THE RING OF SYMMETRIC POLYNOMIALS

GRAYSON GRAHAM

Abstract. In this paper we will define the ring of symmetric polynomials, and build up sequentially six bases of this ring proving connections between them along the way. In doing so we define certain objects such as Young tableaux, and Schur functions which are relevant to not just the study of symmetric polynomials, but also such fields as representation theory. We also prove certain identities involving the symmetric polynomials such as Newton's identities and Pieri's formula.

Contents

1. Introduction 1
2. Basic Definitions 2
3. Bases of Symmetric Polynomials 3
4. Schur Functions 7
5. Acknowledgements 12
References 12

1. Introduction

Symmetric polynomials in n variables are polynomials which are invariant under any permutation of their variables. They form a ring with many interesting properties, relevant to different areas of mathematics. One such instance of this is in representation theory, where symmetric polynomials arise naturally. For example Schur functions are characters of symmetric group representations and the number of semistandard Young tableaux is the dimension of irreducible symmetric group representations.

In §2, we give necessary definitions for the rest of the paper.

In §3, we define four of the five types of symmetric polynomials shown in this paper and prove that they are bases for the ring of symmetric polynomials. In addition we define and prove a formulation of Newton's identities, a set of formulae with importance not only to the study of symmetric polynomials but also such fields as Galois theory, and combinatorics, as well as outside mathematics in such fields as general relativity.

In §4, We define the fifth type of symmetric polynomial shown in this paper, the Schur functions. We do this in two different ways, algebraically and combinatorially. We then prove that the algebraically defined Schur functions form a basis for the ring of symmetric polynomials and prove the equivalence of our two definitions for the Schur functions. We then define an inner product structure on the symmetric polynomials, so that the Schur functions are an orthonormal basis. In fact we find any other orthonormal basis of the ring of symmetric polynomials is a collection of Schur functions up to a signed permutation. We conclude by proving the desired Pieri formula.

2. Basic Definitions

Definition 2.1. A *permutation* of a set S is a bijective map $\phi : S \to S$.

Definition 2.2. The *symmetric group*, denoted by S_n, is defined as the set of permutations of the set $\{1, 2, ..., n\}$, with group law being composition.

Definition 2.3. The ring of symmetric polynomials in n variables is the subring of $\mathbb{Z}[x_1, ..., x_n]$ consisting of polynomials invariant under the standard symmetric group action: $\sigma \in S_n, P(x_1, ..., x_n) \in \mathbb{Z}[x_1, ..., x_n]$,

$$\sigma \cdot P(x_1, ..., x_n) = P(x_{\sigma^{-1}(1)}, ..., x_{\sigma^{-1}(n)}).$$

We denote this ring of symmetric polynomials by

$$\Lambda_n = \mathbb{Z}[x_1, ..., x_n]^{S_n}.$$

Definition 2.4. Denote the set of all homogeneous symmetric functions of degree n, with an arbitrary number of variables by Λ^n. We define the *ring of symmetric functions* to be the ring defined by

$$\Lambda = \Lambda^0 \oplus \Lambda^1 \oplus ...$$

Note, Λ^n is the nth graded component of Λ. Also we will typically work with finitely many variables, i.e. in Λ_n, although many of our results can be phrased more generally in Λ.

Definition 2.5. A *partition*

$$\lambda = (\lambda_1, \lambda_2, ..., \lambda_n, 0, 0, ...)$$

is a sequence of non-negative integers integers arranged in nonincreasing order where there is a finite number of nonzero terms. *Length* of a partition, denoted $l(\lambda)$, refers to the number of nonzero entries in a given partition.

Definition 2.6. A *partition of k* is a partition whose norm $|\lambda| = \lambda_1 + ... + \lambda_n$, is equal to k.

Definition 2.7. The *Young diagram* associated to a partition is a set of 1×1 boxes where the centers are $(i, j) \in \mathbb{Z}^2$ such that $1 \leq j \leq \lambda_i$.

We often write λ to mean a tableau or a partition since they are in bijective corespondence.

Definition 2.8. The *transpose* of a partition λ, denoted by λ', is defined as the partition corresponding to the transpose of the diagram of λ.

Definition 2.9. A *standard Young tableau* is a numbering of the boxes of a Young tableau by positive integers, which are nondecreasing along both rows and columns.

Definition 2.10. A *semistandard Young tableau* is a standard Young tableau in which the numbering of columns is strictly increasing. We denote by $SSYT(\lambda)$ the collection of all tableaux of shape λ which are semistandard.

