1 \wedge \cdots \wedge \psi_n) \to \Box\neg\phi).$$

By modus ponens,

$$\vdash_\Lambda \Box(\psi_1 \wedge \cdots \wedge \psi_n) \to \Box\neg\phi.$$

Next, note that $(\psi_1 \wedge \cdots \wedge \psi_n) \to (\psi_1 \wedge \cdots \wedge \psi_n)$ is a propositional tautology. Hence by Lemma 3.5 we have $\vdash_\Lambda (\Box\psi_1 \wedge \cdots \wedge \Box\psi_n) \to \Box(\psi_1 \wedge \cdots \wedge \psi_n)$. By propositional logic, $\vdash_\Lambda (\Box\psi_1 \wedge \cdots \wedge \Box\psi_n) \to \Box\neg\phi$.

Now, since $\Box\psi_1, \dots \Box\psi_n \in w$ and w is a maximal Λ-consistent set, by Proposition 2.10 we have $\Box\psi_1 \wedge \cdots \wedge \Box\psi_n \in w$. Once more by Proposition 2.10, $\Box\neg\phi \in w$, and so by substitution we have $\neg\Diamond\phi \in w$. But this contradicts the fact that $\Diamond\phi \in w$ and w is Λ-consistent. Thus v^- must be Λ-consistent.

Finally, by Lindenbaum's Lemma, there exists a maximal Λ-consistent set v such that $v^- \subseteq v \in W^\Lambda$. By the construction of v, for any formula ψ, if $\Box\psi \in w$, then $\psi \in v$. Therefore, by Lemma 3.7, $R^\Lambda wv$. Moreover, since $\phi \in v^-$, $\phi \in v$. $\square$

Lemma 3.9 (Truth Lemma). *Let Λ be a normal modal logic and let ϕ be an arbitrary modal formula. Then $\mathfrak{M}^{\Lambda}, w \Vdash \phi$ if and only if $\phi \in w$.*

Proof. We proceed by induction on the degree of ϕ. That is, we prove the bi-implication holds for an arbitrary formula ϕ by proving it holds for a formula that contains any number of logical connectives.

Hence, for the base case, suppose that ϕ contains no logical connectives. Then ϕ is either a propositional variable p or the constant $\perp$. Suppose that ϕ is a propositional variable p. By the definition of V^{Λ}, $\mathfrak{M}^{\Lambda}, w \Vdash p$ if and only if $w \in V(p)$ if and only if $p \in w$, the equivalence we wanted. Suppose instead that ϕ is the constant $\perp$. Then $\mathfrak{M}^{\Lambda}, w \Vdash \perp$ never, and since w is Λ-consistent, $\perp \notin w$. Thus the desired bi-implication goes through for the $\perp$ constant because neither hypothesis may hold.

For the inductive hypothesis, we fix $n \in \mathbb{N}$, let A be any formula with at most n connectives, and suppose that $\mathfrak{M}^{\Lambda}, w \Vdash A$ if and only if $A \in w$. It is also worth noting that a formula with at most $n+1$ connectives takes the form $\neg B$, $B \wedge C$, or $\Diamond B$ (where B and C each have at most n connectives). It remains to show that the proof goes through for these formulas.

Let ϕ be an arbitrary formula with at most n connectives and consider $\neg\phi$, a formula with at most $n+1$ connectives. By our inductive hypothesis, $\mathfrak{M}^{\Lambda}, w \Vdash \phi$ if and only if $\phi \in w$. Thus by the definition of satisfaction, not $\mathfrak{M}^{\Lambda}, w \Vdash \neg\phi$. And by Proposition 2.10, since $\phi \in w$, $\neg\phi \notin w$. Therefore the two directions of the implication obtain as desired, because neither antecedent ever holds.