Example 2.11. Examples of Young diagram, standard Young tableau, and semistandard Young tableau respectively, corresponding to the partition $\lambda = (5, 4, 2)$ are as follows:

1	2	2	5	6
1	4	4	5	
5	6			

1	2	2	5	6
2	4	4	8	
5	6			

Definition 2.12. Let λ and μ be partitions such that $\mu_i \leq \lambda_i$ for all i. Define the *semistandard Young tableau of skew shape* λ/μ to be an array $T = T_{ij}$ of positive integers such that $1 \leq i \leq l(\lambda)$, $\mu_i < j \leq \lambda_i$, where the numbering of rows is nondecreasing and columns increasing. We denote by $SSYT(\lambda/\mu)$ the collection of all semistandard tableaux of skew shape λ/μ. The *Young diagram of skew shape* λ/μ can be defined analogously and bijective correspondence means we can fit each tableau in such a diagram.

For the skew shape $\lambda/\mu = (5, 4, 2)/(3, 1)$ an example of a semistandard Young tableau is

			1	1
	2	2	3	
3	7			

.

Note that the shape λ/μ is equivalent to the shape $\lambda - \mu$ and they may be used interchangeably.

3. Bases of Symmetric Polynomials

In this section we describe four bases of the ring of symmetric polynomials and prove classical identities relating them. The first most natural basis we consider is the monomial symmetric.

Definition 3.1. For $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}_{\geq 0}$, let x^α denote the monomial

$$x^\alpha = x_1^{\alpha_1} ... x_n^{\alpha_n}.$$

Given partition λ of length at most n, we define the *monomial symmetric polynomial*

$$m_\lambda(x_1, .., x_n) = \sum_\alpha x^\alpha,$$

where the sum is over distinct permutations of λ.

Example 3.2. For partitions λ such that $|\lambda| = 3$. The associated functions $m_\lambda[x_1, x_2, x_3]$ are as follows

$$m_3 = x_1^3 + x_2^3 + x_3^3,$$

$$m_{2,1} = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_1 + x_3^2 x_2,$$

$$m_{1,1,1} = x_1, x_2, x_3.$$

The monomial symmetric polynomials m_λ are by construction a $\mathbb{Z}$-basis for Λ_n since λ runs through all permutations of length $l(\lambda) \leq n$.

Definition 3.3. For all nonnegative integers r, define the rth *elementary symmetric polynomial*

$$e_r = \sum_{i_1<i_2<...<i_r} x_{i_1}...x_{i_r} = m_{(1^r)}$$

and given a partition $\lambda = (\lambda_1, \lambda_2, ...)$ define

$$e_\lambda = e_{\lambda_1}e_{\lambda_2}...$$

We observe the e_r satisfy the following identity involving the generating function

$$\sum_{r\geq 0} e_r t^r = \prod_{i\geq 1}(1+x_i t).$$

Example 3.4. For partitions λ such that $|\lambda| = 3$. The associated functions $e_\lambda[x_1, x_2, x_3]$ are as follows:

$$e_3 = x_1^3 + x_2^3 + x_3^3,$$

$$e_{2,1} = x_1^3 + x_2^3 + x_3^3 + x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2.$$

$$e_{1,1,1} = x_1^3 + x_2^3 + x_3^3 + 3x_1^2x_2 + 3x_1^2x_3 + 3x_2^2x_1 + 3x_2^2x_3 + 3x_3^2x_1 + 3x_3^2x_2 + 6x_1x_2x_3,$$

In order to show that the elementary symmetric polynomials form a $\mathbb{Z}$-basis for the ring of symmetric polynomials, we must first show that they may be expressed in terms of monomial symmetric polynomials.