Next, take $\phi \wedge \psi$, where ψ also has at most n connectives. By our inductive hypothesis, $\mathfrak{M}^{\Lambda}, w \Vdash \phi$ if and only if $\phi \in w$ and $\mathfrak{M}^{\Lambda}, w \Vdash \psi$ if and only if $\psi \in w$. Suppose first that $\mathfrak{M}^{\Lambda}, w \Vdash \phi \wedge \psi$. Then $\mathfrak{M}^{\Lambda}, w \Vdash \phi$ and $\mathfrak{M}^{\Lambda}, w \Vdash \psi$, and it follows that $\phi \in w$ and $\psi \in w$. Hence by Proposition 2.10, $\phi \wedge \psi \in w$. Suppose instead that $\phi \wedge \psi \in w$. By Proposition 2.10, $\phi \in w$ and $\psi \in w$. By assumption, therefore, $\mathfrak{M}^{\Lambda}, w \Vdash \phi$ and $\mathfrak{M}^{\Lambda}, w \Vdash \psi$, and it follows that $\mathfrak{M}^{\Lambda}, w \Vdash \phi \wedge \psi$.

Finally, take $\Diamond\phi$. Assume first that $\mathfrak{M}^{\Lambda}, w \Vdash \Diamond\phi$. Then there exists a world v such that $R^{\Lambda}wv$ and $\mathfrak{M}^{\Lambda}, v \Vdash \phi$. By our inductive hypothesis, then, $\phi \in v$. Hence by the definition of R^{Λ}, $\Diamond\phi \in w$. Assume next that $\Diamond\phi \in w$. By the Existence Lemma, there exists a world v such that $R^{\Lambda}wv$ and $\phi \in v$. By our inductive hypothesis, then, $\mathfrak{M}^{\Lambda}, v \Vdash \phi$, and so $\mathfrak{M}^{\Lambda}, w \Vdash \Diamond\phi$. $\square$

Using the previous lemmas, we may easily prove the canonical model theorem, an interesting result in its own right that we will adjust later to get our completeness proofs.

Theorem 3.10 (Canonical Model Theorem). *Let Λ be a normal modal logic. Then Λ is strongly complete with respect to its canonical model $\mathfrak{M}^{\Lambda}$.*

Proof. Let Σ be an arbitrary Λ-consistent set of formulas. By Lindenbaum's Lemma there exists a maximal Λ-consistent set of formulas Σ' such that $\Sigma \subseteq \Sigma'$. Note that Σ' is thus a world in the canonical model $\mathfrak{M}^{\Lambda}$. Hence by the Truth Lemma, $\mathfrak{M}^{\Lambda}, \Sigma' \Vdash \Sigma$. Therefore, by Proposition 3.3, Λ is strongly complete with respect to $\mathfrak{M}^{\Lambda}$. $\square$

The Canonical Model Theorem only tells us that every normal modal logic is strongly complete on a single, contrived structure, its canonical model. The theorem therefore does not immediately yield the interesting strong completeness results we are looking for. However, using the implications of the Canonical Model Theorem together with Proposition 3.3, we will prove that certain normal modal logics are strongly complete on whole classes of frames–in ways that have surprising connections to first-order logic.

4. Completeness Results

We now provide completeness proofs for several normal modal logics.

Definition 4.1. First, we present a few axioms in the basic modal language.

- (T) $p \to \Diamond p$.
- (B) $p \to \Box\Diamond p$.
- (4) $\Diamond\Diamond p \to \Diamond p$.

These axioms have been used often throughout the development of modal logic. Part of the motivation for their use is that they appear plausible under the interpretation of modal logic according to which $\Diamond p$ means "possibly p" and $\Box p$ means "necessarily p" [1]. Hence the axiom T, for example, simply says "if p is true, then it is possible that p," which seems like a rather reasonable principle. Similarly, B says "if p is true, then it is necessarily possible that p" which again seems likely to be true about possibility and necessity. The 4 axiom, however, seems somewhat harder to translate, and much less evaluate, using this method ("if it is possible that p is possible, then p is possible?"). This is partly why the mathematical perspective is helpful in parsing these axioms–it enables us to see whether they create logics that are clean and useful, without requiring an ordinary language interpretation.

That aside, it is also worth noting that if a normal modal logic Γ contains no other formulas besides K, we may call Γ *the normal modal logic*. If a normal modal logic Λ contains only K and, for example, the T axiom, we may call Λ the logic *generated* (or *axiomatized*) by T.

Definition 4.2. We next define a number of normal modal logics using combinations of the axioms:

1. **K** is the normal modal logic.
2. **KT** is the logic generated by the T axiom.
3. **KB** is the logic generated by the B axiom.
4. **K4** is the logic generated by the 4 axiom.
5. **S4** is the logic generated by the T and 4 axioms.
6. **S5** is the logic generated by the T, B, and 4 axioms.

We will show that each of these logics is strongly complete with respect to a unique class of frames.

The work we have done so far allows us to take care of the logic **K** fairly easily. For clarity, we reinterpret the Canonical Model Theorem as a new lemma.

Lemma 4.3. *Let Λ be a normal modal logic and let Γ be a Λ-consistent set of modal formulas. Then there exists a maximal Λ-consistent set of formulas Γ' such that $\mathfrak{M}^{\Lambda}, \Gamma' \Vdash \Gamma$.*

Proof. Contained in the proof of the Canonical Model Theorem. $\Box$

Theorem 4.4 (Completeness of K)**.** *The normal modal logic* ***K*** *is strongly complete with respect to the class of all frames.*

Proof. Let Γ be a **K**-consistent set of modal formulas. We need only find any model that can satisfy Γ. By Lemma 4.3, $\mathfrak{M}^{\mathbf{K}}, \Gamma' \Vdash \Gamma$, and so by Proposition 3.3, **K** is strongly complete on the class of all frames. $\square$

With the strong completeness of **K** done, we arrive at a crucial lemma. This will enable us to reinterpret the axioms named in Definition 4.1 as properties of the relations of canonical frames.

Lemma 4.5. *If a normal modal logic Λ_T contains the axiom T, then the relation R^{Λ_T} on its canonical frame $\mathfrak{F}^{\Lambda_T}$ must be reflexive. An analogous result holds for the B axiom and symmetry, and the 4 axiom and transitivity.*

Proof. Let Λ_T be a normal modal logic containing the T axiom. Let w be an arbitrary world in the canonical model of Λ_T and let ϕ be a modal formula in w. Since w is a maximal Λ_T-consistent set, it contains the axioms of the logic Λ_T. Hence $\phi \to \Diamond\phi \in w$ (the T axiom). By the closure of modus ponens on maximal consistent sets, $\Diamond\phi \in w$, and so $R^{\Lambda_T}ww$. That is, R^{Λ_T} is a reflexive relation.

Next, let Λ_B be a normal modal logic containing the B axiom. Let w and v be arbitrary worlds in the canonical model of Λ_B and let ϕ be a modal formula in w. Suppose as well that $R^{\Lambda_B}wv$. Since w is a maximal Λ_B-consistent set, $\phi \to \Box\Diamond\phi \in w$ (the B axiom), and by modus ponens, $\Box\Diamond\phi \in w$. By Lemma 3.7, $\Diamond\phi \in v$, and so $R^{\Lambda_B}vw$. Hence R^{Λ_B} is a symmetric relation.

Finally, let Λ_4 be a normal modal logic containing the 4 axiom. Let w, v, and z be arbitrary worlds in the canonical model of Λ_4 where $R^{\Lambda_4}wv$ and $R^{\Lambda_4}vz$. Let ϕ be a modal formula with $\phi \in z$. By the fact that $R^{\Lambda_4}vz$, $\Diamond\phi \in v$. By the fact that $R^{\Lambda_4}wv$, $\Diamond\Diamond\phi \in w$. Since w is a maximal Λ_4-consistent set, $\Diamond\Diamond\phi \to \Diamond\phi \in w$ (the 4 axiom), and by modus ponens, $\Diamond\phi \in w$. Hence $R^{\Lambda_4}wz$, and so R^{Λ_4} is a transitive relation. $\square$

Theorem 4.6. *The normal modal logics* **KT***,* **KB***,* **K4***,* **S4***, and* **S5** *are strongly complete with respect to the classes of frames listed below.*

Proof. All of these results can be proved using a similar argument. We prove strong completeness for **S5** as an example and then include the rest of the results in the table below.