Theorem 3.5. *For all partitions λ, with conjugate λ', there exists an integer $a_{\mu\lambda} \geq 0$ such that*

$$e_{\lambda'} = m_\lambda + \sum_{\mu<\lambda} a_{\mu\lambda}m_\mu.$$

Proof. By definition $e_{\lambda'} = e_{\lambda'_1}e_{\lambda'_2}...$, which yields a polynomial whose monomials are of the form

$$(x_{i_1}x_{i_2}...)(x_{j_1}x_{j_2}...)... = x^\alpha.$$

Note that $i_1 < i_2 < ... < i_{\lambda'_1}, j_1 < j_2 < ... < j_{\lambda'_2}$ and so on. When these indices are arranged in a Young tableau of partition λ, the $i_1, i_2...$ occupy the first column, $j_1, j_2, ...$ occupy the second and so on. Since these indices are arranged in increasing order all indices $\leq r$ must appear in the top r rows of the diagram for all integers $r \geq 1$. Consequently we have that for all r $\alpha_1 + \alpha_2 + ... + \alpha_r \leq \lambda_1 + \lambda_2 + ... + \lambda_r$, so $\alpha < \lambda$ and by [1]

$$\alpha \leq \lambda \Rightarrow e_{\lambda'} = \sum_{\mu\leq\lambda} a_{\lambda\mu}m_\mu.$$

We also have that the term x^λ occurs exactly once in the polynomial since this monomial can only occur when each $\alpha_i = \lambda_i$, so its coefficient $a_{\lambda\lambda} = 1$.

□

Corollary 3.6. *For all partitions $\lambda = (\lambda_1, ..., \lambda_n)$ the e_λ form a $\mathbb{Z}$-basis for Λ_n.*

Proof. By Theorem 3.5, the change of basis matrix from m_λ to e_λ has integer coefficients and is an upper triangular matrix with 1's along the diagonal, thus it is invertible. □

Another basis for the ring of symmetric polynomials is the complete symmetric polynomials.

Definition 3.7. For $r \geq 0$, define the *complete symmetric polynomials* to be the sum of degree r monomials in terms of $x_1, x_2...$:

$$h_r = \sum_{|\lambda|=r} m_\lambda = \sum_{i_1 \leq ... \leq i_r} x_{i_1}...x_{i_r}.$$

Additionally for a partition $\lambda = (\lambda_1, \lambda_2, ...)$ define

$$h_\lambda = h_{\lambda_1} h_{\lambda_2}...$$

Example 3.8. For partitions λ such that $|\lambda| = 3$. The associated functions $h_\lambda[x_1, x_2, x_3]$ are as follows:

$$h_3 = x_1^3 + x_2^3 + x_3^3 + x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2 + x_1x_2x_3,$$

$$h_{2,1} = x_1^3 + x_2^3 + x_3^3 + x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2 + 3x_1x_2x_3,$$

$$h_{1,1,1} = x_1^3 + x_2^3 + x_3^3 + 3x_1^2x_2 + 3x_1^2x_3 + 3x_2^2x_1 + 3x_2^2x_3 + 3x_3^2x_1 + 3x_3^2x_2 + 6x_1x_2x_3.$$

The generating function for complete symmetric polynomials is given by

$$H(t) = \sum_{r \geq 0} h_r t^r = \prod_{i \geq 1} (1 - x_i t)^{-i}.$$

Remark 3.9. By multiplying the generating functions for elementary and complete symmetric polynomials,

$$H(t)E(-t) = \prod_{i \geq 1} \frac{(1 - x_i t)^i}{(1 - x_i t)^i} = 1.$$

This is equivalent to the sum

$$\sum_{r=0}^{n} (-1)^r e_r h_{n-r} = 0,$$

for all $n \geq 1$.

Corollary 3.10. *For all partitions $\lambda = (\lambda_1, ..., \lambda_n)$ the h_λ form a $\mathbb{Z}$-basis for Λ_n.*

Proof. Define a homomorphism $\omega : \Lambda_n \to \Lambda_n$, such that for all $r \geq 0$

$$\omega(e_r) = h_r.$$

The symmetric relation between the e_r and h_r which we derive from the sum in Remark 3.9, means that ω is an involution. Consequently ω is an automorphism of Λ_n, and

$$\Lambda_n = \mathbb{Z}[h_1, h_2, ...].$$

□

The last basis which we will define in this section is power sum polynomials.

Definition 3.11. For all $r \geq 1$, define the rth *power sum polynomial* to be the polynomial

$$p_r = \sum x_i^r = m_r.$$

For a partition $\lambda = (\lambda_1, \lambda_2, ...)$, define

$$p_\lambda = p_{\lambda_1} p_{\lambda_2}...$$

The generating function for power sum polynomials is given by

$$P(t) = \sum_{r\geq 1} p_r t^{r-1} = \sum_{i\geq 1}\sum_{r\geq 1} x_i^r t^{r-1} = \sum_{i\geq 1} \frac{x_i}{1-x_i t} = \sum_{i\geq 1} \frac{d}{dt}\log(1-x_i t)^{-1}.$$

Note that this implies that

$$P(-t) = \frac{d}{dt}\log\prod_{i\geq 1}(1+x_i t)^{-1} = \frac{d}{dt}\log E(t) = \frac{E'(t)}{E(t)}.$$

Using this relationship we can derive the formulation of Newton identities which expresses power sum polynomials in terms of elementary symmetric polynomials.

Theorem 3.12. *For all $n \geq 1$*

$$ne_n = \sum_{r=1}^{n}(-1)^{r-1}p_r e_{n-r}.$$

Proof. We have that

$$P(-t) = \frac{E'(t)}{E(t)} \Rightarrow P(-t)E(t) = E'(t).$$

Differentiating $E(t)$ yields the power series

$$E'(t) = \sum_{r\geq 1} r e_r t^{r-1}.$$

By then multiplying the power series $P(-t)E(t)$ we obtain an equivalent power series whose nth term is given by

$$\sum_{r=1}^{n}(-1)^{r-1}p_r e_{n-r}.$$

Since the power series $E'(t)$ and $P(-t)E(t)$ are equal, their coefficients are necessarily also equal, therefore the nth term in the sequence corresponding to each power series is thus equal to

$$ne_n = \sum_{r=1}^{n}(-1)^{r-1}p_r e_{n-r}.$$

□

Corollary 3.13. *For all partitions λ of length $l(\lambda) \leq n$, the p_λ form a $\mathbb{Q}$ basis for Λ_n.*

Proof. From our formulation of Newton's identities we have that

$$\mathbb{Q}[e_1, ..., e_n] = \mathbb{Q}[p_1, ..., p_n].$$

Since the e_r are algebraically independent over $\mathbb{Q}$ they are also algebraically independent over $\mathbb{Z}$ and

$$\Lambda_n = \mathbb{Q}[p_1, p_2, ...].$$

□

By Theorem 3.12, the change of basis matrix from p_λ to e_λ has rational coefficients. Hence the p_λ such that λ has length $l(\lambda) \leq n$ are not a $\mathbb{Z}$-basis of Λ_n.

4. Schur Functions

In this section we will define the Schur functions both combinatorially and algebraically, prove that these definitions are equivalent, and use the property of orthogonality to construct a sixth basis for the ring of symmetric polynomials. This basis will be the result of defining an inner product structure on the symmetric polynomials, such that this sixth basis is an orthonormal basis defined by the Schur functions, and all other orthonormal bases are a collection of Schur functions up to a signed permutation. The orthonormal basis defined by the Schur functions will then be used to prove Pieri's rule.

In order to define and study the Schur functions combinatorially, it will be important to lay out a few more definitions which allow us to more fully describe semistandard Young tableaux.

Definition 4.1. A semistandard Young tableau T has *type* α if there are $\alpha_i = \alpha_i(T)$ parts of it equal to i.

Definition 4.2. We define the *Kostka number* $K_{\lambda,\alpha}$ to be the number of semistandard Young tableaux with shape λ and type α.

With this foundation we will now begin by defining the Schur functions combinatorially.

Definition 4.3. We define the *skew Schur functions* $S_{\lambda/\mu}$ to be the formal power series

$$S_{\lambda/\mu} = \sum_{T \in SSYT(\lambda/\mu)} x^T = \sum_{T \in SSYT(\lambda/\mu)} x_1^{\alpha_1(T)} ... x_n^{\alpha_n(T)}.$$

If the partition $\mu = \emptyset$, then $\lambda/\mu = \lambda$, and S_λ is the *Schur function* of shape λ.

From this definition it is not immediately clear that the Schur function would be symmetric, however with a clever bijection we will show just that.

Theorem 4.4. *For any skew shape λ/μ, the associated skew Schur function $S_{\lambda/\mu}$ is symmetric.*

Proof. Given the permutation

$$\begin{pmatrix} 1 & . & . & . & n \\ \sigma(1) & . & . & . & \sigma(n) \end{pmatrix},$$

we can write the Schur function as a product of transpositions of the form $(i, i+1)$. Thus it suffices to show that the function is invariant when interchanging x_i and x_{i+1}. Let $|\lambda - \mu| = n$ and define $\alpha = (\alpha_1, \alpha_2, ...)$ to be a weak composition of n. That is to say that the terms of the sequence α sum to n where some term α_i may equal 0. Define α' such that

$$\alpha' = (\alpha_1, \alpha_2, ..., \alpha_{i-1}, \alpha_{i+1}, \alpha_i, \alpha_{i+2}, ...).$$