Let Γ be an **S5**-consistent set of modal formulas. By Lemma 4.3, there exists a maximal **S5**-consistent set of formulas Γ' where $\mathfrak{M}^{\mathbf{S5}}, \Gamma' \Vdash \Gamma$. Since **S5** contains the axioms T, B, and 4, by Lemma 4.5, $R^{\mathbf{S5}}$ is reflexive, symmetric, and transitive–an equivalence relation. Therefore, by Proposition 3.3, **S5** is strongly complete with respect to the class of all frames with an equivalence relation.

Parallel arguments will reveal the following completeness results:

KT	Strongly complete on the class of reflexive frames
KB	Strongly complete on the class of symmetric frames
K4	Strongly complete on the class of transitive frames
S4	Strongly complete on the class of reflexive and transitive frames

$\square$

At this point, one might suspect that any normal modal logic is strongly complete with respect to a particular class of frames. However, this is false. The normal modal logics we have discussed have these completeness proofs because they share an attractive feature: their additional axioms 'define' classes of frames in a precise sense. We explain in the following definition.

Definition 4.7 (Frame Definability)**.** Let F be a class of frames and let ϕ be a modal formula. Then ϕ *defines* F if for any frame $\mathfrak{F}$, $\mathfrak{F}$ is in F if and only if $\mathfrak{F}, w \Vdash \phi$.

Hence we may show that, for instance, the axiom T defines reflexivity.

Theorem 4.8. *Let $\mathfrak{F}$ be a frame. Then $\mathfrak{F}, w \Vdash p \to \Diamond p$ if and only if $\mathfrak{F} \models \forall x\ Rxx$ (i.e. $\mathfrak{F}$ is reflexive).*

Proof. For the right to left direction, let $\mathfrak{F}$ be a reflexive frame. Let p be a propositional variable, let V be a valuation, and let w be a world where $(\mathfrak{F}, V), w \Vdash p$. Since the relation on $\mathfrak{F}$ is reflexive, w is accessible from w. Thus, since p holds at w, $\Diamond p$ holds at w (there exists an accessible world, w, where p holds). Hence $\mathfrak{F}, w \Vdash p \to \Diamond p$.

For the left to right direction, we prove the contrapositive. Let $\mathfrak{F}$ be a frame whose relation is not reflexive. Then there exists a world w that is not accessible from itself. Let p be a propositional variable and let $V(p) = \{w\}$. In other words, p is only satisfied at w. And since w is not accessible from itself, no world that is accessible from w can satisfy p, and so $(\mathfrak{F}, V), w \nVdash \Diamond p$. It follows that $(\mathfrak{F}, V), w \nVdash p \to \Diamond p$. $\square$

A key observation here is that the axiom T corresponds to a statement in first order logic, reflexivity, and the analogous claims can be made about the B and 4 axioms. This is not the case for every modal formula, however. For example, the formula $\Box(\Box p \to p) \to \Box p$ has no equivalent formula in first-order logic. Hence $\Box(\Box p \to p) \to \Box p$ cannot define any class of frames, and so canonical models cannot be used to prove completeness for a normal modal logic augmented with $\Box(\Box p \to p) \to \Box p$ as an axiom. As it turns out, the normal modal logic with the aforementioned axiom, which we may call **KL**, is not strongly complete on any class of frames [1, p.211]. For what it's worth, however, **KL** is weakly complete on the class of finite transitive trees.[2]

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