Let $\tau_{\lambda/\mu,\alpha}$ denote the set of semistandard Young tableaux of shape T and type α. In order to prove symmetry we will seek to establish a bijection,

$$\phi : \tau_{\lambda/\mu,\alpha} \to \tau_{\lambda/\mu,\alpha'}.$$

Choose a Young tableau $T \in \tau_{\lambda/\mu,\alpha}$. Consider the elements of T equal to i or $i+1$. Since the columns of semistandard Young tableaux are strictly increasing there are three possibilities for a given column. There could be no elements equal to i or $i+1$, there could be one element equal to i or $i+1$, or there could be one element equal to i and one element equal to $i+1$ in the column. We will restrict our attention solely to those columns which contain one element equal to i or $i+1$. A row containing consecutive such columns will result in a number r of i's followed by a number s of $i+1$'s where r and s depend on the specific row. Such a portion of T with $r=1$ and $s=5$ could for example look like this

$$\begin{array}{|c|c|c|c|c|c|c|} \hline i & i & i+1 & i+1 & i+1 & i+1 & i+1 \\ \hline i+1 & \multicolumn{6}{c}{} \\ \cline{1-1} \end{array}.$$

In each such portion of the tableau, exchange the r i's and s $i+1$'s with s i's and r $i+1$'s

$$\begin{array}{|c|c|c|c|c|c|c|} \hline i & i+1 & i & i & i & i & i \\ \hline i+1 & \multicolumn{6}{c}{} \\ \cline{1-1} \end{array}.$$

The resulting tableau $\phi(T)$ is then in the set $\tau_{\lambda/\mu,\alpha'}$, so the desired bijection is fulfilled.

□

Example 4.5. For partitions λ such that $|\lambda| = 3$. The associated functions $S_\lambda[x_1, x_2, x_3]$ are as follows:

$$S_3 = x_1^3 + x_2^3 + x_3^3 + x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2 + x_1x_2x_3,$$
$$S_{2,1} = x_1^2x_2 + x_2^2x_1 + x_1^2x_3 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2 + 2x_1x_2x_3,$$
$$S_{1,1,1} = x_1 + x_2 + x_3.$$

Alternatively we may define the Schur function algebraically, however before doing so we must lay some groundwork first. We will begin by providing a definition that will be necessary to define the Schur functions algebraically.

Definition 4.6. The *Vandermonde determinant* is the determinant of the *Vandermonde matrix* denoted V,

$$V = \begin{pmatrix} x_1^{n-1} & x_1^{n-2} & . & . & . & x_1 & 1 \\ x_2^{n-1} & x_2^{n-2} & . & . & . & x_2 & 1 \\ . & & & & & & \\ . & & & & & & \\ . & & & & & & \\ x_n^{n-1} & x_n^{n-2} & . & . & . & x_n & 1 \end{pmatrix}$$

Theorem 4.7. *The Vandermonde determinant is equal to*

$$\det\left(x_i^{n-j}\right) = \prod_{i,j}(x_i - x_j).$$

Proof. The determinant of the Vandermonde matrix is a polynomial of degree $\binom{n}{2}$ since $1 + 2 + ... + n - 1 = \binom{n}{2}$. Additionally using Cramer's rule we find that for $i \neq j$, when $x_i = x_j$ the determinant is zero. This implies that the polynomial is of the form

$$\det\left(x_i^{n-j}\right) = C \prod_{1 \leq i,j \leq n} (x_i - x_j)$$

where C is some constant. This constant is equal to one since the product of diagonal entries of the Vandermonde matrix has coefficient 1. □

Next, suppose a finite set of variables $x_1, ..., x_n$, let x^α be a monomial and a_α the polynomial obtained by antisymmetrizing x^α. As a sum this operation is represented by

$$a_\alpha = \sum_{w \in S_n} \epsilon(w) w(x^\alpha)$$

with $\epsilon(w)$ representing the sign of the permutation w. The polynomial a_α is skew symmetric for any permutation in the symmetric group

$$w(a_\alpha) = \epsilon(w) a_\alpha.$$

Hence a_α is zero unless the α_i are pairwise distinct. Let λ be a partition of length $l(\lambda) \leq n$. and let $\delta = (0, ..., n-2, n-1)$. Supposing that $\alpha_n \geq ... > \alpha_2 > \alpha_1 > 0$. Set $\alpha = \lambda + \delta$ such that

$$a_\alpha = a_{\lambda+\delta} = \sum_{w \in S_n} \epsilon(w) w(x^{\lambda+\delta}) = \det\left(x_i^{\lambda_j+n-j}\right)_{1 \leq i,j \leq n}.$$

For each $x_i - x_j, a_{\lambda+\delta}$ is divisible by their product which by Theorem 4.7 is equal to the Vandermonde determinant

$$\prod_{1 \leq i,j \leq n} (x_i - x_j) = \det\left(x_i^{n-j}\right) = a_\delta.$$

$a_{\lambda+\delta}$ is then divisible in $\mathbb{Z}[x_1, ..., x_n]$, and we may define the Schur function algebraically.

Definition 4.8. The Schur function of a partition λ, s_λ, is the quotient

$$s_\lambda = \frac{a_{\lambda+\delta}}{a_\delta}.$$

By construction we can see from this definition that s_λ is symmetric since both sides of the quotient are antisymmetric. Additionally with this definition we may provide a short proof that the Schur functions form a basis for Λ_n:

Theorem 4.9. *For all partitions $\lambda = (\lambda_1, ..., \lambda_n)$, the s_λ form a $\mathbb{Z}$-basis for Λ_n.*

Proof. Let A_n be the $\mathbb{Z}$-module of skew symmetric polynomials in n variables $x_1, ..., x_n$. When λ runs through all partitions of length $l(\lambda) \leq n$ the polynomials $a_{\lambda+\delta}$ form a basis for this module. Define an isomorphism from Λ_n to A_n defined by multiplicaton by a_δ. Division by a_δ then provides the inverse map and the Schur functions $s_\lambda[x_1, ..., x_n]$ corresponding to partitions of length $l(\lambda) \leq n$, thus form a $\mathbb{Z}$-basis for Λ_n. □

These definitions of the Schur functions at first would appear distinct, however we will prove via a formula on skew Schur functions that the definitions are indeed equivalent. In order to define the skew Schur function algebraically however it will prove necessary to derive the orthonormal basis for Λ_n using s_λ. To do this, define an inner product structure on Λ_n over $\mathbb{Z}$ such that if λ and μ are partitions of n:

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}.$$

Theorem 4.10. *The Schur functions s_λ form an orthonormal basis for Λ_n such that*

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}.$$

Proof. From [1] for two sequences of variables which may or may not be infinite $x = (x_1, x_2, ...), y = (y_1, y_2, ...)$, the series expansion of the product $\prod_{i,j}(1-x_iy_j)^{-1}$ has the following identities

$$\sum_\lambda h_\lambda(x)m_\lambda(y) = \sum_\lambda m_\lambda(x)h_\lambda(y) = \sum_\lambda s_\lambda(x)s_\lambda(y).$$

Suppose that u_λ and v_λ are $\mathbb{Z}$-bases of Λ_n and ρ, σ are partitions of n such that

$$u_\lambda = \sum_\rho a_{\lambda\rho}h_\rho, \quad v_\mu = \sum_\sigma b_{\mu\sigma}m_\sigma.$$

We immediately have that

$$\langle u_\lambda, v_\mu\rangle = \sum_\rho a_{\lambda\rho}b_{\mu\rho} = \delta_{\lambda\mu}.$$

Furthermore, by our identities we derive the relation

$$\sum_\lambda u_\lambda(x)v_\lambda(y) = \sum_\rho h_\rho(x)m_\rho(y) = \prod_{i,j}(1 - x_iy_j)^{-1}.$$

By definition of our inner product, this relation is equivalent to

$$\sum_\lambda a_{\lambda\rho}b_{\lambda\sigma} = \delta_{\rho\sigma}.$$

Thus for all $\mathbb{Z}$-bases of Λ_n,

$$\sum_\lambda u_\lambda(x)v_\lambda(y) = \prod_{i,j}(1 - x_iy_j)^{-1} \Leftrightarrow \langle u_\lambda, v_\mu\rangle = \delta_{\lambda\mu}.$$

We then conclude by our identities that

$$\langle s_\lambda, s_\mu\rangle = \delta_{\lambda\mu}.$$

□

From this, we see that any other orthonormal basis is a collection of Schur functions up to a signed permutation. This is because any other orthonormal basis must be obtained from s_λ by transforming it by an orthogonal integer matrix, which must necessarily be a signed permutation matrix.

Using this inner product structure we may now define the skew Schur functions in algebraic terms, and prove that our algebraic and combinatorial definitions of Schur functions are equivalent.

Definition 4.11. Let λ, μ be partitions. Define the *skew Schur functions* by the relation

$$\langle s_{\lambda/\mu}, s_v\rangle = \langle s_\lambda, s_\mu s_v\rangle.$$

Theorem 4.12. *For all partitions λ, μ such that $\mu_i \le \lambda_i$ for all i, we have that*

$$S_{\lambda/\mu} = s_{\lambda/\mu}.$$

Proof. Define a sequence of partitions $(\nu) = (\nu^{(0)}, ..., \nu^{(n)})$ where $\nu^{(0)} = \mu$ and $\nu^{(n)} = \lambda$, and $\nu^{(0)} \subset \nu^{(1)} \subset ... \subset \nu^{(n)}$. We have by [1] that

$$s_{\lambda/\mu}(x_1, ..., x_n) = \sum_{(\nu)} \prod_{i=1}^{n} s_{\nu^{(i)}/\nu^{(i-1)}} x_i,$$

and

$$s_{\lambda/\mu} = \det\Big(e_{\lambda'_i-\mu'_j-i+j}\Big)_{1\leq i,j\leq m},$$

where the length of λ' is $\leq m$ and v is any partition. Suppose that $\lambda'_r - \mu'_r > n$ for some $r \geq 1$. This would mean that $e_{n+1} = e_{n+2} = ... = 0$ which would create a 0 on the main diagonal of the associated matrix, causing the determinant and hence $s_{\lambda/\mu}$ to be equal to 0. In the case of a single variable x this implies that if $\lambda - \mu$ is a horizontal r-strip, then

$$s_{\lambda/\mu} = 0.$$

In all other cases we have

$$s_{\lambda/\mu} = x^{|\lambda-\mu|}.$$

Consequently each of the products in our sum are monomials of the form $x_1^{\alpha_i}...x_n^{\alpha_n}$, where each exponent $\alpha_i = \left|\nu^{(i)} - \nu^{(i-1)}\right|$. From this we can express the skew Schur functions $s_{\lambda/\mu}(x_1, ..., x_n)$ as a sum of monomials x^α with each monomial corresponding to a distinct tableau T of shape λ/μ,

$$s_{\lambda/\mu} = S_{\lambda/\mu} = \sum_{T\in SSYT(\lambda/\mu)} x^T$$

□

Now that we have defined our orthonormal basis on the symmetric polynomials, and have proven the equality of the algebraically and combinatorially defined Schur functions, we have the necessary constructions to prove Pieri's rule, stated as follows:

Theorem 4.13.

$$S_\mu h_r = \sum_\lambda S_\lambda$$

where the sum is over all λ such that $\lambda - \mu$ is a horizontal r strip.

Proof. We have by definition that

$$S_{\lambda/\mu} = \sum_{T\in SSYT(T)} x^T$$

where the sum consists of all tableaux T with shape λ/μ. We will alternatively state this using the Kostka number such that

$$S_{\lambda/\mu} = \sum_\nu K_{\lambda-\mu,\nu} m_\nu.$$

This gives us that $K_{\lambda-\mu,\nu} = \langle S_{\lambda/\mu}, h_\nu\rangle = \langle S_\lambda, S_\mu h_\nu\rangle$, which directly implies that

$$S_\mu h_\nu = \sum_\lambda K_{\lambda-\mu,\nu} S_\lambda.$$

Supposing that $\nu = (r)$ is a partition with only one non zero part then if $\lambda - \mu$ is a horizontal r-strip $K_{\lambda-\mu,(r)} = 1$. If $\lambda - \mu$ is not a horizontal r strip then $K_{\lambda-\mu,(r)} = 0$. Consequently we obtain

$$S_\mu h_r = \sum_\lambda S_\lambda.$$

□

5. Acknowledgements

I would like to thank Peter May for organizing the REU program, as well as my mentor, Nikolay Grantcharov for assisting me with learning the topic and preparing my paper.

References

[1] Ian Grant Macdonald. *Symmetric functions and Hall polynomials*. Oxford university press, 1998.

[2] Richard P Stanley. "Enumerative Combinatorics Volume 2 second edition". In: *Cambridge studies in advanced mathematics* (2011).